The Nash embedding theorem

Khang Manh Huynh

March 13, 2018

Abstract

This is an attempt to present an elementary exposition of the Nash embedding theorem for the graduate student who at least knows what a vector field is. We mainly rely on [Tao16] and [How99].

1 Preliminary definitions

Definition 1. A **Riemannian manifold** (M, g) is a smooth manifold M equipped with a smooth **Riemannian metric** g. In other words, for any vector fields X, Y (i.e. $X, Y \in \mathfrak{X}M$), $g(X, Y)$ is a smooth function on M and for any p in M, there is a positive-definite inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ such that $g(X, Y)(p) = g_p(X_p, Y_p)$. As a consequence, for any $f \in C^\infty(M) : g(fX, Y) = fg(X, Y)$.

An **isometric embedding** ϕ from (M_1, g_1) to (M_2, g_2) (both Riemannian manifolds) is a smooth embedding $\phi : M_1 \to M_2$ that preserves the metric, i.e.

$$\forall X, Y \in \mathfrak{X}M_1, \forall p \in M_1 : (g_1)_p (X_p, Y_p) = (g_2)_{\phi(p)} (d\phi \cdot X_p, d\phi \cdot Y_p)$$

We write $e = \langle \cdot, \cdot \rangle$ for the Euclidean metric on $\mathbb{R}^n$, formed by the usual Euclidean dot product. Unless indicated otherwise, $\mathbb{R}^n$ is always equipped with e and we write $\mathbb{R}^n$ for $(\mathbb{R}^n, e)$.

Throughout this note, everything we work with is assumed to be smooth unless indicated otherwise. Every metric is a Riemannian metric unless indicated otherwise.

Now we can state the main theorem:

Theorem 2 (Nash embedding)**.** *Any compact Riemannian manifold (M, g) without boundary can be isometrically embedded into $\mathbb{R}^n$ for some n.*

2 Sketch of proof

We will first give a sketch of the proof, leaving the technical details as lemmas to be proven later.

The first tool we require is Whitney embedding. For any $u \in C^\infty(M, \mathbb{R}^n)$ and $v \in C^\infty(M, \mathbb{R}^k)$, we can define $u \oplus v : p \mapsto (u(p), v(p))$. Glueing chart functions (properly cut off) leads to Whitney.

Lemma 3 (Baby Whitney)**.** *Any compact smooth manifold can be smoothly embedded into $\mathbb{R}^n$ for some n.*

Because of this, and because M is compact, we can embed M into a torus, which will simplify our calculations greatly.

Lemma 4 (Torus embedding)**.** *WLOG, we can assume $M = \mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$, i.e. the m-dimensional torus, equipped with an arbitrary Riemannian metric g.*

Then, we introduce some definitions:

Definition 5. Let Sym denote the set of **symmetric tensors** on $\mathbb{T}^m$. In other words, $h \in Sym$ when $h = (h_{\alpha\beta})_{1\leq\alpha,\beta\leq m}$ is a smooth function from $\mathbb{T}^m$ into $Sym_m(\mathbb{R})$ (the set of symmetric $m \times m$ matrices). Any metric g of $\mathbb{T}^m$ can be considered a symmetric tensor and $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$ where (∂_α) are the standard coordinate vector fields on $\mathbb{T}^m$. Conversely, if $h \in Sym$ and $h > 0$ (i.e. $h(p)$ is a positive matrix $\forall p$), h is a metric.

Let $Map = \cup_{n\in\mathbb{N}} C^\infty(\mathbb{T}^m, \mathbb{R}^n)$ be the set of all maps from $\mathbb{T}^m$ into some Euclidean space $\mathbb{R}^n$. We define the function $Q : Map \to Sym$ such that for any $u \in Map$:

$$Q(u)_{\alpha\beta} = \langle \partial_\alpha u, \partial_\beta u \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual Euclidean dot product. We note that $Q(u) \geq 0$.

A metric g on $\mathbb{T}^m$ is called **good** when we can find $u \in Map$ such that $Q(u) = g$ (and such u would have to be immersions, though not necessarily embeddings). We write $Good$ for the set of good metrics. Because $Q(u \oplus v) = Q(u) + Q(v)$, $Good$ is closed under addition.

Let $Emb \subset Map$ be the set of maps that are embeddings. Nash's theorem says every metric is in $Q(Emb)$. However, *if every metric is good, Nash's theorem is proven*. Indeed, let g be a metric and $W \in Emb$ be a Whitney embedding. By rescaling W, WLOG $Q(W) < g$. Then $g = Q(W) + g'$ where g' is a metric, therefore good and $g' = Q(v)$ for some $v \in Map$. Then $W \oplus v \in Emb$ and $Q(W \oplus v) = g$. We're done.

Before we can prove every metric is good, we can prove it is approximately good by finding a very clever symmetric tensor.

Lemma 6 (Approximation)**.** *For any metric g, there is $h \in Sym$ such that $g + \varepsilon^2 h \in Good$ $\forall \varepsilon > 0$.*

This reduces Nash's theorem into a local perturbation problem, where Nash made his fundamental contribution. We need another definition:

Definition 7. (Gromov-Rohklin) An injective smooth map $\phi : \mathbb{T}^m \to \mathbb{R}^n$ is called **free** when $\{\partial_\alpha \phi(p)\}_\alpha \cup \{\partial_\alpha \partial_\beta \phi(p)\}_{\alpha\leq\beta}$ is linearly independent for any $p \in \mathbb{T}^m$. Note that injective immersions on a compact manifold are automatically embeddings. Let $Free \subset Emb$ be the set of embeddings that are free.

We can also define free maps on open submanifolds of $\mathbb{T}^m$ and $\mathbb{R}^m$.

Now we can state the perturbation lemma. Recall that $h \xrightarrow{C^\infty} 0$ means $h \xrightarrow{C^k} 0$ $\forall k$.

Lemma 8 (Perturbation)**.** *Let $u \in Free$. Then for any $h \in Sym$ small enough (in the C^∞ topology), $Q(u) + h \in Good$.*

We use the flexibility of free maps make the symmetric tensor good. Now let us prove Nash's theorem.

Proof. Let g be any metric on $\mathbb{T}^m$. We first find a free map.

Because $\mathbb{T}^m$ is just like $\mathbb{R}^m$ locally, we can find a finite open cover (U_i) of $\mathbb{T}^m$ with cutoffs $\psi_i \in C_c^\infty(U_i), 0 \leq \psi_i \leq 1$ such that $V_i = \text{int}\{\psi_i = 1\}$ also form an open cover of $\mathbb{T}^m$, and there are diffeomorphisms $\phi_i : U_i \to B_{\mathbb{R}^m}(0,2)$ such that $\phi_i(V_i) = B_{\mathbb{R}^m}(0,1)$ and $d\phi_i \cdot \partial_\alpha = \partial_\alpha$ $\forall \alpha$.

Obviously ψ_i and $\psi_i\phi_i$ can be defined on $\mathbb{T}^m$ by zero extension. Then we define $\Phi = \bigoplus_i (\psi_i \oplus \psi_i\phi_i)$. So Φ is an injective immersion, therefore embedding from $\mathbb{T}^m$ into $\mathbb{R}^k$ for some k.

Then define the **Veronese embedding** $\iota_k : \mathbb{R}^k \to \mathbb{R}^{k+\frac{k(k+1)}{2}}$, $(x^\alpha)_{\alpha=1}^k \mapsto (x^\alpha, x^\alpha x^\beta)_{1\leq\alpha\leq\beta\leq k}$ and let $u = \iota_k \circ \Phi$. By looking at $u|_{V_i}$, it's easy to see u is a free embedding.

By rescaling u, and by the compactness of $\mathbb{T}^m$, we can WLOG assume $Q(u)(p) < g_p$ $\forall p$ (as positive matrices). Then $g = g' + Q(u)$ where g' is a metric.

By approximation, there is $h \in Sym$ such that $g' + \varepsilon^2 h \in Good$ $\forall \varepsilon > 0$. By perturbation, for ε small enough, $Q(u) - \varepsilon^2 h \in Good$. So $g = (g' + \varepsilon^2 h) + (Q(u) - \varepsilon^2 h) \in Good$.

So every metric is good and Nash's theorem is proven. □

So we now only need to prove our lemmas.

3 Embedding into the torus

Proof of baby Whitney. Let M be any compact manifold. We can find a finite open cover (U_i) of M with cutoffs $\psi_i \in C_c^\infty(U_i), 0 \le \psi_i \le 1$ such that $V_i = \operatorname{int}\{\psi_i = 1\}$ also form an open cover of M, and there are diffeomorphisms $\phi_i : U_i \to B_{\mathbb{R}^m}(0,2)$ such that $\phi_i(V_i) = B_{\mathbb{R}^m}(0,1)$. Obviously ψ_i and $\psi_i\phi_i$ can be defined on M by zero extension. Then we define $W = \bigoplus_i (\psi_i \oplus \psi_i\phi_i)$. So W is an injective immersion, therefore embedding from M into $\mathbb{R}^k$ for some k. □

Proof of torus embedding. Let (M, g) be any compact Riemannian manifold. By Whitney, there is an embedding $W : M \to \mathbb{R}^m$ for some m.

Because M is compact, by translation and rescaling, WLOG assume $W(M) \subset (0,1)^m$. So WLOG, $M \subset \mathbb{T}^m$. We want to extend g from M to $\mathbb{T}^m$.

We first do this locally. Let $p \in M$. As M is a regular submanifold of $\mathbb{T}^m$, there is a neighborhood $\overline{U}$ containing p, open in $\mathbb{T}^m$ with a diffeomorphism $\overline{\Phi} : \overline{U} \to B_{\mathbb{R}^m}(0,1)$ such that for $U = \overline{U} \cap M$, $\overline{\Phi}(U) = B_{\mathbb{R}^l}(0,1) \times \{0\}$ where $l = \dim M$. Parametrize $\mathbb{R}^m = \{(y,z) : y \in \mathbb{R}^l, z \in \mathbb{R}^{m-l}\}$, then WLOG, via the diffeomorphism, $\overline{U} = \{(y,z) : |y|^2 + |z|^2 < 1\}$ and $U = \{(y,0) : |y|^2 < 1\}$ where U is equipped with a metric g. Then simply define $\overline{g}$ on $\overline{U}$: $\overline{g}_{(y,z)}((a_1,b_1),(a_2,b_2)) = g_y(a_1,b_1) + \langle b_1, b_2\rangle$ where $\langle \cdot,\cdot\rangle$ is the usual Euclidean dot product. Then going back via the diffeomorphism we get $\overline{g}$ on the original $\overline{U}$.

Finally we use partition of unity to extend g globally. We can find a finite collection of Riemannian manifolds $(\overline{U}_i, \overline{g}_i)_{i=1}^N$ such that $\overline{U}_i$ are open in $\mathbb{T}^m$ and cover M, while $\overline{g}_i$ are extensions of $g|_{M\cap\overline{U}_i}$. Let $\overline{U}_0 = \mathbb{T}^m \backslash M$. Then $\overline{U}_0$ is open and $(\overline{U}_0, \overline{g}_0)$ is a Riemannian manifold where $\overline{g}_0$ is the usual Euclidean metric. Then we can find a partition of unity $(\psi_i)_{i=0}^N$ subordinate to $\left(\overline{U}_i\right)_{i=0}^N$ and we define $\overline{g} = \sum_{i=0}^N \psi_i \overline{g}_i$.

So we can find an isometric embedding $(M, g) \to (\mathbb{T}^m, \overline{g})$. So we just need to prove Nash's theorem for the torus. □

4 Approximating the metric

Firstly, by the Gram-Schmidt process, for any p, we can find an orthonormal basis for the inner product space $\left(T_p\mathbb{T}^m, g_p\right)$. Identify $T_p\mathbb{T}^m$ with $\mathbb{R}^m$. Let $v = (v^\alpha) \in \mathbb{R}^m$ be any vector, then define the rank-1 tensor $v \otimes v = vv^T = (v^\alpha v^\beta)_{\alpha\beta} \in Sym_m(\mathbb{R})$. So there are vectors $(v_i)_{i=1}^m$ such that $g_p = \sum_i v_i \otimes v_i$.

We want to do this in a stable way around p.

Lemma 9 (Stable decomposition)**.** *Let g be a metric on $\mathbb{T}^m$ and $p \in \mathbb{T}^m$. Then on a neighborhood U of p, we can find a finite collection of vectors v_i in $\mathbb{R}^m$ and smooth functions a_i such that $a_i > 0$ and $g = \sum_i a_i (v_i \otimes v_i)$*

Proof. The principle is simple, and best illustrated by a detailed example in the case when $m = 3$. Because we are working locally, WLOG $p = 0 \in U \subset \mathbb{R}^m$. By a change of basis, WLOG $g_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

By shrinking U if needed, WLOG $g|_U = \begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$ where $|x|, |y|, |z|, |a-1|, |b-1|, |c-1| < \frac{\varepsilon}{2}$ and $\varepsilon \in (0, \frac{1}{100})$ is a small positive constant we can choose.

Let e_1, e_2, e_3 be the basis vectors of $\mathbb{R}^3$. Define $w_i = e_i \otimes e_i$ and $w_{ij}^+ = (e_i + e_j) \otimes (e_i + e_j)$, $w_{ij}^- = (e_i - e_j) \otimes (e_i - e_j)$. These are all the rank-1 tensors we need.

For example, $v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $w_{13}^- = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. Then because $\frac{\varepsilon}{2} > x, y, z > -\frac{\varepsilon}{2}$ on U and $1 \gg \varepsilon$, there are unique smooth functions $a_1, a_2, a_3, a_{12}, a_{13}, a_{23}$ positive on U such that

$$g = \varepsilon \sum_{i<j} w_{ij}^- + \sum_{i<j} a_{ij} w_{ij}^+ + \sum_i a_i w_i \quad \text{on } U$$

□

Now we can prove the approximation lemma.

Proof of approximation. For any metric g, there is $h \in Sym$ such that $g + \varepsilon^2 h \in Good\ \forall \varepsilon > 0$.

Consider the previous lemma. As $a_i > 0$ on U, we can rewrite $a_i = b_i^2$ where b_i is smooth on U. As $\mathbb{T}^m$ is compact, we can find a finite open cover (U^j) such that the above lemma applies for all U^j and $g = \sum_i \left(b_i^j\right)^2 \left(v_i^j \otimes v_i^j\right)$ on U^j.

For any $u \in Map$, write Du for $(\partial_\alpha u)_\alpha$. Because v_i^j are constant vectors, and by shrinking U^j if needed, we can find functions $u_i^j \in C^\infty(U^j)$ such that $Du^j = v_i^j$ on U^j. Then $g|_{U^j} = \sum_i \left(b_i^j\right)^2 Q(u_i^j)$. There exists a partition of unity (ψ^j) subordinate to (U^j). Define $\widetilde{\psi}^j = \frac{\psi^j}{\sqrt{\sum_l (\psi^l)^2}}$. Then $((\widetilde{\psi}^j)^2)$ is another partition of unity subordinate to (U^j) and by zero extension, $(\widetilde{\psi}^j)^2 g = \sum_i \left(\widetilde{\psi}^j b_i^j\right)^2 Q(u_i^j)$ on $\mathbb{T}^m$. Therefore $g = \sum_{i,j} \left(\widetilde{\psi}^j b_i^j\right)^2 Q(u_i^j)$ on $\mathbb{T}^m$.

To simplify notation, WLOG, we can reindex and rewrite this as $g = \sum_j (\eta_j)^2 Q(u_j)$ where $\eta_j \in C_c^\infty(\mathbb{T}^m)$. This is called the **Kuratowski-Weyl-Nash decomposition**.

Then we introduce the very clever **spiral map**

$$u_j^\varepsilon = \varepsilon \eta_j \left(\cos\left(\frac{u_j}{\varepsilon}\right), \sin\left(\frac{u_j}{\varepsilon}\right)\right)$$

Then $Q(u_j^\varepsilon) = (\eta_j)^2 Q(u_j) + \varepsilon^2 Q(\eta_j)$ and $g = \sum_j \left(Q(u_j^\varepsilon) - \varepsilon^2 Q(\eta_j)\right) = Q\left(\bigoplus_j u_j^\varepsilon\right) - \varepsilon^2 Q\left(\bigoplus_j \eta_j\right)$.

Let $h = Q\left(\bigoplus_j \eta_j\right) \geq 0$. Then $g + \varepsilon^2 h \in Good\ \forall \varepsilon > 0$. □

5 Perturbation

Nash's original method to prove the perturbation lemma is now called the *Nash-Moser iteration scheme*, which has played an important role in nonlinear PDE. Here, we present Günther's approach which uses elliptic operators to reduce the lemma into a usual contraction mapping problem.

Although we've tried to make the note as self-contained as possible, here we will need to borrow one important tool from the theory of elliptic operators. See, for instance, chapter 6 of [GT01].

Theorem 10 (Schauder estimates)**.** *For any $k \geq 0$ and $a \in (0,1)$, the linear operator $\Delta - 1 : C^{k+2,a}(\mathbb{T}^m) \to C^{k,a}(\mathbb{T}^m)$ is a homeomorphism. In other words, we can define the continuous linear operator $(\Delta-1)^{-1}$ (also called the Bessel potential) from $C^{k,a}(\mathbb{T}^m)$ to $C^{k+2,a}(\mathbb{T}^m)$. We say $(\Delta-1)^{-1}$ is of order -2. Note that because $(\Delta-1)^{-1}$ commutes with derivatives, the constants for the bounds do not depend on k.*

***Proof of perturbation**.* Let $u \in Free$. Then for any $h \in Sym$ small enough (in the C^∞ topology), $Q(u) + h \in Good$.

For the whole problem, $u \in C^\infty(\mathbb{T}^m, \mathbb{R}^n)$ is fixed. Then we want to find $v \in C^\infty(\mathbb{T}^m, \mathbb{R}^n)$ such that $Q(u+v) = Q(u) + h$.

To do more algebra with Q, we define its bilinear symmetric form. For any $k \geq 1$, define $B : C^\infty(\mathbb{T}^m, \mathbb{R}^k) \times C^\infty(\mathbb{T}^m, \mathbb{R}^k) \to Sym$ such that for any $v_1, v_2 \in C^\infty(\mathbb{T}^m, \mathbb{R}^k)$:

$$B(v_1, v_2)_{\alpha\beta} = \left\langle \partial_\alpha v_1, \partial_\beta v_2 \right\rangle + \left\langle \partial_\beta v_1, \partial_\alpha v_2 \right\rangle = \partial_\beta \left\langle \partial_\alpha v_1, v_2 \right\rangle + \partial_\alpha \left\langle \partial_\beta v_1, v_2 \right\rangle - 2 \left\langle \partial_{\alpha\beta} v_1, v_2 \right\rangle$$

Then $B(v,v) = 2Q(v,v)$ and $Q(u+v) - Q(u) = Q(v) + B(u,v)$. So we define $L : C^\infty(\mathbb{T}^m, \mathbb{R}^n) \to Sym, v \mapsto B(u,v)$. We want $Q(v) + L(v) = h$.

It turns out that we can get rid of L by exploring its algebraic properties. The crucial fact is that u is free, so we can freely control quantities like $\langle \partial_\alpha u, v\rangle$ and $\left\langle \partial_{\alpha\beta} u, v\right\rangle$ by using the duals of $\partial_\alpha u$ and $\partial_{\alpha\beta} u$. More precisely:

Lemma 11. *There are smooth functions $\{w_\alpha\}_\alpha, \{w_{\alpha\beta}\}_{\alpha\leq\beta} : \mathbb{T}^m \to \mathbb{R}^n$ such that when $\alpha \leq \beta, \alpha' \leq \beta'$:*

$$\begin{cases} \langle w_\alpha, \partial_{\alpha'} u \rangle & = \delta_{\alpha\alpha'} \\ \left\langle w_{\alpha\beta}, \partial_{\alpha'\beta'} u \right\rangle & = \delta_{\alpha\alpha'}\delta_{\beta\beta'} \\ \left\langle w_\alpha, \partial_{\alpha'\beta'} u \right\rangle & = \langle w_{\alpha\beta}, \partial_{\alpha'} u \rangle = 0 \end{cases}$$

Proof. Consider the linearly independent set $\{\partial_\alpha u(p)\}_\alpha \cup \{\partial_{\alpha\beta} u(p)\}_{\alpha\leq\beta}$. We can give it an order so that, say, $\partial_1 u(p)$ comes last. Then apply the Gram-Schmidt process to the ordered sequence, and we will have an orthogonal sequence of vectors, in which the last vector is, say, $\widetilde{w}_1(p)$. Then $\langle \widetilde{w}_1(p), \partial_1 u(p)\rangle \neq 0$, $\langle \widetilde{w}_1(p), \partial_\alpha u(p)\rangle = 0 \ \forall \alpha \neq 1$ and $\langle \widetilde{w}_1(p), \partial_{\alpha\beta} u(p)\rangle = 0 \ \forall \alpha, \beta$. As the process is stable, $\widetilde{w}_1$ is smooth. Define $w_1 = \frac{\widetilde{w}_1}{\langle \widetilde{w}_1, \partial_1 u\rangle}$. Then $\langle w_1, \partial_{\alpha'} u\rangle = \delta_{1\alpha'}$. Do similar things to get all of $\{w_\alpha\}_\alpha, \{w_{\alpha\beta}\}_{\alpha\leq\beta}$. ☐

Note that these functions depend on u, which is fixed for this problem. Now we can explore L.

Lemma 12. *There is an operator $M : Sym \to C^\infty(\mathbb{T}^m, \mathbb{R}^n)$ such that $LM = Id_{Sym}$. Moreover, M is of order zero, i.e. $\forall k \geq 1, \forall f \in Sym : ||Mf||_{C^{k,a}} \leq C||f||_{C^{k,a}}$ for some constant $C = C(u, m, a, k)$.*

Proof. Let $f \in Sym$. We want $f = LMf = B(u, Mf)$. This means

$$f_{\alpha\beta} = B(u, Mf)_{\alpha\beta} = \partial_\beta \langle \partial_\alpha u, Mf\rangle + \partial_\alpha \left\langle \partial_\beta u, Mf \right\rangle - 2\left\langle \partial_{\alpha\beta} u, Mf\right\rangle$$

Then define $Mf = \sum_{\alpha\leq\beta} \frac{-1}{2} f_{\alpha\beta} w_{\alpha\beta}$. Then $\langle \partial_\alpha u, Mf\rangle = 0$ and for any $\alpha \leq \beta$, $-2\left\langle \partial_{\alpha\beta} u, Mf\right\rangle = \left\langle \partial_{\alpha\beta} u, f_{\alpha\beta} w_{\alpha\beta}\right\rangle = f_{\alpha\beta}$. Obviously M is of order zero. ☐

The next step is Günther's contribution. The use of smoothing elliptic operators helps make the operator zero-order (preventing the *loss of derivatives* phenomenon).

Lemma 13 (Günther's lemma)**.** *There is a zero-order operator $Q_0 : C^\infty(\mathbb{T}^m, \mathbb{R}^n) \to C^\infty(\mathbb{T}^m, \mathbb{R}^n)$ such that $Q = LQ_0$ on $C^\infty(\mathbb{T}^m, \mathbb{R}^n)$. Moreover, it has a bilinear symmetric form B_0 such that $B_0(v,v) = 2Q_0(v)$ and*

$\forall v_1, v_2 \in C^\infty(\mathbb{T}^m, \mathbb{R}^n), \forall k \geq 2, \forall a \in (0,1)$:

$$||B_0(v_1, v_2)||_{C^{k,a}} \leq C_1 \left(||v_1||_{C^{k,a}}||v_2||_{C^{2,a}} + ||v_2||_{C^{k,a}}||v_1||_{C^{2,a}}\right) + C_2 \left(||v_1||_{C^{k-1,a}}||v_2||_{C^{k-1,a}}\right)$$

for some constants $C_1 = C_1(u, m, a)$, $C_2 = C_2(u, m, a, k)$. The fact that C_1 does not depend on k is crucial.

Proof. We want $Q(v) = LQ_0(v) = B(u, Q_0(v))$ $\forall v \in C^\infty(\mathbb{T}^m, \mathbb{R}^n)$. Then observe that

$$\Delta \langle \partial_\alpha v, \partial_\beta v \rangle = \langle \partial_\alpha \Delta v, \partial_\beta v \rangle + \langle \partial_\alpha v, \partial_\beta \Delta v \rangle + 2 \langle \partial_\alpha Dv, \partial_\beta Dv \rangle = B(\Delta v, v)_{\alpha\beta} + 2B(Dv, Dv)_{\alpha\beta}$$

Therefore

$$\begin{aligned}((\Delta - 1)\, Q(v))_{\alpha\beta} &= B(\Delta v, v)_{\alpha\beta} + 2B(Dv, Dv)_{\alpha\beta} - \frac{1}{2}B(v, v)_{\alpha\beta} \\ &= \partial_\alpha \langle \Delta v, \partial_\beta v \rangle + \partial_\beta \langle \Delta v, \partial_\alpha v \rangle - 2 \langle \Delta v, \partial_{\alpha\beta} v \rangle + 2B(Dv, Dw)_{\alpha\beta} - \frac{1}{2}B(v, v)_{\alpha\beta}\end{aligned}$$

By defining $Q'_\alpha(v) = (\Delta - 1)^{-1} \langle \Delta v, \partial_\alpha v \rangle$ and $Q'_{\alpha\beta}(v) = (\Delta - 1)^{-1} \left(2 \langle \Delta v, \partial_{\alpha\beta} v \rangle - 2B(Dv, Dv)_{\alpha\beta} + \frac{1}{2}B(v, v)_{\alpha\beta}\right)$, we have

$$Q(v)_{\alpha\beta} = \partial_\alpha Q'_\beta(v) + \partial_\beta Q'_\alpha(v) - Q'_{\alpha\beta}(v)$$

We want $Q(v)_{\alpha\beta} = B(u, Q_0(v))_{\alpha\beta} = \partial_\alpha \langle \partial_\beta u, Q_0(v) \rangle + \partial_\beta \langle \partial_\alpha u, Q_0(v) \rangle - 2 \langle \partial_{\alpha\beta} u, Q_0(v) \rangle$. So the obvious definition for Q_0 is

$$Q_0(v) = \sum_\alpha Q'_\alpha(v) w_\alpha + \frac{1}{2} \sum_{\alpha \leq \beta} Q'_{\alpha\beta}(v) w_{\alpha\beta}$$

Now we define the bilinear symmetric forms $B'_\alpha(v_1, v_2) = (\Delta - 1)^{-1} \left(\langle \Delta v_1, \partial_\alpha v_2 \rangle + \langle \Delta v_2, \partial_\alpha v_1 \rangle\right)$ and

$$B'_{\alpha\beta}(v_1, v_2) = \sum_{\{i,j\}=\{1,2\}} (\Delta - 1)^{-1} \left(2 \langle \Delta v_i, \partial_{\alpha\beta} v_j \rangle - 2B(Dv_i, Dv_j)_{\alpha\beta} + \frac{1}{2}B(v_i, v_j)_{\alpha\beta}\right)$$

Then we can define

$$B_0(v_1, v_2) = \sum_\alpha B'_\alpha(v_1, v_2) w_\alpha + \frac{1}{2} \sum_{\alpha \leq \beta} B'_{\alpha\beta}(v_1, v_2) w_{\alpha\beta}$$

To estimate B_0, we note the following product estimate on $C^{k,a}(\mathbb{T}^m, \mathbb{R}^n)$ (by Leibniz's product rule):

$$|| \langle v_1, v_2 \rangle ||_{C^{k,a}} \leq C(m, a) \left(||v_1||_{C^{k,a}}||v_2||_{C^{0,a}} + ||v_2||_{C^{k,a}}||v_1||_{C^{0,a}}\right) + C(m, a, k) \left(||v_1||_{C^{k-1,a}}||v_2||_{C^{k-1,a}}\right)$$

Then due to Schauder estimates, we have the estimate:

$$\begin{aligned}||B'_\alpha(v_1, v_2)||_{C^{k,a}} &\leq C(u, m, a)\, ||\langle \Delta v_1, \partial_\alpha v_2 \rangle + \langle \Delta v_2, \partial_\alpha v_1 \rangle||_{C^{k-2,a}} \\ &\leq C_1 \left(||v_1||_{C^{k,a}}||v_2||_{C^{2,a}} + ||v_2||_{C^{k,a}}||v_1||_{C^{2,a}}\right) + C_2 \left(||v_1||_{C^{k-1,a}}||v_2||_{C^{k-1,a}}\right)\end{aligned}$$

where $C_1 = C_1(u, m, a)$, $C_2 = C_2(u, m, a, k)$. Same story for $B'_{\alpha\beta}$ and B_0. □

To sum up, we have $Q = LQ_0$ and $LM = 1$. Then the equation $h = Q(v) + L(v)$ becomes

$$LMh = LQ_0 v + Lv = L(Q_0 v + v)$$

So we can forget about L and only need to find $v \in C^\infty(\mathbb{T}^m, \mathbb{R}^n)$ such that $Mh = Q_0 v + v$.

Define the (nonlinear) operator $\Phi_h : C^\infty(\mathbb{T}^m, \mathbb{R}^n) \to C^\infty(\mathbb{T}^m, \mathbb{R}^n), v \mapsto Mh - Q_0 v$. We want to find a fixed point of Φ_h by contraction mapping. Unfortunately there isn't a convenient Banach norm for C^∞. However, if we fix $a \in (0,1)$, by Günther's lemma we have the estimate

$$||\Phi_h(v_1) - \Phi_h(v_2)||_{C^{2,a}} = \frac{1}{2}||B_0(v_1 - v_2, v_1) + B_0(v_2, v_1 - v_2)||_{C^{2,a}} \leq R||v_1 - v_2||_{C^{2,a}} \left(||v_1||_{C^{2,a}} + ||v_2||_{C^{2,a}}\right)$$

where $R = R(u, m, a)$ is a constant, for any $v_1, v_2 \in C^\infty(\mathbb{T}^m, \mathbb{R}^n)$. We also note that $||\Phi_h(0)||_{C^{2,a}} = ||Mh||_{C^{2,a}} \leq r||h||_{C^{2,a}}$ where $r = r(u, m, a)$ is a constant. So there is $\varepsilon = \varepsilon(u, m, a) > 0$ small enough that when $||h||_{C^{2,a}} < \frac{\varepsilon}{2r}$ and $||v_1||_{C^{2,a}}, ||v_2||_{C^{2,a}} \leq \varepsilon$:

$$\begin{aligned} ||\Phi_h(v_1) - \Phi_h(v_2)||_{C^{2,a}} &\leq 2\varepsilon R||v_1 - v_2||_{C^{2,a}} \leq \frac{1}{2}||v_1 - v_2||_{C^{2,a}} \\ ||\Phi_h(v_1)||_{C^{2,a}} &\leq ||\Phi_h(v_1) - \Phi_h(0)||_{C^{2,a}} + ||\Phi_h(0)||_{C^{2,a}} \leq R||v_1||_{C^{2,a}}||v_1||_{C^{2,a}} + \frac{\varepsilon}{2} \leq R\varepsilon^2 + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

So for any $h \in Sym$ such that $||h||_{C^{2,a}} < \frac{\varepsilon}{2r}$, there is a unique continuous (in the $C^{2,a}$ topology) extension of Φ_h that makes it a contraction mapping on $\mathrm{cl}\left(B_{C^{2,a}}(0, \varepsilon)\right)$ (closed ball), which is a complete metric space. Then by the Banach fixed point theorem, the sequence of smooth functions $\left((\Phi_h)^i(0)\right)_{i \in \mathbb{N}}$ eventually converges in the $C^{2,a}$ norm to some $v_h \in \mathrm{cl}\left(B_{C^{2,a}}(0, \varepsilon)\right)$. We wish to show that v_h is actually smooth. Since $\mathbb{T}^m$ is compact, by Arzelà–Ascoli, it suffices to show that the sequence is in $B_{C^{k,a}}(0, \varepsilon_k)$ for some $\varepsilon_k > 0$ for any $k \geq 2$. Note that the case $k = 2$ is already done. So we use induction on k. Assume $k > 2$ and ε_{k-1} is already found. Then once again we rely on Günther's lemma:

$$\begin{aligned} ||(\Phi_h)^{i+1}(0)||_{C^{k,a}} = ||Mh - Q_0(\Phi_h)^i(0)||_{C^{k,a}} &\leq ||Mh||_{C^{k,a}} + C_1||(\Phi_h)^i(0)||_{C^{k,a}}||(\Phi_h)^i(0)||_{C^{2,a}} + C_2||(\Phi_h)^i(0)||^2_{C^{k-1,a}} \\ &\leq ||Mh||_{C^{k,a}} + \varepsilon C_1||(\Phi_h)^i(0)||_{C^{k,a}} + C_2\varepsilon^2_{k-1} \end{aligned}$$

where $C_1 = C_1(u, m, a)$ and $C_2 = C_2(u, m, a, k)$. Note that C_1 does not depend on k, and neither does ε. As we can make ε as small as we want, WLOG $\varepsilon C_1 < \frac{1}{2}$. Then we note that $||Mh||_{C^{k,a}}$ is constant in i, so if we define $A_i = ||(\Phi_h)^{i+1}(0)||_{C^{k,a}}$, and $b = ||Mh||_{C^{k,a}} + C_2\varepsilon^2_{k-1}$, we have the recurrence inequality:

$$A_{i+1} \leq \frac{A_i}{2} + b$$

The final step is to show that the nonnegative sequence (A_i) has an upper bound as $i \to \infty$ by the standard bootstrap argument: if $A_i \leq 2b$ then $A_{i+1} \leq 2b$, while if $A_i > 2b$ then $A_{i+1} \leq A_i$. So $A_i \leq \max\{A_0, 2b\}$ $\forall i$. We are finally done. □

References

[GT01] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. U.S. Government Printing Office, 2001. ISBN: 9783540411604. URL: `https://books.google.com/books?id=eoiGTf4cmhwC`.

[How99] Ralph Howard. *Notes on Gunther's Method and the Local Version of the Nash Isometric Embedding Theorem*. 1999. URL: `http://people.math.sc.edu/howard/Notes/nash.pdf`.

[Tao16] Terence Tao. *Notes on the Nash embedding theorem*. 2016. URL: `https://terrytao.wordpress.com/2016/05/11/notes-on-the-nash-embedding-theorem`.