Problem 1. Consider the differential equation system $y'(t) = y(t), y(0) = 1$.

This is the most important differential equation of all. Its solution is called the exponential function $exp(t)$, or $e^t$. But from this definition, how do we calculate the function?

Use the Taylor series:

$$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + ... = y(0) + y(0)t + \frac{y'(0)}{2!}t^2 + ... = 1 + t + \frac{1}{2!}t^2 + ...$$

So $exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!}t^k$. This series allows us to calculate $exp(t)$ to arbitrary precision.

Many differential equation systems will eventually boil down to this one.

Remark. A finer point we did not go into is why the Taylor series works. Not every smooth (or infinitely differentiable) function can be written as a Taylor series. The ones that can be written as such are called analytic functions. Luckily, we know the exponential function is analytic, but for now we will accept that fact as a blackbox (to visit later).

Problem 2. Solve the differential equation system $y'(t) = 4y(t), y(0) = 3$.

First method Use the separation of variables trick

$$\frac{dy}{y} = 4 \ dt \implies \ln(y) = 4t + C \implies y(t) = \tilde{C}e^{4t}$$

Plug in $y(0) = 3$ and get $\tilde{C} = 3$. So $y(t) = 3e^{4t}$. One important assumption we have to make is that $y$ must always be nonzero. That is not always known, and this trick, despite being useful for quick calculations, is not a real solution or proof until we can establish that.

Note that $dy$ and $dt$ aren’t numbers, but “infinitesimals” (really small changes) which are linked together by $y = y(t)$ and $dy = y'(t) \ dt$. This is really sloppy notation we borrow from physics, but it is the kind of helpful, intuitive nonsense that works. In higher maths, we will make these notions more logical and precise (exterior derivatives, non-standard analysis etc.). For now we shall be grateful that this blackbox works.

Second method Separation of variables isn’t always possible. In fact, more often than not, it isn’t (and dividing by 0 is a constant concern). A more elementary (and rigorous) method works for this case. We note that the equation is similar to Problem 1, just with different constants like 4 and 3. I will overexplain how this method works because change of variables is an important, foundational skill (for physics and differential equations in general).

To turn 3 into 1, let $z(t) = \frac{y(t)}{3}$. After this change of variable, the first equation becomes

$$z'(t) = \frac{y'(t)}{3} = \frac{4y(t)}{3} = 4z(t)$$
So the system becomes \( z'(t) = 4z(t), z(0) = 1 \). We note that the first equation stays the same. We say that we have exploited a \textbf{symmetry} of the first equation (i.e. a transformation that actually doesn’t change something, like reflecting a circle through its center). This allowed us to change only the second equation. The kind of symmetry we used is called \textbf{linearity}, and it is indeed connected to linear algebra (more on that later).

Then to finally turn 4 into 1, we must change the variable \( t \) itself. Define

\[
\tilde{z}(\tau) = z\left(\frac{T}{4}\right)
\]

Then differentiate in \( \tau \):

\[
\tilde{z}'(\tau) = \frac{1}{4}z'\left(\frac{T}{4}\right) = \frac{1}{4}\cdot 4z\left(\frac{T}{4}\right) = z\left(\frac{T}{4}\right) = \tilde{z}(\tau)
\]

So \( \tilde{z} \) satisfies \( \tilde{z}'(\tau) = \tilde{z}(\tau), \tilde{z}(0) = 1 \). So \( \tilde{z}(\tau) = e^\tau \). Now we work our way back to \( y(t) \). Let \( t = \frac{T}{4} \), then

\[
z(t) = \tilde{z}(4t) = e^{4t}
\]

and \( y(t) = 3z(t) = 3e^{4t} \).

\textbf{Remark.} We didn’t have to introduce a new symbol \( \tau \), and could have just written \( \tilde{z}(t) = z\left(\frac{t}{4}\right) \) and differentiated in \( t \). But it is customary (at least in physics) to introduce new symbols when we change coordinates / times. This allows us to think of \( z, t \) and \( \tau \) as 3 different linked entities (in particular, \( z \) is no longer a function), and write the intuitive nonsense \( dz, dt, d\tau \) where \( dz = 4z \, dt \) and \( d\tau = 4 \, dt \), so

\[
dz = z \, d\tau, z = 1 \text{ when } \tau = 0
\]

This yields \( z = e^{\tau} = e^{4t} \). This is another way to do change of variables. Physicists love to write things in this sloppy way (see thermodynamics for instance), where the functions are hidden and only the variables matter. It depends on one’s taste. Some like how elegant it is, despite the sloppiness.

In summary, to simplify constant multipliers, instead of looking at \( y(t) \), look at \( z(t) = y(Ct) \) or \( z(t) = Cy(t) \) where \( C \) is some constant. Differentiate to see what equations \( z \) satisfies.