Counting dimensions and Stokes’ theorem

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The general Stokes’ theorem, and its derivatives such as the divergence theorem (3-dimensional region in $\mathbb{R}^3$), Kelvin–Stokes theorem (2-dimensional surface in $\mathbb{R}^3$), and Green’s theorem (2-dimensional region in $\mathbb{R}^2$), are simply fancier versions of the fundamental theorem of calculus (FTC) in one dimension. They also help determine the algebraic formulas for div and curl, which might look arbitrary, but just remember that they all come from applying FTC repeatedly.

To see how the theorem can be derived, imagine a 2-dimensional region in $\mathbb{R}^2$ or a 2-dimensional surface in $\mathbb{R}^3$ as a sheet of square grid paper, so it can be seen as a collection of small and (approximately) flat 2-dimensional squares. Similarly, a 3-dimensional region in $\mathbb{R}^3$ can be seen as a collection of small cubes. In short, the formula for Stokes’ theorem comes from the case of flat squares and cubes. It is a good exercise to check Green’s theorem on the unit square on $\mathbb{R}^2$ and the divergence theorem on the unit cube in $\mathbb{R}^3$ (try it).

To attain the general Stokes’ theorem and put together the simple shapes, we need more precise tools from algebraic topology (which is, of course, beyond the scope of this course). But we can learn what Stokes’ theorem means, and how it can be used to simplify many calculations.

General Stokes’ theorem

We shall work in $\mathbb{R}^3$, as Green’s theorem in $\mathbb{R}^2$ is just a special case of the Kelvin-Stokes theorem in $\mathbb{R}^3$, and FTC in $\mathbb{R}$ is a special case of the fundamental theorem of line integrals (FTLI) in $\mathbb{R}^3$, so the results here can carry to $\mathbb{R}^2$ and $\mathbb{R}$ with some minor tweaking.

Consider the diagram

$$
\begin{array}{c}
f \rightarrow \nabla \rightarrow F \rightarrow \text{curl} \rightarrow G \rightarrow \text{div} \rightarrow h
\end{array}
$$

where $f$ and $h$ are real-valued functions and $F, G$ are vector fields. The diagram tells us that $\nabla$ turns a real-valued function into a vector field, curl turns a vector field into a vector field, and div turns a vector field into a real-valued function. But this diagram, along with Stokes’ theorem, also contains the hidden idea of dimensions. To make it more obvious, we will rewrite this diagram in terms of differential forms:

$$
\begin{array}{c}
0\text{-form} \rightarrow 1\text{-form} \rightarrow 2\text{-form} \rightarrow 3\text{-form}
\end{array}
$$

In $\mathbb{R}^3$, we can define 0-forms and 3-forms to be real-valued functions, while 1-forms and 2-forms are vector fields. The subtlety is that even though 1-forms and 2-forms are both vector fields, they are meant to be used in different kinds of integrals, which we will see shortly. Anyhow, $d$ is a special operator that generalizes the operators you have learned in this course, called the exterior derivative, that turns $n$-forms into $(n + 1)$-forms. In $\mathbb{R}^3$, it coincides with $\nabla$, curl, and div when $n$ is 0, 1, 2.

On the other hand, all the special geometric objects we have learned such as points, curves, surfaces and regions are collectively called manifolds. More specifically, in $\mathbb{R}^3$, 0-manifolds are just isolated points;
a 1-manifold is a curve; a 2-manifold is a surface, and a 3-manifold is a region. Roughly speaking, an \( n \)-manifold is a geometric object which, if we zoom in closely enough, looks flat locally like \( \mathbb{R}^n \). Think about the surface of the earth, which looks flat like a plane when we zoom in.

To parametrize any \( n \)-manifold, we will need \( n \) variables (where \( n \) is called the \textit{dimension} of the manifold). Any \( n \)-manifolds \( \Omega \) (well, the nice ones at least) should have a boundary \( \partial \Omega \) which is an \((n-1)\)-manifold. For example, the boundary of a solid ball (3-dimensional) is a sphere (2-dimensional). Sometimes the boundary might be empty, e.g. the boundary of the sphere is empty. We call \( \partial \) the \textbf{boundary operator}, and the diagram for \( n \)-manifolds in \( \mathbb{R}^3 \) looks like this:

\[
\begin{align*}
\text{3-manifold} & \xrightarrow{\partial} \text{2-manifold} & \xrightarrow{\partial} \text{1-manifold} & \xrightarrow{\partial} \text{0-manifold}
\end{align*}
\]

Sometimes an \( n \)-manifold can be \textbf{oriented} (not always). If an \( n \)-manifold can be oriented, it will induce a compatible orientation on the boundary (the rules are taught in lectures and the textbook).

Here is the central idea:

\textit{n-forms can only be integrated on oriented \( n \)-manifolds.}

To be more specific:

- When \( n = 0 \), the integration is simply a sum over points.
- When \( n = 1 \), it is the \textit{work} of a vector field over an oriented curve (vector line integral).
- When \( n = 2 \), it is the \textit{flux} of a vector field across an oriented surface (vector surface integral).
- When \( n = 3 \), it is the scalar integral of a function over a region in \( \mathbb{R}^3 \).

We make a small remark about orientation on an \( n \)-manifold: When \( n = 0 \) or 3, the orientation is simply scalar multiplication by 1 or \(-1\) (recall the 0-dimensional endpoints in the fundamental theorem of calculus). When \( n = 1 \), the orientation is defined by a unit tangent vector field over the curve. When \( n = 2 \), the orientation is defined by a unit normal vector field over the surface. The difficulty in orienting a manifold in \( \mathbb{R}^3 \) is that the orientation has to be “consistent” (i.e. the vectors and scalars for orientation must change continuously over the manifold, so you can’t, say, suddenly switch between pointing up and pointing down). For instance, we have the famous Möbius band which is unorientable.

Let \( \Omega \) be an oriented \( n \)-manifold and \( \omega \) be a \((n-1)\)-form. Write \( \partial \Omega \) for the boundary of \( \Omega \) (possibly empty). Then, loosely speaking, the \textbf{general Stokes’ theorem} (not the same as Kelvin-Stokes) says

\[
\int_{\partial \Omega} \omega = \int_{\Omega} d\omega
\]

So loosely speaking, Stokes’ theorem allows us to switch between integrating in \( n-1 \) dimensions and \( n \) dimensions. Just like FTC, it also leads to integration by parts in higher dimensions. Note carefully that the dimensions must match: \( \omega \) and \( \partial \Omega \) are \((n-1)\)-dimensional, while \( d\omega \) and \( \Omega \) are \( n \)-dimensional, as the \textbf{exterior derivative} \( d \) adds 1 dimension while the \textbf{boundary operator} \( \partial \) subtracts 1 dimension.

We have seen the special cases of the theorem for specific values of \( n-1 \) and \( n \):

- Going between 0 and 1: fundamental theorem of line integrals, \( f(\gamma(1)) - f(\gamma(0)) = \int_{\gamma} \nabla f \cdot ds \)
- Going between 1 and 2: Kelvin-Stokes, \( \int_{\partial \Omega} \mathbf{F} \cdot ds = \int \int_{\Omega} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} \)
- Going between 2 and 3: divergence theorem, \( \int \int_{\partial E} \mathbf{G} \cdot d\mathbf{S} = \int \int \int_{E} \text{div}(\mathbf{G}) \, dV \)
A curious thing we note about \( d \) and \( \partial \) is that \( dd\omega = 0 \) for any differential form \( \omega \), and similarly \( \partial \partial \Omega = 0 \) for any oriented manifold \( \Omega \) (by 0 here we mean the empty set, which integrates to zero with any differential forms). We have seen many specific cases:

- For any function \( f \): \( \text{curl}(\nabla f) = \nabla \times (\nabla f) = 0 \).
- For any vector field \( \mathbf{F} \): \( \text{div}(\text{curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F}) = 0 \).
- The boundary of the unit ball (with the default orientation of \( \mathbb{R}^3 \)) is the sphere oriented outwards.

The boundary of the oriented sphere is empty.

Closed

We call an \( n \)-form \( \omega \) **closed** if \( d\omega = 0 \). So a closed 0-form is a scalar function with zero gradient (locally constant). A closed 1-form is an **irrotational** vector field (curl = zero). A closed 2-form is an **incompressible** vector field (div = zero). Any 3-form in \( \mathbb{R}^3 \) is considered to be closed by default (because we can’t go to 4-forms in \( \mathbb{R}^3 \) which only has 3 dimensions, so \( d\omega = 0 \) for every 3-form \( \omega \)).

Similarly, an \( n \)-manifold \( \Omega \) is called **closed** if \( \partial \Omega = 0 \). For instance, a closed 1-manifold is a **loop**, while a closed 2-manifold is a **closed surface** (how imaginative). Loosely speaking, when we talk about a closed surface, think about something like the sphere, or the donut (I mean the surface of the torus). By now you should see, from the common terminology, the striking relationship between \( n \)-manifolds and \( n \)-forms. This is part of the so-called **duality principle** in mathematics, where you can study an object by looking at its dual object, like Alice stepping through the Looking Glass from the world of manifolds to the world of differential forms. Stokes’ theorem is then the symmetric pairing of these dual objects, and \( d \) is the dual image of \( \partial \).

A simple application of Stokes’ theorem is **vanishment**:

\[
\text{If either } d\omega = 0 \text{ or } \partial \Omega = 0, \text{ then } \int_{\partial \Omega} \omega = \int_{\Omega} d\omega = 0.
\]

Stokes’ theorem also implies some useful **deformation criteria** for closed 1-forms and closed 2-forms. For example, if \( \omega \) is a closed 1-form, then over any 2-manifold \( \Omega \): \( \int_{2\Omega} \omega = \int_{\Omega} d\omega = 0 \). A stereotypical application of this fact is when \( \partial \Omega \) consists of 2 oriented loops \( \gamma_1 \) and \( \gamma_2 \) (for instance, when \( \Omega \) is a 2D annulus in the \( Oxz \) plane and \( \partial \Omega \) are 2 circles). Then Stokes’ theorem implies \( \int_{\gamma_1} \omega + \int_{\gamma_2} \omega = 0 \) and \( \int_{\gamma_1} \omega = - \int_{\gamma_2} \omega \), so the vector line integrals over those 2 loops are the same (after messing around with the orientation to make sure the sign is right). We can continuously deform one loop into the other by wiggling through \( \Omega \), so this leads to the deformation criteria for closed 1-forms which we encountered previously:

When integrating a **closed** 1-form over a **closed** 1-manifold (loop), we can deform the manifold without changing the integral.

Note that this does NOT imply the so-called “path-independence” / “zero-over-loops” condition. Think about the vortex field \( \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \) on the **punctured plane** (\( \mathbb{R}^2 \) minus the origin), which is irrotational (closed 1-form). Integrating it over any circle centered at 0, going counter-clockwise, will lead to \( 2\pi \) because of deformation. But obviously, it does not satisfy the “path-independence” or “zero-over-loops” condition as \( 2\pi \neq 0 \). However, if the domain we are working on is **simply-connected** (any loop can be deformed into a constant point), then the integral of the closed 1-form over any loop will be zero, and we do get the “zero-over-loops” condition.

Similarly, when integrating a **closed** 2-form over a **closed** 2-manifold, we can deform the manifold without changing the integral. The stereotypical example is the model of the gravity field, \( \mathbf{F}(x, y, z) = \frac{1}{(x^2+y^2+z^2)^{3/2}} (x, y, z) \)
which is incompressible (closed 2-form). Calculating the flux of $\mathbf{F}$ across the surface of a closed cylinder centered at the origin (exercise 17.5.36 in the textbook) is somewhat tricky, but by using the deformation criterion, we can deform the closed cylinder into the unit sphere, on which $\frac{1}{(x^2+y^2+z^2)^{3/2}}$ is just 1. That certainly makes our life easier.

**Exact**

We call an $n$-form $\omega$ **exact** on the domain $U$ when we can find a $(n-1)$-form $\nu$ on the domain $U$ such that $d\nu = \omega$. When this happens, because $d\omega = d(d\nu) = 0$, we see that every exact $n$-form must be closed, but the converse is not always true (for instance, the vortex field).

An exact 1-form $\mathbf{F}$ is called a **conservative** vector field, associated with a **scalar potential** $f$ such that $\nabla f = \mathbf{F}$. There isn’t a common catchy name for an exact 2-form $\mathbf{G}$, but it is a vector field associated with a **vector potential** $\mathbf{F}$ such that $\nabla \times \mathbf{F} = \mathbf{G}$. Exact 3-forms aren’t commonly discussed.

In the special case of 1-forms, a 1-form is exact if and only if it satisfies 2 conditions: being closed and “zero-over-loops”. The vortex field, for instance, is a closed 1-form, but does not satisfy “zero-over-loops”, and therefore it is not exact. On simply-connected domains, closed 1-form $= \text{exact 1-form}$.

If closed $n$-forms have their deformation criteria, then exact $n$-forms have their **path-independence** and **surface-independence** (which is stronger than the deformation criteria). You can guess what they mean by the name. Stare at the equation of Stokes’ theorem

$$\int_{\partial \Omega} \omega = \int_{\Omega} d\omega$$

This means that, if we replace the manifold $\Omega$ by a different manifold $\tilde{\Omega}$, as long as $\partial \Omega = \partial \tilde{\Omega}$ (with the same orientation), then the left-hand side stays the same, and we get $\int_{\Omega} d\omega = \int_{\tilde{\Omega}} d\omega$. For any exact $n$-form $\nu$, we can find $\omega$ such that $d\omega = \nu$, so we have

$$\int_{\Omega} \nu = \int_{\tilde{\Omega}} \nu \text{ whenever } \nu \text{ is exact and } \partial \Omega = \partial \tilde{\Omega}$$

There is a trivial consequence (which is also part of the vanishment idea from above):

$$\int_{\Omega} \nu = 0 \text{ whenever } \nu \text{ is exact and } \partial \Omega = 0$$

When $\nu$ is a conservative vector field (exact 1-form) and $\Omega$ is a curve (1-manifold), then this is the usual path-independence condition (the curve does not matter as long as the endpoints are the same). When we go to exact 2-forms, we call it surface independence (the surface does not matter as long as the boundary is the same). Note here that independence is much stronger than the deformation criteria, because $\Omega$ does not have to be closed, and we do not require there to be a way to deform $\Omega$ into $\tilde{\Omega}$. For instance, in the punctured plane, the unit circle cannot be deformed into a single point, but path-independence implies the integral over the unit circle must be zero.

**Mantra to remember**

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Some exercises and key principles

Problem 1. Let \( F(x, y, z) = (y, 2x, x \sqrt{y^2 + 1}) \) and \( E = \{(x, y, z) : z = x^2 + y^2, 1 \leq z \leq 3\} \) be the surface of an open bowl (with bottom and top lids missing), oriented outwards. Find \( \int \int_E \text{curl}(F) \cdot \text{d}S \).

Solution. We are definitely using Kelvin-Stokes. Note that \( \partial E \) consists of 2 circles (circling the two lids). We can parametrize these curves and calculate. However, we quickly note this method can be time-consuming.

Let \( F \) be the surface consisting of the bottom and the top lids (so \( F \) is the union of two 2-dimensional disks). Then \( F \) and \( E \) have the same boundary, and if we orient \( F \) the right way, we will have

\[
\int \int_E \text{curl}(F) \cdot \text{d}S = \int \int_F \text{curl}(F) \cdot \text{d}S = \int \int_F (\text{curl}(F) \cdot n) \, \text{d}S
\]

To determine the right orientation for \( F \), a good trick is to think of putting both \( E \) and \( F \) together to form a closed surface (a closed bowl), oriented outwards. Then the total flux of \( \text{curl}(F) \) over this closed surface is 0 (vanishment from Stokes’ theorem). Just like how \( A + B + C = 0 \) implies \( A = -B - C \), we will then orient \( F \) by letting the top lid be oriented downwards and the bottom lid be oriented upwards. Then the unit normal vector for the top lid is \( (0, 0, -1) \), and for the bottom lid is \( (0, 0, 1) \). Note that \( \text{curl}(F) = \left( \frac{2y}{\sqrt{y^2 + 1}}, -\sqrt{y^2 + 1}, 1 \right) \), so calculating the flux over the lids is trivial.

Problem 2. Let \( F(x, y, z) = \left( \frac{4yz}{r^3}, -\frac{xz}{r^3}, -\frac{3xy}{r^3} \right) \) where \( r(x, y, z) = \sqrt{x^2 + y^2 + z^2} \). Let \( E = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\} \) be the sphere of radius \( R > 0 \) (a constant), oriented outwards. Find \( \int \int_E F \cdot \text{d}S \).

Solution. We can parametrize the sphere and calculate. We quickly note this method can be time-consuming. HOWEVER, we can NOT use divergence theorem even though \( \text{div}(F) \) seems to be 0. The answer is because Stokes’ theorem only works on a nice \( n \)-manifold whose boundary is a well-defined \( (n-1) \)-manifold. Here, the \( n \)-manifold on which we want to invoke the divergence theorem is the punctured ball (missing the origin), so its boundary is the sphere PLUS the origin. The ball is 3-dimensional, yet the origin is not 2-dimensional, therefore the punctured ball does not have the nice kind of boundary to use Stokes’ theorem, which allows us to switch between \( n \)-dimensional integrals and \( (n-1) \)-dimensional integrals.

Not to mention that, Stokes’ theorem implicitly requires the differential forms to be smoothly defined up to the boundary. But \( \left( \frac{4yz}{r^3}, -\frac{xz}{r^3}, -\frac{3xy}{r^3} \right) \) can’t be extended continuously to the origin (proving it requires some knowledge from Math 32A), let alone smoothly (i.e. extending the derivatives as well). So for multiple reasons, Stokes’ theorem fails here.

However, we can just calculate directly, without parametrization or Stokes’. We use the abstract formula \( \int \int_E F \cdot \text{d}S = \int \int_E (F \cdot n) \, \text{d}S \). By some thinking, at a point \((x, y, z)\) on the sphere \( E \), \( n \) must be parallel to the vector \((x, y, z)\) itself (pointing from the origin). As we require \( ||n|| = 1 \), we must have \( n(x, y, z) = \frac{1}{r}(x, y, z) \). Then on \( E \),

\[
F \cdot n = \frac{1}{r^3} (4yz, -xz, -3xy) \cdot (x, y, z) = 0
\]

So no matter how we parametrize the sphere, the integral should yield zero.
**Area by Green’s theorem**

Let $\gamma$ be a loop in $\mathbb{R}^2$ that, for instance, encloses a 2-dimensional region $E$, then by Green’s theorem we have

$$\text{Area}(E) = \int \int_E 1 \, dx \, dy = \left| \int_\gamma (0, x) \cdot ds \right| = \left| \int_\gamma (-y, 0) \cdot ds \right| = \frac{1}{2} \left| \int_\gamma (-y, x) \cdot ds \right|$$

We don’t really care about the orientation for $\gamma$ because we know the area has to be nonnegative. If we get the wrong orientation for $\gamma$, our integral will yield a negative result and we just need to take its absolute value.

Note the meanings of notation here. Assuming $\gamma : I \rightarrow \mathbb{R}^2, t \mapsto (x(t), y(t))$ is a parametrization of the loop, where $I$ is an interval (usually $(0, 1)$ or $(0, +\infty)$). Then

$$\int_\gamma (0, x) \cdot ds = \int_\gamma x \, dy = \int_I x(t)y'(t) \, dt$$

Here $\int_I x \, dy$ does not mean integrating $x$ over a region $\gamma$ in $\mathbb{R}$. Essentially, $x \, dy$ gives a recipe for parametrization (after you parametrize $x$ by $x(t)$ and $y$ by $y(t)$, then $x \, dy$ becomes $x(t)y'(t) \, dt$). Also note that $x(t)$ and $y'(t)$ are both real numbers, not vectors.

**Example 3.** Find the area of the region $E$ in $\mathbb{R}^2$ which is enclosed by the curve $(x^2 + y^2)^2 = 4xy$.

![Graph of the curve](image)

**Solution.** We note the curve is symmetric over the origin, i.e. if $(x_0, y_0)$ is on the curve, then $(-x_0, -y_0)$ is also. The only point on the curve with $x = 0$ is $(0, 0)$. So we can surmise that the curve actually forms 2 loops symmetric over the origin, and they meet at the origin. We only need to focus where $x, y > 0$. By defining $E^+ = \{(x, y) \in E : x > 0, y > 0\}$, we have $\text{Area}(E) = 2\text{Area}(E^+)$.

To parametrize the curve, let $t = \frac{y}{x}$ (recall that now, $x$ and $y$ are positive). As $x$ and $y$ vary over the loop, $t$ is in $(0, +\infty)$. We aim to use $t$ to express both $x$ and $y$ on the curve.

Then the equation for the curve becomes $(x^2 + t^2x^2)^2 = 4tx^2$. As $x > 0$, we can divide by $x^2$ and get $x^2(1 + t^2)^2 = 4t$, which gives $x = \frac{2\sqrt{t}}{1 + t^2}$ and $y = tx = \frac{2\sqrt{t}}{1 + t^2}$. So let $\gamma(t) = \left(\frac{2\sqrt{t}}{1 + t^2}, \frac{2\sqrt{t}}{1 + t^2}\right)$ where $0 < t < +\infty$. Then

$$\text{Area}(E^+) = \left| \int_\gamma y \, dx \right| = \left| \int_0^\infty \gamma(t)x'(t) \, dt \right| = \left| \int_0^\infty \left(\frac{2\sqrt{t}}{1 + t^2}\right) \left(\frac{2\sqrt{t}}{1 + t^2}\right)' \, dt \right|$$

$$= \left| \int_0^\infty \left(\frac{2\sqrt{t}}{1 + t^2}\right) \frac{1 - 3t^2}{\sqrt{t}(1 + t^2)^2} \, dt \right| = \left| \int_0^\infty \frac{2t(1 - 3t^2)}{(1 + t^2)^3} \, dt \right|$$

$$= \left| \int_0^\infty \frac{4 - 3u}{u^3} \, du \right| = \left| 2 - 3 \right| = 1$$
Then $\text{Area}(E) = 2\text{Area}(E^+) = 2$.

**Trick:** For any problem where we set $t = \frac{y}{x}$, instead of using Green directly as above, we could also add the following trick:

$$x(t)^2 = x(t)^2 \partial_t \left( \frac{y(t)}{x(t)} \right) = x(t) \frac{2x(t)y'(t) - x'(t)y(t)}{x(t)^2} = x(t)y'(t) - x'(t)y(t) = x(t) \frac{dy}{dt}(t) - y(t) \frac{dx}{dt}(t)$$

Writing it in differential forms for Green’s theorem, this means $x \, dy - y \, dx = x^2 \, d\left(\frac{y}{x}\right) = x^2 \, dt$ as we set $t = \frac{y}{x}$ in the problem above. So

$$\text{Area}(E^+) = \frac{1}{2} \int_\gamma x \, dy - y \, dx = \frac{1}{2} \int_\gamma x^2 \, dt = \frac{1}{2} \int_0^\infty x(t)^2 \, dt$$

and we get $\text{Area}(E) = 2\text{Area}(E^+) = 2$ as before. It does simplify the calculations by 1 step, but remember that it only works when you set $t = \frac{y}{x}$ where $t$ is the time variable used to parametrize the curve.

**Mantra to remember (repeated)**

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**Poincaré lemma (cheat)**

Recall that an exact 1-form = a closed 1-form with path-independence. So on a simply connected domain (any loop can be deformed to a point), closed 1-form = exact 1-form. So when is a closed 2-form exact? The answer is somewhat more subtle (in algebraic topology), but there is a good, simple tool we can use:

**Lemma (Poincaré).** On $\mathbb{R}^n$ or a solid ball in $\mathbb{R}^n$, closed form = exact form.

The reason why this is true is somewhat funny. It is because in the world of topology, you can, morally speaking, deform things in special ways without changing certain topological properties. The solid ball can be smoothly “contracted” within itself to become a single point, which is a trivial space, where closed = exact. It sounds like some topology nonsense, but we will see quickly how powerful this lemma is. In fact, it is powerful enough that you are probably not allowed to use it, just that you secretly rely on it to know there is some kind of potential, and you must calculate that potential.

**Example.** Let $F(x, y, z) = k = (0, 0, 1)$. Then $F$ is defined on $\mathbb{R}^3$. Obviously $\text{curl}(F) = 0$ and $\text{div}(F) = 0$. So $F$ is a closed 1-form and closed 2-form. By Poincaré, $F$ is an exact 1-form and an exact 2-form. So we can find a scalar potential $f$ and a vector potential $G$ such that $\nabla f = F = \text{curl}(G)$. For instance, $f(x, y, z) = z$ and $G(x, y, z) = (0, x, 0)$ (there are other choices).

**Example.** Let $F(x, y, z) = (e^y, 2xe^{x^2}, 0)$. Let $S$ be the upper hemisphere of the unit sphere, oriented upwards. Calculate $\int_S F \cdot dS$.

**Solution.** We can try to parametrize and use spherical coordinates, before we cry and hate mathematics because of $e^{x^2}$ etc. A better way is to notice that $\text{div}(F) = 0$, or in other words, $F$ is a closed 2-form. Because $F$ is defined on the whole space $\mathbb{R}^3$, by Poincaré lemma we know it is an exact 2-form with a vector potential.
even without calculating. In fact, it is a good exercise to find a vector potential $G$ such that $\text{curl}(G) = F$. For instance, $G = (2xe^{x^2}, -ze^y, 0)$ is one solution.

Anyhow, the point is that we know $F$ is an exact 2-form, therefore by the mantra, we have surface-independence (not path-independence, which is 1D). So we find a simpler surface with the same boundary. The obvious choice is the 2D disk in the $Oxy$ plane: $\{(x, y, 0) : x^2 + y^2 \leq 1\}$, oriented upwards. Obviously, the normal vector is $(0, 0, 1)$, and when we take the dot product with $F$, we get zero so the flux is just 0.

Once again, we stress that Poincaré lemma can not be used directly in exams, but only as a cheat to help you know a form is exact, and you must calculate the scalar potential / vector potential directly. There are many domains where the Poincaré lemma does not apply, such as the punctured space $\mathbb{R}^3 \setminus \{0\}$ or the sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ (this is because the sphere cannot be “contracted” within itself to become a point, if you can visualize it).

Mix-up examples

**Example.** Let $F(x, y, z) = \left(\frac{4yz}{r^3}, -\frac{xz}{r^3}, -\frac{3xy}{r^3}\right)$ where $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Let $S$ be the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outwards. Calculate $\int \int_S F \cdot dS$.

**Solution.** We parametrize and cry because of $r^3$. Or we realize that $\text{div}(F) = 0$, $F$ is closed 2-form on a closed 2-manifold, so by the mantra, we deform the shape into something where $r$ is simple. The obvious choice is the unit sphere, where $r = 1$. It is then trivial to check the flux is 0. Remember that neither the divergence theorem nor Poincaré lemma can work here because the domain is $\mathbb{R}^3 \setminus \{0\}$.

**Example.** Let $F(x, y, z) = (e^y, 2xe^{x^2}, z^2)$. Let $S$ be the upper hemisphere of the unit sphere, oriented upwards. Calculate $\int \int_S F \cdot dS$.

**Solution.** We parametrize and cry because of $e^y$ and $e^{x^2}$ etc. Or we split $F = F_1 + F_2$, where $F_1 = (e^y, 2xe^{x^2}, 0)$ and $F_2 = (0, 0, z^2)$. Then flux of $F = \text{flux of } F_1 + \text{flux of } F_2$. As before, the flux of $F_1 = 0$. The flux of $F_2$, unfortunately, has to be calculated manually by parametrization. $F_2$ is a closed 1-form, therefore an exact 1-form, but that does not help here because the flux is a 2D integral. The good thing is that we no longer have to deal with “unintegrable” formulas like $e^y \sin \theta \sin \phi$. 
