• $\mathbb{R} = (-\infty, \infty)$ is the set of real numbers. $\mathbb{N}$ is the set of natural numbers.

• For a vector $\mathbf{v}$ in $\mathbb{R}^n$, the length can be denoted by $|\mathbf{v}|$ or $||\mathbf{v}||$.

• Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^3$, then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Since $\mathbf{a} \perp \mathbf{a} \times \mathbf{c}$, we have $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) = 0$. This is equivalent to the fact that a square matrix has zero determinant whenever 2 rows (or 2 columns) are exactly the same.

• Similarly, for vector fields $\mathbf{F}, \mathbf{G}$, we have

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \det \begin{pmatrix} \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{pmatrix}$$

Note that this is purely a symbolic matrix and not an actual matrix containing real numbers. Be careful in symbolic calculation and treating symbolic partials like $\partial_x$ as real numbers, because there is Leibniz’s product rule: $\partial_z (F_2 G_3) = (\partial_z F_2) G_3 + F_2 (\partial_z G_3)$, which certainly doesn’t hold if you replace $\partial_x$ by a real number. In certain cases, when there is no possibility of Leibniz’s rule and the symbols commute like products of real numbers (e.g. $\partial_x \partial_y = \partial_y \partial_x$), you can treat the symbols like real numbers, for instance in the formulas

$$\nabla \times (\nabla f) = (\nabla \times \nabla) f = 0$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \det \begin{pmatrix} \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{pmatrix} = 0$$

The second equation holds because 2 rows are exactly the same.

• For a given vector field $\mathbf{F}$, checking the cross-partial (or mixed-partial) condition is just checking $\nabla \times \mathbf{F} = 0$ (i.e. closed). If for some reason this doesn’t hold, like $\partial_z F_1 - \partial_x F_3 \neq 0$, then the intuition is that something is wrong on the $Oxz$ plane. Perhaps integrating $\mathbf{F}$ over a circle (aka loop) on the $Oxz$ plane will yield a nonzero result, meaning that FTC (fundamental theorem of calculus) fails for $\mathbf{F}$ and $\mathbf{F}$ cannot be conservative/exact.
1 The Algebraic Method for switching the order of integration

The chief use of the algebraic method is that you should never have to draw pictures again when switching the order of integration. This is valuable when the region is defined by complicated equations, or dealing with high dimensions. The idea is simple, and best illustrated by detailed examples.

Any logical statement, such as \( x \leq y \), is either true or false. We write the logical statement in square brackets, e.g. \([x \leq y]\), to mean the number 1 when the statement is true, and 0 when the statement is false. Using this notation, we can write equations like

\[
\int_1^3 f(x) \, dx = \int_{\mathbb{R}} f(x)[1 < x < 3] \, dx
\]

In the equation above, we can replace \([1 < x < 3]\) by \([1 \leq x \leq 3]\) or \([1 < x < 3]\) etc. The idea is that the Riemann integral does not care about values at isolated points. So the values of \( f \) at the endpoints don’t really matter and will not change the result of the integral.

We use this kind of notation because in order to switch the order of integration (in a double or triple integral), we require the region to be a “boxed” region, i.e. a product of fixed subsets of \( \mathbb{R} \) (usually intervals). So \([0, 2] \times [-2, 6]\) and \( \mathbb{R} \times \mathbb{R} \) are boxed regions, while \{(x, y) : x < y\} is not a boxed region. Therefore it is a mistake to write things like

\[
\int_{\mathbb{R}} \int_{-\infty}^{y} f(x, y) \, dx \, dy
\]

Notice that the integral on the right hand side does not even make sense, because once we have integrated in \( y \), \( \int_{\mathbb{R}} f(x, y) \, dy \) becomes a function in \( x \) only, so what is \( y \) in the symbol \( \int_{-\infty}^{y} \)?

The correct way to do it is to turn the region into \( \mathbb{R} \times \mathbb{R} \) by our brackets

\[
\int_{\mathbb{R}} \int_{-\infty}^{y} f(x, y) \, dx \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)[x < y] \, dxdy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)[x < y < \infty] \, dxdy
\]

\[
= \int_{\mathbb{R}} [x < \infty] \int_{x}^{\infty} f(x, y) \, dy \, dx = \int_{\mathbb{R}} \int_{x}^{\infty} f(x, y) \, dy \, dx
\]

The factor \([x < \infty]\) is called the domain condition. It is there because

\[
[x < y < \infty] = [x < y < \infty][x < \infty]
\]

As \([x < \infty]\) is a constant function in \( y \), it is thrown out when integrating in \( y \):

\[
\int_{\mathbb{R}} f(x, y)[x < y < \infty][x < \infty] \, dy = \int_{x}^{\infty} f(x, y)[x < \infty] \, dy = [x < \infty] \int_{x}^{\infty} f(x, y) \, dy
\]

The implicit assumption is that in the algebraic method, to make things consistent, we want the upper limit of an integral to be greater than the lower limit. Sometimes they contain log or square roots, so they are only defined for certain values of \( x, y \) or \( z \). So whenever we use our square brackets to get the endpoints for an integral, we must always check that the upper limit is greater than the lower limit, and that they are both well-defined.

Obviously, the example above is too simple for us to see the usefulness of the domain condition \((x < \infty\) is always true), so we turn to a more difficult example.

Let us rewrite \( A = \int_0^2 \int_0^{2-z} \int_0^{y^2} f(x, y, z) \, dx \, dy \, dz \) into \( \cdots dz \, dy \, dx \)

For convenience’s sake, we write \( f \) instead of \( f(x, y, z) \). Then
A = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(0 < z < 2) [0 < y < 2 - z] [0 < x < yz] \, dx \, dy \, dz
\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(0 < z < 2) [0 < y < 2 - z] [0 < x < yz] \, dy \, dx \, dz

The 1st step is simple, turning the region into the boxed region \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) so that we can switch the order of integration. The 2nd step is to find the endpoints for \( z \). Note that the region is defined by 6 inequalities. But some of them do not contain \( z \), such as \( [0 < y] \), so they are constant in \( z \), and thus thrown out.

\[
A = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(0 < z < 2) [0 < y < 2 - z] [0 < x < x] \, dx \, dy \, dz
\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(0 < z < 2) [0 < y < 2 - z] [0 < x < yz] \, dy \, dx \, dz
\]

Note that we could divide \( x \) by \( y \) because \( y > 0 \). By some thinking, we see that

\[
[0 < z < 2] [z < 2 - y] \frac{x}{y} < z = [\max(0, \frac{x}{y}) < z < \min(2, 2 - y)]
\]

But in this problem, \( x \) and \( y \) are positive, so \( \max(0, \frac{x}{y}) = \frac{x}{y} \) and \( \min(2, 2 - y) = 2 - y \). Therefore

\[
A = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(0 < z < 2) \frac{x}{y} < z < 2 - y \, dx \, dy \, dz
\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(0 < z < 2) \frac{x}{y} < 2 - y \, \frac{2 - y}{\frac{x}{y}} \, dx \, dy \, dz
\]

Again, \( \frac{x}{y} < 2 - y \) is the domain condition (upper limit > lower limit, both well defined), and it is there because \( [\frac{x}{y} < z < 2 - y] [\frac{x}{y} < 2 - y] \). Technically we should have had to include \( [y > 0] \) to make sure the lower limit \( \frac{x}{y} \) is well-defined, but we already have the factor \( [y > 0] \).

Next is integrating in \( y \). We need to turn the inequality \( \frac{x}{y} < 2 - y \) into acceptable values of \( y \) for integration. Assuming \( x \) is a constant, for which \( y \) do we have \( \frac{x}{y} < 2 - y \)? Because \( y > 0 \) we have

\[
[0 < y] [\frac{x}{y} < 2 - y] = [0 < y] [x < 2y - y^2] = [0 < y] [(y - 1)^2 < 1 - x] = [-\sqrt{1-x}+1 < y < \sqrt{1-x}+1] [1-x \geq 0]
\]

Note that we implicitly used the fact that \( 0 < -\sqrt{1-x} + 1 \), rendering \( [0 < y] \) redundant in the end, and obviously we need \( 1 - x \geq 0 \) for the square roots to be well defined. Then

\[
A = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(-\sqrt{1-x}+1 < y < \sqrt{1-x}+1) [1-x \geq 0] \, dx \, dy \, dz
\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(-\sqrt{1-x}+1 < y < -\sqrt{1-x}+1) \, \frac{2-y}{\frac{1-x}{y}} \, dx \, dy \, dz
\]

Again, \( [1-x \geq 0] [\sqrt{1-x}+1 > -\sqrt{1-x}+1] \) is the domain condition (upper limit > lower limit, both well defined), and it is there because blah blah blah. We see that \( 1 - x \geq 0 \) guarantees both upper and lower limits are well-defined.
Luckily, \( \sqrt{1-x} + 1 > -\sqrt{1-x} + 1 \) is always true, so

\[
A = \int_0^1 \int_{\sqrt{1-x}+1}^{2-y} f \, dz \, dy \, dx.
\]

Technically, we should have had to write down the domain condition \([1 > 0]\), but I trust that you know why it’s unnecessary. In fact, once we get used to the algebraic method, we can skip many obvious steps, but for beginners, it’s better to be foolproof and follow the procedure.

2 Conservative vector fields

Let \( \Omega \) be a region in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Let \( F \) be a vector field on \( \Omega \).

\( F \) is called irrotational or closed when \( \nabla \times F = 0 \). \( F \) is called conservative or exact when there is a potential function \( f \) such that \( \nabla f = F \). Exact always implies closed.

The words closed and exact come from the study of differential forms, which are outside the scope of this course (though they are intrinsically related to Stokes’ theorem, which is covered later in this course).

If a closed vector field satisfies FTC/path-independence (i.e. integrates to zero over loops) then it is conservative/exact. Note that there are usually infinitely many loops in \( \Omega \), so it’s not practical to test every loop. There is a useful criterion: any two loops that can be continuously deformed into each other can be considered to be equivalent (i.e. you only need to test 1 of them). This criterion is a consequence of Stokes’ theorem, which is covered later in this course.

A simply connected region is a region where every loop is equivalent to a constant point, so FTC is always satisfied (even without testing any loops). Consequently, on simply connected regions, closed = exact.

Example 1. Let \( \Omega = \mathbb{R}^2 \setminus \{0\} \) (the punctured plane) and \( F \) be a closed vector field on \( \Omega \). Let \( S^1 \) be the unit circle oriented counterclockwise. Then \( F \) is conservative if and only if \( \int_{S^1} F \cdot dr = 0 \).

Remark. We only need to test 1 loop for the punctured plane, or the annulus, or the solid torus. Note that \( \mathbb{R}^3 \setminus \{0\} \) is actually simply connected so no tests are needed.

Problem 2. Let \( \Omega = \mathbb{R}^3 \) and \( F(x, y, z) = \begin{pmatrix} g(y, z) \\ 2xyz \\ xy^2 \end{pmatrix} \). Find all \( g \) such that \( F \) is conservative, and find all the potential functions \( f \).

Solution. As \( \Omega \) is simply connected, closed = exact. Consequently, we only need \( \nabla \times F = 0 \), i.e.

\[
\begin{align*}
\partial_y g(y, z) &= \partial_x (2xyz) = 2yz \\
\partial_z g(y, z) &= \partial_x (xy^2) = y^2 \\
2xy &= 2xy \quad \text{(always true)}
\end{align*}
\]

By the 1st equality, we conclude \( g(y, z) = \int 2yz \, dy = y^2z + C(z) \) where \( C \) is constant in \( y \) (though it is a function of \( z \), i.e. it might vary in \( z \)). But the 2nd equality then implies

\[
y^2 + C'(z) = y^2
\]

so \( C'(z) = 0 \) and \( C \) is constant in both \( y \) and \( z \) (therefore a true constant). So \( g(y, z) = y^2z + C \) where \( C \) is a constant real number (each choice for \( C \) will give a different solution \( g \), and all solutions for \( g \) will look like this).
Then we want to find \( f \) such that \( \nabla f = \mathbf{F} \), i.e.

\[
\begin{align*}
\partial_x f(x, y, z) &= y^2 z + C \\
\partial_y f(x, y, z) &= 2xyz \\
\partial_z f(x, y, z) &= xy^2
\end{align*}
\]

Again, by the 1st equality, we have \( f(x, y, z) = xy^2z + Cx + D(y, z) \) where \( D \) is constant in \( x \) (though it may vary in \( y \) and \( z \)).

The 2nd equality implies \( \partial_y D(y, z) = 0 \), while the 3rd implies \( \partial_z D(y, z) = 0 \) so \( D \) is constant in both \( y \) and \( z \). So the solution for \( f \) is

\[ f(x, y, z) = xy^2z + Cx + D \]

where \( C, D \) are any constant real numbers.

## 3 Surface integrals

Recall that for a change of variable \( G(u, v) = (x, y) \), where \( G \) is a one-to-one map between two 2-dimensional regions in \( \mathbb{R}^2 \), we have the Jacobian

\[
\frac{dxy}{dudv} = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \det \begin{pmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{pmatrix} \right| = ||\partial_u G \times \partial_v G||
\]

where \( \partial_u G = (\partial_u x, \partial_u y) \) and \( \partial_v G = (\partial_v x, \partial_v y) \).

Examples include polar coordinates \( G(r, \theta) = (x, y) \) where the Jacobian is \( ||\partial_r G \times \partial_\theta G|| = r \) and \( dxy = r \, dr \, d\theta \).

This can be generalized to surface integrals where \( G(u, v) = (x, y, z) \) is a one-to-one map from a 2-dimensional region \( D \) in \( \mathbb{R}^2 \) to a 2-dimensional surface \( \Omega \) in \( \mathbb{R}^3 \):

\[
dS = ||\partial_u G \times \partial_v G|| \, dudv
\]

where \( \partial_u G = (\partial_u x, \partial_u y, \partial_u z) \) and \( \partial_v G = (\partial_v x, \partial_v y, \partial_v z) \). We define the normal vector to be \( \mathbf{N}(u, v) = \partial_u G(u, v) \times \partial_v G(u, v) \), which is perpendicular to the surface. Also, the unit normal vector is \( \mathbf{n}(u, v) = \frac{\mathbf{N}(u, v)}{||\mathbf{N}(u, v)||} \).

Do not confuse between \( \mathbf{n} \) and \( \mathbf{N} \). Review the formulas for scalar surface integral and vector surface integral

\[
\int \int_{\Omega} f \, dS = \int \int_{D} f(G(u, v)) ||\mathbf{N}(u, v)|| \, dudv
\]

\[
\int \int_{\Omega} \mathbf{F} \cdot dS = \int \int_{\Omega} (\mathbf{F} \cdot \mathbf{n}) \, dS = \int \int_{D} (\mathbf{F}(G(u, v)) \cdot \mathbf{n}(u, v)) ||\mathbf{N}(u, v)|| \, dudv = \int \int_{D} \mathbf{F}(G(u, v)) \cdot \mathbf{N}(u, v) \, dudv
\]

Consequently, when the surface is parametrized by \( G \), we need \( ||\mathbf{N}(u, v)|| \) to calculate the scalar surface integral and \( \mathbf{N}(u, v) \) to calculate the vector surface integral. We don’t actually use \( \mathbf{n}(u, v) \) even though it appears in \( \int \int_{\Omega} (\mathbf{F} \cdot \mathbf{n}) \, dS \).

**Problem 3.** Let \( \Omega \) be the the portion of the plane \( x + 2y + z = 8 \) lying above the rectangle \( 1 \leq x \leq 2, 0 \leq y \leq 2 \), with normal vector pointing upwards. Let \( \mathbf{F} = zk \). Find the area of \( \Omega \) and the flux of \( \mathbf{F} \) across \( \Omega \).
Solution. We first note that \( F(x, y, z) = (0, 0, z) \).

As \( \Omega \) is a 2-dimensional surface, we want to use 2 variables to parametrize it. The obvious choice here is \( x \) and \( y \) (as we already have bounds for \( x \) and \( y \), and \( z \) can easily be expressed in \( x \) and \( y \)). We define

\[
G : [1, 2] \times [0, 2] \to \Omega \\
(x, y) \mapsto (x, y, 8 - x - 2y)
\]

Then the normal vector is \( \mathbf{N}(x, y) = \partial_x G \times \partial_y G = (1, 0, -1) \times (0, 1, -2) = (1, 2, 1) \) (which is actually constant), and the “Jacobian” is \( ||\mathbf{N}|| = \sqrt{6} \).

So the area is

\[
\int \int_\Omega 1 \, dS = \int_0^2 \int_1^2 ||\mathbf{N}(x, y)|| \, dxdy = \sqrt{6} \int_0^2 \int_1^2 1 \, dxdy = 2\sqrt{6}
\]

and the flux of \( \mathbf{F} \) across \( \Omega \) is

\[
\int \int_\Omega \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_1^2 \mathbf{F}(G(x, y)) \cdot \mathbf{N}(x, y) \, dxdy = \int_0^2 \int_1^2 (0, 0, 8 - x - 2y) \cdot (1, 2, 1) \, dxdy = 16 - 3 - 4 = 9
\]

Problem 4. Let \( \Omega \) be the sphere in \( \mathbb{R}^3 \) centered at 0 with radius \( R > 0 \).

1. Find the 2 planes parallel to the \( Oxy \) plane that divide \( \Omega \) into 3 parts of equal area.

2. Gauss’s law of gravity: Calculate the flux of \( \mathbf{F}(r) = \frac{\mathbf{r}}{||\mathbf{r}||^3} \) across \( \Omega \), where \( \Omega \) is oriented with normal vector pointing outwards. See that it does not depend on \( R \).

Solution. Parametrize \( \Omega \) by defining

\[
G : (0, \pi) \times (0, 2\pi) \to \Omega \\
(\phi, \theta) \mapsto (R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi)
\]

Then \( \mathbf{N}(\phi, \theta) = \partial_\phi G \times \partial_\theta G = R^2 \sin \phi (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) = R \sin \phi G(\phi, \theta) \) and \( ||\mathbf{N}(\phi, \theta)|| = R^2 \sin \phi \). Also trivially note that \( ||G|| = R \).

Note that we deliberately chose \( (\phi, \theta) \) instead of \( (\theta, \phi) \) because \( \partial_\phi G \times \partial_\theta G \) points outwards while \( \partial_\theta G \times \partial_\phi G \) points inwards. There is the small matter of \( G \) not being well-defined near the poles \( (\phi = 0 \text{ or } \pi, \text{ any } \theta \) is acceptable, but \( G \) needs to be one-to-one), or when \( \theta = 0 \text{ or } 2\pi \) (again, \( G \) needs to be one-to-one). But because the Riemann integral does not care about endpoints, we can work on the interior \( (0, \pi) \times (0, 2\pi) \).

1. The intersection of a plane parallel to \( Oxy \) and the sphere \( \Omega \), assuming they do meet, will be a circle on \( \Omega \) corresponding to \( \{ \phi = a \} \) for some \( a \) in \( [0, \pi] \).

Among the 2 planes, there will be an upper plane and a lower plane. By symmetry over the \( Oxy \) plane, we just need to find the upper plane. So we want to find \( a \) such that

\[
\int_0^{2\pi} \int_0^a ||\mathbf{N}(\phi, \theta)|| \, d\phi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^\pi ||\mathbf{N}(\phi, \theta)|| \, d\phi d\theta
\]
This is equivalent to
\[ 2\pi R^2 \int_0^\alpha \sin \phi \, d\phi = \frac{1}{3} 2\pi R^2 \int_0^\pi \sin \phi \, d\phi \]
\[ \iff 1 - \cos \alpha = \frac{2}{3} \]
\[ \iff \alpha = \arccos \left( \frac{1}{3} \right) \]

The equation of the upper plane will then be \( z = R \cos \alpha = \frac{R}{3} \). By symmetry, the lower plane is \( z = -\frac{R}{3} \).

2. The flux is
\[
\int_0^{2\pi} \int_0^\pi F(G(\phi, \theta)) \cdot N(\phi, \theta) \, d\phi d\theta = \int_0^{2\pi} \int_0^\pi \left( \frac{G}{||G||^3} \cdot R \sin \phi G \right) \, d\phi d\theta = \int_0^{2\pi} \int_0^\pi \sin \phi \left( \frac{||G||^2}{||G||^3} \right) \, d\phi d\theta
\]
\[
= \int_0^{2\pi} \int_0^\pi R \sin \phi \left( \frac{||G||^2}{||G||^3} \right) \, d\phi d\theta = \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi d\theta
\]
\[
= 2\pi \int_0^\pi \sin \phi \, d\phi = 4\pi.
\]

Remark 5 (Optional). Stokes’ theorem actually guarantees that for any oriented closed surface in \( \mathbb{R}^3 \setminus \{0\} \), the flux of \( \frac{r}{||r||^3} \) across the surface is \( 4\pi, -4\pi \) or 0. The flux is zero if and only if the surface does not “enclose” the origin. This ability to “detect” the location of the origin is quite surprising.

4 Stokes’ theorem (Fundamental theorem of vector calculus)

The general Stokes’ theorem, and its derivatives such as the divergence theorem (3-dimensional region in \( \mathbb{R}^3 \)), Kelvin–Stokes theorem (2-dimensional surface in \( \mathbb{R}^3 \)), and the Green’s theorem (2-dimensional region in \( \mathbb{R}^2 \)), are simply fancier versions of the fundamental theorem of calculus in one dimension. They also help determine the algebraic formulas for div and curl.

To see how the theorem can be derived, imagine a 2-dimensional region in \( \mathbb{R}^2 \) or a 2-dimensional surface in \( \mathbb{R}^3 \) as a sheet of square grid paper, so it can be seen as a collection of small and (approximately) flat 2-dimensional squares. Similarly, a 3-dimensional region in \( \mathbb{R}^3 \) can be seen as a collection of small cubes. In short, the formula for Stokes’ theorem comes from the case of flat squares and cubes. It is a trivial exercise to check the Green’s theorem on the unit square on \( \mathbb{R}^2 \) and the divergence theorem on the unit cube in \( \mathbb{R}^3 \) (try it).

To attain the general Stokes’ theorem and put together the simple shapes, we need more precise tools from algebraic topology (which is, of course, beyond the scope of this course).

Area by Green’s theorem  Let \( \gamma \) be a loop in \( \mathbb{R}^2 \) that, for instances, encloses a 2-dimensional region \( E \), then by Green’s theorem we have
\[
\text{Area}(E) = \iint_E 1 \, dx \, dy = \left| \int_{\gamma} (0, x) \cdot ds \right| = \left| \int_{\gamma} (-y, 0) \cdot ds \right| = \frac{1}{2} \left| \int_{\gamma} (-y, x) \cdot ds \right|
\]

We don’t really care about the orientation for \( \gamma \) because we know the area has to be nonnegative. If we get the wrong orientation for \( \gamma \), our integral will yield a negative result and we just need to take its absolute
value.

Note the meanings of notation here. Assuming \( \gamma : I \to \mathbb{R}^2, t \mapsto (x(t), y(t)) \) is a parametrization of the loop, where \( I \) is an interval (usually \((0,1)\) or \((0, +\infty)\)). Then

\[
\int_{\gamma} (0, x) \cdot \mathbf{ds} = \int_{\gamma} x \, dy = \int x(t)y'(t) \, dt
\]

Here \( \int_{\gamma} x \, dy \) does not mean integrating \( x \) over a region \( \gamma \) in \( \mathbb{R} \). We call \( x \, dy \) a differential form for integrating over a curve, and it essentially gives a recipe for parametrization (after you parametrize \( x \) by \( x(t) \) and \( y \) by \( y(t) \), then \( x \, dy \) becomes \( x(t)y'(t) \, dt \)). Also note that \( x(t) \) and \( y'(t) \) are both real numbers, not vectors.

**Example 6.** Find the area of the region \( E \) in \( \mathbb{R}^2 \) which is enclosed by the curve \((x^2 + y^2)^2 = 4xy\).

\[\text{Solution.} \quad \text{We note the curve is symmetric over the origin, i.e. if } (x_0, y_0) \text{ is on the curve, then } (-x_0, -y_0) \text{ is also. The only point on the curve with } x = 0 \text{ is } (0,0). \text{ So we can surmise that the curve actually forms 2 loops symmetric over the origin, and they meet at the origin. We only need to focus where } x, y > 0. \text{ By defining } E^+ = \{ (x, y) \in E : x > 0, y > 0 \}, \text{ we have } \text{Area}(E) = 2 \text{Area}(E^+).
\]

To parametrize the curve, let \( t = \frac{y}{x} \) (recall that now, \( x \) and \( y \) are positive). As \( x \) and \( y \) vary over the loop, \( t \) is in \((0, +\infty)\). We aim to use \( t \) to express both \( x \) and \( y \) on the curve.

Then the equation for the curve becomes \( (x^2 + t^2x^2)^2 = 4tx^2 \). As \( x > 0 \), we can divide by \( x^2 \) and get \( x^2(1 + t^2)^2 = 4t \), which gives \( x = \frac{2\sqrt{t}}{1 + t^2} \) and \( y = tx = \frac{2t\sqrt{t}}{1 + t^2} \). So let \( \gamma(t) = \left( \frac{2\sqrt{t}}{1 + t^2}, \frac{2t\sqrt{t}}{1 + t^2} \right) \) where \( 0 < t < +\infty \).

Then

\[
\text{Area}(E^+) = \left| \int_{\gamma} y \, dx \right| = \left| \int_{0}^{\infty} y(t)x'(t) \, dt \right| = \left| \int_{0}^{\infty} \left( \frac{2t\sqrt{t}}{1 + t^2} \right)' \, dt \right|
\]

\[
= \left| \int_{0}^{\infty} \left( \frac{2t\sqrt{t}}{1 + t^2} \right) \frac{1 - 3t^2}{\sqrt{t}(1 + t^2)^2} \, dt \right| = \left| \int_{0}^{\infty} \frac{2t(1 - 3t^2)}{(1 + t^2)^3} \, dt \right|
\]

\[
= \left| \int_{1}^{\infty} \frac{4 - 3u}{u^3} \, du \right| = |2 - 3| = 1
\]

Then \( \text{Area}(E) = 2 \text{Area}(E^+) = 2 \).

**Trick:** For any problem where we set \( t = \frac{y}{x} \), instead of using Green directly as above, we could also add the following trick:

\[
x(t)^2 = x(t)^2 \frac{dy(t)}{x(t)} = x(t)^2 \frac{y(t) - x(t)y(t)}{x(t)^2} = x(t)y'(t) - x'(t)y(t) = x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}
\]
Writing it in differential forms for Green’s theorem, this means \( x \, dy - y \, dx = x^2 \, d\left(\frac{y}{x}\right) = x^2 \, dt \) as we set \( t = \frac{y}{x} \) in the problem above. So

\[
\text{Area}(E^+) = \left| \frac{1}{2} \int \gamma x \, dy - y \, dx \right| = \left| \frac{1}{2} \int_\gamma x^2 \, dt \right| = \left| \frac{1}{2} \int_0^\infty x(t)^2 \, dt \right|
\]

\[
= \left| \int_0^\infty \frac{2t}{(1+t^2)^2} \, dt \right| = \left| \int_1^\infty \frac{1}{u^2} \, du \right| = 1
\]

and we get \( \text{Area}(E) = 2\text{Area}(E^+) = 2 \) as before. It does simplify the calculations by 1 step, but remember that it only works when you set \( t = \frac{y}{x} \) where \( t \) is the time variable used to parametrize the curve.

## Center of mass, moment of inertia

Just reviewing some formulas. Let \( E \) be a 3-dimensional region in \( \mathbb{R}^3 \) with the mass density function \( \delta \). Then

1. The total mass of \( E \) is mass(\( E \)) = \( \int_E \delta = \int \int_E \delta(x, y, z) \, dxdydz \).
2. The mass center of \( E \) is \( c = (c_x, c_y, c_z) \) where \( c_x = \frac{\int \int_E x \delta(x,y,z) \, dxdydz}{\text{mass}(E)} \), \( c_y = \frac{\int \int_E y \delta(x,y,z) \, dxdydz}{\text{mass}(E)} \), \( c_z = \frac{\int \int_E z \delta(x,y,z) \, dxdydz}{\text{mass}(E)} \). It is only when \( \delta \) is constant that we can simplify the common factor \( \delta \) away and get simplified equations such as \( c_x = \frac{\int \int_E x \delta \, d\delta}{\text{volume}(E)} \), etc. where volume(\( E \)) = \( \int \int_E 1 \, dxdydz \). Also, when \( \delta \) is just a constant, \( \delta = \frac{\text{mass}(E)}{\text{volume}(E)} \). Another observation is that when \( E \) has both \( x \)-symmetry and \( y \)-symmetry, we have \( c_x = c_y = 0 \).
3. The moment of inertia of \( E \) when rotating around an axis \( l \) is \( I_l = \int \int_E d(x, y, z)^2 \delta(x, y, z) \, dxdydz \) where \( d(x, y, z) \) is the distance from the point \( (x, y, z) \) to the axis of rotation \( l \). For instance, if the axis \( l \) is chosen to be the axis \( Oz \), then \( I_{Oz} = \int \int_E (x^2 + y^2) \delta(x, y, z) \, dxdydz \).

On the other hand, if \( S \) is a 2-dimensional surface in \( \mathbb{R}^3 \) with known mass density, then we simply make the appropriate changes with surface integrals. Let \( G(u, v) = (x(u,v), y(u,v), z(u,v)) \) be a parametrization of the surface and \( N = \partial_u G \times \partial_v G \). Let \( \delta(u, v) \) be the known mass density at \( G(u, v) \) on the surface.

1. The total mass of \( S \) is mass(\( S \)) = \( \int_S \delta = \int \int \delta(u, v)||N(u, v)|| \, dudv \).
2. The mass center of \( E \) is \( c = (c_x, c_y, c_z) \) where \( c_x = \frac{\int \int_S x \delta \, d\delta}{\text{mass}(S)} \), similarly for \( c_y \) and \( c_z \). It is only when \( \delta \) is constant that we can simplify the common factor \( \delta \) away and get simplified equations such as \( c_x = \frac{\int \int_S x ||N(u,v)|| \, dudv}{\text{area}(S)} \), where area(\( S \)) = \( \int \int ||N(u, v)|| \, dudv \). Also, when \( \delta \) is just a constant, \( \delta = \frac{\text{mass}(S)}{\text{area}(S)} \). Another observation is that when \( S \) has both \( x \)-symmetry and \( y \)-symmetry, we have \( c_x = c_y = 0 \).
3. The moment of inertia of \( E \) when rotating around an axis \( l \) is

\[
I_l = \int \int_S d^2 \delta \, dS = \int \int (G(u, v))^2 \delta(u, v)||N(u, v)|| \, dudv
\]

where \( d(x, y, z) \) is the distance from the point \( (x, y, z) \) to the axis of rotation \( l \). For instance, if the axis \( l \) is chosen to be the axis \( Oz \), then \( I_{Oz} = \int \int (x(u, v)^2 + y(u, v)^2) \delta(u, v)||N(u, v)|| \, dudv \).
**REMEMBER NOT TO FORGET THE JACOBIAN** $||N(u,v)||$ when parametrizing the surface integrals.

A small remark is that you should not write integrals such as $\int \int_S 1 \, dx \, dy$. They do not make sense. You are probably thinking of $\int \int_S 1 \, dS$.

**Example.** Assume a surface cone $S$ (as opposed to a solid cone) of radius 1, height 2, and constant mass density 3. Assume that the cone surface only consists of the lateral surface without the base. Then find its mass center and the moment of inertia around the main axis of the cone.

**Solution.** Position the cone $S$ in $\mathbb{R}^3$ such that the origin is the tip of the cone and the base points upwards. We think of the surface of the cone as a collection of horizontal circles of radius $r$, where $r$ is in $(0,1)$. But because the height of the cone is twice its radius, the height of any horizontal circle (aka the distance from its center to the origin of $\mathbb{R}^3$) should also be twice its radius. So one way to parametrize the surface is

$$G : (0,1) \times (0,2\pi) \to S$$ $$\quad \quad \quad \quad \quad (r,\theta) \mapsto (r \cos \theta, r \sin \theta, 2r)$$

Then the Jacobian is $||N|| = ||\partial_r G \times \partial_{\theta} G|| = ||(\cos \theta, \sin \theta, 2) \times (-r \sin \theta, r \cos \theta, 0)|| = \sqrt{5} r$. Note that we can also use cylindrical coordinates and parametrize by $G(z,\theta) = \left(\frac{z}{2} \cos \theta, \frac{z}{2} \sin \theta, z\right)$ where $z$ is in $(0,2)$ and $\theta$ is in $(0,2\pi)$.

Let $c = (c_x, c_y, c_z)$ be the mass center. Then $c_x = c_y = 0$ by symmetry. As $\delta$ is constant, we can simplify away $\delta$ and get

$$c_z = \frac{\int_0^{2\pi} \int_0^1 z(r,\theta)||N(r,\theta)|| \, dr \, d\theta}{\int_0^{2\pi} \int_0^1 ||N(r,\theta)|| \, dr \, d\theta} = \frac{\int_0^{2\pi} \int_0^1 2r \sqrt{5} r \, dr \, d\theta}{\int_0^{2\pi} \int_0^1 \sqrt{5} r \, dr \, d\theta} = \frac{\int_0^1 2r^2 \, dr}{\int_0^1 r \, dr} = 4 \cdot \frac{1}{3} = \frac{4}{3}.$$

We could also have used the formula $c_z = \frac{\int_0^{2\pi} \int_0^1 z(r,\theta)\delta(r,\theta)||N(r,\theta)|| \, dr \, d\theta}{\text{mass}(S)}$. It should give the same answer.

Then the mass center is $(0,0,\frac{4}{3})$.

On the other hand, the moment of inertia is

$$I_{Oz} = \int_0^{2\pi} \int_0^1 (x(r,\theta)^2 + y(r,\theta)^2)\delta(r,\theta)||N(r,\theta)|| \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^1 r^2 \sqrt{5} r \, dr \, d\theta = \frac{3\sqrt{5}}{2} \pi$$

**REMEMBER NOT TO FORGET THE JACOBIAN** $||N(r,\theta)||$ when parametrizing the surface integrals.