Math 133

Khang Manh Huynh

Notation  As analysts, we will often be using \( \lesssim, \gtrsim \) and \( \sim \) to throw away unimportant constants. For instance, instead of \(|x_n| \leq 2|y_n|\), we can just write \(|x_n| \lesssim |y_n|\). Furthermore, \( A \lesssim_{x,y} B \) means \( A \leq CB \) where \( C > 0 \) depends on \( x \) and not \( y \). Similarly, \( A \sim_{x,y} B \) means \( A \lesssim_{x,y} B \) and \( B \lesssim_{x,y} A \). Depending on the context, the dependencies might not need to be made explicit.

Throughout the note, let \( F \) be either \( \mathbb{R} \) or \( \mathbb{C} \).

People interested in homework should pay attention to Lemma 5, Lemma 13, and Definition 26.

1 Normed vector spaces

Definition 1. \( V \) be a vector space over \( F \).
We say \( ||\cdot|| : V \to [0,\infty) \) is a norm and \( V \) a normed vector space (NVS) when
1. \( \|\lambda u\| = |\lambda|\|u\| \forall \lambda \in F, \forall u \in V \)
2. \( \forall u \in V : \|u\| = 0 \iff u = 0 \)
3. \( \forall u, v \in V : \|u + v\| \leq \|u\| + \|v\| \)

For a sequence \( (v^j)_{j \in \mathbb{N}} \) in \( V \), we say \( v^j \to v \) when \( \|v^j - v\| \to 0 \).

For 2 different norms \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \), we say they are equivalent when \( \|v^j - v\|_1 \to 0 \iff \|v^j - v\|_2 \to 0 \). This is the same as having \( \|v\|_1 \sim \|v\|_2 \forall v \in V \).

Proof. \( \Leftarrow \) is obvious by the squeeze theorem. So assume \( (\|v_j - v\|_1 \to 0 \iff \|v_j - v\|_2 \to 0) \) is true. We only need to show \( \|v\|_2 \lesssim \|v\|_1 \). Assume towards a contradiction that we have \( (v_j)_{j \in \mathbb{N}} \)
such that \( \|v_j\|_2 > j \|v_j\|_1 \forall j \in \mathbb{N} \). Let \( \bar{v}_j = \frac{v_j}{\|v_j\|_2} \) and observe that \( \|\bar{v}_j\|_1 \to 0 \) while \( \|\bar{v}_j\|_2 = 1 \).

Definition 2. NVS \( V \) is said to be (Cauchy) complete when every Cauchy sequence converges. A Cauchy complete NVS is also called a Banach space.

If \( V \) is a Banach space with a compatible inner product \( (\|v\| = \sqrt{\langle v, v \rangle}) \), then we say \( V \) is a Hilbert space. As an example, \( (\mathbb{R}^n, \|\cdot\|_E) \) is a Hilbert space where \( \|\cdot\|_E \) is the usual Euclidean norm. \( (\|v\|_E = \sqrt{\sum_{i=1}^{n} \|v_i\|^2}) \)

Theorem 3 (Continuity). Let \( V_1 \) and \( V_2 \) be NVS and \( f : V_1 \to V_2 \) be a linear map. Then the following are equivalent:
1. \( f \) is continuous \( (v_j \to v \implies f(v_j) \to f(v)) \)
2. \( v_j \to 0 \implies f(v_j) \to 0 \)
3. \( \|f(v)\|_{V_2} \lesssim \|v\|_{V_1} \forall v \in V_1 \)
Proof. (1) \implies (2) is obvious. (2) \implies (1) follows because \(v_j \to v \implies v_j - v \to 0 \implies f(v_j) \to f(v)\).

\(T\) \implies (2) is just the squeeze theorem. For (2) \implies (3), mimic the proof above. Assume there is \((v_j)_{j \in \mathbb{N}}\) such that \(\|f(v_j)\|_V > j\|v_j\|_V\) \forall j \in \mathbb{N}. Let \(v_j = \frac{v_i}{\|f(v_i)\|_V}\) and observe that \(\|v_j\|_V \to 0\) while \(\|f(v_j)\|_V = 1\). \(\square\)

**Theorem 4.** All norms on \(\mathbb{R}^n\) are equivalent.

**Proof.** Let \(\|\cdot\|_E\) be the usual Euclidean norm and \((e_j)_{j=1}^n\) be the standard basis. Let \(\|\cdot\|_N\) be any norm on \(\mathbb{R}^n\). Observe that the nonlinear function \(\|\cdot\|_N\) is continuous on \((\mathbb{R}^n,\|\cdot\|_E)\) by the triangle inequality and \(\|e_j\|_N \leq y_j 1\). We just need \(\|v\|_N \sim \|v\|_E \forall v \in \mathbb{R}^n\). By scalar multiplication, this is the same as \(\|v\|_N \sim 1 \forall v \in S^{n-1}\) where \(S^{n-1} = \{v : \|v\|_E = 1\}\). But by Bolzano-Weierstrass on each coordinate, the sphere \(S^{n-1}\) is compact (any sequence has a convergent subsequence), so \(\|\cdot\|_N\) has a positive maximum and minimum on \(S^{n-1}\). \(\square\)

**Remark.** We won’t directly use this elegant theorem, but the idea of considering the compact sphere will reappear shortly.

**Lemma 5** (Homogeneous polynomial). Let \(P : \mathbb{R}^n \to \mathbb{R}\) be a homogeneous polynomial of order \(k\) in \(n\) variables. Then \(|P(x)| \leq \rho \|x\|^k_E \forall x \in \mathbb{R}^n\). If \(P(x) > 0 \forall x \neq 0\) then \(|P(x)| \sim \rho \|x\|^k_E \forall x \in \mathbb{R}^n\).

**Proof.** Note \(P(\lambda x) = \lambda^k P(x) \forall \lambda \in \mathbb{R}\). Consider the compact sphere again. Obviously \(P\) is continuous. \(\square\)

**Theorem 6** (Transitivity of density). Let \(V_1, V_2, V_3\) be NVS. Let \(T_1 : V_1 \to V_2\) and \(T_2 : V_2 \to V_3\) be functions. Assume \(T_2\) is continuous. If \(\text{Im} \ T_1\) and \(\text{Im} \ T_2\) are dense, then \(\text{Im} \ T_2 \circ T_1\) is also dense.

**Proof.** Let \(z \in V_3\) and \(\varepsilon > 0\). We want \(x \in V_1\) such that \(\|T_2 T_1 x - z\|_{V_3} < \varepsilon\). As \(\text{Im} \ T_2\) is dense, \(\exists y \in V_2 : \|T_2 y - z\|_{V_3} < \frac{\varepsilon}{2}\). Then there is \((x_j)_{j \in \mathbb{N}}\) in \(V_1\) such that \(\|T_1 x_j - y\|_{V_2} \to 0\), which implies \(\|T_2 T_1 x_j - T_2 y\|_{V_3} \to 0\). Easily pick \(x = x_j\) for some large \(j\). \(\square\)

**Remark.** A linear, injective, continuous map with dense image is called a dense immersion. In particular, the composition of 2 dense immersions is a dense immersion.

## 2 Function spaces

**Definition 7.** Let \(T\) be the 1-dimensional torus, which can also be written as the unit circle \(S^1\), or \([-\pi, \pi]\) where \(\pi\) is identified with \(-\pi\) (forming a loop). Other authors might instead choose \([0, 1]\), where 1 is glued into 0. Define \(C(T, \mathbb{F})\) as the NVS of continuous \(\mathbb{F}\)-valued functions on \(T\) (they must be bounded and uniformly continuous since \(T\) is compact), where the norm is \(\|f\|_{C(T)} = \sup_{x \in T} |f(x)|\) (sup norm). Then for \(p \in [1, \infty)\), \(L^p(T, \mathbb{F})\) is the metric completion of \(C(T, \mathbb{F})\), where \(\|f\|_{L^p(T)} = (\int_T |f|^p) \frac{1}{p} \forall f \in C(T, \mathbb{F})\) (defined by the Riemann integral).

Define \(C_0(\mathbb{R}, \mathbb{F})\) as the NVS of continuous \(\mathbb{F}\)-valued functions on \(\mathbb{R}\) which vanish at \(\pm\infty\) (\(f(x) \to 0\) as \(x \to \pm\infty\)). Such functions must be bounded and uniformly continuous (easily proven by vanishing at infinity), and \(\|f\|_{C_0(\mathbb{R})} = \sup_{x \in \mathbb{R}} |f(x)|\). Also, they are Riemann integrable on \([-N, N] \forall N > 0\). In particular, the
Gaussian functions are in $C_0(\mathbb{R}, F)$. Then for $p \in [1, \infty)$, $L^p(\mathbb{R}, F)$ is the metric completion of \( \{ f \in C_0(\mathbb{R}, F) : \| f \|_{L^p(\mathbb{R})} < \infty \} \), where \( \| f \|_{L^p(\mathbb{R})} = \left( \lim_{N \to \infty} \int_{-N}^N |f|^p \right)^{1/p} \) $\forall f \in C_0 \cap L^p$ (defined by the Riemann integral).

Define $c_0(\mathbb{Z}, F)$ as the space of $F$-valued functions on $\mathbb{Z}$ (i.e. sequences) which vanish at $\pm \infty$ ($f(x) \to 0$). Such functions must be bounded, and $\| f \|_{c_0(\mathbb{Z})} = \sup_{x \in \mathbb{Z}} |f(x)|$. Then for $p \in [1, \infty)$, define $L^p(\mathbb{Z}, F)$ as $\{ f \in c_0(\mathbb{Z}, F) : \| f(x) \|_{L^p(\mathbb{Z})} < \infty \}$, where $\| f \|_{L^p(\mathbb{Z})} = \left( \sum_{i \in \mathbb{Z}} |f(i)|^p \right)^{1/p}$ (defined by series). Unlike the above cases, we also define $L^\infty(\mathbb{Z}, F) = \{ \langle f : \mathbb{Z} \to F \rangle : \sup |f| < \infty \}$ with $\| \cdot \|_\infty = \| \cdot \|_{\sup}$. Observe that the constant sequence $(1)_{j \in \mathbb{Z}}$ is in $L^\infty$ but not $c_0$.

Let $\Omega \in \{ \mathbb{T}, \mathbb{R}, \mathbb{Z} \}$. For convenience’s sake, we write $L^p$ for $L^p(\Omega, F)$, and $C_0$ for $C(\mathbb{T}, F)$, $C_0(\mathbb{R}, F)$ or $c_0(\mathbb{Z}, F)$. Define the integral $\int_\Omega$ as either $\int_\mathbb{T}$, $\lim_{N \to \infty} \int_{-N}^N$ or $\sum_{i \in \mathbb{Z}}$ as above.

$L^p$ is the metric completion of $C_0 \cap L^p, \| \cdot \|_{L^p}$ where $C_0 \cap L^p = \{ f \in C_0 : \int_{\Omega} |f|^p < \infty \}$

In particular, even though metric completion is an abstract construction based on Cauchy sequences, we can canonically identify $C_0 \cap L^p$ as a subset of $L^p$ (by using constant sequences), and so we can say $\| f \|_{L^p} = \left( \int_\Omega |f|^p \right)^{1/p} \forall f \in C_0 \cap L^p$.

Note that when $\Omega = \mathbb{T}$, $C_0 \cap L^p = C(\mathbb{T}, F)$.

We say $E \subset \Omega$ is an interval when there is $\varepsilon > 0$ such that $E = (-\varepsilon, \varepsilon) \cap \Omega$ in the case $\Omega \in \{ \mathbb{Z}, \mathbb{R} \}$, or $E = S^1 \cap \{ z \in \mathbb{C} : |z-1| < \varepsilon \}$ in the case $\Omega = \mathbb{T} = S^1$.

For simplicity, we will only discuss $L^1, L^2$ and $C_0$ in this course. We also define $L^1 \cap L^2$ as the metric completion of $\{ (C_0 \cap L^1) \cap (C_0 \cap L^2) : \| f \|_{L^1 \cap L^2} \}$ where $\| f \|_{L^1 \cap L^2} = \max(\| f \|_{L^1}, \| f \|_{L^2})$.

In particular, $L^2$ has a canonical Hilbert space structure by setting $\langle f, g \rangle = \int_{\Omega} f \overline{g} \forall f, g \in C_0$ and then defining $\langle \cdot, \cdot \rangle$ on $L^2$ by density (past homework). Note that this means $\langle \cdot, \cdot \rangle$ on $L^2$ cannot be defined by Riemann integrals in the case $\Omega \in \{ \mathbb{T}, \mathbb{R} \}$.

It is a trivial exercise to make analogous definitions when the domain is $\mathbb{T}^n, \mathbb{R}^n$ or $\mathbb{Z}^n$.

**Theorem 8 (Inequalities).** Show that

1. **Holder’s inequality:**
   \[
   \| fg \|_{L^p} \leq \| f \|_{L^r} \| g \|_{L^s} \text{ for } f \in C_0 \cap L^p, g \in C_0, p \in \{1, 2\}
   \]
   \[
   \| fg \|_{L^1} \leq \| f \|_{L^2} \| g \|_{L^2} \text{ for } f, g \in C_0 \cap L^2
   \]
   \[
   \| fg \|_{L^2} \leq \| f \|_{L^1} \| g \|_{L^\infty} \text{ for } f \in C_0 \cap L^1
   \]

2. **Minkowski’s inequality:**
   \[
   \| f + g \|_{L^p} \leq \| f \|_{L^p} + \| g \|_{L^p} \text{ for } f, g \in C_0
   \]
   \[
   \| f + g \|_{L^p} \leq \| f \|_{L^p} + \| g \|_{L^p} \text{ for } f, g \in C_0 \cap L^p, p \in \{1, 2\}.
   \]

**Proof.**

1. \[
\| fg \|_{L^p} = \left( \int |fg|^p \right)^{1/p} \leq \left( \int |f|^p \right)^{1/p} \sup |g| = \| f \|_{L^p} \| g \|_{\sup}
\]
   \[
\| fg \|_{L^1} = \| f \|_{L^1} \| g \|_{\sup} \leq \| f \|_{L^2} \| g \|_{L^2} \text{ (Cauchy-Schwarz)}
\]
   \[
\| fg \|_{L^2} = \| f \|^2 \leq \sup \| f \| \| f \|
\]

2. Trivial. \( \| f + g \|_{L^2} \leq \| f \|_{L^2} + \| g \|_{L^2} \) comes from Cauchy-Schwarz.

**Remark.** Cauchy-Schwarz is a special case of Holder’s inequality. Note that we needed these inequalities in order to even define $(C_0 \cap L^p, \| \cdot \|_{L^p})$ as a metric space.

We can replace $C_0 \cap L^p$ by $L^p$ in the statements of these inequalities, but we must be careful how things are defined. For instance, let $f \in L^p, g \in C_0$. This means $\langle f \rangle = \{(f_j)_{j \in N} \}$ (equivalence class) where $(f_j)_{j \in N}$
is a Cauchy sequence in \((C_0 \cap L^p, \| \cdot \|_{L^p})\). Then by the theorem, \((f_jg_j)_{j \in \mathbb{N}}\) is also a Cauchy sequence in 
\((C_0 \cap L^p, \| \cdot \|_{L^p})\). Then define \([fg] = [(f_jg_j)_{j \in \mathbb{N}}]\) and conclude \(\|fg\|_{L^p} \leq \|f\|_{L^p} \|g\|_{sup}\).

Similarly, for \([f] = [(f_j)_{j \in \mathbb{N}}], [g] = [(g_j)_{j \in \mathbb{N}}] \in L^2\), define \([fg] = [(f_jg_j)_{j \in \mathbb{N}}] \in L^1\) etc.

**Definition 9.** Define \(C_c(\mathbb{R}, F) = \{ f \in C_0(\mathbb{R}, F) : \text{sup} f \text{ bounded}\}\) and \(c_c(\mathbb{Z}, F) = \{ f \in c_0(\mathbb{Z}, F) : \text{sup} f \text{ bounded}\}\). Obviously \(C_c(\mathbb{T}, F) = C(\mathbb{T}, F)\). Write \(C_c\) for \(C_c(\mathbb{R}, F), c_c(\mathbb{Z}, F)\) or \(C(\mathbb{T}, F)\).

We collect some basic properties:

**Lemma 10.** Let \(p \in \{1, 2\}\).

1. **Simple step functions** (finite sum of indicator functions \(1_E\) where \(E \subset \Omega\) is a bounded interval) are dense in \(L^p\).

2. \((C_0, \| \cdot \|_{sup})\) and \(L^p\) are Banach spaces. Note that \((C_0 \cap L^p, \| \cdot \|_{L^p})\) is **NOT** complete.

3. \((C_c, \| \cdot \|_{sup}) \hookrightarrow (C_0 \cap L^p, \| \cdot \|_{L^p})\) is dense immersion.

4. \((C_c, \| \cdot \|_{L^p}) \hookrightarrow L^p\) is dense immersion. Same for \((C_c, \| \cdot \|_{L^1 \cap L^2}) \hookrightarrow L^1 \cap L^2\).

5. \((C(\mathbb{T}, F), \| \cdot \|_{sup}) \hookrightarrow L^2(\mathbb{T}, F)\) and \(L^2(\mathbb{T}, F) \hookrightarrow L^1(\mathbb{T}, F)\) are dense immersions.

6. \(l^1(\mathbb{Z}, F) \hookrightarrow l^2(\mathbb{Z}, F)\) and \(l^2(\mathbb{Z}, F) \hookrightarrow c_0(\mathbb{Z}, F)\) are dense immersions. Note \(c_0(\mathbb{Z}, F) \hookrightarrow l^\infty(\mathbb{Z}, F)\) does **NOT** have dense image.

**Remark.** \(C(\mathbb{T}, F) \hookrightarrow L^2(\mathbb{T}, F)\) means that uniform convergence (convergence in the sup norm) implies \(L^2\) convergence on \(\mathbb{T}\). Note that this is **NOT** true for \(\mathbb{R}\) or \(\mathbb{Z}\). Also observe that \((C_0, \| \cdot \|_{sup}) \hookrightarrow L^p\) is **NOT** even defined when \(\Omega \in \{\mathbb{R}, \mathbb{Z}\}\).

Observe that, in a sense, \(\mathbb{T}\) and \(\mathbb{Z}\) are “inverses” of each other, where \(\mathbb{T}\) has finite “size” but infinitely small “atoms”, while \(\mathbb{Z}\) has infinite “size” but discrete “atoms”.

It is a trivial search-and-replace exercise to adapt the lemma to \(\mathbb{T}^n, \mathbb{R}^n\) or \(\mathbb{Z}^n\).

**Proof.**

1. Approximate \(C_0 \cap L^p\) functions in the \(L^p\) norm by using simple step functions (done in homework)

2. \(L^p\) is complete as it is metric completion. \((C_0, \| \cdot \|_{sup})\) is complete by Analysis II, and showing that vanishment at infinity is preserved under taking limits.

3. Only worry when \(\Omega \in \{\mathbb{Z}, \mathbb{R}\}\). Multiply functions in \(C_0 \cap L^p\) with a \(C_c\) cutoff.

4. Also by multiplying with a \(C_c\) cutoff, observe \((C_c, \| \cdot \|_{L^p}) \hookrightarrow (C_0 \cap L^p, \| \cdot \|_{L^p})\) has dense image. By definition, \((C_0 \cap L^p, \| \cdot \|_{L^p}) \hookrightarrow L^p\) is dense immersion, so we’re done by transitivity (Theorem 6). Proceed similarly for \(L^1 \cap L^2\).

5. \(\|f\|_{L^2(\mathbb{T})} \leq \|1\|_{L^2(\mathbb{T})} \|f\|_{sup}\) and \(\|f\|_{L^1(\mathbb{T})} \leq \|1\|_{L^2(\mathbb{T})} \|f\|_{L^2(\mathbb{T})}\) so the maps are continuous. They have dense images since \(C_c \hookrightarrow L^p\) has dense image.

6. Note that \(\forall x \in \mathbb{Z} : |f(x)| \leq \left(\sum_{i \in \mathbb{Z}} |f(i)|^2\right)^{\frac{1}{2}}\) and \(|f(x)| \leq \sum_{i \in \mathbb{Z}} |f(i)|\) (infinite sums possibly being \(\infty\)) so \(l^1 \hookrightarrow c_0\) and \(l^2 \hookrightarrow c_0\) are continuous, linear, injective. Then observe that
\[
\left(\sum_{i \in \mathbb{Z}} |f(i)|^2\right)^{\frac{1}{2}} \leq \sup |f|^{\frac{1}{2}} \left(\sum_{i \in \mathbb{Z}} |f(i)|^2\right)^{\frac{1}{2}} = \|f\|_{L^1} \leq \|f\|_{L^2}.
\]
For dense images, simply recall \(C_c \rightarrow L^2\) and \(C_c \rightarrow \left(C_0, \|\cdot\|_{\sup}\right)\) have dense images.

\[\square\]

3 Mollification & convolution

**Definition 11.** We continue to let \(\Omega \in \{\mathbb{T}, \mathbb{R}, \mathbb{Z}\}\) and define \(\int_\Omega\) as before. Note that addition/subtraction is defined on \(\Omega\) (if you’ve taken algebra, this means \(\Omega\) is an abelian group). So **convolution** can be defined, such as \(f \ast g(x) = \int_\Omega f(y)g(x-y) \, dy = \int_\Omega f(x-y)g(y) \, dy\) for \(f, g \in C_c\).

Recall from the proof of Parseval’s theorem for Gaussian functions, we implicitly accepted **Fubini’s theorem**:

\[
\forall f \in C_0 \cap L^1(\Omega^2) : \int_\Omega \int_\Omega f(x,y) \, dx \, dy = \int_\Omega \int_\Omega f(x,y) \, dy \, dx.
\]

In particular, for \(f \in C_0 \cap L^1(\Omega^2), f(\cdot, y) \in C_0 \cap L^1(\Omega) \forall y \in \Omega\) and \(\|f(x,y)\|_{L^1_y} \in L^1_x(\Omega)\). We also have

\[
\|f(x,y)\|_{L^1_y} = \|f(x,y)\|_{L^1_y} \leq \|f(x,y)\|_{L^1_y} \leq \|f(x,y)\|_{L^1_y}.
\]

**Theorem 12** (L^2 Minkowski / interchanging bars). Show that for \(f \in C_c(\Omega^2)\):

\[
\left\|\|f(x,y)\|_{L^1_y}\right\|_{L^2_y} \leq \left\|\|f(x,y)\|_{L^1_y}\right\|_{L^2_y}
\]

**Remark.** Also trivial to check \(\|f(x,y)\|_{L^1_y} \leq \|f(x,y)\|_{L^1_y} \leq \|f(x,y)\|_{L^1_y}\).

**Proof.** By bounded support and uniform continuity, we can prove that \(\|f(x,y)\|_{L^2_y} \in (C_c)_x(\Omega)\) and \(\|f(x,y)\|_{L^1_y} \in (C_c)_y(\Omega)\), so both sides are well-defined. Define \(F(y) = \|f(x,y)\|_{L^1_y}\), then observe that

\[
\|F(y)\|_{L^2_y} = \sup_{g \in C_c(\Omega)} \langle F, |g| \rangle
\]

This is a consequence of Cauchy-Schwarz (CS), and that \(g\) can be \(F\). Then

\[
\left\|\|f(x,y)\|_{L^1_y}\right\|_{L^2_y} = \sup_{\|g\|_{L^2_y}=1} \left\|\|f(x,y)\|_{L^1_y} g(y)\right\|_{L^2_y} \overset{\text{CS}}{\leq} \sup_{\|g\|_{L^2_y}=1} \left\|\|f(x,y)\|_{L^1_y} g(y)\right\|_{L^2_y} \sup_{\|g\|_{L^2_y}=1} \left\|\|f(x,y)\|_{L^1_y} g(y)\right\|_{L^2_y}.
\]

**Remark.** If we pretend the \(L^1\) integral is just a sum, this is simply the triangle inequality. Playing around with norm bars \(\|\cdot\|\), using Minkowski, Holder, Fubini etc. is also what much of analysis will look like.
Equation (1), the most important insight which reduces an $L^2$ problem into an $L^1$ problem, is a particularly elegant use of duality, a universal theme in mathematics which lies outside the scope of this course.

**Lemma 13** (Young’s convolution inequality). Show that for $f, g \in C_c(\Omega)$:

1. $\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$ for $p \in \{1, 2\}$.
2. $\|f * g\|_{\sup} \leq \|f\|_{\sup} \|g\|_{L^1}$

**Proof.**

1. Play with bars. The reader should justify each step:

$$\|f * g\|_{L^1} \leq \|f(x - y)g(y)\|_{L^1_{x,y}} \leq \|f(x - y)\|_{L^1_x} \|g(y)\|_{L^1_y}$$

$$= \|f(x - y)\|_{L^p_x} \|g(y)\|_{L^1_y} = \|f\|_{L^p_x} \|g\|_{L^1_y}$$

For the 2nd step, when $p = 2$, we need Theorem 12. But when $p = 1$, we only need Fubini.

2. $\forall x \in \Omega : |f * g(x)| \leq \|f(x - y)g(y)\|_{L^1_y} \leq \|f\|_{\sup} \|g\|_{L^1_y}$

**Remark.** As usual, we can extend the result by density. For instance, $\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$ for $f \in L^p(\Omega), g \in L^1(\Omega)$, where $f * g$ is defined by a Cauchy sequence etc.

We next introduce 2 simple, yet endlessly useful tools:

**Theorem 14** (Unique extension). Let $X, Y$ be NVS and $X_0 \subset X$ be a dense subspace. Let $T : X_0 \to Y$ be a continuous linear map. If $Y$ is Banach (i.e. Cauchy complete), then there is a unique extension of $T$ to a continuous linear map $\tilde{T} : X \to Y$ such that $\tilde{T}\big|_{X_0} = T$.

**Remark.** This theorem is extremely important. The general Fourier transform on $L^2$ is only defined thanks to it.

**Proof.** Recall from Theorem 3 that $T$ being continuous means $\|Tv\|_Y \leq \|v\|_X \forall v \in X_0$.

Let $x \in X$. Then there is $(x_j)_{j \in \mathbb{N}}$ in $X_0$ such that $x_j \to x$. Simply define $\tilde{T}x = \lim_{j \to \infty} Tx_j$.

There are multiple things one needs to check: that the choice of $(x_j)_{j \in \mathbb{N}}$ does not matter, or that $\lim_{j \to \infty} Tx_j$ exists ($Y$ being Banach is necessary), or that $T$ is linear. But they should be trivial.

**Lemma 15** (Dense convergence). Let $X, Y$ be Banach spaces and $X_0 \subset X$ be a dense subspace. Let $(T_n)_{n \in \mathbb{N}}$ and $T$ be linear maps from $X$ to $Y$ such that $\exists M > 0 : \|T_n x\| + \|T x\| \leq M \|x\| \forall x \in X, \forall n \in \mathbb{N}$.

If $T_n x_0 \to T x_0 \forall x_0 \in X_0$ (dense convergence) then $T_n x \to T x \forall x \in X$ (total convergence)

**Proof.** Obviously $T_n$ and $T$ are continuous. Let $x \in X$. There is $(x_j)_{j \in \mathbb{N}}$ in $X_0$ such that $x_j \to x$.

Then $\forall j, n \in \mathbb{N}$:

$$\|Tx - T_n x\| \leq \|Tx - T x_j\| + \|T x_j - T_n x_j\| + \|T_n x_j - T_n x\| \leq M \|x - x_j\| + \|T x_j - T_n x_j\|$$
So \( \limsup_{n \to \infty} \| T x - T_n x \| \leq M \| x - x_j \| + \limsup_{n \to \infty} \| T x_j - T_n x_j \| = M \| x - x_j \| . \) True for all \( j \), so \( \limsup_{n \to \infty} \| T x - T_n x \| = 0. \)

**Corollary 16** (Continuity of translation). Let \( p \in \{ 1, 2 \} \). For \( h \in \Omega \), define \( \tau_h f(x) = f(x + h) \forall f \in C(\Omega) \). As \( \| \tau_h f \|_{L^p} = \| f \|_{L^p} \forall f \in C_0 \cap L^p \), there is a unique extension to make \( \tau_h : L^p \to L^p \) an isometry (translation invariance). Then show that

1. \( \tau_h f \xrightarrow{\| \cdot \|_{L^p}} f \forall f \in C_0(\Omega) \)
2. \( \tau_h f \xrightarrow{L^p_{h\downarrow 0}} f \forall f \in L^p(\Omega) \)

**Remark.** The problem is trivial when \( \Omega = \mathbb{Z} \), as a sequence going to 0 is just 0 after some finite time.

**Proof.**

1. Uniform continuity.

2. By dense convergence, we just need \( \tau_h f \xrightarrow{L^p_{h\downarrow 0}} f \forall f \in C_c(\Omega) \). Observe that \( \tau_h f \xrightarrow{\| \cdot \|_{L^p}} f \). We can find a bounded interval \( E \subset \Omega \) such that \( \text{supp} f \cup \text{supp} \tau_h f \subset E \) for all \( h \) small. Then \( \| \tau_h f - f \|_{L^p} \xrightarrow{\text{holder}} 0 \). Note the similarity to \( C(\mathbb{T}) \to L^p(\mathbb{T}) \).

**Corollary 17** (Mollification). Let \( p \in \{ 1, 2 \} \) and \( \Omega \in \{ \mathbb{T}, \mathbb{R} \} \).

Let \( \phi \in C_0 \cap L^1(\mathbb{R}) \) such that \( \int_{\Omega} \phi = 1 \). Define \( \phi_n(x) = n\phi(nx) \forall n \in \mathbb{N} \). If \( \Omega = \mathbb{T} \), further require \( \text{supp} \phi \subset [-\frac{1}{2}, \frac{1}{2}] \). Canonicallly identify \( [-\frac{1}{2}, \frac{1}{2}] = \mathbb{T} \).

Then \( \phi_n \in C_0 \cap L^1(\Omega) \) and \( \int_{\Omega} \phi_n = 1 \) (change of variables). Show that

1. \( \phi_n \ast f \xrightarrow{n \to \infty} f \forall f \in C_0(\Omega) \)
2. \( \phi_n \ast f \xrightarrow{L^p_{n \to \infty}} f \forall f \in L^p(\Omega) \)

We can therefore conclude that \( C_c^\infty(\Omega) \) is dense in \( C_0 \) and \( L^p \).

**Proof.** The chief insights are simply \( f(x) = \int_{\Omega} f(y) \phi_n(y) dy = \int_{\Omega} f(x) \phi(y) dy \) while \( \phi_n \ast f(x) = \int_{\Omega} f(x - \frac{y}{n}) \phi(y) dy \) (change of variables)

1. For \( x \in \Omega \):

\[
|\phi_n \ast f(x) - f(x)| \leq \int_{\Omega} \left| f\left(x - \frac{y}{n}\right) - f(x)\right| \phi(y) dy \leq \| \tau_{-\frac{x}{n}} f - f \|_{L^p_{\sup}} \| \phi \|_{L^1_{\sup}} \xrightarrow{n \to \infty} 0
\]

2. By Young’s convolution inequality: \( \| \phi_n \ast f \|_{L^p} \leq \| \phi_n \|_{L^1} \| f \|_{L^p} = \| \phi \|_{L^1} \| f \|_{L^p} \). By dense convergence, we just need \( \phi_n \ast f \xrightarrow{L^p_{n \to \infty}} f \forall f \in C_c(\Omega) \). Mimic the previous proof: observe that \( \phi_n \ast f \xrightarrow{\| \cdot \|_{L^p}} f \). Again, find a bounded interval \( E \subset \Omega \) such that \( \text{supp} f \cup (\text{supp} \phi_n \ast f) \subset E \) for all \( n \) large. Then \( \phi_n \ast f \xrightarrow{L^p_{n \to \infty}} f \) by Holder.
4 Fourier transforms

Definition 18. Let $\mathcal{F}$ be the Fourier transform. Depending on the choice of the spatial domain $\Omega$, the frequency domain $\hat{\Omega}$ can be different. For instance, $\hat{T} = \mathbb{Z}, \hat{Z} = T$ and $\hat{\mathbb{R}} = \mathbb{R}$. Note that $\hat{\Omega}$, which is a very deep fact in harmonic analysis.

Define $C_{\text{loc}} \cap L^1(\Omega) = \{ f : f$ continuous on $\Omega, \int_{\Omega} |f| < \infty \}$. Note that we don’t require $f$ to be bounded, but $f$ is locally bounded so in the case $\Omega = \mathbb{R}, \lim_{N \to \infty} \int_{-N}^{N} |f| \text{ is still well-defined by the Riemann integral.}$ Analogously, we can define $C_{\text{loc}} \cap L^2(\Omega)$ and $C_{\text{loc}} \cap L^1 \cap L^2(\Omega)$. Note that continuous functions on $T$ must be bounded.

We also define $C \cap L^1(\Omega) = \{ f : f$ continuous and bounded on $\Omega, \int_{\Omega} |f| < \infty \}$. In general, $C$ denotes continuous and bounded functions, while $C_{\text{loc}}$ denotes continuous and locally bounded functions. (Beware that other authors might instead use $BC$ and $C$ respectively)

If we stick to the convention $T = [-\frac{1}{2}, \frac{1}{2}]$, then we can define

$$\forall f \in C_{\text{loc}} \cap L^1(\Omega), \forall \xi \in \hat{\Omega} : \mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{\Omega} f(x)e^{-2\pi i x \xi}dx$$

Easily check that the Riemann integral is well-defined since $\int_{\Omega} |f(x)e^{-2\pi i x \xi}|dx = \int_{\Omega} |f(x)|dx < \infty$. Also define the inverse Fourier transform

$$\forall f \in C_{\text{loc}} \cap L^1(\Omega), \forall \xi \in \hat{\Omega} : \mathcal{F}^{-1}(f)(\xi) = \mathcal{F}^*(f)(\xi) = \hat{f}(\xi) := \int_{\Omega} f(x)e^{2\pi i x \xi}dx$$

Lemma 19 (Riemann-Lebesgue lemma). There is a unique extension turning $\mathcal{F}$ into $\mathcal{F}_{L^1} : L^1(\Omega) \to C_0(\hat{\Omega})$ which is continuous and linear. Same for $\mathcal{F}^*_{L^1} : L^1(\Omega) \to C_0(\hat{\Omega})$.

Proof. First observe that $\forall f \in C_c(\Omega), \hat{f}(\xi) = \int_{\Omega} f(x)e^{-2\pi i x \xi}dx$ is continuous in $\xi$ by uniform continuity of $f$. Then note

$$\forall f \in C_c(\Omega), \xi \in \hat{\Omega} : |\hat{f}(\xi)| = \left| \int_{\Omega} f(x)e^{-2\pi i x \xi}dx \right| \leq \int_{\Omega} |f(x)|dx = \|f\|_{L^1}$$

Since $C_c$ is dense in $L^1$, by unique extension, we have $\mathcal{F}_{L^1} : L^1(\Omega) \to \left( C \left( \hat{\Omega} \right), \| \cdot \|_{\text{sup}} \right)$ continuous and linear. We are done if we can find $S$ dense in $L^1$ such that $\mathcal{F}_{L^1}(S) \subset C_0(\hat{\Omega})$ (vanishing at infinity).

When $\Omega = \mathbb{Z}$: nothing to do since $C_0(\hat{\Omega}) = C \left( \hat{\mathbb{Z}} \right) = C(T)$.

When $\Omega = T$: let $S = C(T)$. Then $\forall f \in S : \left\| \hat{f}(\xi) \right\|_{L^2(\mathbb{Z})} \sim \|f\|_{L^2},$ so $\hat{f}(\xi) \xrightarrow{|\xi| \to \infty} 0$.

When $\Omega = \mathbb{R}$: let $S$ be the set of Gaussian functions.

Recall from lectures that we proved Parseval’s theorem

$$\|\mathcal{F}f\|_{L^2(\hat{\Omega})} = \|f\|_{L^2(\Omega)} \forall f \in C_{\text{loc}} \cap L^1 \cap L^2(\Omega)$$

\[\square\]
In particular, when \( \Omega \in \{ T, Z \} \), this is just Pythagoras. For \( \Omega = R \), we first proved for Gaussian functions, which are dense in \( (C_{\text{loc}} \cap L^2, \| \cdot \|_{L^2}) \) so the equality follows by density.

In fact, as \( (C_{\text{loc}} \cap L^2(\Omega), \| \cdot \|_{L^2}) \) is itself dense in \( L^2(\Omega) \), we conclude there is a unique extension making \( F_{L^1} : L^2(\Omega) \to L^2(\hat{\Omega}) \) an isometry (continuous, linear, bijective, norm-preserving). Same for \( F_{L^2}^* \).

In summary:

\[
F_{L^1} \text{ extends from } F|_{C_{\text{loc}} \cap L^1} \text{ while } F_{L^2} \text{ extends from } F|_{C_{\text{loc}} \cap L^1 \cap L^2}
\]

BEWARE: different conventions exist, such as \( T = [-\pi, \pi] \), or using \( e^{-i\xi x} \) instead of \( e^{-i2\pi x} \), or using sine and cosine... In such cases, we might only have \( \| \hat{f} \|_{L^2(\hat{\Omega})} \sim \| f \|_{L^2(\Omega)} \) and \( \| \hat{f} \|_{\text{sup}} \lesssim \| f \|_{L^1(\Omega)} \). The implicit constants do not change the theory, but they are an annoyance one must remember.

Note that for \( f \in C_{\text{loc}} \cap L^1 \), \( \mathcal{F} f(\xi) = \int_\Omega f(x) e^{-i2\pi \xi x} dx \) while for \( f \in C_{\text{loc}} \cap L^2 \), in general we do not have an integral form for \( F_{L^2} f(\xi) \).

Interestingly, we can show \( F_{L^1} \) and \( F_{L^2} \) agree on \( L^1 \cap L^2 \), and both map to the same member in \( C_0 \cap L^2 \). This statement is more subtle than it appears (mostly due to the strange way we define \( L^p \) spaces). In particular, since we proceed similarly as in Theorem 14, we need to show \( (C_0 \cap L^2, \| \cdot \|_{C_0 \cap L^2}) \) is a Banach space, where \( \| f \|_{C_0 \cap L^2} = \max \left( \| f \|_{\text{sup}}, \| f \|_{L^2} \right) \). Let this be a non-trivial exercise for the reader.

### 4.1 Inversion

Recall from lectures that we also proved Fourier inversion:

\[
f(x) = \left( \hat{f} \right) \mathcal{F} (x) = \int_{\hat{\Omega}} \hat{f}(\xi) e^{i2\pi \xi x} d\xi
\]

in some special cases. For instance, when \( \Omega = R \) and \( f \) is a Gaussian. Or when \( \Omega = T \) and \( f \in C^1(T) \) \( (\hat{f} \in l^1 \) via integration by parts, then uniform convergence). A more general version is true.

**Theorem 20** (\( L^2 \) isomorphism). \( F^*_{L^1} F_{L^2} f = f \forall f \in L^2 \)

**Proof.** When \( \Omega = R \), recall that \( F^* F f = f \) for all Gaussians \( f \). As the Gaussians are dense in \( L^2(R) \) (lecture), conclude \( F^*_{L^2} F_{L^2} f = f \forall f \in L^2(R) \).

When \( \Omega = T \), recall that \( F^* F f = f \) for all \( f \in C^1(T) \). Done as \( C^1(T) \) is dense in \( L^2(T) \).

When \( \Omega = Z \), observe that for \( f \in c_c(Z) \), \( F f \) is a finite sum of \( e^{-i2\pi nx} \), therefore smooth. Then \( F^* F f(n) = \int_T F f(x) e^{i2\pi nx} dx = \langle F f, e^{-i2\pi n(\cdot)} \rangle = f(n) \) by orthogonality. Done as \( c_c(Z) \) is dense in \( l^2(Z) \).

\[ \square \]

**Corollary 21** (Inversion 2). Let \( g \) be continuous on \( \Omega \) and \( \int_\Omega |g| + |g|^2 < \infty \) (so \( g \in C_{\text{loc}} \cap L^1 \cap L^2 \), but \( g \) might be unbounded). Assume \( \hat{g} \in L^1 \). Then \( F^* F g = g \).

**Proof.** Play with \( F^* \). Obviously \( \hat{g} = F g \in C_0 \cap L^1 \) by Riemann-Lebesgue, so \( \hat{g} \in L^2 \) by Holder’s inequality. Then define \( G = F^* \hat{g} \in C_0 \cap L^2 \) (Riemann-Lebesgue). Observe that \( F_{L^2} G = \hat{g} = F_{L^2} g \), so \( G \leq L^2 \). As they are both continuous, if \( \exists x : g(x) \neq G(x) \) then \( \int_\Omega |G - g|^2 > 0 \) (contradiction). Then we must have \( g = G \) pointwise.

\[ \square \]

The arguments in Corollary 21 crucially rely on the Riemann-Lebesgue lemma (actually just its first half, \( F_{L^1} : L^1(\Omega) \to C(\hat{\Omega}) \)), and the fact that \( g \in L^2 \). The proof of Riemann-Lebesgue (or just its first half) is short enough that it is not a burden to include.
We must remember that we can only Riemann-integrate a function once we’ve proved the function is Riemann integrable locally. Also note that \( G \overset{L^2}{=} g \) is not the same as \( G(x) = g(x) \ \forall x \in \Omega \) (Riemann integration and continuity were used in Corollary 21). This is a subtle point many students might overlook.

There is a more general version which forgoes the condition \( g \in L^2 \), and does not require the Riemann-Lebesgue lemma. However, it will use Fubini’s theorem and some approximation steps require justification:

**Theorem 22** (Inversion 3). Let \( g \in C_{\text{loc}} \cap L^1(\Omega) \). Assume \( \hat{g} \in L^1 \). Then \( \mathcal{F}^* \mathcal{F} g = g \).

**Proof.** If \( \Omega = \mathbb{T}, C_{\text{loc}} \cap L^1(\Omega) = C \cap L^1 \cap L^2(\Omega) \) and we can use Corollary 21. If \( \Omega = \mathbb{Z} \), recall \( l^1(\mathbb{Z}) \hookrightarrow l^2(\mathbb{Z}) \) and we’re back to Corollary 21. So we only care when \( \Omega = \mathbb{R} \).

Let \( \phi \) be the even Gaussian function such that \( \phi = \phi \) and \( \int_{\mathbb{R}} \phi = \phi(0) = 1 \). Define \( \phi_n(x) = \phi \left( \frac{x}{n} \right) \). Then \( \phi_n \to 1 \) locally uniformly and \( \hat{\phi}_n(x) = n \hat{\phi}(nx) \), so \( \left( \hat{\phi}_n \right)_{n \in \mathbb{N}} \) can serve as mollifiers. Then by Fubini, we can show that

\[
g \ast \hat{\phi}_n(0) = \int_{\mathbb{R}} g \hat{\phi}_n = \int_{\mathbb{R}} \hat{g} \phi_n
\]

We leave it as an exercise for the reader to show \( \int_{\mathbb{R}} g \hat{\phi}_n \to g(0) \) and \( \int_{\mathbb{R}} \hat{g} \phi_n \to \int_{\mathbb{R}} \hat{g} \). The rest follows as in the textbook (Theorem 1.9, page 141).

**Remark.** The two approximations at the end of the proof require justification. Even the first one, which looks like mollification, has not yet been proved since \( g \) is not yet bounded. The second one is a case of interchanging \( \lim \) with \( \int \), needing justification which students might overlook. In the textbook, \( g \) is a Schwartz function (so mollification does work, or alternatively we can just use Corollary 21). Here however, we do not have Schwartz functions. Nevertheless we can prove the approximations manually by splitting \( \mathbb{R} \) into \([-N, N]\) and \( \mathbb{R} \setminus [-N, N] \) as usual.

## 5 Sobolev spaces

First, some formulas:

**Lemma 23** (Basic Fourier identities).

1. \( \int \hat{f} \ast \hat{g} = \hat{f} \hat{g} \) when \( f, g \in L^1(\Omega) \). (using \( \mathcal{F}_{L^1} \))

2. \( \hat{f}'(x)(\xi) = M \xi \hat{f}(x)(\xi) \) when \( \Omega \in \{ \mathbb{T}, \mathbb{R} \} \), \( f \in C^1(\Omega) \), and \( f, f' \in C_0 \cap L^1(\Omega) \). Here \( M \in \mathbb{Z} \) is a constant depending on the conventions chosen for the Fourier transform. If, furthermore, \( f, f' \in C_0 \cap L^2(\Omega) \), then \( \| f'(x) \|_{L^2} \sim \| \xi \hat{f}(x) \|_{L^2} \).

3. \( \hat{f}'(x)'(\xi) = M \xi f(x)(\xi) \) when \( \Omega \in \{ \mathbb{Z}, \mathbb{R} \} \) and \( f(x), x f(x) \in (C_0 \cap L^1)_x \). Here \( M \in \mathbb{Z} \) is, again, some constant. Note that this implies \( \hat{f} \in C^1 \left( \frac{\Omega}{0} \right) \) with \( \hat{f}, \hat{f}' \in C_0 \left( \frac{\Omega}{0} \right) \). If, furthermore, \( f(x), x f(x) \in (C_0 \cap L^2)_x \), then \( \| x f(x) \|_{L^2} \sim \| \partial_x \hat{f}(x) \|_{L^2} \).

4. Define \( \mathcal{R} f(x) = f(-x) \) (\( \mathcal{R} \) is **time reversal**), then \( \mathcal{R} : L^2 \to L^2 \) is an isometry defined by unique extension. Then \( \mathcal{F}^2 = \mathcal{R} \) on \( L^2(\Omega) \).

5. \( \langle \mathcal{F}_{L^2} f, \mathcal{F}_{L^2} g \rangle = \langle f, g \rangle \forall f, g \in L^2(\Omega) \).
Remark. As $F^4 = \text{Id}_{L^2(\Omega)}$ and $F$ is an isometry, the Fourier transform can be thought of as a 90-degree rotation on $L^2$ which interchanges derivatives and polynomials (regularity and decay).

BEWARE: We defined $(f, g) = \int f \overline{g} \forall f, g$ to make it a complex inner product (or Hermitian). As a consequence, $(Ff, g) = (f, F^*g)$ (justifying the adjoint notation $F^*$). Observe that $F^* \overline{g} = \overline{F g}$.

However, other authors might instead choose $(f, g) = \int f g$ to avoid complex conjugation (but it’s no longer Hermitian), and then $(Ff, g) = (f, Fg)$.

Note that we always have $\int f g = \int f \overline{g}$ no matter the convention for $\langle \cdot, \cdot \rangle$.

Proof.

1. By density and Riemann-Lebesgue (Lemma 19), enough to prove for $f, g \in C_c(\Omega)$. Then

$$\overline{\hat{f} \ast g}(\xi) = \int f(x - y) g(y)dy \overset{\text{Fubini}}{=} \int f(x) e^{-i2\pi \xi x} dx \int g(y) e^{-i2\pi \xi y} dy = \int f(x) e^{-i2\pi \xi x} dx \int g(y) e^{-i2\pi \xi y} dy = \hat{f}(\xi) \overline{\hat{g}(\xi)}$$

2. Integrate by parts:

$$\overline{\hat{f}'(x)}(\xi) = \int f'(x) e^{-i2\pi \xi x} dx = -\int f(x) \partial_x (e^{-i2\pi \xi x}) dx = i2\pi\xi \int f(x) e^{-i2\pi \xi x} dx = (i2\pi) \xi \hat{f}(\xi)$$

The rest is Parseval.

3. By Riemann-Lebesgue (RL), $\hat{f} \in C_0(\widehat{\Omega})$. Then

$$\overline{\hat{f}(x)}(\xi) = \int xf(x) e^{-i2\pi \xi x} dx = \frac{1}{i2\pi} \int f(x) \partial_\xi (e^{-i2\pi \xi x}) dx = \frac{1}{-i2\pi} \partial_\xi \left( \int f(x) e^{-i2\pi \xi x} dx \right)$$

The last equality requires proper justification. Observe that $\left| -\frac{1}{i2\pi h} (e^{-i2\pi (\xi + h)x} - e^{-i2\pi \xi x}) - xe^{-i2\pi \xi x} \right| \lesssim |x|$ for $h \in \widehat{\Omega}$ small. Use the squeeze theorem to justify.

4. Recall $F^{-1} = RF$ by definition. So $F = RF^{-1}$ and $F^2 = RF^{-1}F = R$.

5. By isometry, $(F_{L^2}(f + g), F_{L^2}(f + g)) = (f + g, f + g) \forall f, g \in L^2(\Omega)$. Let $R$ and $\Im$ denote real/imaginary parts. So $R(F_{L^2}f, F_{L^2}g) = R(f, g) \forall f, g \in L^2(\Omega)$. We’re done since $\Im(F_{L^2}f, F_{L^2}g) = R(F_{L^2}\frac{i}{2}f, F_{L^2}g) = R(\frac{i}{2}f, g) = \Im(f, g)$.

\[\square\]

Definition 24. Let $\Omega \in \{T, \mathbb{R}\}$. For $m \in \mathbb{N}$, define the Sobolev space $W^{m, 2}(\Omega)$ as the metric completion of $C^\infty_c(\Omega)$ under the norm

$$\|f\|_{W^{m, 2}} = \sum_{j \leq m} \|f^{(j)}\|_{L^2(\Omega)}$$

Motivated by Lemma 23, we introduce the Fourier-Sobolev space $H^m(\Omega)$ (also called Bessel potential space) as the metric completion of $C^\infty_c(\Omega)$ under the norm

$$\|f\|_{H^m} = \sum_{j \leq m} \|\xi^j \hat{f}(\xi)\|_{L^2_\xi}$$

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Note that \( \|f\|_{W^{m,2}} \sim \|f\|_{H^m} \forall f \in C_c^\infty (\Omega) \) by Lemma 23, therefore the two metric completions are the same and \( W^{m,2} = H^m \). The advantage of \( H^m \) is that it features no derivatives.

Next, we introduce the strange, but surprisingly useful **Japanese bracket**

\[
\langle x \rangle = \sqrt{1 + |x|^2} \forall x \in \mathbb{R}
\]

Easy to check that \( \langle x \rangle \sim 1 + |x| \forall x \in \mathbb{R} \). The key insight is that \( \langle x \rangle \sim 1 \) for \( |x| < 1 \), while \( \langle x \rangle \sim |x| \) for \( |x| \geq 1 \). Also, \( |x| \) is not smooth at 0, but \( \langle x \rangle \) is smooth everywhere, which is important since the Fourier transform likes smooth multipliers. We also see that for \( f \in C_c^\infty (\Omega) \):

\[
\|f\|_{H^1} = \|\hat{f}(\xi)\|_{L^2_\xi} + \|\xi|\hat{f}(\xi)|\|_{L^2_\xi} \sim \|\langle \xi \rangle |\hat{f}(\xi)|\|_{L^2_\xi} \sim \|\langle \xi \rangle \hat{f}(\xi)\|_{L^2_\xi}
\]

Similarly, \( \|f\|_{H^m} \sim \|\langle \xi \rangle^m \hat{f}(\xi)\|_{L^2_\xi} \forall f \in C_c^\infty (\Omega) \).

**Theorem 25** (Sobolev embedding). Let \( \Omega \in \{T, \mathbb{R}\} \). Show that \( H^1 (\Omega) \hookrightarrow C_0 (\Omega) \) is continuous.

**Proof.** By density, it is enough to show \( \|f\|_{\text{sup}} \lesssim \|f\|_{H^1} \forall f \in C_c^\infty (\Omega) \). Observe that

\[
\left\| \hat{f} \right\|_{L^1} \overset{CS}{\lesssim} \left\| \langle \xi \rangle \hat{f}(\xi) \right\|_{L^2_\xi} \left\| \frac{1}{\langle \xi \rangle} \right\|_{L^2_\xi} \lesssim \left\| \langle \xi \rangle \hat{f}(\xi) \right\|_{L^2_\xi} \sim \|f\|_{H^1}.
\]

So \( \hat{f} \in C_0 \cap L^1 \), by inversion \( f = \mathcal{F}^* \hat{f} \) and \( \|f\|_{\text{sup}} = \|\mathcal{F}^* \hat{f}\|_{\text{sup}} \overset{RL}{\lesssim} \|\hat{f}\|_{L^1} \). \( \square \)

**Remark.** Note that a function in \( H^1 (\Omega) \) is not necessarily smooth since we defined \( H^1 \) by metric completion.

The argument can be reiterated for higher derivatives. For instance, \( H^2 (\Omega) \hookrightarrow C^1 (\Omega) \) is continuous since

\[
\|f\|_{C^1} = \|f\|_{\text{sup}} + \|f'\|_{\text{sup}} \lesssim \|f\|_{H^1} + \|f'\|_{H^1} \sim \|f\|_{W^{1,2}} + \|f'\|_{W^{1,2}} \lesssim \|f\|_{W^{2,2}} \forall f \in C_c^\infty (\Omega)
\]

The key dimensional-dependent estimate is \( \left\| \frac{1}{\langle \xi \rangle} \right\|_{L^2_\xi (\tilde{\Omega})} \lesssim 1 \), which follows from the usual integral test (now written in a new form)

\[
\int_{\mathbb{R}} \frac{1}{\langle x \rangle^\alpha} \, dx < \infty \forall \alpha > 1
\]

This is why the Japanese bracket is useful, since we wouldn’t be able to say \( \left\| \frac{1}{\langle \xi \rangle} \right\|_{L^2_\xi (\tilde{\Omega})} \lesssim 1 \) due to the singularity at 0. Generalizations to higher dimensions completely depend on this.

**Definition 26** (Laplacian in higher dimensions). We briefly note that for \( m, n \in \mathbb{N} \), \( \Omega \in \{\mathbb{T}^n, \mathbb{R}^n\} \) and \( f \in C_c^\infty (\Omega) \): \( \|\Delta^m f\|_{L^2(\Omega)} \sim_m \left\| \langle \xi \rangle^{2m} \hat{f}(\xi) \right\|_{L^2_\xi (\tilde{\Omega})} \) and

\[
\|f\|_{H^{2m}(\Omega)} \sim_m \left\| \langle \xi \rangle^{2m} \hat{f}(\xi) \right\|_{L^2_\xi (\tilde{\Omega})} = \left( \|1 + |\xi|^2\|^m \right) \left\| \hat{f}(\xi) \right\|_{L^2_\xi (\tilde{\Omega})} \sim_m \|\Delta^m f\|_{L^2(\Omega)}
\]

\( (1 - \Delta)^m \) is also called the **Bessel potential** when we generalize to have \( m \in \mathbb{R} \) (then \( H^m \) becomes **fractional Sobolev spaces**). Indeed, since the definition of \( H^m \) never used derivatives, \( m \) does not need to be a natural number (unlike with \( W^{m,2} \)). People also define \( (\nabla)^m = (1 - \Delta)^{\frac{m}{2}} \) (using the Japanese bracket),
which allows us to neatly write
\[ \|f\|_{H^m(\Omega)} \sim_m \left\| \langle \xi \rangle^{2m} \hat{f}(\xi) \right\|_{L^2_\xi(\hat{\Omega})} \sim_m \left\| \langle \nabla \rangle^{2m} f \right\|_{L^2(\Omega)} \]

**Polar coordinates** Let \( dS \) be the “infinitesimal” surface area on the sphere \( S^{n-1} \), and \( |S^{n-1}| \) be its area. From multivariable calculus, we know that for a function \( f \in C(\mathbb{R}^n) \), we can use polar coordinates for integration:
\[ \int_{B(0,R)} f(x) \, dx = \int_0^R \left( \int_{S^{n-1}} f(r\theta) \, dS(\theta) \right) r^{n-1} \, dr \]
where \( B(0,R) = \{ x \in \mathbb{R}^n : |x| < R \} \).

Let \( m \in \mathbb{N} \). Then
\[ \int_{B(0,R)} \frac{1}{(1+|x|^2)^m} \, dx = \int_0^R \left( \int_{S^{n-1}} \frac{1}{(1+r^2)^m} \, dS(\theta) \right) r^{n-1} \, dr = |S^{n-1}| \int_0^R \frac{r^{n-1}}{(1+r^2)^m} \, dr \]
If \( m > \frac{n}{2} \), then \( \lim_{R \to \infty} \int_{B(0,R)} \frac{1}{(1+|x|^2)^m} \, dx \) exists since \( \int_0^R \frac{r^{n-1}}{(1+r^2)^m} \, dr \lesssim \int_0^R \frac{r^{n-1}}{(r^2)^m} \, dr \).

**Corollary 27** (Sobolev embedding). Let \( \Omega \in \{ \mathbb{T}^n, \mathbb{R}^n \} \). Then \( \hat{\Omega} \in \{ \mathbb{Z}^n, \mathbb{R}^n \} \) and \( H^m(\Omega) \hookrightarrow C^0(\Omega) \) is continuous when \( m > \frac{n}{2} \).

**Proof.** Everything else generalizes. Only need to check \( \left\| \langle \xi \rangle \right\|_{L^2_\xi(\hat{\Omega})} \lesssim 1 \).

When \( \Omega = \mathbb{R}^n \), use polar coordinates as above.
When \( \Omega = \mathbb{T}^n \) and \( \hat{\Omega} = \mathbb{Z}^n \), mimic the integral test: for \( \xi \in \prod_{j=1}^n [k_j, k_j+1) \) where \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \), simply note \( \langle \xi \rangle \sim |\xi| \sim |k| \) for large enough \( |\xi| \).

**Remark.** Proving Sobolev embedding without the Fourier transform would be more laborious (an understatement). Once again, it is easy to reiterate the argument for higher derivatives. In particular, \( H^\infty(\Omega) \subset C^\infty(\Omega) \) where \( H^\infty = \cap_{m \in \mathbb{N}} H^m \).

The “derivatives” Sobolev spaces offer, under metric completion, are called weak derivatives, \( L^2 \) derivatives or distributional derivatives. Note that we require more Sobolev regularity to get true regularity as the dimension increases. Sobolev embedding is a key tool in the analysis of partial differential equations, where Sobolev solutions are called weak solutions. It is common to work in the abstract Sobolev setting where many tools are available (elliptic regularity, Lax-Milgram, energy method, Ehrling’s inequality etc.), and then hopefully recover true solutions in the end.