Previously, integral sums were defined. In the case of zero measure, $0 = \emptyset \subseteq \mathbb{R}^2$.

Prove that if $E$ is a closed, $x \in \mathbb{R}^n$, and $x$ can be written as a countable union of closed sets $\{x_n\}$, then there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{E_n} f(x, y) \, dx = f(x)$. Let $g = \text{countable union of } f$ closed.

$$f = \bigcup_{n=1}^{\infty} \{ (x_n, y_n) : f(x_n, y_n) \theta(x) \}$$

For $f$, there is an increasing sequence $x_1 \subseteq x_2 \subseteq \ldots$ such that $\lim_{n \to \infty} x_n = x$. If $f(x_n, y_n)$ is continuous, $f(x, y) = \lim_{n \to \infty} f(x_n, y_n)$.

Proof:

1. Suppose $f(x_n, y_n) = f(x_n, y_n)$. If $f(x_n, y_n)$ is continuous, $f(x, y) = \lim_{n \to \infty} f(x_n, y_n)$.
2. If $f(x_n, y_n)$ is not continuous, then $f(x, y) = \lim_{n \to \infty} f(x_n, y_n)$.
(3) Let $a, b, c, d, C, D, c, ...$

a sequence of Jordan sets.

Suppose that the set $\mathbb{U} \mathbb{G}_a$ is Jordan measurable.

Prove that $\lim_{N \to \infty} V(\mathbb{G}_a) = V(\mathbb{G}_b).

\textbf{Proof:}

1. Assume $\mathbb{G}_a, \mathbb{G}_b$ are all similar sets.

2. Jordan measurable = almost simple.

Then from:

$\lim_{N \to \infty} V(\mathbb{G}_a) - V(\mathbb{G}_b) \leq \epsilon$,

forall $\epsilon > 0$.

Replace $\mathbb{G}_a, \mathbb{G}_b$ by simple sets.

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(a) The ball $\{x \mid \|x\| \leq r\} \subseteq \mathbb{R}^n$ is Jordan.

(b) For every $p > 0$

\[ E_p = \{ (x_1, ..., x_n) / \sum_{i=1}^{n} x_i^p \leq 1 \} \subseteq \mathbb{R}^n \]

is Jordan measurable in $\mathbb{R}^n$.

and $V(E_p)$ is strictly increasing in $p$.

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5. For $\mathbb{B}_r = \{ x \mid \|x\| \leq r \} \subseteq \mathbb{R}^n$, prove:

(a) $V(\mathbb{B}_r) = r^n V(n)$.

(b) $V(\mathbb{B}_r) < e^{\gamma(n-1)^n} V(E_r)$ for $r < 1$.

[In high dim the vol. of a ball is concentrated near the sphere!]

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\[ V(\mathbb{B}_r) = \frac{r^n}{n!} V(\mathbb{S}_r) \]

\[ V(\mathbb{S}_r) = \frac{r^n}{n!} \]

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\[ r^n < e^{\alpha_n r} \quad \text{for} \quad r < 1 \]

\[ n B(r) < r (1-r) \]

\[ B(r) < r^{n+1} \]

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\[ B_r \subseteq \mathbb{S}_r \]