a \leq b \implies a \leq Cb \text{ for some constant } C > 0
\implies a \sim b \text{ mean } \exists \alpha, \beta \geq 0 \text{ s.t. } a = \alpha b + \beta \text{ and } b \sim a
a \preceq x b \text{ means } a \leq C_x b \text{ where } C_x \text{ depends on variable } x.

\text{Example: } \mathbf{A} \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n) \text{ is invertible, } \forall x, y \in \mathbb{R}^n \implies ||Ax|| \leq ||x||
\implies ||A^{-1}y|| \leq ||y||.
So \exists \|x\| \sim \|y\|
\implies (r, \theta) \sim (r, \phi)
\implies \|x\| \leq r
\implies \|y\| \leq r
\implies (x, \theta) \sim (y, \phi)

\text{Do you prove theorems yourself?}

\text{Rudin may be terse}

\textbf{Inverse Function Theorem:}

Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be \( C^1 \), \( f(x_0) \) invertible.

\( \forall \delta \exists \varepsilon \text{ s.t. } ||x-x_0|| < \varepsilon \implies ||f(x)-f(x_0)|| < \delta \) \( \forall x \sim x_0 \).

\( f \) is \( C^1 \) -diffeomorphism near \( x_0 \), \( (f, f^{-1}) \in C^1 \).

\text{Proof:}
\text{1st step: differentiaton = composition, } (f(x_0) - f(0)) \cdot x = 0
\text{So assume } f is linear, \text{ pivot } (f(x) - f(x_0) \cdot x)
\text{Then } f is a linear isomorphism \( \mathbb{R}^n \to \mathbb{R}^m \).
\text{Also } f(\text{open set}) = \text{open as } f^{-1} \text{ is cont.}
\text{where } \|f\| = \|f^{-1}\| = 1.

Linear isomorphisms \textit{trivially} satisfy theorem.

\text{2nd step: Look at composition } g = (f^{-1})^{-1} \circ f \text{ Then } g(0) = I \text{ (identity matrix)}
\text{WLOG } f'(0) = I \implies \|f'(x) - I\| \text{ small for } x \text{ near } 0
\text{Most important fact?}
\text{Which will make } f(x) \approx x \text{ for small } x
\text{3rd step: Injectivity.}
\text{Restrict } x \in U \text{ open, so that } \|f'(x) - I\| \leq \varepsilon \text{ (random and } \eta) \text{ (open & trivial)}
\text{Then by FTC, } \forall x, y \in U, \text{ } f(x) - f(y) = \int_{x}^{y} f'(z) \, dz \approx (I_n + M_z)(x-y)
\text{where } \|M_z\| \leq \varepsilon \text{ small error matrix}
\text{Then } \frac{\varepsilon}{\eta} \|x-y\| \leq \|f(x) - f(y)\| \leq \frac{\varepsilon}{\eta} \|x-y\| \text{ for all } x, y \in U.
\text{If } f(x) - f(y) \leq \|x-y\| \implies f \text{ injective}

\text{4th step: } f(U) \text{ open}
\text{Iteration, each step moves closer to } y \text{. First WLOG } U = B(0, r) \text{ (shrink it)}
\text{Indeed, } (x, y) \in B(0, \frac{r}{2})
\text{First let } x = 0 \text{, } \varepsilon_x = \|y - f(x)\| \leq \frac{r}{2}
\text{Then } x \to y \text{, } \varepsilon_x \to 0
\text{Is } f \text{ a good pick?}
\text{We are done}
\[ f \circ l_{z} = l_{z} \quad \text{for all } z \in X \]

\[ x \mapsto y = f(x) \quad \text{and} \quad \| y - f(x) \| \leq \frac{C}{10} < 10 \]

So \( f \) is close to linear near point \( p \).

** Iterate:** 
\[ E_{n} = \left\{ \frac{1}{10^{n}} \rightarrow 0 \text{ so } \frac{f(x_{n})}{x_{n}} \rightarrow y \right\} \]

\[ \| x_{n+1} - x_{n} \| = E_{n} \leq \frac{C}{10^{n}} \text{ so } \{ x_{n} \} \text{ Cauchy (Cauchy inequality)} \]

Since \( x_{0} \) is an interior point \( x_{n} \rightarrow x_{0} \) where \( \| x_{n} \| \leq \sum_{n=0}^{N} \frac{1}{10^{n}} \leq r \),

Then \( f(x_{0}) = y \), \( x_{0} \in B(0, r) \). Done.

** Case:** 
\[ \mathcal{g}^{-1} : f(U) \rightarrow U \text{ is } C^{2} \]

Just recall FTC: \[ \mathcal{g}^{-1}(x) = \left( I + \mathcal{M}_{L} \right)(x - y) \]

\( I + \mathcal{M}_{L} \) near \( I_{0} \) and invertible. Differentiate \( \mathcal{g}^{-1} \) by definition.

** Philosophy:** For most problems in analysis, we can build a solution directly, but we start with an approximation then iterate towards the true solution.

- PDE, ODE (Pecorari, Construction mapping)
- Open mapping theorem (Functional analysis)
- Nash embedding theorem (Riemannian geometry)
- Calabi-Yau theorem (Complex geometry/string theory)

** Implicit Function Theorem = Inverse Function Theorem **

\[ f : \mathbb{R}^{m} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{m} \text{ where } \partial_{y} f(x_{0}, y) \text{ is bijective} \]

Then \( (x_{0}, y_{0}) \), \( \{ (x_{0}, y) \} \text{ is bijective if and only if } \partial_{y} f(x_{0}, y) \text{ is bijective} \)

\[ \partial_{y} f(x_{0}, y) = \begin{pmatrix} f_{1}(x_{0}, y) & \cdots & f_{n}(x_{0}, y) \end{pmatrix} \]

\[ \begin{pmatrix} f_{1}(x_{0}, y) & \cdots & f_{n}(x_{0}, y) \end{pmatrix} \begin{pmatrix} x_{1} \\ \vdots \\ x_{n} \end{pmatrix} = y \]

\[ y \begin{pmatrix} x_{1} \\ \vdots \\ x_{n} \end{pmatrix} = f(x_{0}, y) \]

\[ f(x_{0}, y) = \begin{pmatrix} f_{1}(x_{0}, y) \\ \vdots \\ f_{n}(x_{0}, y) \end{pmatrix} \]

Then \( \partial_{y} f(x_{0}, y) = \begin{pmatrix} f_{1}(x_{0}, y) & \cdots & f_{n}(x_{0}, y) \end{pmatrix} \)

\[ \partial_{y} f(x_{0}, y) \text{ is invertible } \Rightarrow \text{ inverse theorem.} \]

\[ R \text{ is trivial.} \]
Implicit versus the Foundation of manifolds

Analytic versus the geometry

Example, \( f: \mathbb{R}^3 \to \mathbb{R} \)
\[(x, y, z) \mapsto x^2 + y^2 + z^2 \]
1 degree of freedom
3 - 1 = 2 degrees left

Then the set \( \{ f = 1 \} \) can be parametrized by 2 variables locally.
and the sphere in \( \mathbb{R}^3 \) is a 2-dimensional manifold

(*this is the definition*)

Exercises:

1. Let \( \Phi: (-\pi, \pi) \to \mathbb{R}^n \) be \( \Phi'() \neq 0 \).

Show that \( \exists \varepsilon > 0 \), \( \forall r < \varepsilon \):

\[ \{ \Phi(\varepsilon) : \| \Phi(t) - \Phi(\varepsilon) \| < r \} \]

is an interval.

Is it true when \( \Phi'(0) = 0 \)?

\( \dot{\Phi}(t) = t^4 \sin(1/t) \)

\( \dot{\Phi}''(t) = 4t^3 \sin(1/t) - t^5 \cos(1/t) \)

\( \dot{\Phi}(0) = 0 \)

\( \| \Phi'(0) \| = 1 \)

\( \Phi'(0) = (1, 0, 0, \ldots) \)

\( \Phi(t) = \int_0^t \Phi'(s) \, ds \)

\( \Phi(t) - \Phi(0) = \Phi'(t) \cdot t + \\varphi'(t) \cdot t \)

\( \varphi'(t) \to 0 \)

\( \frac{2}{\partial t} (\| \Phi(t) \|^2) = 2 \frac{\partial t}{\partial t} < \Phi'(t), \Phi'(t) > = < \Phi'(t), \Phi'(t) > \)

\( \frac{\partial}{\partial t} \| \Phi(t) \|^2 \approx 2 \quad \| \Phi'(t) \|^2 \approx 1 \quad \| \Phi(t) \|^2 \)

\( \| \Phi'(t) \| ^2 \) is increasing as \( t \) increase.

\( g(t) = t \) then \( f \) is decreasing on \((0, \infty)\)

With this monotonicity, you can prove \( \log \) does not happen. \( \log \) is trivial.

2. Show that for \( f(x, y, z) = (x + y - z, x^2 + y^2 - yz) \)

\( \{ f = (1, 0) \} \) is one-dimensional near \((-1, 1, 0)\)
\[
\begin{align*}
\text{Prof:} \\
\left( \begin{array}{cc}
\partial x & \partial y \\
\partial z_1 & \partial z_2
\end{array} \right) = \left( \begin{array}{cc}
2 & 1 \\
3x^2 & 3x^2 - 2
\end{array} \right) \\
\Rightarrow f(-1, 0, 0) \Rightarrow \left( \begin{array}{cc}
0 & 1 \\
3 & 3
\end{array} \right) \text{ invertible} \\
\Rightarrow (x, y) = (x(z), y(z)) \text{ near } y \text{ for } \{f(z, c)\}
\end{align*}
\]