STABILITY OF AN INVERSE PROBLEM IN POTENTIAL SCATTERING ON THE REAL LINE

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1 Introduction

In this paper we are going to study some problems arising in inverse scattering theory for the Schrödinger operator $H_p u = -u'' + pu$ on the real line. Here p is a real-valued potential. It will be assumed that $x \leq 0$ in the support of p and that

$$\int_{-\infty}^{0} (1 + |x|)\, |p(x)|\; dx < \infty.$$

In the later part of the paper we shall assume in addition that p decays exponentially at $-\infty$.

The support properties of the potential imply that the right reflection coefficient $r(k)$ admits a meromorphic continuation to the upper half-plane $\mathbf{C}_+ = \{k \in \mathbf{C}, \operatorname{Im} k > 0\}$ with at most finitely many poles. The inverse problem under consideration consists of recovery of the potential p from the sequence $\{r(in), n = 1, 2 \ldots\}$. This sequence will be referred to as "the data". We remark that part of the motivation for studying this problem comes from an inverse boundary value problem for the Schrödinger operator with radially symmetric potential in two dimensions. We refer to the paper by J.

Sylvester [17] for a study of the conductivity equation in the radially symmetric case. Working in the scattering theoretical framework, we first prove that the data determine the potential uniquely.

The main part of the paper is devoted to the problem of recovery of the potential from some partial data. The question we are addressing is that of stability of the inverse problem. The general stability estimates in inverse boundary value problems and inverse scattering at a fixed energy are due to Alessandrini and Stefanov, see [1], [16], and also [15]. These are logarithmic continuous dependence results (see [18] for the precise formulation), and therefore predict quite a weak form of continuous dependence. In the paper [5] a related problem is studied and logarithmic type estimates are established. The main purpose of the present work is to derive a stronger stability estimate of Hölder type. The estimate obtained applies in particular to the radially symmetric inverse boundary value problem in two dimensions. It may be interesting to notice that it was believed that, for this problem, a logarithmic stability estimate would be optimal, see [9].

Our main Theorem 2.2 is obtained by combining two results, which are independent of each other: Theorem 2.4 and Theorem 2.5. Here in Theorem 2.4 we establish a general stability estimate for potentials in terms of their reflection coefficients. In Theorem 2.5 we derive Hölder type estimates for the reflection coefficients. This is possible to do thanks to the exponential fall-off of our potentials, which guarantees the meromorphic continuation of the scattering matrix to a strip in the lower half-plane.

The plan of the paper is as follows. In Section 2 we present our main results. Section 3 is devoted to a review of the basic facts of one-dimensional scattering theory. In particular, we recall the representation of the reflection coefficients, as given in [13]. We also show identifiability of the potential from the data. Theorem 2.4 is proved in Section 4 and Theorem 2.5—in Section 5. Finally, in Section 6, as an application of the results, we present an inverse boundary value problem for the Schrödinger operator, and in the case of radial potential reduce it to the inverse problem, studied in the main part of the paper.

2 Main results

First we shall briefly describe the main notation, used in the paper. Most distributions that we shall consider will be real-valued. The Fourier transform will be normalized so that it is given by

$$\hat{u}(k) = \int_{-\infty}^{\infty} u(x) e^{-ixk}\, dx, \quad k \in \mathbf{R},$$

when u is in the Schwartz space, and it is extended to the space of temperate distributions in the usual way. The notation L is used for the space of real-valued measurable potentials p such that

$$||\, p\, || = \int_{-\infty}^{\infty} (1+|x|)\, |p(x)|\; dx < \infty.$$

When p is a potential in L, then the right reflection coefficient $r = r_p$ is defined (the precise definition will be recalled in Section 3).

We let now $a \geq 0$ be a given number and $q \geq 0$ be a function such that $e^{2a|x|}q(x) \in L$ and $x \leq 0$ in $\operatorname{supp}(q)$. We shall work with the following class of potentials.

Definition 2.1 *M_q is the set of all $p \in L$ such that $|p| \leq q$ and H_p has no bound states.*

We shall also consider potentials having some additional regularity. When m is a nonnegative integer, introduce the set $M_q^{(m)}$ which consists of all $p \in M_q$ such that $p^{(j)} \in L$ when $j \leq m$ and $\left|p^{(j)}\right| \leq q$, $0 \leq j \leq m$.

For $p \in M_q$, set

$$||\, p\, ||_a = \int_{-\infty}^{0} (1+|x|) e^{2a|x|}\, |p(x)|\; dx.$$

When $a = 0$, we shall write $||\, p\, ||_0 = ||\, p\, ||$.

The notation $C(a)$ will be used for various constants, depending only on a, and constants which depend only on $||\, q\, ||$ will be denoted $C(q)$.

The main result of the paper is the following stability estimate.

Theorem 2.2 *Assume that $0 < a \leq 1/2$ and let p_1 and $p_2 \in M_q$ be two potentials such that*

$$r_{p_1}(in) = r_{p_2}(in), \quad n = 1, \ldots N.$$

Then the estimate

$$||\, \langle x \rangle^{-\delta}(P_2 - P_1)\, ||_{L^2} \leq C N^{-a}, \quad \langle x \rangle = (1+x^2)^{1/2}, \;\; \delta = \frac{5}{2},$$

holds. Here

$$P_j(x) = \int_x^{\infty} p_j(t)\, dt, \quad j = 1, 2$$

and

$$C = C(a) C(q) \left(1 + ||\, q\, ||_a\right)^2.$$

Remark. If we assume in addition that the potentials are uniformly Hölder continuous, then we can also derive estimates for $p_1 - p_2$ in the L^∞ norm on each compact set.

For potentials in $M_q^{(m)}$ we have a stronger estimate, provided that m is sufficiently large.

Theorem 2.3 *Assume that $a > 0$ and p_1, $p_2 \in M_q^{(m)}$ are such that*

$$r_{p_1}(in) = r_{p_2}(in), \quad n = 1, \dots N.$$

Then if $m \geq 2a - 1$ the estimate

$$|| \langle x \rangle^{-\delta} (P_2 - P_1) ||_{L^2} \leq C N^{-a}, \quad \langle x \rangle = (1 + x^2)^{1/2}, \ \delta = \frac{5}{2}$$

holds. Here

$$C = C(a) C(2^m q) \left(1 + || q ||_a\right)^2 .$$

Theorems 2.2 and 2.3 are immediate consequences of the following two results.

Theorem 2.4 *Assume that $a \geq 0$ and p_1, $p_2 \in M_q$. Then we have*

$$|| \langle x \rangle^{-\delta} (P_2 - P_1) ||_{L^2} \leq C(q) || r_{p_2} - r_{p_1} ||_{L^2}, \ \delta = \frac{5}{2}.$$

Theorem 2.5 *Assume that $a > 0$ and p_1, $p_2 \in M_q$ are such that*

$$r_{p_1}(in) = r_{p_2}(in), \quad n = 1, \dots N.$$

If $a \leq 1/2$ then the estimate

$$|| r_{p_2} - r_{p_1} ||_{L^2} \leq C(a) C(q) \left(1 + || q ||_a\right)^2 N^{-a}$$

holds. When p_1, $p_2 \in M_q^{(m)}$ with $m \geq 2a - 1$ we have

$$|| r_{p_2} - r_{p_1} ||_{L^2} \leq C(a) C(2^m q) \left(1 + || q ||_a\right)^2 N^{-a}.$$

Remark. The number δ above may be replaced by any number > 2.

3 Review of scattering theory

We begin by recalling some important results of scattering theory on the line. The present state of the theory of (inverse) scattering for the Schrödinger equation on the real line is described in the articles [2], [3], [13] and the monograph [11]. Our basic reference here is the paper [13].

Consider the Schrödinger equation

$$H_p u := -u'' + pu = k^2 u, \quad k \in \mathbf{R}, \tag{3.1}$$

where $p \in L$. We shall not assume in these discussions that $x \leq 0$ in the support of p.

There exist two functions $f(x,k)$ and $g(x,k)$, such that f and g solve (3.1), and

$$f(x,k) = e^{ixk} + o(1), \quad x \to +\infty,$$
$$g(x,k) = e^{-ixk} + o(1), \quad x \to -\infty.$$

We shall say that f and g are the Jost functions. For $k \in \mathbf{R} \setminus \{0\}$, $f(x,k)$ and $\overline{f(x,k)} = f(x,-k)$ are solutions of the same equation (3.1), but with different boundary conditions at $+\infty$, so they are linearly independent. Therefore, we can write

$$ikg(x,k) = a(k)f(x,-k) + b(k)f(x,k), \tag{3.2}$$

where $a(k)$ and $b(k)$ are uniquely determined. One finds that $\overline{a(k)} = a(-k)$, $\overline{b(k)} = b(-k)$, and that

$$k^2 + |b(k)|^2 = |a(k)|^2. \tag{3.3}$$

A combination of (3.2) with its complex conjugate then shows that

$$ikf(x,k) = a(k)g(x,-k) + b(-k)g(x,k). \tag{3.4}$$

We shall now introduce the elements of the scattering matrix of p when $k \neq 0$. When doing it we notice that, since $a(k) \neq 0$ by (3.3), it follows that the functions f and g form a basis of solutions of (3.1). Moreover, they extend to analytic functions of k in the upper half-plane, continuous in the closure of that set. Their complex conjugates $\overline{f(x,k)} = f(x,-k)$ and $\overline{g(x,k)} = g(x,-k)$ have natural analytic extensions to the lower half-plane instead. The equations (3.2) and (3.4) may now be rewritten in the form

$$f(x,-k) = \frac{ik}{a(k)} g(x,k) - \frac{b(k)}{a(k)} f(x,k), \tag{3.5}$$

$$g(x,-k) = -\frac{b(-k)}{a(k)} g(x,k) + \frac{ik}{a(k)} f(x,k),$$

where we have expressed the solutions of (3.1) with analytic extensions in k to $\mathbf{C}_-$ as linear combinations of those with analytic extensions to $\mathbf{C}_+$. Since $f(x,k)$ was normalized by boundary conditions at $+\infty$, we shall call

$$r(k) = \frac{b(k)}{a(k)} \tag{3.6}$$

the *right reflection coefficient*. For similar reasons, $b(-k)/a(k)$ is called the left reflection coefficient, and the function

$$t(k) = \frac{ik}{a(k)} \tag{3.7}$$

is the *transmission coefficient*.

The assertions about analyticity in $k \in \mathbf{C}_+$ of the functions f and g are consequences of their integral representations in terms of the intertwining operators between H_p and H_0, which we proceed to discuss following [13]. Associated to p, there are two operators $A_+ = I + R_+$ and $A_- = I + R_-$, with $H_p A_\pm = A_\pm H_0$, such that $\pm(y-x) \geq 0$ in the support of $A_\pm$. (Here and in what follows we identify operators with their distribution kernels.) The functions $R_\pm$ are continuous up to the boundary in the sets $\pm(y-x) > 0$, and

$$||\, R_\pm(x,\cdot)\,||_{L^1} = \int |R_\pm(x,y)|\; dy < \infty, \tag{3.8}$$

for any x. Moreover, $||\, R_\pm(x,\cdot)\,||_{L^1} \to 0$ as $x \to \pm\infty$. It follows from the properties of $R_\pm$ that $f(x,-k)$ (resp. $g(x,k)$) is the Fourier transform of $A_+(x,y)$ (resp. $A_-(x,y)$) with respect to the second variable. More precisely,

$$f(x,k) = e^{ixk} + \int_x^\infty R_+(x,y) e^{iyk}\, dy, \tag{3.9}$$

and

$$g(x,k) = e^{-ixk} + \int_{-\infty}^x R_-(x,y) e^{-iyk}\, dy, \tag{3.10}$$

where $k \in \mathbf{R}$.

Apart from functions that are continuous on the whole of $\mathbf{R}^2$, we have

$$R_\pm(x,y) \equiv R_{\pm,0}(x,y),$$

where

$$R_{+,0}(x,y) = \left(\frac{1}{2}\right) \theta_+(y-x) \int_{(x+y)/2}^\infty p(t)\, dt,$$

and

$$R_{-,0}(x,y) = \left(\frac{1}{2}\right) \theta_+(x-y) \int_{-\infty}^{(x+y)/2} p(t)\,dt, \tag{3.11}$$

$$\theta_+(t) = 1 \quad \text{when } t \geq 0 \quad \text{and 0 otherwise.}$$

These are the leading terms in $R_\pm$, and one has that the $R_\pm$ satisfy the equations

$$R_\pm = R_{\pm,0} + L_{p,\pm} R_\pm, \tag{3.12}$$

where $L_p = L_{p,-}$ is given by

$$L_p T(x,y) = \iint E(x-x', y-y') p(x') T(x', y')\, dx'\, dy',$$

$$E(x,y) = \frac{1}{2} \quad \text{when } x > 0,\ |y| < |x| \quad \text{and 0 otherwise.}$$

There is a similar expression for $L_{p,+}(x,y)$. In what follows we shall write $R_{-,0}(x,y) = R_0(x,y)$, $R_-(x,y) = R(x,y)$, when no confusion seems possible. The reason for this is that later the attention will be restricted to potentials vanishing for $x > 0$.

In order to describe the growth properties of $R(x,y)$, and, in particular, to sharpen (3.8), we introduce the increasing functions

$$u(x) = \int_{-\infty}^{x} |p(t)|\, dt, \quad v(x) = \int_{-\infty}^{x} u(t)\, dt.$$

The solution R of (3.12) is obtained by inverting the operator $I - L_p$;

$$R = \sum_{k=0}^{\infty} L_p^k R_0. \tag{3.13}$$

Estimating the partial sums in (3.13), one can show that the estimate

$$|R(x,y)| \leq \frac{1}{2} u\left(\frac{x+y}{2}\right) \exp\left(v(x) - v\left(\frac{x+y}{2}\right)\right), \quad y < x, \tag{3.14}$$

is true, see [11]. It implies that

$$||\, R(x,\cdot)\, ||_{L^1} = \int_{-\infty}^{x} |R(x,y)|\, dy \leq e^{v(x)} - 1.$$

Another important result ([13], Lemma 4.2) is that

$$p(x) R(x,y) \in L^1(\mathbf{R}^2). \tag{3.15}$$

Notice also that if $x \geq a$ in the support of p, then it is immediate from (3.14) that

$$2a - x \leq y \leq x \quad \text{in the support of } \; R(x,y). \tag{3.16}$$

We introduce now the following representations for the functions a and b, given by Melin [13]: There exist temperate real-valued distributions X and Y such that

$$a(k) = \hat{X}(k) \quad \text{and} \quad b(k) = \hat{Y}(k), \tag{3.17}$$

where X and Y are given by the following explicit formulas

$$X(y) = \delta'(y) - \left(\frac{1}{2}\right)\left(\int_{-\infty}^{+\infty} p(z)dz\right)\delta(y) - \left(\frac{1}{2}\right)\int_{-\infty}^{+\infty} p(z)R(z, z+y)\, dz, \tag{3.18}$$

$$Y(y) = \left(\frac{1}{4}\right)p\left(\frac{y}{2}\right) + \left(\frac{1}{2}\right)\int_{-\infty}^{+\infty} p(z)R(z, y-z)\, dz. \tag{3.19}$$

We remark that the expressions for X and Y in [13] were given in terms of the kernel $R_+(x,y)$, but using the identities $R_+(x,y,\check{p}) = R(-x,-y,p)$, $X_{\check{p}} = X_p$ and $Y_{\check{p}} = \check{Y}_p$, which appear in [13], formula (5.14), it is easy to see that the representations (3.18) and (3.19) are valid.

For future reference we rewrite now (3.2) in the form

$$ikg(x,k) = \hat{X}(k)f(x,-k) + \hat{Y}(k)f(x,k). \tag{3.20}$$

A combination of (3.16) with (3.18) and (3.19) shows that

$$\operatorname{chsupp}(Y) \subset \operatorname{chsupp}(p(\cdot/2)) \tag{3.21}$$

and

$$\operatorname{chsupp}(X) \subset [-2d, 0], \tag{3.22}$$

if d is the diameter of the support of p. Furthermore,

$$X(y) - \delta'(y) + \left(\frac{1}{2}\right)\left(\int_{-\infty}^{+\infty} p(z)dz\right)\delta(y) \in L^1 \cap L^\infty, \tag{3.23}$$

and

$$Y(y) - \left(\frac{1}{4}\right)p\left(\frac{y}{2}\right) \in L^1 \cap L^\infty. \tag{3.24}$$

It follows from (3.22) and (3.23) that $\hat{X}$ extends to an analytic function in $\operatorname{Im} k > 0$, continuous up to the boundary. We also know that $\hat{X}(k)$ has finitely many zeros in $\operatorname{Im} k > 0$, all of them simple and situated on the imaginary axis. Furthermore, $i\beta$ is a zero precisely when $-\beta^2$ is an eigenvalue of H_p.

We shall now give some symmetry properties of the distributions X and Y with respect to the one-parameter groups $\delta_\lambda p(x) = \lambda^2 p(\lambda x)$, $\lambda > 0$ and $\tau_h p(x) = p(x+h)$, $h \in \mathbf{R}$. It follows from (3.11) and (3.13) that $R_{\delta_\lambda p}(x,y) = \lambda R_p(\lambda x, \lambda y)$. Therefore, the mappings

$$p \to X_p \quad \text{and} \quad p \to Y_p$$

commute with the action of the dilatation group. In other words,

$$X_{\delta_\lambda p} = \delta_\lambda X_p, \quad Y_{\delta_\lambda p} = \delta_\lambda Y_p. \tag{3.25}$$

For the translation group we have instead,

$$X_{\tau_h p} = X_p, \quad Y_{\tau_h p} = \tau_{2h} Y_p. \tag{3.26}$$

We have already observed the important result (3.15), valid for all potentials $p \in L$. Later on we shall concentrate the discussion on the case of exponentially decaying potentials, and the following proposition will then be our starting point.

Proposition 3.1 *Let $e^{2a|x|}p(x) \in L$ for some $a > 0$, and set*

$$q(x) = e^{2a|x|}\left|p(x)\right|.$$

Then

$$e^{a|y|}\int_{-\infty}^{\infty} |p(z)R_p(z,z+y)|\,dz \le \int_{-\infty}^{\infty} q(z)R_q(z,z+y)\,dz \in L^1 \cap L^\infty, \tag{3.27}$$

and

$$e^{a|y|}\int_{-\infty}^{\infty} |p(z)R_p(z,y-z)|\,dz \le \int_{-\infty}^{\infty} q(z)R_q(z,y-z)\,dz \in L^1 \cap L^\infty. \tag{3.28}$$

Proof: Since $|R_p| \le R_{|p|}$, we may assume that $p \ge 0$. First we shall prove that

$$p(x)R_p(x,y) \le e^{-2a|x|+a(x+y)}q(x)R_q(x,y), \tag{3.29}$$

for this clearly implies (3.27). We have

$$R_{0,p}(x,y) \le \frac{1}{2}\int_{-\infty}^{(x+y)/2} e^{-2a|t|}q(t)\,dt \le e^{a(x+y)}R_{0,q}(x,y), \quad y \le x,$$

since $|t| \ge -(x+y)/2$. It suffices then to prove that

$$L_p^k R_{0,p}(x,y) \le e^{a(x+y)} L_q^k R_{0,q}(x,y). \tag{3.30}$$

However, we have that the following sharper bound is valid,

$$L_p^k R_{0,p}(x,y) \le e^{a(x+y)} L_p^k R_{0,q}(x,y). \tag{3.31}$$

We have just seen that it is true when $k = 0$, so we assume that $k \ge 1$ and (3.31) has been proved for lower values of k. We have

$$\begin{aligned} L_p^k R_{0,p}(x,y) &= \iint E(x-x', y-y') p(x') L_p^{k-1} R_{0,p}(x',y')\, dx'\, dy' \\ &\le \iint E(x-x', y-y') e^{a(x'+y')} p(x') L_p^{k-1} R_{0,q}(x',y')\, dx'\, dy', \end{aligned}$$

and we only have to notice that

$$x' + y' \le x + y$$

when $x \ge y$ and (x', y') is in the support of the integrand. This gives (3.27).

When proving (3.28), we obtain from (3.29)

$$p(z)R_p(z, y-z) \le e^{-2a|z|+ay} q(z) R_q(z, y-z) \le e^{-a|y|} q(z) R_q(z, y-z),$$

when $y < 0$. When $y \ge 0$, we notice that $y \le 2z$ in the support of $p(z)R_p(z, y-z)$, and estimating R_p by R_q, we get

$$p(z)R_p(z, y-z) \le e^{-2a|z|} q(z) R_q(z, y-z) \le e^{-a|y|} q(z) R_q(z, y-z).$$

The proof is complete. □

It follows that when $e^{2a|x|}p(x) \in L$, $\hat{X} = \hat{X}_p$ and $\hat{Y} = \hat{Y}_p$ extend to analytic functions in the strip $S = \{k; |\operatorname{Im} k| < a\}$, continuous up to the boundary of this set. The relation (3.3) extends to S as

$$k^2 + \hat{Y}(k)\hat{Y}(-k) = \hat{X}(k)\hat{X}(-k), \tag{3.32}$$

since $\overline{\hat{X}(k)} = \hat{X}(-k)$, $\overline{\hat{Y}(k)} = \hat{Y}(-k)$, $k \in \mathbf{R}$.

We shall finish this section by discussing potentials p such that $x \le b$ in the support of p for some $b < \infty$. Then (3.21) gives that $x \le 2b$ in the support of Y, and it follows that the reflection coefficient

$$r(k) = \frac{\hat{Y}(k)}{\hat{X}(k)}$$

has a continuation off the real axis to a meromorphic function in $\operatorname{Im} k > 0$. We have the following

Theorem 3.2 *If $p \in L$ and $x \leq b$ in the support of p for some b, then p is uniquely determined by the sequence $\{r(in), n = N, N+1, \ldots\}$, where N is an arbitrary positive integer.*

Proof: In view of (3.26) we may assume that $b = 0$. Let p_1 and p_2 be two potentials, satisfying the assumptions above and such that $r_1(in) = r_2(in)$, $n = N, N+1, \ldots$. Put

$$h(k) = r_2(k + iM) - r_1(k + iM),$$

for a sufficiently large positive integer $M \geq N$. Then it follows from (3.18), (3.19) and (3.23), (3.24) that h is a bounded analytic function in $\mathbf{C}_+$. Now it is well known that the zeros $\{z_j\}$ of a function in $H^\infty(\mathbf{C}_+)$, which is not identically zero, satisfy the condition for convergence of a Blaschke product,

$$\sum_{j=1}^{\infty} \frac{\operatorname{Im} z_j}{1 + |z_j|^2} < \infty,$$

see [4]. Since $h(in) = 0$, $n \geq N$, we have that h must vanish identically. Therefore, $r_1(k) = r_2(k)$ in $\operatorname{Im} k \geq 0$, $k \neq 0$. The proof is therefore completed by an application of the following proposition. □

Proposition 3.3 *If $p \in L$ and $x \leq b$ in the support of p, then p is uniquely determined by the right reflection coefficient $r(k)$.*

Proof: This result was first proved in [12], as was kindly pointed out to the author by Roman Novikov. For the sake of completeness we present here the simple argument.

We may assume again that $b = 0$, and then we write

$$r(k) = \frac{\hat{Y}(k)}{\hat{X}(k)}, \quad \operatorname{Im} k \geq 0, \quad k \neq 0.$$

The support properties of p imply that $x \leq y \leq -x$ in the support of $R_+(x, y)$ and (3.9) therefore gives that $f(x, k)$ is an entire analytic function of k. The relation (3.20) extends then to $\operatorname{Im} k > 0$ and it follows that $\hat{Y}(k)$ does not vanish at the zeros of $\hat{X}(k)$. Thus $-\beta^2$ is an eigenvalue of H_p precisely when $i\beta$ is a pole of $r(k)$ and the eigenvalues of H_p are determined by $r(k)$.

Consider now the Gelfand-Levitan equation for the right scattering data,

$$R_+(x, y) + Q(x + y) + \int_x^{+\infty} R_+(x, t) Q(y + t) dt = 0, \quad y > x, \tag{3.33}$$

where the kernel $Q(x)$ is given by

$$Q(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\frac{\hat{Y}(k)}{\hat{X}(k)}dk + \sum_{j=1}^{N} c_j^+ e^{-\beta_j x}, \tag{3.34}$$

and c_j^+ are the norming constants, see [13]. It is easy to see that $Q(x) = 0$ for $x > 0$ if and only if $R_+(x,y) = 0$ for $x > 0$. The last condition holds, since $\operatorname{supp}(p) \subset \overline{\mathbf{R}_-}$. Now it is clear that the fact that $Q(x) = 0$ for $x > 0$ determines the c_j^+ uniquely. The uniqueness in the proposition follows, since any $p \in L$ is determined by the right reflection coefficient together with the poles β_j and the norming constants c_j^+. □

Remark. We can also find the expression for the norming constants in terms of the reflection coefficient. Recall from [13] that

$$c_j^+ = \left(\int_{-\infty}^{\infty} f(x, i\beta_j)^2\, dx\right)^{-1}.$$

It follows from (3.20) that when $\hat{X}(i\beta_j) = 0$, we have

$$g(x, i\beta_j) = \frac{-\hat{Y}(i\beta_j)}{\beta_j} f(x, i\beta_j),$$

and, thus,

$$c_j^+ = \frac{-i\hat{Y}(i\beta_j)}{i\beta_j \int_{-\infty}^{\infty} f(x, i\beta_j) g(x, i\beta_j)\, dx}.$$

In view of [13], formula (5.36), we get

$$c_j^+ = \frac{-i\hat{Y}(i\beta_j)}{\hat{X}'(i\beta_j)} = -i\operatorname{Res}(r(k), i\beta_j). \tag{3.35}$$

4 Proof of Theorem 2.4

When $p \in L$, the Gelfand-Levitan kernel $Q = Q_p$ of p is a continuous function on $\mathbf{R}$ which solves the equation

$$R_+(x,y) + Q(x+y) + \int_x^{\infty} R_+(x,t) Q(y+t)\, dt = 0, \quad y > x.$$

We notice that the part of Q which is linear in p equals

$$-R_{+,0}(x, x+0) = -R_+(x, x+0) = -\frac{1}{2}\int_x^{\infty} p(y)\, dy = -\frac{1}{2}P(x).$$

We let Q also denote the operator with integral kernel $Q(x+y)$. It will be clear from the context which interpretation we have chosen. If $A_{+,j} = A_{+,p_j}$ correspond to $p_j \in L$, $j = 1,2$ and Q_1, Q_2 are the corresponding Gelfand-Levitan operators, we have the following general identity.

Proposition 4.1 *Assume that $p_j \in L$, $j = 1,2$. Then*

$$\frac{1}{2}\left(P_1(x) - P_2(x)\right) = \left(A_{+,1}(Q_2 - Q_1)A^*_{+,2}\right)(x,x). \tag{4.1}$$

Proof: We shall make use of the calculus for some classes of operators developed in Section 3 of [13]. This calculus makes it meaningful to study products $A_{+,p_j}\ (I\ +\ Q_k)\ A^*_{+,p_l}$ for arbitrary $j,k,l \in \{1,2\}$.

Recall the operator formulation of the Gelfand-Levitan equation, as given in [13]

$$A_{+,j}(I + Q_j)A^*_{+,j} = I, \quad j = 1,2. \tag{4.2}$$

Then

$$A_{+,1}(I + Q_1)A^*_{+,2} = (A^*_{+,1})^{-1}A^*_{+,2}.$$

It follows from the calculus developed in [13] that $x \geq y$ in the support of (the kernel of) the operator $A_{+,1}(I + Q_1)A^*_{+,2}$, and since this is a continuous function up to the boundary in the sets $\pm(y - x) > 0$, we have

$$\left(A_{+,1}(I + Q_1)A^*_{+,2}\right)(x, x + 0) = 0.$$

With $A_{+,j} = I + R_{+,j}$, $j = 1,2$ this implies that

$$\begin{aligned} 0 &= \left((I + Q_1)A^*_{+,2}\right)(x, x+0) + (R_{+,1}(I + Q_1)A^*_{+,2})(x, x+0) \\ &= \left(Q_1 A^*_{+,2}\right)(x, x+0) + (R_{+,1}A^*_{+,2})(x, x+0) + \left(R_{+,1}Q_1\right)(x,x) \\ &\quad + \left(R_{+,1}Q_1R^*_{+,2}\right)(x,x) = \left(Q_1A^*_{+,2}\right)(x,x) + \left(R_{+,1}R^*_{+,2}\right)(x,x) \\ &\quad + R_{+,1}(x, x+0) + \left(R_{+,1}Q_1\right)(x,x) + \left(R_{+,1}Q_1R^*_{+,2}\right)(x,x). \end{aligned}$$

Here we have written, for example, $\left(R_{+,1}R^*_{+,2}\right)(x, x+0) = \left(R_{+,1}R^*_{+,2}\right)(x,x)$, since it follows easily from the results in [13] that the function $\left(R_{+,1}R^*_{+,2}\right)(x,y)$ is continuous in $\mathbf{R}^2$.

Interchanging p_1 and p_2 and taking the difference of the equations, we obtain

$$\begin{aligned} R_{+,1}(x, x+0) - R_{+,2}(x, x+0) &= \left(Q_2A^*_{+,1}\right)(x,x) - \left(Q_1A^*_{+,2}\right)(x,x) \\ &\quad + \left(R_{+,2}Q_2\right)(x,x) - \left(R_{+,1}Q_1\right)(x,x) \\ &\quad + \left(R_{+,1}(Q_2 - Q_1)R^*_{+,2}\right)(x,x). \end{aligned}$$

Here we have used that $\left(R_{+,1}R^*_{+,2}\right)(x,x) = \left(R_{+,2}R^*_{+,1}\right)(x,x)$ and

$$\left(R_{+,2}Q_2R^*_{+,1}\right)(x,x) = \left(R_{+,1}Q_2R^*_{+,2}\right)(x,x).$$

We get

$$\begin{aligned} R_{+,1}(x,x+0) - R_{+,2}(x,x+0) &= (Q_2 - Q_1)(x,x) + \left(Q_2R^*_{+,1}\right)(x,x) \\ &- \left(Q_1R^*_{+,2}\right)(x,x) + (R_{+,2}Q_2)(x,x) - (R_{+,1}Q_1)(x,x) \\ &+ \left(R_{+,1}(Q_2-Q_1)R^*_{+,2}\right)(x,x) = (Q_2-Q_1)(x,x) \\ &+ (R_{+,1}(Q_2-Q_1))(x,x) + (R_{+,2}(Q_2-Q_1))(x,x) \\ &+ \left(R_{+,1}(Q_2-Q_1)R^*_{+,2}\right)(x,x) = \left(A_{+,1}(Q_2-Q_1)A^*_{+,2}\right)(x,x), \end{aligned}$$

since $(R_{+,2}(Q_2-Q_1))(x,x) = \left((Q_2-Q_1)R^*_{+,2}\right)(x,x)$. The proof is complete. □

Let us write

$$\begin{aligned} \frac{1}{2}(P_1(x) - P_2(x)) &= (Q_2 - Q_1)(2x) \\ &+ \int R_{+,1}(x,y)\,(Q_2(x+y) - Q_1(x+y))\,dy \\ &+ \int R_{+,2}(x,y)\,(Q_2(x+y) - Q_1(x+y))\,dy \\ &+ \int R_{+,1}(x,y)\left\{\int R_{+,2}(x,z)\,(Q_2(y+z) - Q_1(y+z))\,dz\right\}dy. \end{aligned} \tag{4.3}$$

From now on we assume that $a = 0$ and $p_1,\, p_2 \in M_q$. Since H_{p_j}, $j = 1,2$ has no bound states, (3.34) gives that

$$r_{p_j}(k) = \widehat{Q}_j(k). \tag{4.4}$$

Hence $Q_j \in L^2(\mathbf{R})$ and we have that

$$\left|\int R_{+,j}(x,y)\,(Q_2(x+y) - Q_1(x+y))\,dy\right| \leq ||\,R_{+,j}(x,\cdot)\,||_{L^2}||\,Q_2 - Q_1\,||_{L^2}.$$

Estimating the last integral in the right-hand side of (4.3), we see that

$$\begin{aligned} &\left|\int R_{+,1}(x,y)\left\{\int R_{+,2}(x,z)\,(Q_2(y+z) - Q_1(y+z))\,dz\right\}dy\right| \\ \leq\ &||\,R_{+,2}(x,\cdot)\,||_{L^2}\int_x^\infty |R_{+,1}(x,y)|\left\{\int_x^\infty |Q_2(y+z) - Q_1(y+z)|^2 dz\right\}^{1/2} dy \\ \leq\ &||\,R_{+,1}(x,\cdot)\,||_{L^2}||\,R_{+,2}(x,\cdot)\,||_{L^2}||\,Q_x\,||_{L^2}, \end{aligned}$$

where $Q_x(y,z) = Q_2(y+z) - Q_1(y+z)$ when $x \le y$, $x \le z$ and $Q_x = 0$ otherwise. Since $y + z \le 0$ in the support of Q_x, we have

$$\begin{aligned} \|\, Q_x \,\|_{L^2}^2 &= \iint_{x\le y, x\le z, y+z\le 0} |Q_2(y+z) - Q_1(y+z)|^2 \, dy\, dz \\ &= \iint_{2x\le t\le 0, x\le z\le |x|} |Q_2(t) - Q_1(t)|^2 \, dt\, dz \le 2\,|x|\, \|\, Q_2 - Q_1 \,\|_{L^2}^2 . \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |P_2(x) - P_1(x)| \;\le\; & 2\,|Q_2(2x) - Q_1(2x)| \\ & +2\,(\|\, R_{+,1}(x,\cdot)\,\|_{L^2} + \|\, R_{+,2}(x,\cdot)\,\|_{L^2})\,\|\, Q_2 - Q_1\,\|_{L^2} \\ & +2\,(2\,|x|)^{1/2}\,\|\, R_{+,1}(x,\cdot)\,\|_{L^2}\|\, R_{+,2}(x,\cdot)\,\|_{L^2}\|\, Q_2 - Q_1\,\|_{L^2}. \end{aligned} \tag{4.5}$$

We now come to estimate $\|\, R_+(x,\cdot)\,\|_{L^2} = \|\, R_{+,p}(x,\cdot)\,\|_{L^2}$ when $p \in M_q$. From (3.12) we recall that

$$R_+ = R_{+,0} + L_{p,+} R_+,$$

where

$$R_{+,0}(x,y) = \frac{1}{2}\theta_+(y-x) \int_{(x+y)/2}^{\infty} p(t)\, dt,$$

and

$$L_{p,+}T(x,y) = \iint E(x-x', y-y') p(x') T(x',y')\, dx'\, dy',$$

where

$$E(x,y) = \frac{1}{2} \quad \text{when } x<0,\ |y|<|x| \quad \text{and 0 otherwise.}$$

Using the Minkowski inequality and the fact that

$$\|\, E(x,\cdot)\,\|_{L^2} \le |x|^{1/2},$$

we get

$$\|\, R_+(x,\cdot)\,\|_{L^2} \le \|\, R_{+,0}(x,\cdot)\,\|_{L^2} + \iint_{x\le x'\le 0} |x-x'|^{1/2}\, |p(x') R_+(x',y')|\, dx'\, dy'.$$

Since

$$R_{+,0}(x,y) = \frac{1}{2} \int p(z) \chi_{x,z}(y)\, dz,$$

where $\chi_{x,z}$ is the characteristic function of the set $\{y; x \le y \le 2z - x\}$, we have

$$\|\, R_{+,0}(x,\cdot)\,\|_{L^2} \le \|\, p\,\|\, |x|^{1/2}.$$

We obtain the estimate

$$\begin{aligned} \| R_+(x,\cdot) \|_{L^2} &\leq \left(\| p \| + \iint |p(x')R_+(x',y')| \; dx' \, dy' \right) |x|^{1/2} \qquad (4.6) \\ &\leq C(q) \, |x|^{1/2}, \quad x < 0, \quad p \in M_q. \end{aligned}$$

Here we have used (3.15) together with Lemma 4.2 in [13], which shows that for any $p \in M_q$ we have

$$\iint |p(x)R_+(x,y)| \; dx \, dy \leq \varphi(\| q \|),$$

where $\varphi(s) = 2e^s(2s^2 + s^3 e^s)$. In fact, it follows from the analysis in [7] that

$$\iint |p(x)R_{+,p}(x,y)| \; dx \, dy \leq \| p \|_{L^1} \psi \left(\int |x| \, |p(x)| \; dx \right),$$

where $\psi(s) = 2e^s s(1+s)$. This estimate has the merit of being invariant under scaling, see (3.25). We shall, however, not be concerned with the exact values of the constants, depending only on $\| q \|$ and as in Section 2 we use the notation $C(q)$ to denote such constants.

Combining (4.5) and (4.6) we obtain

$$|P_2(x) - P_1(x)| \leq 2 \, |Q_2(2x) - Q_1(2x)| + C(q)(1+|x|)^{3/2} \| Q_2 - Q_1 \|_{L^2}.$$

This gives

$$\| \langle x \rangle^{-\delta} (P_2 - P_1) \|_{L^2} \leq C(q) \| Q_2 - Q_1 \|_{L^2}, \quad \delta = \frac{5}{2},$$

and in view (4.4) the proof of Theorem 2.4 is complete.

5 Proof of Theorem 2.5

Let $a > 0$ be a given number and $q \geq 0$ be a function such that $e^{2a|x|}q(x) \in L$ and $x \leq 0$ in supp(q). The number a and the function q will be kept fixed in the considerations. In the beginning of this section we shall work with potentials in the set M_q, introduced in Section 2. We recall the notation

$$\| p \|_a = \int_{-\infty}^0 (1+|x|) e^{2a|x|} \, |p(x)| \; dx,$$

and $\| p \| = \| p \|_0$.

If $p \in M_q$ we let r_p denote the corresponding right reflection coefficient. We are going to derive estimates for $r_{p_2} - r_{p_1}$ when p_1, $p_2 \in M_q$ and

$$r_{p_1}(in) = r_{p_2}(in), \quad n = 1, \ldots N. \tag{5.1}$$

When $p \in M_q$ then $r_p = \hat{Y}_p / \hat{X}_p$ is meromorphic in $\operatorname{Im} k > -a$. It is analytic in the upper half-plane, continuous in $\operatorname{Im} k \geq 0$, $k \neq 0$ and it has at most a pole at the origin. Since $|r_p(k)| < 1$ when $k \in \mathbf{R}\backslash\{0\}$ in view of (3.3) and (3.17), we have that r_p is analytic at the origin. We notice that $\hat{Y}_p$ is bounded in $\operatorname{Im} k \geq 0$ and vanishes at ∞ in view of the Riemann-Lebesgue lemma, while $\left|\hat{X}_p(k)/k\right| \to 1$ as $|k| \to \infty$, $\operatorname{Im} k \geq 0$. Moreover, $\left|\hat{X}_p(k)\right| \geq |k|$ when $k \in \mathbf{R}$ in view of (3.3) and (3.17). It follows therefore from the maximum principle that

$$|r_p(k)| \leq 1, \ \ \operatorname{Im} k \geq 0,$$

and

$$\left|\hat{X}_p(k)\right| \geq |k|\,, \quad |k r_p(k)| \leq ||\, Y_p \,||_{L^1}, \ \ \operatorname{Im} k \geq 0. \tag{5.2}$$

Consider

$$g(k) = \frac{\hat{X}_1(k)\hat{X}_2(k)}{k} \rho(k) = \frac{\hat{Y}_2(k)\hat{X}_1(k) - \hat{Y}_1(k)\hat{X}_2(k)}{k}, \tag{5.3}$$

when $X_j = X_{p_j}$, $Y_j = Y_{p_j}$, $j = 1, 2$ and $\rho = r_{p_2} - r_{p_1}$. It follows from (3.18) and (3.19) that $\hat{X}_j(0) + \hat{Y}_j(0) = 0$ and therefore g is analytic in $\operatorname{Im} k > -a$. It is bounded and continuous in the closure of this set and $g \to 0$ as $|k| \to \infty$, $\operatorname{Im} k + a \geq 0$. It is important for us to have an explicit bound on g in terms of the potentials.

Lemma 5.1 *We have*

$$|g(k)| \leq C(a) C(q) \left(1 + ||\, q \,||_a\right)^2, \quad \operatorname{Im} k + a \geq 0.$$

Proof: It suffices to estimate $|g|$ from above along the line $\operatorname{Im} k = -a$. When $p \in M_q$ and

$$f(y) = f_p(y) = \int_{-\infty}^{0} p(x) R(x, x + y)\, dx,$$

we shall estimate $\hat{f}(k)$ on $\operatorname{Im} k = -a$. We have

$$\int_{-\infty}^{0} |f(y)|\, e^{a|y|}\, dy \leq \int_{-\infty}^{0} e^{ax}\, |p(x)| \left(\int_{-\infty}^{x} |R(x, y)|\, e^{-ay}\, dy \right) dx.$$

Using (3.14), we see that the inner integral does not exceed $\exp(v(x) + ax)$ times

$$\int_{-\infty}^{x} u(y)e^{-2ay}\,dy \le \int_{-\infty}^{x} \left(\int_{-\infty}^{y} e^{-2at}\,|p(t)|\,dt \right) dy$$
$$= \int_{-\infty}^{x} (x-t)e^{-2at}\,|p(t)|\,dt \le \int_{-\infty}^{0} |t|\,e^{-2at}\,|p(t)|\,dt \le ||\,p\,||_a.$$

Therefore,

$$\left|\hat{X}_p(k) - ik\right| \le \frac{1}{2}||\,p\,|| + \frac{1}{2}e^{||\,p\,||}||\,p\,||\,||\,p\,||_a, \quad \operatorname{Im} k = -a.$$

Similarly we get

$$\left|\hat{Y}_p(k)\right| \le \frac{1}{2}||\,p\,||_a + \frac{1}{2}e^{||\,p\,||}||\,p\,||\,||\,p\,||_a, \quad \operatorname{Im} k = -a.$$

Then,

$$\begin{aligned}
|g(k)| &\le \left|\hat{Y}_1(k)\right| + \left|\hat{Y}_2(k)\right| + \frac{1}{a}\left(\left|\hat{Y}_1(k)(\hat{X}_2(k) - ik)\right| + \left|\hat{Y}_2(k)(\hat{X}_1(k) - ik)\right| \right) \\
&\le \frac{a+1}{a}\left(\left|\hat{Y}_1(k)\right| + \left|\hat{X}_1(k) - ik\right| + 1 \right)\left(\left|\hat{Y}_2(k)\right| + \left|\hat{X}_2(k) - ik\right| + 1 \right) \\
&\le \frac{a+1}{a}\left(||\,p_1\,||_a(1 + ||\,p_1\,||e^{||\,p_1\,||}) + 1 \right)\left(||\,p_2\,||_a(1 + ||\,p_2\,||e^{||\,p_2\,||}) + 1 \right) \\
&\le \frac{a+1}{a}\left(1 + e^{||\,q\,||}||\,q\,||\right)^2 (1 + ||\,q\,||_a)^2 = C(a)C(q)\,(1 + ||\,q\,||_a)^2.
\end{aligned}$$

The proof is complete. □

The following result is now crucial.

Proposition 5.2 *The estimate*

$$|g(k)| \le C(a)C(q)\,(1 + ||\,q\,||_a)^2 \left(\frac{|k+i|}{N} \right)^{2a}, \quad k \in \mathbf{R}$$

holds.

Proof: Set $k_n = i(a+n)$, $n = 1, \dots N$ and

$$B_N(k) = \prod_{n=1}^{N} \frac{k - k_n}{k - \overline{k_n}},$$
$$h(k) = g(k - ia). \tag{5.4}$$

Then it follows from (5.1) that $h(k)/B_N(k)$ is analytic in $\operatorname{Im} k > 0$ and continuous in its closure. In view of Lemma 5.1 we have that

$$\left|\frac{h(k)}{B_N(k)}\right| \leq C(a)C(q)\,(1 + ||\,q\,||_a)^2$$

when $k \in \mathbf{R}$, and it follows from the maximum principle then that

$$|h(k)| \leq C(a)C(q)\,(1 + ||\,q\,||_a)^2\,|B_N(k)|\,, \quad \operatorname{Im} k \geq 0.$$

Hence

$$|g(k)| \leq C(a)C(q)\,(1 + ||\,q\,||_a)^2\,|B_N(k + ia)|\,, \quad k \in \mathbf{R},$$

and the proposition follows from Lemma 5.3 below. □

Lemma 5.3 *We have*

$$|B_N(k)| \leq \left(\frac{e\,|k + i(a+1)|}{N}\right)^{2\operatorname{Im} k}, \quad \operatorname{Im} k > 0.$$

Proof: The inequality

$$-\log t \geq 1 - t, \quad t > 0,$$

gives

$$-\log\left|\frac{k - k_j}{k - \overline{k_j}}\right|^2 \geq 1 - \left|\frac{k - k_j}{k - \overline{k_j}}\right|^2 = 4\operatorname{Im} k\frac{\operatorname{Im} k_j}{\left|k - \overline{k_j}\right|^2}.$$

Therefore,

$$\begin{aligned}
\log|B_N(k)|^2 &\leq -4\operatorname{Im} k\sum_{j=1}^{N}\frac{\operatorname{Im} k_j}{\left|k - \overline{k_j}\right|^2} \leq -4\operatorname{Im} k\sum_{j=1}^{N}\frac{j}{|k + ia + ij|^2} \\
&= -4\operatorname{Im} k\,\varphi_N(k + ia), \quad k \in \mathbf{C}_+,
\end{aligned}$$

where

$$\varphi_N(k) := \sum_{j=1}^{N}\frac{j}{|k + ij|^2}.$$

Writing $k = \alpha + i\beta$, we get

$$\begin{aligned}
\sum_{j=1}^{N}\frac{j}{|k + ij|^2} &\geq \int_0^N \frac{t}{\alpha^2 + (t + \beta + 1)^2}\,dt \\
&= \int_{\beta+1}^{\beta+1+N}\frac{t}{\alpha^2 + t^2}\,dt - (\beta + 1)\int_{\beta+1}^{(\beta+1+N}\frac{dt}{\alpha^2 + t^2} \\
&\geq \frac{1}{2}\log\left(\frac{\alpha^2 + (\beta + 1 + N)^2}{\alpha^2 + (\beta + 1)^2}\right) - (\beta + 1)\int_{\beta+1}^{\infty}\frac{dt}{t^2} \\
&= \log\left(\left|\frac{k + i(N+1)}{k + i}\right|\right) - 1 \geq \log\left(\frac{N}{|k + i|}\right) - 1.
\end{aligned}$$

Since

$$|B_N(k)| \leq \exp\left(-2\mathrm{Im}\, k\varphi_N(k+ia)\right), \quad \mathrm{Im}\, k > 0,$$

the estimate in the lemma follows. □

Remark. Proposition 5.2 can be generalized to the case of "inaccurate" partial scattering data. Assume that instead of (5.1) we only have

$$|r_{p_2}(in) - r_{p_1}(in)| \leq \delta, \quad n = 1, \ldots N \tag{5.5}$$

for some small δ, and define the functions g and h as in (5.3) and (5.4). By the Pick-Nevanlinna theory (see [4], p. 138) we can find a bounded analytic function f in $\mathrm{Im}\, k > 0$ such that f interpolates the values of h on the sequence $\{i(a+n)\}$, $n = 1, \ldots N$, and which has the minimal $H^\infty-$norm. Then the function $h - f$ vanishes on the sequence $\{i(a+n)\}$, $n = 1, \ldots N$, and as in Proposition 5.2 this gives us a bound on $h - f$ in $\mathrm{Im}\, k > 0$. The problem is then to control the function f. We have to estimate the constant of interpolation

$$M_N = \sup_{||\, b_n\, ||_\infty \leq 1} \inf\{||\, f\, ||_\infty; f(ia+in) = b_n, n = 1 \ldots N, f \in H^\infty(\mathbf{C}_+)\},$$

and this can be done by means of Carleson's interpolation theorem, see [4]. The following estimate holds

$$\frac{1}{\rho} \leq M_N \leq \frac{A}{\rho}\left(1 + \log\frac{1}{\rho}\right), \tag{5.6}$$

where A is some absolute constant, and

$$\rho = \inf_{1 \leq j \leq N} \prod_{l; l \neq j} \left|\frac{k_j - k_l}{k_j - \bar{k}_l}\right|, \quad k_j = i(a+j). \tag{5.7}$$

It is known that except for the value of the numerical constant A, the upper bound for M_N given by (5.6) is sharp, see [4]. Estimating (5.7) we get that $\rho \geq 1/Ce^{-CN}$ and then $M_N \leq Ce^{CN}$ for some $C = C(a)$. This leads to estimates of g on the real line under the assumption (5.5).

We also notice that since the constant of interpolation M_N is a characteristic of the uniform grid of points $\{in + ia\}$, $n = 1, \ldots N$, the bounds on M_N which follow from (5.6) illustrate the degree of ill-posedness of the inverse problem, which is due to the limited accuracy of the measurements of the data.

We shall now use Proposition 5.2 to obtain an L^2 estimate of the function ρ on a compact subset of the real line. We recall that $C(a)$ denotes various constants, which depend only on a.

Proposition 5.4 *The estimate*

$$\int_{|k|\leq R} |\rho(k)|^2 \, dk \leq C(a)C(q)\,(1+||\,q\,||_a)^4 \left(\frac{\psi_a(R)}{N^{4a}} + \frac{1}{N^{2a}}\right) \tag{5.8}$$

is true for any $R > 1$. *Here*

$$\psi_a(R) = \begin{cases} R^{4a-1}, & a > 1/4 \\ \log R, & a = 1/4 \\ 1, & 0 < a < 1/4. \end{cases}$$

Proof: It follows from (5.2) that

$$\left|\hat{X}_p(k)\right| \geq |k|\,, \quad k \in \mathbf{R}, \;\; p \in M_q,$$

and then by (5.3) and Proposition 5.2 we get

$$|\rho(k)| \leq |k|^{-1}\,|g(k)| \leq C(a)C(q)\,(1+||\,q\,||_a)^2\,\frac{(|k|+1)^{2a}}{N^{2a}\,|k|}. \tag{5.9}$$

We notice that, when $0 < a < 1/4$, then the right-hand side of (5.9) is in L^2 near infinity. Then, when $\varepsilon \in (0,1)$ and $R > 1$, we have

$$\begin{aligned} \int_{\varepsilon\leq|k|\leq R} |\rho(k)|^2 \, dk \;\; &\leq \;\; \int_\varepsilon^1 \frac{C(a)C(q)\,(1+||\,q\,||_a)^4}{N^{4a}k^2}\,dk \\ &\quad + \int_1^R \frac{C(a)C(q)\,(1+||\,q\,||_a)^4\,k^{4a-2}}{N^{4a}}\,dk \\ &\leq \;\; C(a)C(q)\,(1+||\,q\,||_a)^4 \left(\frac{1}{N^{4a}\varepsilon} + \frac{\psi_a(R)}{N^{4a}}\right). \end{aligned}$$

Combining this inequality with the fact that $|\rho(k)| \leq 2$ for $|k| \leq \varepsilon$ and choosing $\varepsilon = N^{-2a}$, we get therefore

$$\int_{-R}^{R} |\rho(k)|^2 \, dk \leq C(a)C(q)\,(1+||\,q\,||_a)^4 \left(\frac{\psi_a(R)}{N^{4a}} + \frac{1}{N^{2a}}\right). \tag{5.10}$$

The proof is complete. □

We shall finally obtain an estimate of the function $\rho(k)$ in the whole $L^2(\mathbf{R})$. When doing this we recall from (5.2) that

$$|r_p(k)| \leq \frac{||\,Y_p\,||_{L^1}}{|k|} \leq \frac{C(q)}{|k|}, \quad k \neq 0.$$

Combining this with (5.8) gives

$$||\,\rho\,||_{L^2}^2 \le C(a)C(q)\,(1+||\,q\,||_a)^4 \left(\frac{\psi_a(R)}{N^{4a}} + \frac{1}{N^{2a}} + \frac{1}{R}\right), \quad R > 1.$$

When $1/4 \le a \le 1/2$ we balance the terms in the right-hand side by choosing $R = N$ and when $0 < a < 1/4$ we take $R = +\infty$. This leads directly to the first statement of Theorem 2.5.

Let us now recall the set $M_q^{(m)}$, $m \ge 0$, introduced in Section 2. In the remaining part of this section we shall complete the proof of Theorem 2.5, deriving stability estimates for $r_{p_2} - r_{p_1}$ when p_1, $p_2 \in M_q^{(m)}$ and (5.1) holds.

Proposition 5.5 *When $p \in M_q^{(m)}$ then*

$$||\,Y_p^{(\beta)}\,||_{L^1} \le C(2^m q), \quad 0 \le \beta \le m.$$

Proof: It follows from the results in [13] that when $p \in M_q^{(m)}$ we have $R_p(x,y) = \theta_+(x-y)G_p(x,y)$ with $G_p \in C^m(\mathbf{R}^2)$. Our starting point is the estimate

$$|(\partial_x + \partial_y)\,R_p| \le R_{2q}, \;\; p \in M_q^{(m)}, \;\; m \ge 1. \tag{5.11}$$

When proving it we let $R_{p,j}$ denote the contribution to R_p which is homogeneous of degree $j \ge 1$ in p. In order to prove (5.11) it suffices to show that

$$|(\partial_x + \partial_y)\,R_{p,j}| \le jR_{q,j} \le R_{2q,j}, \quad j \ge 1. \tag{5.12}$$

We notice that the last inequality follows since $R_{q,j}$ is homogeneous of degree j in q and $2^j \ge j$.

We have

$$R_{p,1}(x,y) = \left(\frac{1}{2}\right)\theta_+(x-y)\int_{-\infty}^{(x+y)/2} p(t)\,dt,$$

and since $|p(x)| \le \int_{-\infty}^{x} |p'(t)|\,dt$, it follows that

$$|(\partial_x + \partial_y)\,R_{p,1}(x,y)| \le \left(\frac{1}{2}\right)\theta_+(x-y)\int_{-\infty}^{(x+y)/2} q(t)\,dt = R_{q,1}(x,y), \;\; p \in M_q^{(m)}.$$

We may assume therefore that $j > 1$ and (5.12) has already been proved for lower values of j. Since

$$R_{p,j}(x,y) = \iint E(x-x', y-y')p(x')R_{p,j-1}(x',y')\,dx'\,dy',$$

we have

$$|(\partial_x+\partial_y)\,R_{p,j}(x,y)| \le \iint E(x-x',y-y')\,|p'(x')R_{p,j-1}(x',y')|\,dx'\,dy'$$

$$+\quad \iint E(x-x',y-y')\,|p(x')\,(\partial_x+\partial_y)\,R_{p,j-1}(x',y')|\,dx'\,dy$$

$$\le \iint E(x-x',y-y')q(x')R_{q,j-1}(x',y')\,dx'\,dy$$

$$+\quad \iint E(x-x',y-y')q(x')(j-1)R_{q,j-1}(x',y')\,dx'\,dy = jR_{q,j}(x,y),$$

and this gives (5.12) and therefore (5.11). Similarly, using Leibniz' rule, we obtain

$$\left|(\partial_x+\partial_y)^\beta R_{p,j}\right| \le j^\beta R_{q,j} \le R_{2^\beta q,j},\quad p\in M_q^{(m)},\quad \beta\le m,$$

and then

$$\left|(\partial_x+\partial_y)^\beta R_p\right| \le R_{2^\beta q},\qquad p\in M_q^{(m)},\quad \beta\le m. \tag{5.13}$$

We shall now compute the derivatives of Y_p and when doing this we write $R_p(x,y)=\theta_+(x-y)G_p(x,y)$, where $G_p\in C^m(\mathbf{R}^2)$. It follows from (3.19) that

$$Y_p(y)=\frac14 p\left(\frac y2\right)+\frac12\int_{y/2}^0 p(z)G_p(z,y-z)\,dz,$$

and then

$$\begin{aligned}
Y_p'(y)-\frac18 p'\left(\frac y2\right) &= -\frac14 p\left(\frac y2\right)G_p\left(\frac y2,\frac y2\right)+\frac12\int_{y/2}^0 p(z)\partial_y G_p(z,y-z)\,dz\\
&= -\frac14 p\left(\frac y2\right)G_p\left(\frac y2,\frac y2\right)+\frac14\int_{y/2}^0 p(z)\left((\partial_x+\partial_y)G_p\right)(z,y-z)\,dz\\
&\quad -\frac14\int_{y/2}^0 p(z)\left((\partial_x-\partial_y)G_p\right)(z,y-z)\,dz = -\frac14 p\left(\frac y2\right)G_p\left(\frac y2,\frac y2\right)\\
&\quad +\frac14\int_{y/2}^0 p(z)\left((\partial_x+\partial_y)G_p\right)(z,y-z)\,dz-\frac14\int_{y/2}^0 p(z)\partial_z G_p(z,y-z)\,dz\\
&= \frac14\int_{y/2}^0 p(z)\left((\partial_x+\partial_y)G_p\right)(z,y-z)\,dz+\frac14\int_{y/2}^0 p'(z)G_p(z,y-z)\,dz\\
&= \frac14\int p(z)\left((\partial_x+\partial_y)R_p\right)(z,y-z)\,dz+\frac14\int p'(z)R_p(z,y-z)\,dz.
\end{aligned}$$

It follows then by induction that

$$Y_p^{(\beta)}(y)=\frac{1}{2^{\beta+2}}p^{(\beta)}\left(\frac y2\right)+\frac{1}{2^{\beta+1}}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\int p^{(\gamma)}(z)\left((\partial_x+\partial_y)^{\beta-\gamma}R_p\right)(z,y-z)\,dz, \tag{5.14}$$

when $p \in M_q^{(m)}$, $\beta \leq m$. Combining (5.13) and (5.14) we see that $Y_p^{(\beta)} \in L^1$ when $\beta \leq m$ and we have bounds

$$|| Y_p^{(\beta)} ||_{L^1} \leq || Y_{2^m q} ||_{L^1} \leq C(2^m q), \quad \beta \leq m.$$

The proof is complete. □

It follows from (5.2) and Proposition 5.5 that when $p \in M_q^{(m)}$ we have

$$(1 + |k|)^{m+1} |r_p(k)| \leq C(2^m q),$$

and combining it with (5.8) we get

$$|| \rho ||_{L^2}^2 \leq C(a) C(2^m q) (1 + || q ||_a)^4 \left(\frac{1}{N^{2a}} + \frac{\psi_a(R)}{N^{4a}} + \frac{1}{R^{2m+1}} \right).$$

We may assume now that $a \geq 1/2$. Then $\psi_a(R) = R^{4a-1}$ and we choose R so that $R^{4a-1} = N^{2a}$. If $2m + 1 \geq 4a - 1$ it follows that

$$|| \rho ||_{L^2} \leq C(a) C(2^m q) (1 + || q ||_a)^2 \frac{1}{N^a}.$$

This completes the proof of Theorem 2.5.

6 A radial inverse boundary value problem

The purpose of this section is to present an inverse boundary value problem for the Schrödinger operator, and, in the radially symmetric case, relate the Dirichlet to Neumann map to the right reflection coefficient of one-dimensional scattering theory. This establishes a link between this inverse problem and the problem, studied above. In particular, the identifiability of a radial potential becomes the contents of Theorem 3.2.

6.1 Basic facts on the Dirichlet to Neumann map. A simple example

Let Ω be a smooth bounded domain in $\mathbf{R}^n$, $n \geq 2$, and consider the following elliptic boundary value problem

$$\begin{cases} (-\Delta + v)u = 0 \ \text{ in } \Omega, \\ u = f \ \text{ on } \ \partial\Omega. \end{cases} \tag{6.1}$$

In order to define the Dirichlet to Neumann map, we recall the following solvability result.

Theorem 6.1 *Let Ω be a bounded domain in $\mathbf{R}^n$ with a smooth boundary. Suppose that $v \in L^\infty(\Omega)$ is real-valued, and that zero is not a Dirichlet eigenvalue of $-\Delta + v$ in Ω. Then for every f in $H^{3/2}(\partial\Omega)$ there is a unique u in $H^2(\Omega)$ which solves* (6.1). *The Poisson operator $P(v)f =: u$ satisfies*

$$||P(v)f||_{H^2(\Omega)} \leq C||f||_{H^{3/2}(\partial\Omega)}. \tag{6.2}$$

Proof: The proof follows from the general theory of elliptic boundary value problems and can in this particular case be found in Grisvard [6]. □

If the conditions of the theorem are fulfilled, we may introduce the Dirichlet to Neumann map

$$N(v): \quad H^{3/2}(\partial\Omega) \ni f \longmapsto \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \in H^{1/2}(\partial\Omega),$$

where n is the outer unit normal to the boundary. The Sobolev trace theorem together with (6.2) shows that

$$||\, N(v)f \,||_{H^{1/2}(\partial\Omega)} \leq C||\, f \,||_{H^{3/2}(\partial\Omega)}.$$

It is known that when $v \in C^\infty(\overline{\Omega})$, $N(v)$ is a first order classical elliptic pseudo-differential operator and it follows from Green's theorem that $N(v)$ is symmetric. Hence the self–adjoint extension of $N(v)$ has discrete real spectrum. We refer the reader to Lee and Uhlmann [10] where the full symbol of $N(v)$ is analyzed in some local coordinates.

The inverse boundary value problem consists of the study of the map $v \longmapsto N(v)$ and its inverse, when it exists. Let us illustrate it by studying a very simplified version of the inverse problem. We shall take $v = \lambda$, a constant, and $\Omega = \mathbf{D}$, the unit disc, and examine to what extent knowledge of the Dirichlet to Neumann map $N(v)$ allows us to determine this constant. In order to have the unique solvability of the boundary value problem (6.1) we assume that $-\lambda$ lies below the spectrum of the Dirichlet Laplacian on $\mathbf{D}$.

We set now $v(x) = \lambda > 0$ and consider

$$\begin{cases} -\Delta u + \lambda u = 0 \text{ in } \mathbf{D}, \\ u = f \text{ on } \partial\mathbf{D}. \end{cases} \tag{6.3}$$

Assuming that f is constant, we shall find the radial solution to this problem.

Let J_0 denote the Bessel function of the first kind of order 0 and I_0 denote the modified Bessel function, corresponding to J_0 (i.e. $I_0(r) = J_0(ir)$). The function I_0 is positive on the positive real axis and solves the equation

$$-\frac{d^2 I_0}{dr^2} - \frac{1}{r}\frac{dI_0}{dr} + I_0 = 0.$$

It follows that the function

$$u_\lambda(r) = \frac{f}{I_0(\sqrt{\lambda})} I_0(\sqrt{\lambda} r)$$

is the solution of the Dirichlet problem (6.3). We have also

$$I_0(\sqrt{\lambda}) = \sum_{p=0}^{\infty} \frac{\lambda^p}{4^p (p!)^2}.$$

Then

$$N(v)f = (\partial_r u_\lambda)(1) = \psi(\lambda) f,$$

where

$$\psi(\lambda) = \frac{I_0'(\sqrt{\lambda})\sqrt{\lambda}}{I_0(\sqrt{\lambda})}$$

Thus the Dirichlet to Neumann map, restricted to constant functions, is given as a multiplication by $\psi(\lambda)$. We have the following theorem.

Theorem 6.2 *$\psi(\lambda)$ is strictly increasing on the interval* $(0, \infty)$.

Proof: Set $\lambda = e^s$ and $g(s) = \varphi(\lambda) = I_0(\sqrt{\lambda})$. Then

$$\frac{1}{2}\psi(\lambda) = \frac{\lambda \varphi'(\lambda)}{\varphi(\lambda)} = \frac{g'(s)}{g(s)} = (\log g(s))',$$

and it is enough to show that $\log g$ is strictly convex. In view of Lemma 1.2.8 in [8] this follows from the fact that the function

$$e^{cs} g(s) = \sum_{j=0}^{\infty} \frac{e^{(c+j)s}}{4^j (j!)^2},$$

is strictly convex for any c. □

6.2 The Dirichlet to Neumann map in the radial case

Consider

$$\begin{cases} -\Delta u + v(r)u = 0 \text{ in } \mathbf{D}, \\ u = f \quad \text{on } \partial \mathbf{D}. \end{cases} \tag{6.4}$$

We assume that v is a bounded radial function, such that 0 is not a Dirichlet eigenvalue of $-\Delta + v$ in $\mathbf{D}$.

For $n \in \mathbf{Z}$, we let Π_n be the orthogonal projection of $L^2(\mathbf{D})$ onto the subspace $L^2_n(\mathbf{D})$ consisting of functions which in polar coordinates have the form $\rho(r)e^{in\varphi}$. We notice that

$$(\Pi_n u)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(ze^{i\theta})e^{-in\theta}\,d\theta, \quad z \in \mathbf{D}.$$

The corresponding spaces of functions on the boundary are denoted $L^2(\partial\mathbf{D})$ and $L^2_n(\partial\mathbf{D})$. The latter space is one-dimensional. We also let $\Pi^0_n : L^2(\partial\mathbf{D}) \to L^2_n(\partial\mathbf{D})$ be the orthogonal projection.

The operator Π_n commutes with $-\Delta + v(r)$, and the Sobolev spaces $H^s(\mathbf{D})$ are invariant under Π_n for each $s \geq 0$. Since

$$\left(\frac{\partial}{\partial r}\Pi_n u\right)\Big|_{\partial\mathbf{D}} = \Pi^0_n\left(\frac{\partial u}{\partial r}\Big|_{\partial\mathbf{D}}\right),$$

it is also true that Π^0_n commutes with the Dirichlet to Neumann map $N(v)$. We may therefore write $N(v)$ as a direct sum

$$N(v) = \bigoplus_{n\in\mathbf{Z}} N_n(v),$$

where $N_n(v)$, for each n, is a linear operator on the one-dimensional space $L^2_n(\partial\mathbf{D})$, and is therefore given by a number. These numbers are real and $N_n(v) = N_{-n}(v)$, since v is real. We may compute $N_n(v)$ by noticing that

$$-\Delta u + vu = \left(-\rho''(r) - \frac{1}{r}\rho'(r) + (n^2/r^2 + v(r))\rho(r)\right)e^{in\varphi},$$

when $u \in L^2_n(\mathbf{D})$ equals $\rho(r)e^{in\varphi}$ in polar coordinates. If $-\Delta u + vu = 0$, then we have $\rho'(1) = N_n(v)\rho(1)$.

It is convenient now to introduce new coordinates. We write $r = e^s$, $-\infty < s < 0$, and set $h(s) = \rho(r)$. Then we have

$$-h''(s) + p(s)h(s) = -n^2 h(s), \quad -\infty < s < 0. \tag{6.5}$$

where $p(s) = e^{2s}v(e^s)$. A basis $h^{\pm}_n$ of solutions of the equation (6.5) for $n \neq 0$ is defined by the condition $h^{\pm}_n(s) \sim e^{\pm ns}$, as $s \to -\infty$. When $n = 0$, then a basis $h^{\pm}_0$ satisfies $h^+_0 \sim 1$ and h^-_0 grows linearly at $-\infty$. If $n \geq 0$, then, since u is locally bounded, we have that $u(re^{i\varphi})$ is proportional to $h^+_n(s)e^{in\varphi}$, if $-\Delta u + vu = 0$, and $u \in L^2_n(\mathbf{D})$. We write therefore $h_n(s) = h^+_n(s)$, $n \geq 0$, and it follows that $h'_n(0) = N_n(v)h_n(0)$. Since we have assumed that $\rho(1) \neq 0$, we have the following proposition.

Proposition 6.3 *In the radially symmetric case the Dirichlet to Neumann map $N(v)$ and the sequence*

$$\left\{ \frac{h_n'(0)}{h_n(0)} \right\}_{n=0}^{\infty} \tag{6.6}$$

carry the same information.

Using Proposition 6.3 we shall now relate the Dirichlet to Neumann map to the right reflection coefficient of the potential $p(s)$, extended as zero to the positive half line. Recall the left Jost function $g(x,k)$, defined by (3.10). We observe that the restriction of $g(x,in)$ to the negative axis solves (6.5) for $n = 0,1,2,\ldots$, and since $g(x,in) \sim e^{nx}$ at $-\infty$, the knowledge of the Dirichlet to Neumann map $N(v)$ is equivalent to the knowledge of the sequence

$$\mu(in) = \frac{g'(0,in)}{g(0,in)}, \quad n = 0,1,2\ldots$$

Recall from (3.20) that

$$ik\, g(x,k) = \hat{X}(k)\, f(x,-k) + \hat{Y}(k)\, f(x,k), \quad k \in \mathbf{R}. \tag{6.7}$$

Since

$$f(x,k) = e^{ixk} \quad \text{for} \quad x > 0,$$

we immediately obtain

$$ikg(0,k) = \hat{X}(k) + \hat{Y}(k),$$

$$g_x'(0,k) = \hat{Y}(k) - \hat{X}(k).$$

This extends to $\operatorname{Im} k > 0$, since both sides here are analytic in $\operatorname{Im} k > 0$ and continuous in the closure of this set. We get

$$r(k) = \frac{ik + \mu(k)}{ik - \mu(k)}, \quad \operatorname{Im} k > 0. \tag{6.8}$$

The discussion above can be summarized in the following proposition.

Proposition 6.4 *If $v = v(r)$ and we consider $-\Delta + v(r)$ on the unit disc in two dimensions, then the knowledge of the Dirichlet to Neumann map $N(v)$ implies the knowledge of the sequence $\{r_p(in)\}$, where $n = 1,2\ldots$ and $p(s) = e^{2s}v(e^s)$ for $s \leq 0$ and zero otherwise.*

Remark. The exponential decay of the potential p is of fundamental importance when deriving the stability estimates in Section 5.

From Theorem 3.2 we now get a uniqueness result for the radially symmetric inverse boundary value problem.

Theorem 6.5 *If* $\mathbf{D}$ *is the unit disc in the plane and* $v(r) \in L^\infty(\mathbf{D})$ *is a radial potential, then* v *is uniquely determined by its Dirichlet to Neumann map* $N(v)$*.*

Remarks:
1. The uniqueness was proved in [14] for potentials of the form $v = \Delta u/u$, with $u \in W^{2,p}(\Omega)$, $u > 0$, for some $p > 1$ on arbitrary domains Ω. However, as pointed out in [19], the corresponding result for a general potential $v \in L^\infty(\Omega)$ in two dimensions is unknown.

2. We notice that Theorem 2.2 together with Propositions 6.3 and 6.4 provides us with Hölder type stability estimates for the radially symmetric inverse boundary value problem. More precisely, suppose that a bounded radial potential v is such that the Schrödinger operator H_p, with p defined as in Proposition 6.4, has no bound states. (This is always the case if v is nonnegative, for example.) Then the partial Dirichlet to Neumann data

$$\left\{ \frac{h_n'(0)}{h_n(0)} \right\}_{n=0}^{N}.$$

determines v (in a suitable norm as given by Theorem 2.2) up to an error of order $N^{-1/2}$.

3. Assume that the radial potential $0 \leq v(r)$ is such that $v \in C^k[0,1]$, $v^{(j)}(1) = 0$, $j \leq k$, for some k, and v vanishes near the origin. Then in view of Theorem 2.3 together with Proposition 6.4 it follows that the exponent in the stability estimates for v can be taken as large as we wish provided that k is sufficiently large. This can be considered to reflect the fact that the part of a smooth potential near the centre of the disc is the most difficult to recover.

ACKNOWLEDGEMENTS

The author is deeply grateful to Professor Anders Melin for his invaluable advice and encouragement during the preparation of this paper.

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