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Properties of the Scattering Transform on the Real Line

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Continuity properties of the scattering transform associated to the Schrödinger operator on the real line are studied. Stability estimates of Lipschitz type are derived for the scattering and inverse scattering transforms. © 2001 Academic Press

1. INTRODUCTION

The purpose of this paper is to analyze some of the mapping properties of the scattering transform associated to the Schrödinger operator $H_p u = -u'' + pu$ on the real line. Here p belongs to the space $\mathscr{L}^+$ of real-valued measurable potentials such that

$$\int_a^\infty (1+|x|)|p(x)|\,dx < \infty,$$

for any $a > -\infty$. We establish Lipschitz type estimates for the scattering and inverse scattering trasforms on bounded sets in $\mathscr{L}^+$.

The scattering transform S appears naturally in connection with the Marchenko equation. Also, it is precisely this transform that linearizes the Korteweg–de Vries (KdV) flow. Several authors (see [4, 11, 13]) have observed that p and $S(p)$ have similar behavior at $+\infty$. Moreover, $S(p) - p$ is continuous and has better bounds at $+\infty$ than p. When p is also well behaved at $-\infty$, there is a simple relation between $S(p)$ and the scattering matrix of p, given essentially by the Fourier transformation. In [17] the range of S on $\mathscr{L}^+$ was characterized and it was proved that S maps $\mathscr{L}^+$ homeomorphically onto $S(\mathscr{L}^+)$. In the context of half-line scattering, a similar observation was made in [19]. Here we shall sharpen these

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results by establishing the Lipschitz continuity of the scattering and inverse scattering transformations. (See [8, 9] for stability analyses of the Cauchy problem for the KdV equation by means of scattering transforms.) It should also be mentioned that stability estimates for potentials directly in terms of their scattering data have been studied in [14, 12] (see also [2, 6]).

The plan of the paper is as follows. In Section 2, after recalling the intertwining operators and the scattering transform, we state our main result in Theorem 2.1. The Lipschitz continuity of the forward transform is then established in Section 3, and that of the inverse transform in Section 4. Throughout the paper we make use of the operator theoretical approach to inverse scattering developed in [17]. This approach has recently been used by the author (see [6, 7]) when studying error estimates in inverse scattering and the distribution of resonances in one dimension. Some of the results obtained in these papers will be useful here as well.

2. THE SCATTERING TRANSFORM AND STATEMENT OF THE RESULTS

First, we shall briefly describe the main notation used in the paper. The space $\mathscr{L}^+$ of real-valued measurable functions p such that

$$\|p\|_z = \int_z^\infty (1+|x|)|p(x)|\,dx < \infty, \quad z \in \mathbf{R}, \tag{2.1}$$

was already introduced in Section 1. This is a Fréchet space under the seminorms in (2.1). We shall also use the Banach space L of real-valued measurable functions p on $(0,\infty)$ such that $\|p\|_0 < \infty$, and let L_c denote its complexification. For $p \in L$, write $\|p\|_0 = \|p\|$.

We shall now recall the intertwining operators and the scattering transform, as presented in [17]. Consider the Schrödinger operator

$$H_p u = -u'' + pu,$$

where $p \in \mathscr{L}^+$. Associated to p is the operator $A = I + R$ with

$$H_p A = A H_0, \tag{2.2}$$

and $y \geq x$ in the support of the distribution kernel of A. The function $R(x, y)$ is continuous up to the boundary in the set where $y > x$ and

$$\|R(x,\cdot)\|_{L^1} = \int |R(x,y)|\,dy < \infty, \tag{2.3}$$

for any x. Moreover, $\|R(x,\cdot)\|_{L^1} \to 0$ as $x \to +\infty$. (Much more precise estimates of R are given later.) Apart from functions that are continuous on the whole of $\mathbf{R}^2$,

$$R(x,y) \equiv R_0(x,y),$$

where

$$R_0(x, y) = \left(\frac{1}{2}\right)\theta_+(y - x)\int_{(x+y)/2}^{\infty} p(t)\,dt, \tag{2.4}$$

and

$$\theta_+(t) = 1 \quad \text{when } t \geq 0 \quad \text{and 0 otherwise.}$$

This is the leading term in R, and R satisfies the equation

$$R = R_0 + L_p R, \tag{2.5}$$

where L_p is given by

$$L_p R(x, y) = \int\int E(x - x', y - y')p(x')R(x', y')\,dx'\,dy', \tag{2.6}$$

$$E(x, y) = \frac{1}{2} \quad \text{when } x < 0, |\,y| < |x| \quad \text{and 0 otherwise.} \tag{2.7}$$

The solution of (2.5) is obtained by iteration,

$$R = \sum_{k=0}^{\infty} L_p^k R_0. \tag{2.8}$$

It follows in view of (2.4) and (2.8) that the function $R(x, \cdot)$, for any x, depends only on the restriction of p to the interval (x, ∞). Also note that the foregoing construction of R is valid for complex-valued potentials p satisfying (2.1).

To introduce the scattering transform, the kernel of the Marchenko equation $Q(x, y) = Q(x, y, p)$ is also needed. When $p \in \mathscr{L}^+$, this is a continuous function on $\mathbf{R}^2$ that satisfies $Q(x, y) = Q(y, x)$ and solves the equation

$$R(x, y) + Q(x, y) + \int_x^{\infty} R(x, t)Q(t, y)\,dt = 0, \quad y > x. \tag{2.9}$$

Note that the part of $Q(x, x)$ that is linear in p is equal to

$$-R_0(x, x + 0) = -R(x, x + 0) = -\frac{1}{2}\int_x^{\infty} p(y)\,dy. \tag{2.10}$$

Let Q denote also the operator with integral kernel $Q(x, y)$. It is clear from the context which interpretation is chosen at every formula.

For future reference, we shall now state the operator formulation of the Marchenko equation (2.9) as given in [17],

$$A(I + Q)A^{\tau} = I. \tag{2.11}$$

Here A^τ is the transpose of A, $A^\tau(x, y) = A(y, x)$. [We remark here that the operator calculus developed in Section 3 of [17] makes it meaningful to study products as in (2.11).]

It is then well known that we have the following representation for the kernel Q,

$$Q(x, y) = -\frac{1}{2}\int_{(x+y)/2}^{\infty} q(s)\, ds, \tag{2.12}$$

with $q \in \mathscr{L}^+$ (see [13, 15, 11, 17]). This gives the map $S : p \to q$ from $\mathscr{L}^+$ to $\mathscr{L}^+$ obtained by solving (2.11) for Q. We shall say that S is the scattering transform. According to Theorem 4.9 of [17], S is a homeomorphism between $\mathscr{L}^+$ and its range. Note that in view of (2.10), S may be thought as a nonlinear perturbation of the identity on $\mathscr{L}^+$. We refer to Sections 2 and 7 of [17], where it is shown that the KdV equations of all orders are linearized when passing from p to q.

We remark that when p is complex valued, the kernel Q obtained by solving (2.11) is a function of $x + y$, and the representation (2.12) is true. This follows from the proof of Theorem 4.7 in [17]. In what follows, however, we are concerned mainly with real-valued potentials.

It follows from (2.9), (2.10), and (2.12) [see also (3.23)] that the restriction of q to any interval (z, ∞) depends only on the restriction of p to this set, and, conversely, the restriction of q to (z, ∞) determines p on this interval. Restricting to the positive half-axis, S thus may be regarded as an injective mapping on L. (See also Remarks 5 and 7.)

THEOREM 2.1. *Let p_1 and $p_2 \in \mathscr{L}^+$ be two potentials such that*

$$\|p_j\| \le M, \quad j = 1, 2,$$

for some $M > 0$. Then the following holds:

1. *There exists a constant C_1 depending only on M such that*

$$\|S(p_1) - S(p_2)\| \le C_1\|p_1 - p_2\|.$$

2. *The estimate*

$$\|p_1 - p_2\| \le C_2\|S(p_1) - S(p_2)\|$$

is true. Here $C_2 = C_2(M)$ depends only on M.

Remark 1. In the proofs we shall obtain explicit bounds for the size of the Lipschitz constants in terms of the bounded sets in question.

Remark 2. It follows easily from (2.8), (2.9), and (2.12) that the mapping $S : \mathscr{L}^+ \to \mathscr{L}^+$ commutes with the action of the translation group $\tau_h p(x) = p(x+h)$, $h \in \mathbf{R}$, so that $\tau_h S = S\tau_h$. It follows therefore from Theorem 2.1 that when $\mathscr{M}$ is a bounded set in $\mathscr{L}^+$ we have

$$\|S(p_2) - S(p_1)\|_z \le C\|p_2 - p_1|_z, \quad p_1, p_2 \in \mathscr{M},$$

for any z, where C depends only on z and $\mathscr{M}$. Similarly, it is true that

$$\|p_2 - p_1\|_z \le C(z, \mathscr{M})\|S(p_2) - S(p_1)\|_z, \quad p_1, p_2 \in \mathscr{M},$$

for any $z \in \mathbf{R}$.

3. ESTIMATES FOR THE FORWARD TRANSFORM

In this section we shall prove the first part of Theorem 2.1. In our analysis we make use of the nonincreasing functions

$$f(x) = \int_x^\infty |p(t)|\,dt, \qquad g(x) = \int_x^\infty f(t)\,dt, \tag{3.1}$$

associated with a potential $p \in \mathscr{L}^+$. Then the estimate

$$|R(x,y)| \le \frac{1}{2} f\left(\frac{x+y}{2}\right) \exp\left(g(x) - g\left(\frac{x+y}{2}\right)\right), \quad y > x, \tag{3.2}$$

is true, (see [13]).

When considering two potentials p_1 and $p_2 \in \mathscr{L}^+$, set $\delta p = p_2 - p_1$ and

$$f_j(x) = \int_x^\infty |p_j(x)|\,dx, \qquad g_j(x) = \int_x^\infty f_j(x)\,dx, \quad j = 1, 2. \tag{3.3}$$

Our starting point is the following result. Although giving only $L^\infty(x, \infty)$ bounds on the difference of the intertwining kernels, it will be sufficient for our purposes.

LEMMA 3.1. *Suppose that* $p_1, p_2 \in \mathscr{L}^+$, *and let* $I + R_j$, $j = 1, 2$, *be the corresponding intertwining operators. Then the estimate*

$$|\delta R(x,y)| \le \frac{1}{2} \exp\Bigg(g_1(x) + g_2(x) - g_2\left(\frac{x+y}{2}\right)\Bigg) \int_x^\infty |\delta p(t)|\,dt, \quad y > x, \tag{3.4}$$

holds. Here $\delta R = R_2 - R_1$.

Proof. From the proof of Lemma 3.1.1 in [13], recall that if a function $H_j(u,v)$, $j=1,2$, is defined by $H_j(u,v)=R_j(u-v,u+v)$, then H_j satisfies

$$H_j(u,v)=\frac{1}{2}\int_u^\infty p_j(t)\,dt+\int_u^\infty\left(\int_0^v p_j(\alpha-\beta)H_j(\alpha,\beta)\,d\beta\right)d\alpha,\qquad v>0. \tag{3.5}$$

Writing $\delta H=H_2-H_1$, we get

$$\delta H(u,v)=\frac{1}{2}\int_u^\infty \delta p(t)\,dt+\int_u^\infty\left(\int_0^v p_2(\alpha-\beta)\delta H(\alpha,\beta)\,d\beta\right)d\alpha$$
$$+\int_u^\infty\left(\int_0^v \delta p(\alpha-\beta)H_1(\alpha,\beta)\,d\beta\right)d\alpha,\quad v>0. \tag{3.6}$$

Consider

$$\rho(u,v)=\int_u^\infty\left(\int_0^v \delta p(\alpha-\beta)H_1(\alpha,\beta)\,d\beta\right)d\alpha.$$

Bound (3.2) means that

$$|H_j(u,v)\leq\frac{1}{2}f_j(u)\exp(g_j(u-v)-g_j(u)),$$

and, therefore,

$$\begin{aligned}
|\rho(u,v)| &\leq \frac{1}{2}\int_u^\infty f_1(\alpha)\left(\int_0^v|\delta p(\alpha-\beta)|\exp(g_1(\alpha-\beta)-g_1(\alpha))\,d\beta\right)d\alpha\\
&\leq \frac{1}{2}\int_u^\infty f_1(\alpha)\exp(g_1(\alpha-v)-g_1(\alpha))\left(\int_0^v|\delta p(\alpha-\beta)|\,d\beta\right)d\alpha\\
&\leq \frac{1}{2}\int_u^\infty f_1(\alpha-v)\exp(g_1(\alpha-v)-g_1(\alpha))\left(\int_{\alpha-v}^\infty|\delta p(s)|\,ds\right)d\alpha\\
&\leq \frac{1}{2}\left(\int_{u-v}^\infty f_1(t)\exp(g_1(t))\,dt\right)\int_{u-v}^\infty|\delta p(s)|\,ds\\
&= \frac{1}{2}\left(\exp(g_1(u-v))-1\right)\int_{u-v}^\infty|\delta p(s)|\,ds.
\end{aligned}$$

Setting

$$h(u,v)=\frac{1}{2}\int_u^\infty \delta p(t)\,dt+\rho(u,v),$$

it follows that

$$\begin{aligned}
|h(u,v)| &\leq \frac{1}{2}\int_u^\infty|\delta p(t)|\,dt+\frac{1}{2}(\exp(g_1(u-v))-1)\int_{u-v}^\infty|\delta p(t)|\,dt\\
&\leq \frac{1}{2}\exp\left(g_1(u-v)\right)\int_{u-v}^\infty|\delta p(t)|\,dt.
\end{aligned}$$

Solving (3.6) by iteration as in [13], Lemma 3.1.1, yields

$$|\delta H(u,v)| \leq \frac{1}{2}\exp(g_1(u-v))\left(\int_{u-v}^{\infty} |\delta p(t)|\, dt\right)\exp\left(g_2(u-v) - g_2(u)\right).$$

This completes the proof. ∎

Remark 3. It follows from the proof of Lemma 3.1 that the following sharper bound is valid:

$$|R_2(x,y) - R_1(x,y)| \leq \frac{1}{2}\left(\int_{(x+y)/2}^{\infty} |\delta p(t)|dt + (\exp(g_1(x))-1)\int_x^{\infty} |\delta p(t)|dt\right)$$
$$\times \exp\left(g_2(x) - g_2\left(\frac{x+y}{2}\right)\right), \quad y > x.$$

We also remark that an L^1 estimate on $\delta R(x,\cdot)$ can be obtained from the bound

$$|\delta R(x,y)| \leq \frac{1}{2}\left(\int_{(x+y)/2}^{\infty} |\delta p(t)|\, dt + e^{g_1(x)} f_1\left(\frac{x+y}{2}\right)\int_x^{\infty} (t-x)|\delta p(t)|\, dt\right)$$
$$\times \exp\left(g_2(x) - g_2\left(\frac{x+y}{2}\right)\right),$$

which follows from a small modification of the proof of Lemma 3.1. Setting $p_1 = 0$ in these bounds, (3.2) is recovered.

In addition to basic estimate (3.2) and Lemma 3.1, similar bounds are required for $B = A^{-1}$, the second intertwining operator. These bounds will be needed to estimate the kernel Q in (2.11) in terms of the potential. It follows from the results in Section 3 of [17] that $y \geq x$ in the support of B, and $B = I + T$ with T continuous in this set. Moreover, $T = \sum_{k=1}^{\infty}(-1)^k R^k$. (See [17] for the definition and properties of the relevant Fréchet algebra of distribution kernels containing R and T.)

PROPOSITION 3.2. *The estimate*

$$|T(x,y)| \leq \frac{1}{2} f\left(\frac{x+y}{2}\right)\exp\left(g\left(\frac{x+y}{2}\right) - g(y)\right), \quad y > x \tag{3.7}$$

is true. Moreover, if p_1 *and* $p_2 \in \mathscr{L}^+$, *and* T_j *correspond to* p_j, $j = 1, 2$, *then*

$$|T_2(x,y) - T_1(x,y)| \leq \frac{1}{2}\left(\int_{(x+y)/2}^{\infty} |\delta p(s)|\, ds\right)$$
$$\times \exp\left(g_1\left(\frac{x+y}{2}\right) + g_2\left(\frac{x+y}{2}\right)\right). \tag{3.8}$$

Proof. We shall make use of the intertwining relation (2.2). Approximating the potential $p \in \mathscr{L}^+$ by test functions and making use of some arguments of Section 4 in [17], clearly B satisfies $BH_p = H_0 B$, and the corresponding equation for the function T is then

$$T(x,y) = T_0(x,y) - \int\int E(x-x', y-y')p(y')T(x',y')\,dx'dy', \tag{3.9}$$

where

$$T_0(x,y) = -\left(\frac{1}{2}\right)\theta_+(y-x)\int_{(x+y)/2}^{\infty} p(t)\,dt,$$

and $E(x,y)$ has been defined in (2.7). It follows, as in the proof of Lemma 3.1.1 of [13], that if a function $G(u,v)$ is defined by $G(u,v) = T(u-v, u+v)$, then G obeys

$$G(u,v) = -\int_u^{\infty} p(t)dt - \int_u^{\infty}\left(\int_0^v p(\alpha+\beta)G(\alpha,\beta)d\beta\right)d\alpha, \quad v>0. \tag{3.10}$$

Equation (3.10) is solved by iteration, and we have

$$T(x,y) = \sum_{n=1}^{\infty} T_n(x,y), \tag{3.11}$$

where T_n is homogeneous of degree n in p. Working with (3.10), a simple inductive argument demonstrates that the estimate

$$|T_n(x,y)| \leq \frac{1}{2} f\left(\frac{x+y}{2}\right)\frac{(g((x+y)/2) - g(y))^{n-1}}{(n-1)!}, \quad y \geq x \tag{3.12}$$

holds, and this gives (3.7). Note also that

$$T(x, x+0) = -\frac{1}{2}\int_x^{\infty} p(t)\,dt. \tag{3.13}$$

When proving (3.8), we argue in the same way as in the proof of Lemma 3.1, using (3.10) instead of (3.5), and this completes the proof. ∎

Remark 4. Note that the bound (3.7) on the kernel T is better than the bound (3.2) on R, which reflects the convexity of the function g. Note also that estimate (3.8) gives both L^1 and L^∞ bounds on $T_2(x,\cdot) - T_1(x,\cdot)$.

Remark 5. Recall the complex Banach space L_c introduced in Section 2. It follows from the proof of Proposition 3.2 (see also Theorem 4.1 in [17]) that the map $p \to T(x,y) = T(x,y,p)$ is analytic from L_c to the Banach space N. Here, following [17], we first introduce the space V of complex-valued functions Q in $\{(x,y); x \geq 0, y \geq 0\}$, tending to 0 as $x+y \to \infty$, that are continuous up to the boundary in the sets $\pm(y-x)>0$ and such that if

$$Q_*(z,w)=\sup_{x\ge z,\, y\ge z,\, (x+y)/2\ge w}|Q(x,y)|, \tag{3.14}$$

then

$$\|Q\|=\sup_{x\ge 0,\, y\ge 0}|Q(x,y)|+2\int_0^\infty Q_*(0,w)\,dw<\infty. \tag{3.15}$$

This is a complex Banach space with the norm just defined, and we then let N be the closed subspace comprising all $Q\in V$ such that $y\ge x$ in their support. Because $T(x,y,p)$ is given as the power series (3.11), where T_n is a homogeneous polynomial of degree n in $p\in L_c$ with values in N, the analyticity of $p\to T$ follows from (3.12), which shows that the series converges in N, and the convergence is uniform on bounded sets in L_c. We have the same result for the map $p\to R(x,y)=R(x,y,p)$.

From (2.5) and (2.6), recall that

$$R(x,y)-R_0(x,y)=\frac{1}{2}\int_x^\infty p(x')\left(\int_{y+x-x'}^{y+x'-x}R(x',y')\,dy'\right)dx'. \tag{3.16}$$

It is then easy to see that the function $R-R_0$ is of class C^1 up to the boundary in the set where $y>x$. Set $R_{(1,0)}(x,y)=(\partial_x R)(x,y)$ and $R_{(0,1)}(x,y)=(\partial_y R)(x,y)$ when $y>x$ and $=0$ when $x>y$. The function

$$U(x,y)=R_{(1,0)}(x,y)-R_{(0,1)}(x,y) \tag{3.17}$$

restricted to the set $y>x$ extends continuously to the closure of this set. Moreover, a simple computation using (3.16) or, alternatively, (3.5) shows that

$$U(x,y)=-\int_x^\infty p(x')R(x',y-x+x')\,dx', \tag{3.18}$$

and using (3.2), we obtain

$$|U(x,y)|\le\frac{1}{2}f(x)f\left(\frac{x+y}{2}\right)\exp\left(g(x)-g\left(\frac{x+y}{2}\right)\right),\quad y>x. \tag{3.19}$$

The following related bound is needed to establish the Lipschitz continuity.

LEMMA 3.3. *If U_j correspond to $p_j\in\mathscr{L}^+$, $j=1,2$, then*

$$|U_2(x,y)-U_1(x,y)|\le\frac{1}{2}\int_x^\infty|\delta p(t)|\,dt\left(f_1\left(\frac{x+y}{2}\right)\exp(g_1(x))\right.$$
$$\left.+\exp(g_1(x)+g_2(x))f_2(x)\right).$$

Proof. From (3.18),

$$U_2(x,y)-U_1(x,y)=-\int_x^\infty p_2(x')\big(R_2(x',y-x+x')-R_1(x',y-x+x')\big)\,dx'$$
$$-\int_x^\infty (p_2(x')-p_1(x'))R_1(x',y-x+x')\,dx',$$

and the result then follows in view of (3.2) and Lemma 3.1. ∎

We now come to estimate the difference of the kernels $Q_2 - Q_1$ corresponding to $p_j \in \mathscr{L}^+$, $j = 1, 2$. The following proposition is the starting point. We remark that a different proof of estimate (3.20) is given in Chapter 3 of [13]. However, because the argument here is short, and in what follows the identity (3.24) is needed, we present the proof.

PROPOSITION 3.4. *The estimate*

$$|Q_j(x, x)| \leq \frac{1}{2} f_j(x) \exp(g_j(x)) \cosh g_j(x), \quad j = 1, 2 \tag{3.20}$$

is true.

Proof. Recall from (2.11) that

$$A_j(I + Q_j)A_j^\tau = I, \quad j = 1, 2. \tag{3.21}$$

Writing $A_j^{-1} = I + T_j$, we have

$$Q_j = A_j^{-1}(A_j^{-1})^\tau - I = (I + T_j)(I + T_j^\tau) - I = T_j + T_j^\tau + T_j T_j^\tau. \tag{3.22}$$

Both sides here are continuous up to the boundary in the sets $\pm(y - x) > 0$, and because T_j^τ is supported in the set where $x \geq y$, (3.22) gives

$$Q_j(x, x + 0) = Q_j(x, x) = T_j(x, x + 0) + \big(T_j T_j^\tau\big)(x, x). \tag{3.23}$$

Here we have written $\big(T_j T_j^\tau\big)(x, x + 0) = \big(T_j T_j^\tau\big)(x, x)$, because this function is continuous in $\mathbf{R}^2$. This gives

$$Q_j(x, x) = T_j(x, x + 0) + \int_x^\infty T_j^2(x, z)\,dz, \tag{3.24}$$

and using (3.7) and (3.13), we obtain

$$\begin{aligned}|Q_j(x, x)| &\leq \frac{1}{2} f_j(x) + \frac{1}{4}\int_x^\infty f_j\left(\frac{x+z}{2}\right)\\ &\quad\times \exp\left(2\left(g_j\left(\frac{x+z}{2}\right) - g_j(z)\right)\right) f_j\left(\frac{x+z}{2}\right)dz\\ &\leq \frac{1}{2} f_j(x) + \frac{1}{4} f_j(x)\int_x^\infty f_j\left(\frac{x+z}{2}\right)\exp\left(2g_j\left(\frac{x+z}{2}\right)\right)dz\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} f_j(x) + \frac{1}{4} f_j(x)(\exp(2g_j(x)) - 1) \\
&= \frac{1}{2} f_j(x)\left(\frac{1}{2}\exp(2g_j(x)) + \frac{1}{2}\right) \\
&= \frac{1}{2} f_j(x)\exp(g_j(x))\cosh g_j(x).
\end{aligned}
$$

The proof is complete. ∎

Remark 6. Note that the map $T \to T + T^\tau + TT^\tau$ is analytic from N to V. Here we have used the Banach spaces introduced in Remark 5. Because $V \ni Q(x, y) \to Q(x, x + 0) \in L^1(0, \infty)$ is continuous, we conclude in view of that remark and Proposition 3.4 that the mapping $p \to Q(x, x) = Q(x, x, p)$ is analytic from L_c to $L^1(0, \infty)$.

LEMMA 3.5.

$$
\begin{aligned}
|(Q_2 - Q_1)(x, x)| \le \frac{1}{2}\int_x^\infty |\delta p(t)|\, dt (1 + \exp(g_1(x) + g_2(x)) \\
\times (\sinh g_1(x) + \sinh g_2(x))).
\end{aligned}
$$

Proof. In view of (3.13) and (3.24),

$$
\begin{aligned}
Q_2(x, x) - Q_1(x, x) = \int_x^\infty (T_2(x, y) - T_1(x, y))(T_1(x, y) + T_2(x, y))\, dy \\
-\frac{1}{2}\int_x^\infty \delta p(t)\, dt.
\end{aligned}
$$

A straightforward estimate using (3.7) and (3.8) then gives the result. ∎

We now come to prove the first part of Theorem 2.1. In doing so, we need the following representation for $q = S(p)$ (see, e.g., Chapter 4 in [11]):

$$
q(x) - p(x) = 4R(x, x + 0)Q(x, x) - 2\int U(x, y)Q(x, y)\, dy. \tag{3.25}
$$

Here the function U has been introduced in (3.17). Note that (3.25) shows that $q - p$ is an absolutely continuous function vanishing at $+\infty$.

When proving (3.25), we observe that it follows from (2.11) that $x \ge y$ in the support of (the kernel of) the operator $(I + R)(I + Q)$, and because this is a continuous function up to the boundary in the sets $\pm(y - x) > 0$, we have

$$
R(x, x + 0) + Q(x, x) + (RQ)(x, x) = 0.
$$

Here we may also notice that $(RQ)(x, y)$ is continuous in $\mathbf{R}^2$. Writing $(RQ)(x, x) = \int_{2x}^\infty R(x, y - x)Q(0, y)\, dy$, we obtain

$$
\partial_x (RQ)(x, x) = -2R(x, x + 0)Q(x, x) + \int U(x, y)Q(x, y)\, dy,
$$

and because

$$-\partial_x(R(x, x+0) + Q(x, x)) = \frac{p(x) - q(x)}{2},$$

the representation (3.25) follows.

If $q_j = S(p_j)$ correspond to $p_j \in \mathscr{L}^+$, $j = 1, 2$, then we have, in view of (3.25),

$$h_2(x) - h_1(x) = I_1 + I_2 + I_3 + I_4, \tag{3.26}$$

where we have written $h_j(x) = q_j(x) - p_j(x)$, $j = 1, 2$, and I_j are given by

$$I_1 = 4(R_2(x, x+0) - R_1(x, x+0))Q_2(x, x),$$

$$I_2 = 4R_1(x, x+0)(Q_2(x, x) - Q_1(x, x)),$$

$$I_3 = -2\int(U_2(x, y) - U_1(x, y))Q_2(x, y)\,dy,$$

and

$$I_4 = -2\int U_1(x, y)(Q_2(x, y) - Q_1(x, y))\,dy,$$

with R_j, Q_j, and U_j corresponding to p_j, $j = 1, 2$.
Now it follows from Proposition 3.4 and Lemma 3.5 that

$$|I_1| \leq \exp(g_2(x)) \cosh g_2(x) f_2(x) \int_x^\infty |\delta p(t)|\,dt \tag{3.27}$$

and

$$|I_2| \leq (1 + \exp(g_1(x) + g_2(x))(\sinh g_1(x) + \sinh g_2(x))) f_1(x) \times \int_x^\infty |\delta p(t)|\,dt. \tag{3.28}$$

Using Lemma 3.3 and Proposition 3.4, we further obtain

$$|I_3| \leq \exp(g_1(x) + g_2(x)) \sinh g_2(x) \left(\int_x^\infty |\delta p(t)|\,dt\right) f_1(x) + \exp\,(g_1(x) + 2g_2(x)) \sinh g_2(x) \left(\int_x^\infty |\delta p(t)|\,dt\right) f_2(x), \tag{3.29}$$

and, finally, combining (3.19) with Lemma 3.5, we arrive at

$$|I_4| \leq (1 + \exp(g_1(x) + g_2(x))(\sinh g_1(x) + \sinh g_2(x))) \times \exp(g_1(x)) f_1(x) \left(\int_x^\infty |\delta p(t)|\,dt\right). \tag{3.30}$$

Because $g_j(x) \leq g_j(0) \leq \|p_j\|$, $j = 1, 2$, when $x \geq 0$, combining (3.26) with (3.27)–(3.30) yields

$$|h_2(x) - h_1(x)| \leq C\left(\int_x^\infty |\delta p(t)|\,dt\right)(f_1(x) + f_2(x)), \quad x > 0.$$

It follows from (3.27)–(3.30) that $C = 2e^M(3e^{2M}\sinh M + 2\cosh M)$ can be taken when M is such that $\|p_j\| \le M$, $j = 1, 2$.

Because $(1+x)f_j(x) \le \|p_j\|$, $x \ge 0$, and

$$\int_0^\infty \left(\int_x^\infty |\delta p(t)|\,dt\right)dx = \int_0^\infty t|\delta p(t)|\,dt \le \|p_2 - p_1\|,$$

recalling the definition of h_j, we finally obtain

$$\|q_2 - q_1\| \le C\|p_2 - p_1\|,$$

when $\|p_j\| \le M$, $j = 1, 2$, and C depends only on M. This completes the proof of the first part of Theorem 2.1.

Remark 7. We shall finish this section by making some remarks concerning analyticity properties of S. When doing this, we shall use the notation and the spaces introduced in Remark 5. Consider first the map $S_1 : L_c \to L_c$ given by

$$S_1(p)(x) = \left(\int_x^\infty p(t)\,dt\right)Q(x, x, p). \tag{3.31}$$

When $p_j \in L_c$, $j = 0, 1$, $\|p_1\| \le 1$ and $\lambda \in \mathbf{C}$, $Q(x, x, p_0 + \lambda p_1)$ is entirely analytic in λ with values in $L^1(0,\infty)$. Therefore,

$$Q(x, x, p_0 + \lambda p_1) = \sum_{j=0}^\infty Q_j(x, x, p_0, p_1)\lambda^j,$$

where Cauchy's inequalities together with (3.20) give that

$$\|Q_j\|_{L^1(0,\infty)} \le \frac{1}{R^j} e^{\|p_0\|+R}\sinh(\|p_0\| + R), \quad R > 0.$$

It follows that with convergence in L_c, we have

$$S_1(p_0 + \lambda p_1)(x) = \sum_{j=0}^\infty S_j(x, p_0, p_1)\lambda^j,$$

where $S_j \in L_c$ equals $(\int_x^\infty p_0(t)\,dt)Q_j + (\int_x^\infty p_1(t)\,dt)Q_{j-1}$, $j \ge 0$, with $Q_{-1} = 0$. Because S_1 is locally bounded, analyticity of S_1 as a map from L_c to L_c follows (see [18]).

It further follows from (3.18) that we have the estimate

$$U_*(z, w) \le \left(\int_z^\infty |p(z')|\,dz'\right)R_*(z, w),$$

and because $z\int_z^\infty |p(z')|\,dz' \le \|p\|$, $z \ge 0$, we also get that $\|V\| \le \|p\|\,\|R\|$, where $V(x, y) = (1+x)U(x, y)$. Using this bound together with the fact that $R(x, y, p)$ depends analytically on p, we obtain that the map $L_c \ni p \to$

$V(x, y) = V(x, y, p) \in N$ is continuously differentiable, and hence analytic. Because the analyticity of $p \to Q(x, y, p)$ has already been observed in Remark 6, it follows that the map

$$p \to \int U(x, y, p) Q(x, y, p)\, dy$$

is analytic from L_c to L_c. From (3.25), we conclude that $q = S(p) \in L$ depends analytically on $p \in L$.

4. ESTIMATES FOR THE INVERSE TRANSFORM

In this section we shall continue to work with potentials in the space $\mathscr{L}^+$. Recall the notation

$$\|p\| = \int_0^\infty (1 + x)|p(x)|\, dx.$$

When $p_1, p_2 \in \mathscr{L}^+$ are such that $\|p_j\| \leq M$, we derive Lipschitz estimates for $p_2 - p_1$ in terms of $S(p_2) - S(p_1)$, thereby completing the proof of Theorem 2.1. When doing this, we shall use the notation C for various positive constants that depend only on M.

Our starting point is the following representation for the potential:

$$p(x) - q(x) = 2R^2(x, x + 0) + \int_x^\infty R(x, y) q\left(\frac{x + y}{2}\right) dy. \tag{4.1}$$

Here the intertwining operator $A = I + R$ corresponding to p and $q = S(p)$ is related by the Marchenko equation

$$I + Q = A^{-1}(A^\tau)^{-1}, \tag{4.2}$$

and Q is defined by q as in (2.12). It follows from Section 4 in [17] that each side in (4.1) is a continuous function of $p \in \mathscr{L}^+$ with values in $\mathscr{L}^+$. When proving (4.1), it therefore may be assumed that $p \in \mathscr{L}^+_\infty = \{p \in \mathscr{L}^+; p^{(j)} \in \mathscr{L}^+, j = 1, 2 \ldots\}$, so that in particular $p \in C^\infty(\mathbf{R})$, and all derivatives tend to 0 at ∞. It follows from the results of [17] that these regularity conditions imply the same regularity of q. From Section 4 of [17], we also know that the kernel of $A(I + Q)$ is a smooth function up to the boundary in the sets $\pm(y - x) > 0$, and because $x \geq y$ in its support, we have

$$(\partial^\alpha((I + R)(I + Q)))(x, x + 0) = 0, \tag{4.3}$$

for any multi-index $\alpha = (\alpha_x, \alpha_y)$. When $\alpha = (0, 1)$,

$$(\partial^\alpha(RQ))(x, x + 0) = \frac{1}{4}\int_x^\infty R(x, y) q\left(\frac{x + y}{4}\right) dy.$$

Now a computation using (2.5) and (3.16) gives

$$(\partial^\alpha R)(x, y) = -\frac{1}{4}p\left(\frac{x+y}{2}\right) + \frac{1}{2}\int_x^\infty p(x')\,R(x', y - x + x')\,dx'$$
$$-\frac{1}{2}\int_x^\infty p(x')\,R(x', x + y - x')\,dx', \quad y > x,$$

so that

$$(\partial^\alpha R)(x, x+0) = -\frac{1}{4}p(x) + \frac{1}{2}\int_x^\infty p(x')\,R(x', x'+0)\,dx'$$
$$= -\frac{1}{4}p(x) + \frac{1}{8}\left(\int_x^\infty p(y)\,dy\right)^2.$$

Here in the last step we have used that

$$R(x, x+0) = \frac{1}{2}\int_x^\infty p(y)\,dy.$$

In view of (4.3), the representation (4.1) now follows. Note that a representation similar to (4.1) can be found in [3], where it is related to the trace formula of inverse scattering.

We now consider two potentials p_1 and $p_2 \in \mathscr{L}^+$ and let the triple R_j, Q_j, q_j correspond to p_j, $j = 1, 2$. We also write $\rho_j = p_j - q_j$ and

$$P_j(x) = \int_x^\infty p_j(t)\,dt = 2R_j(x, x+0), \quad j = 1, 2.$$

Using (4.1), we get

$$\rho_2(x) - \rho_1(x) = I_1 + I_2, \tag{4.4}$$

where $I_j = I_j(x)$, $j = 1, 2$, is given by

$$I_1 = \frac{1}{2}(P_2(x) - P_1(x))(P_2(x) + P_1(x))$$

and

$$I_2 = \int_x^\infty (R_2(x,t) - R_1(x,t))q_2\left(\frac{x+t}{2}\right)dt$$
$$+ \int_x^\infty R_1(x,t)\left(q_2\left(\frac{x+t}{2}\right) - q_1\left(\frac{x+t}{2}\right)\right)dt.$$

Because $(1+x)|P_j(x)| \le \|p_j\|$, $x \ge 0$, $j = 1, 2$, we get that

$$\|I_1\| \le C\|P_2 - P_1\|_{L^1(0,\infty)}.$$

The required bound on $\|I_1\|$ in terms of $\|q_2 - q_1\|$ then becomes the contents of the following lemma.

LEMMA 4.1. *The estimate*

$$\|P_2 - P_1\|_{L^1(0,\infty)} \leq C\|q_2 - q_1\| \tag{4.5}$$

is true.

Proof. We shall make use of the following identity proved in Proposition 4.1 of [6]:

$$\frac{1}{2}(P_1(x) - P_2(x)) = (A_1(Q_2 - Q_1)A_2^\tau)(x, x), \tag{4.6}$$

where $A_j = I + R_j$, $j = 1, 2$. We then have

$$\begin{aligned}\frac{1}{2}(P_1(x)-P_2(x)) = (Q_2-Q_1)(x,x) &+ \int R_1(x,y)(Q_2(x,y)-Q_1(x,y))\,dy\\ &+ \int R_2(x,y)(Q_2(x,y)-Q_1(x,y))\,dy\\ &+ \int R_1(x,y)\left\{\int R_2(x,z)(Q_2(y,z)-Q_1(y,z))\,dz\right\}dy.\end{aligned} \tag{4.7}$$

Because

$$|Q_2(x, y) - Q_1(x, y)| \leq \frac{1}{2}\int_{(x+y)/2}^{\infty} |\delta q(t)|\,dt,$$

where $\delta q = q_2 - q_1$, using (3.2), we get

$$\begin{aligned}\left|\int R_j(x, y)(Q_2(x, y) - Q_1(x, y))\,dy\right| &\leq \frac{1}{2}\|R_j(x, \cdot)\|_{L^1}\int_x^{\infty} |\delta q(t)|\,dt\\ &\leq \frac{1}{2}(\exp(g_j(x)) - 1)\int_x^{\infty} |\delta q(t)|\,dt, \quad j = 1, 2.\end{aligned}$$

Estimating the last integral on the right side of (4.7), we see that

$$\begin{aligned}&\left|\int R_1(x, y)\left\{\int R_2(x, z)(Q_2(y, z) - Q_1(y, z))\,dz\right\}dy\right|\\ &\leq \frac{1}{2}\|R_1(x, \cdot)\|_{L^1}\|R_2(x, \cdot)\|_{L^1}\int_x^{\infty} |\delta q(t)|\,dt\\ &\leq \frac{1}{2}(\exp(g_1(x)) - 1)(\exp(g_2(x)) - 1)\int_x^{\infty} |\delta q(t)|\,dt\\ &\leq (e^{\|p_2\|} - 1)(e^{\|p_1\|} - 1)\int_x^{\infty} |\delta q(t)|\,dt, \quad x \geq 0.\end{aligned}$$

From (4.7), we obtain

$$|P_2(x) - P_1(x)| \leq C\int_x^{\infty} |\delta q(t)|\,dt, \quad x \geq 0,$$

and this completes the proof. ∎

We now come to estimate I_2, the second term in (4.4). When doing this, when $x \in \mathbf{R}$, we consider the operators

$$Q_j^x f(y) = \int_x^\infty Q_j(y,t) f(t)\,dt, \qquad y > x,\ f \in C_0(x,\infty), \quad j = 1,2 \tag{4.8}$$

on (x,∞). These are then Hilbert–Schmidt operators on $L^2(x,\infty)$, and from [17] recall that the fact that q_j is in the range of S means precisely that $I + Q_j^x$ has a positive lower bound on $L^2(x,\infty)$ for any x, $j = 1,2$. The operators $I + Q_j^x$ are thus invertible on $L^2(x,\infty)$, and it is also well known (see [11, 17]) that $I + Q_j^x$ is then an isomorphism on $L^1(x,\infty)$, $j = 1,2$. When deriving bounds on $R_2(x,y) - R_1(x,y)$, the norm of $\left(I + Q_2^x\right)^{-1}$ on $L^1(x,\infty)$ must be controlled. To this end, we introduce the operators

$$\left(R_j^x f\right)(y) = \int_y^\infty R_j(y,t) f(t)\,dt, \quad y > x,$$

and

$$\left((R_j^\tau)^x f\right)(y) = \int_x^y R_j(t,y) f(t)\,dt, \quad y > x,$$

on $L^p(x,\infty)$, $p = 1,2$. Then we have (see [13] and [14]) that

$$\left(I + Q_j^x\right)^{-1} = (I + (R_j^\tau)^x)(I + R_j^x), \quad j = 1,2 \tag{4.9}$$

in these spaces. A straightforward estimate using (3.2) shows that

$$\|I + R_j^x\|_{L_x^1 \to L_x^1} \le 1 + g_j(x)\exp(g_j(x)) \le e^{2g_j(x)}, \quad j = 1,2,$$

where $L_x^1 = L^1(x,\infty)$. The same estimate is valid for the operator $I + \left(R_j^\tau\right)^x$, and in view of (4.9), we get

$$\|\left(I + Q_j^x\right)^{-1}\|_{L_x^1 \to L_x^1} \le \exp(4g_j(x)). \tag{4.10}$$

Setting $S_{j,x}(y) = -Q_j(x,y)$ and $R_{j,x}(y) = R_j(x,y)$, $y > x$, $j = 1,2$, from (2.9) we obtain

$$\begin{aligned} R_{2,x} - R_{1,x} &= \left(I + Q_2^x\right)^{-1} S_{2,x} - \left(I + Q_1^x\right)^{-1} S_{1,x} \\ &= \left(I + Q_2^x\right)^{-1}\left(S_{2,x} - S_{1,x}\right) + \left(\left(I + Q_2^x\right)^{-1} - \left(I + Q_1^x\right)^{-1}\right) S_{1,x} \\ &= \left(I + Q_2^x\right)^{-1}\left(S_{2,x} - S_{1,x}\right) + \left(I + Q_2^x\right)^{-1}\left(Q_1^x - Q_2^x\right)\left(I + Q_1^x\right)^{-1} S_{1,x} \\ &= \left(I + Q_2^x\right)^{-1}\left[S_{2,x} - S_{1,x} + \left(Q_1^x - Q_2^x\right) R_{1,x}\right]. \end{aligned} \tag{4.11}$$

Because

$$|Q_2(x,y) - Q_1(x,y)| \le \frac{1}{2}\int_{(x+y)/2}^\infty |\delta q(t)|\,dt,$$

it follows that the absolute value of the expression in the square brackets in (4.11) does not exceed

$$\frac{1}{2}\int_{(x+y)/2}^{\infty}|\delta q(s)|\,ds+\frac{1}{2}\int_{x}^{\infty}|R_1(x,t)|\left(\int_{(y+t)/2}^{\infty}|\delta q(s)|\,ds\right)dt$$
$$\leq\frac{1}{2}\int_{(x+y)/2}^{\infty}|\delta q(s)|\,ds\big(1+\|R_1(x,\cdot)\|_{L^1}\big)\leq\frac{1}{2}e^{g_1(x)}\int_{(x+y)/2}^{\infty}|\delta q(s)|\,ds,$$

where in the last step we had that

$$\|R_j(x,\cdot)\|_{L^1}\leq\exp(g_j(x))-1,$$

which follows from (3.2). Combining this with (4.10) and (4.11), we get the L^1 bound

$$\|R_{2,x}-R_{1,x}\|_{L^1_x}\leq\exp(4g_2(x))\exp(g_1(x))\int_x^{\infty}(t-x)|\delta q(t)|\,dt. \tag{4.12}$$

Now we have

$$\begin{aligned}R_2(x,y)-R_1(x,y)&\\ =Q_1(x,y)-Q_2(x,y)&\\ +\int_x^{\infty}(Q_1-Q_2)(y,y')R_1(x,y')\,dy'&\\ +\int_x^{\infty}Q_2(y,y')(R_1(x,y')-R_2(x,y'))\,dy',&\quad y>x,\end{aligned} \tag{4.13}$$

and using (4.12), we obtain

$$\begin{aligned}|R_2(x,y)-R_1(x,y)|&\leq|Q_2(x,y)-Q_1(x,y)|\\&\quad+\int_x^{\infty}|Q_2(y,y')-Q_1(y,y')||R_1(x,y')|\,dy'\\&\quad+\int_x^{\infty}|Q_2(y,y')||R_2(x,y')-R_1(x,y')|\,dy'\\&\leq\frac{1}{2}\int_{(x+y)/2}^{\infty}|\delta q(t)|\,dt(1+\|R_1(x,\cdot)\|_{L^1})\\&\quad+\frac{1}{2}\left(\int_{(x+y)/2}^{\infty}|q_2(t)|\,dt\right)\|R_{2,x}-R_{1,x}\|_{L^1_x}\\&\leq\frac{1}{2}\exp(4g_2(x)+g_1(x))\left(\int_{(x+y)/2}^{\infty}|q_2(s)|\,ds\right)\\&\quad\times\int_x^{\infty}(t-x)|\delta q(t)|\,dt\\&\quad+\frac{1}{2}\exp(g_1(x))\int_{(x+y)/2}^{\infty}|\delta q(t)|\,dt.\end{aligned} \tag{4.14}$$

Write

$$I_2 = I_{2,1} + I_{2,2},$$

where

$$I_{2,1} = \int_x^\infty (R_2(x,t) - R_1(x,t)) q_2\left(\frac{x+t}{2}\right) dt$$

and

$$I_{2,2} = \int_x^\infty R_1(x,t)\left(q_2\left(\frac{x+t}{2}\right) - q_1\left(\frac{x+t}{2}\right)\right) dt.$$

Bound (4.14) then gives

$$\begin{aligned}|I_{2,1}| \le \exp(g_1(x))\left(\int_x^\infty |\delta q(t)|\,dt\right)\int_x^\infty |q_2(t)|\,dt \\ + \exp(4g_2(x) + g_1(x))\left(\int_x^\infty |q_2(s)|\,ds\right)^2 \int_x^\infty (t-x)|\delta q(t)|\,dt,\end{aligned}$$

and we immediately get

$$\|I_{2,1}\| \le C\big(\|q_2\| + \|q_2\|^2\big)\|q_2 - q_1\|. \tag{4.15}$$

It follows now from [13] that $\|q_2\| \le C\big(\|p_2\| + \|p_2\|^2\big)$. Therefore, to complete the proof of Theorem 2.1, it remains to estimate $\|I_{2,2}\|$. When doing this, note that in view of (3.2), the estimate

$$|I_{2,2}| \le \exp(g_1(x)) f_1(x) \int_x^\infty |\delta q(t)|\,dt$$

is true, which gives that $\|I_{2,2}\| \le C\|q_2 - q_1\|$. Combining this with (4.4), Lemma 4.1, and (4.15) completes the proof of Theorem 2.1.

Remark 8. We shall finally give a representation formula for the scattering transform of a potential $p \in \mathscr{L}^+$ that vanishes near $+\infty$. It will then be convenient to change the notation somewhat, so that we consider $p \in L^1_{\text{loc}}([0,\infty))$ and take the zero extension of p to the negative half-axis. Then $\check{p}(x) = p(-x) \in \mathscr{L}^+$ vanishes for $x > 0$, and associated with p we introduce the scattering transform from the left, S_-,

$$S_-(p)(x) = S(\check{p})(-x).$$

We also introduce the kernels R_- and T_- so that $R_-(x,y,p) = R(-x,-y,\check{p})$ and $I + T_- = (I + R_-)^{-1}$. It is then true that $S_-(p)$ vanishes on the interval $(-\infty, 0)$, and the restriction of p to any interval of the form $(0,a)$, $a > 0$, determines and is determined by the restriction of $S_-(p)$ to this set. We claim that

$$S_-(p)(x) = -4\frac{\partial}{\partial y}\bigg|_{y=0} T_-(2x,y), \quad x > 0. \tag{4.16}$$

When proving (4.16), recall the Marchenko equation from the left,

$$(I + R_-)(I + Q_-)(I + (R_-)^\tau) = I,$$

where

$$Q_-(x, y) = -\frac{1}{2}\int_{-\infty}^{(x+y)/2} S_-(p)(t)\,dt. \tag{4.17}$$

Arguing as in the proof of Proposition 3.4, we get

$$Q_-(x, y) = T_-(x, y) + \int_{-\infty}^{y} T_-(x, y')T_-(y, y')\,dy', \quad x > y. \tag{4.18}$$

Here also note that $|y| \leq x$ in the support of $T_-(x, y)$ in view of the support properties of p, so that in particular $T(y, y-0) = 0$, $y \leq 0$. Then by (4.17) and (4.18), we get (4.16).

Representation (4.16) is analogous to the expression for the A-amplitude associated with p appears in Section 9 of [5]. We recall here that the A-amplitude has been introduced in [20] as a basic ingredient of an approach to the inverse spectral theory for half-line Schrödinger operators. In [20], A was related to the Weyl m-function by means of a Fourier–Laplace transformation. To recall the expression for A given in [5], set

$$K_-(x, y) = R_-(x, y) - R_-(x, -y), \quad 0 \leq y \leq x,$$

and define the kernel L_- so that $I + L_- = (I + K_-)^{-1}$. It is then true that

$$A(\alpha) = -2\frac{\partial}{\partial y}\bigg|_{y=0} L_-(2\alpha, y) \tag{4.19}$$

(see [5]). It follows from (2.5), (4.16), and (4.19) that the leading terms in $S_-(p)$ and A agree and are both equal to p. (Alternatively, this also follows from (3.25) and the series expansion for A given in Lemma 2.2 of [20]). We also have that $S_-(p) - A$ is (absolutely) continuous, and bounds on this function can now be derived using the results of [20] and this paper.

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