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Existence of resonances in magnetic scattering

Michael Hitrik

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720-3840, USA

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Abstract

The Schrödinger operator with a compactly supported magnetic field is shown to produce infinitely many resonances, in any odd dimension $\geqslant 3$. The proof is based on the Poisson formula for resonances and properties of the magnetic heat invariants.
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1. Introduction

The purpose of this note is to show the existence of infinitely many resonances for the magnetic Schrödinger operator in $\mathbb{R}^n$, $n \geqslant 3$ odd, with a nonvanishing compactly supported magnetic field. As in previous works on the existence of resonances in odd dimensions, see [15–17], this result is a consequence of the Poisson formula for resonances and an explicit expression for the second heat coefficient for the magnetic Schrödinger operator. This expression appears to be essentially well known in the literature—see [5,14,18]. The latter paper also contains a discussion of the physical significance of the first heat coefficients in electromagnetic scattering. In this note we shall take an opportunity to outline how the approach of [13] (see also [6]) leads to closed formulas for all the heat coefficients for the magnetic Schrödinger operator.

When $a_1, \ldots, a_n$ are real-valued C^∞-functions on $\mathbb{R}^n$, n is odd $\geqslant 3$, we introduce the corresponding 1-form $a = \sum_{j=1}^n a_j \, \mathrm{d}x^j$, the magnetic potential, and consider the magnetic Schrödinger operator

$$H_a = \sum_{j=1}^{n} (D_j + a_j)^2, \tag{1.1}$$

E-mail address: hitrik@math.berkeley.edu (M. Hitrik).

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where $D_j = -\mathrm{i}\partial_j$. It describes a scalar non-relativistic particle without a spin in an external magnetic field. The magnetic field here is the 2-form B which is defined as

$$B = \mathrm{d}a = \sum_{j,k=1}^{n} \frac{\partial a_j}{\partial x^k}\, \mathrm{d}x^k \wedge \mathrm{d}x^j = \frac{1}{2}\sum_{j,k=1}^{n} B_{j,k}\, \mathrm{d}x^j \wedge \mathrm{d}x^k,$$

where $B_{kj} = -B_{jk}$. Obviously,

$$B = \sum_{j<k} B_{jk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k,$$

and

$$B_{jk} = \frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k}.$$

We introduce a norm of B defined as follows,

$$\|B\| = \left(\sum_{j<k} |B_{jk}|^2\right)^{1/2}. \tag{1.2}$$

Throughout the paper we shall assume that the magnetic field B is compactly supported, so that $B_{jk} \in C_0^\infty(\mathbb{R}^n)$. The magnetic potential a can then be chosen to be of compact support as well, since the dimension $n \neq 2$. This follows from the fact that the second de Rham cohomology group with compact support $H_c^2(\mathbb{R}^n)$ vanishes when $n \neq 2$—see [2] for this familiar topological result. In what follows we shall therefore work with operator (1.1), with $a_j \in C_0^\infty(\mathbb{R}^n)$, $j = 1,\dots,n$.

The operator H_a, defined on $C_0^\infty(\mathbb{R}^n)$, is essentially self-adjoint, non-negative, and the spectrum of H_a is purely absolutely continuous, filling the positive half-axis $[0,\infty)$. Much more general essentially self-adjoint perturbations of the Euclidean Laplacian $H_0 = -\Delta$ were considered in [4].

Let

$$R(\lambda) = (H_a - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \operatorname{Im}\lambda > 0,$$

be the (modified) resolvent of H_a. It is then well-known, see [19], that $R(\lambda)$ continues meromorphically to the entire complex plane, as an operator

$$R(\lambda) : L^2_{\mathrm{comp}}(\mathbb{R}^n) \to L^2_{\mathrm{loc}}(\mathbb{R}^n),$$

with the poles of finite rank. The poles of $R(\lambda)$ are called *resonances* or *scattering poles*, and they constitute a natural replacement of discrete spectral data for problems on unbounded domains. It is a consequence of the gauge invariance of the operator H_a that the set of resonances of H_a does not depend on the choice of the magnetic potential a, provided that the magnetic field B is kept fixed. Namely, if $a, a' \in C_0^\infty$ are two magnetic potentials with $\mathrm{d}a = \mathrm{d}a' = B$, then by the Poincaré lemma we know that $a' = a + \mathrm{d}\varphi$, where $\varphi \in C^\infty$ is defined up to an additive constant and can be assumed to be real-valued. Then the operators $H_{a'}$ and H_a are related by

$$H_{a'} = \mathrm{e}^{-\mathrm{i}\varphi} H_a \mathrm{e}^{\mathrm{i}\varphi},$$

and thus have the same scattering poles. It is therefore natural that the condition on the existence of resonances should be given in terms of B, rather than a.

Theorem 1.1. *Let $n \geqslant 3$ be odd. Assume that the magnetic field B is a compactly supported smooth 2-form on $\mathbb{R}^n$ which does not vanish identically. Then the operator H_a has infinitely many resonances.*

As in [15–17], the method that we use to prove Theorem 1.1 is non-constructive, and therefore the result is far from the optimal upper bounds on the density of resonances of H_a in large discs, established in [19].

2. Proof of Theorem 1.1

As pointed out in the Introduction, the proof of Theorem 1.1 depends on the explicit expression for the second heat coefficient for H_a. In the beginning of this section we shall therefore recall this expression and make some remarks on how its structure can be deduced from some general considerations.

When $t > 0$, it follows from [14] that the operator

$$\mathrm{e}^{-tH_a} - \mathrm{e}^{-tH_0}$$

is of trace class, and as $t \to 0^+$, there is an asymptotic expansion of the form

$$\operatorname{tr}(\mathrm{e}^{-tH_a} - \mathrm{e}^{-tH_0}) \sim (4\pi t)^{-n/2} \sum_{j=1}^{\infty} b_j t^j. \tag{2.1}$$

We shall refer to the coefficients b_j as the heat invariants, or heat coefficients, of H_a. While for our purposes we shall only need an expression for b_2, the following proposition gives a closed formula for the general heat coefficient.

Proposition 2.1. *We have that $b_j = \int b_j(x)\,\mathrm{d}x$, where $b_j(x) \in C_0^{\infty}(\mathbb{R}^n)$ are given by*

$$b_j(x) = (-1)^j \sum_{k=0}^{j} \binom{j+n/2}{k+n/2} \frac{H_a^{k+j}(\rho_x(y)^{2k})|_{y=x}}{4^k k!(k+j)!}. \tag{2.2}$$

Here $\rho_x(y)$ is the Euclidean distance between the points $x, y \in \mathbb{R}^n$, and the binomial coefficients are given by

$$\binom{j+n/2}{k+n/2} = \frac{\Gamma(j+(n/2)+1)}{(j-k)!\,\Gamma(k+(n/2)+1)}.$$

Proof. This result follows from the proof in [13, Theorem 1.2.1], and we shall merely indicate how to extract it from that paper. In [13], closed formulas are derived for the local heat invariants for the Laplace–Beltrami operator on a smooth Riemannian manifold. This is achieved by means of the Agmon–Kannai expansion for the kernel of (a power of) the resolvent of the Laplace–Beltrami operator, see [1]. Now the same approach can be applied to the Schrödinger operator with a magnetic field—see [6] for applications to non-magnetic Schrödinger operators, also with long-range potentials. An inspection of the arguments of [13] then leads to (2.2). □

An explicit computation using Proposition 2.1 then shows that $b_1 = 0$, in agreement with [18]. When computing the coefficient b_2 by means of Proposition 2.1 in dimension 3, it is convenient to implement Formula (2.2) using the *Maple* software. As a result, we get a several lines long expression, which is readily seen to be equal to

$$b_2 = -\frac{1}{6}\int \|B\|^2 \, \mathrm{d}x. \tag{2.3}$$

Here the norm of B has been defined in (1.2). This result is also in agreement with [18], and moreover from [5,18] we know that (2.3) is true in any dimension. Therefore, $b_2 < 0$, unless the magnetic field vanishes identically.

Remark. The following argument indicates that the expression for the coefficient b_2 in (2.3) has the correct physical dimension to match the factor $t^{-(n/2)+2}$, in front of which it appears in (2.1). Here we use the units related to the heat equation, where the time t has the dimension of units of length squared, $[m]^2$. From (1.1) we know that the magnetic potential has the dimension of $[m]^{-1}$, and it follows from (1.2) and (2.3) that the dimension of $\|B\|^2$ is $[m]^{-4}$. Therefore, the dimension of the second heat coefficient b_2 is $[m]^{-4+n}$, which matches precisely the power $t^{-(n/2)+2}$. Notice also that the heat coefficients b_j are defined through (2.1) and therefore do not depend on the choice of the magnetic potential, provided that the magnetic field B is kept fixed. The local densities $b_j(x)$, defined in (2.2), are therefore polynomials in B_{jk} and its derivatives, and we see immediately that the only contributions to $b_2(x)$, which have non-vanishing integrals and the correct dimension, must be proportional to $|B_{jk}|^2$. This determines the heat coefficient b_2 up to a numerical constant, whose value can be obtained, e.g., from the computation using the *Maple* program, mentioned above.

Remark. In the context of Schrödinger operators without a magnetic field, explicit expressions for the first two heat coefficients can be obtained from the Feynman–Kac formula for the heat kernel by a means of straightforward computation—see [6, Section 2]. It would therefore be interesting to compute the first several heat coefficients for the magnetic Schrödinger operator H_a by means of the Feynman–Kac–Itô formula, see [3]. The computational complexity here could be smaller than in computations based on Proposition 2.1.

We shall now turn to the proof of Theorem 1.1. This will follow closely the arguments of [17] with further improvements from [22], and for the sake of completeness we shall recall the main steps here. The starting point in the proof of Theorem 1.1 is the Poisson formula for resonances, originally established in [9,10] for scattering by obstacles, in the framework of the Lax–Phillips scattering theory. It was then extended in [20] to the general situation of "black box" scattering, which includes the case of the magnetic Schrödinger operator. A proof of the Poisson formula not relying upon the Lax–Phillips theory was given in [21], which is our basic reference.

The property of finite propagation speed for supports of solutions to the wave equation implies that we can define a distribution

$$u(t) = 2\,\mathrm{tr}(\cos t\sqrt{H_a} - \cos t\sqrt{H_0}) \in \mathscr{D}'(\mathbb{R}),$$

so that

$$\langle u(t),\varphi\rangle = 2\,\mathrm{tr}\int \varphi(t)(\cos t\sqrt{H_a}-\cos t\sqrt{H_0})\,\mathrm{d}t,\qquad \varphi\in C_0^\infty(\mathbb{R}).$$

The Poisson formula, see [21], then states that in $\mathscr{D}'(\mathbb{R}\backslash\{0\})$ it is true that

$$u(t)=\sum \mathrm{e}^{-\mathrm{i}\lambda_j|t|}. \tag{2.4}$$

Here the summation is performed over the set of all resonances $\{\lambda_j\}$, counted according to their multiplicity. The sum in (2.4) converges in the sense of distribution theory, in view of the polynomial upper bounds on the number of resonances in large discs, see [19].

Assume now that H_a has no resonances. Then it follows from (2.4) that $u(t)$ is supported by the origin. Now it is well known that the asymptotic expansion of $u(t)$ for small t, which can be obtained from Hadamard's parametrix construction, see [7], is of the form,

$$u(t)\sim \sum_{j=1}^{(n-1)/2} d_j\delta^{(n-1-2j)}(t)+\sum_{j\geqslant (n+1)/2} d_j|t|^{2j-n+1}. \tag{2.5}$$

The second sum therefore vanishes, and (2.5) is an exact equality,

$$u(t)=\sum_{j=1}^{(n-1)/2} d_j\delta^{(n-1-2j)}(t),\qquad t\in\mathbb{R}. \tag{2.6}$$

Using (2.6), we shall now pass to the regularized trace of the heat semigroup,

$$\mathrm{tr}(\mathrm{e}^{-tH_a}-\mathrm{e}^{-tH_0}),\qquad t>0,$$

and relate the heat coefficients b_j in (2.1) to the wave coefficients d_j. We have the well-known transmutation formula of [8],

$$\mathrm{e}^{-tH_0}=\frac{1}{\sqrt{4\pi t}}\int \mathrm{e}^{-s^2/4t}\cos(s\sqrt{H_0})\,\mathrm{d}s, \tag{2.7}$$

and the corresponding formula is also true for H_a. Using (2.6) and (2.7) we get

$$\begin{aligned}\mathrm{tr}(\mathrm{e}^{-tH_a}-\mathrm{e}^{-tH_0}) &= \frac{1}{2\sqrt{4\pi t}}\int \mathrm{e}^{-s^2/4t}u(s)\,\mathrm{d}s\\ &= \frac{1}{2\sqrt{4\pi t}}\sum_{j=1}^{(n-1)/2} d_j\int \mathrm{e}^{-s^2/4t}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{n-1-2j}\delta(s)\,\mathrm{d}s\\ &= \frac{1}{2\sqrt{4\pi t}}\sum_{j=1}^{(n-1)/2}(-1)^{n-1-2j}d_j\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{n-1-2j}\mathrm{e}^{-s^2/4t}\right)(0)\\ &= \frac{1}{\sqrt{t}}\sum_{j=1}^{(n-1)/2}\alpha_j d_j t^{-(n-1-2j)/2} = t^{-n/2}\sum_{j=1}^{(n-1)/2}\alpha_j d_j t^j,\qquad \alpha_j\neq 0.\end{aligned}$$

In this computation we have used Taylor's formula. We have thus proved the following result.

Proposition 2.2. *Assume that the resolvent $R(\lambda)$ of H_a is an entire function. Then*

$$\operatorname{tr}(e^{-tH_a} - e^{-tH_0}) = (4\pi t)^{-n/2} \sum_{j=1}^{(n-1)/2} b_j t^j, \quad t > 0,$$

where $b_j = \beta_j d_j$, $\beta_j = (4\pi)^{n/2}\alpha_j \neq 0$, are the heat coefficients of H_a.

When $n = 3$, Proposition 2.2 together with (2.1) gives that $b_2 = 0$, and it follows from (2.3) that the magnetic field B vanishes. In the case of dimensions $n \geqslant 5$, following [17], we shall analyze the behaviour of the regularized heat trace as $t \to \infty$. To this end we recall the Birman–Krein formula for H_a—see [22] for the relevant version,

$$\operatorname{tr}(e^{-tH_a} - e^{-tH_0}) = \int_0^\infty e^{-t\lambda^2} \sigma'(\lambda)\, d\lambda, \quad t > 0. \tag{2.8}$$

Here $\sigma(\lambda)$ is the scattering phase of H_a, and we have also used the fact that H_a has no point spectrum and that 0 is not a resonance of H_a. Let us recall that the scattering phase $\sigma(\lambda)$ is defined as the argument of the determinant of the scattering matrix of H_a. We refer to [11,21] for further information on this basic object of scattering theory, and here we remark only that heuristically, the scattering phase measures the averaged phase shift of a wave passing through the scatterer.

In order to study the behaviour of the left-hand side of (2.8) as $t \to \infty$, it is natural to study $\sigma'(\lambda)$ near $\lambda = 0$. (See also [12].) In doing so we recall from [22], that in any dimension $n \geqslant 3$, it is true that $\sigma'(\lambda)$ vanishes to the order at least $n - 3$ as $\lambda \to 0^+$, provided that 0 is not a resonance. More precisely, from [22] we get,

$$\sigma'(\lambda) = \lambda^{n-3} f(\lambda), \quad \lambda > 0, \tag{2.9}$$

with f smooth near the origin. It follows therefore from (2.8) and (2.9) that, as $t \to \infty$,

$$\operatorname{tr}(e^{-tH_a} - e^{-tH_0}) = \frac{1}{2}\Gamma\left(\frac{n}{2} - 1\right) f(0) t^{-n/2+1} + \mathcal{O}(t^{-(n/2)+1/2}). \tag{2.10}$$

Combining (2.10) with Proposition 2.2, we get that $b_j = 0$, $j \geqslant 2$, and as before, it follows that $B \equiv 0$.

We conclude that, if B does not vanish identically, there exists at least one resonance. The argument of [11] now shows that the existence of some resonances implies that there are infinitely many of them. Indeed, it follows from (2.4) and (2.5) that for small t,

$$\sum e^{-i\lambda_j |t|} = F(t), \quad |t| > 0,$$

where F is continuous near $t = 0$ and $F(0) = 0$. If there were only finitely many resonances, then their number would be equal to $\lim_{t\to 0} F(t) = 0$, which is a contradiction. This completes the proof of Theorem 1.1.

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