

The complex Poisson kernel on a compact analytic Riemannian manifold

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Abstract

We give a detailed proof of a theorem of L. Boutet de Monvel formulated in 1978 in [\[1\]](#), and we state a "conjecture" on the ramification locus of the Poisson kernel on general analytic Riemannian manifolds with boundary.

Contents

1	Introduction	2
2	Explicit formulas in the flat case	4
3	Symplectic geometry	7
4	A proof of the Boutet de Monvel theorem	11
5	A conjecture on the ramification locus for general Poisson kernels	25
6	Appendix	31
	References	31

1 Introduction

SO

These notes are a written version of a 3 hours course given at Northwestern university in may 2013. The main purpose is to give a detailed proof of a theorem of L. Boutet de Monvel formulated in 1978 in [1]. We have add a "conjecture" on the ramification locus of the Poisson kernel on general analytic Riemannian manifolds with boundary. We hope that this will be motivating for students and researchers in linear partial differential equations and microlocal analysis.

Let (M, g) be a compact, connected, analytic Riemannian manifold of dimension m . Let us recall that the metric g on the tangent bundle TM gives a canonical identification of TM with the cotangent bundle T^*M . Let $d_g x$ be the volume form on M associated to the metric g . The Laplace operator Δ_g on M is defined by the formula

$$\int_M \Delta_g(u) \bar{v} d_g x = - \int_M (du|dv) d_g x \quad (1.1) \quad \boxed{1.1}$$

Here df denotes the differential of the function f so one has by definition $\Delta_g = -d^*d$ where d^* is the adjoint of d for the natural Hilbert structure induced by g on section of T^*M . The unbounded operator $-\Delta_g$ with domain $H^2(M)$ is self-adjoint on $L^2(M, d_g x)$, non negative, with compact resolvent. We will denote by $(e_j)_{j \geq 0}$ an orthonormal basis of $L^2(M, d_g x)$ of real eigenfunctions of $-\Delta_g$ associated to the eigenvalues ω_j^2 , with $\omega_0 = 0 < \omega_1 \leq \omega_2 \leq \dots$, $\lim_{j \rightarrow \infty} \omega_j = +\infty$, so that one has

$$-\Delta_g(e_j) = \omega_j^2 e_j, \quad \int_M e_j e_k d_g x = \delta_{j,k} \quad (1.2) \quad \boxed{1.2}$$

Since Δ_g is a second order elliptic operator with analytic coefficients, the eigenfunctions e_j are real analytic functions on M .

Let X be a complexification of M . This means that X is a complex analytic manifold of complex dimension m , and $M \subset X$ is a totally real submanifold of X (this means $TM \cap iTM = M$ where $M \subset TM$ is view as the zero section). Let $d(x, y)$ be the distance function on $M \times M$. Then $d^2(x, y)$ is an analytic function near the diagonal $Diag_M = \{(x, x), x \in M\} \subset M \times M$, and therefore extends as an holomorphic function in a complex neighborhood of $Diag_M$ in $X \times X$. Let us define $\Phi(z)$ by the formula

$$\Phi(z) = \frac{1}{2} \sup_{y \in M} Re(-d^2(z, y)) \quad (1.3) \quad \boxed{1.3}$$

We will see in section ^{sec3}3 that this function is well defined for $z \in X$ close to M , and is real analytic and strictly pluri-subharmonic. Moreover, one has $\Phi|_M = 0$, $d\Phi|_M = 0$ and the signature of the Hessian of Φ is equal to $(m, 0)$ at any point of M ; in particular, one has $\Phi(z) \geq 0$ and $\Phi(z) = 0$ if and only if $z \in M$. This function allows to define, for $\epsilon > 0$ small enough, the tubular neighborhood B_ϵ of M in X

$$B_\epsilon = \{z \in X, \Phi(z) < \frac{\epsilon^2}{2}\} \quad (1.4) \quad \boxed{1.4}$$

Let us denote by $\mathcal{O}(B_\epsilon)$ the space of holomorphic functions defined on B_ϵ . For $f \in \mathcal{O}(B_\epsilon)$, its boundary value $f|_{\partial B_\epsilon}$ on ∂B_ϵ is well defined as an hyperfunction on ∂B_ϵ which is an

analytic compact real manifold of dimension $2m-1$. This boundary value is a distribution on ∂B_ϵ if and only if the function f satisfies a polynomial growth condition at the boundary of the form $|f(z)| \leq C \text{dist}(z, \partial B_\epsilon)^{-N}$. Let us recall that the Hardy space $H(B_\epsilon)$ is the Hilbert space defined by

$$H(B_\epsilon) = \{f \in \mathcal{O}(B_\epsilon), f|_{\partial B_\epsilon} \in L^2(\partial B_\epsilon)\} \quad (1.5) \quad \boxed{1.5}$$

We can now state the Boutet theorem formulated in [\[1\]](#)^{Boutet1} (in a slightly different but equivalent form). Let us recall that a family $(u_j)_{j \geq 0}$ is a Riesz basis of an Hilbert space H if and only if any $x \in H$ can be written in a unique way as the sum of a convergent serie in H , $x = \sum c_j(x)u_j$ and $\sum |c_j(x)|^2$ is equivalent to $\|x\|_H^2$. We use the classical notation $\langle x \rangle = (1 + x^2)^{1/2}$.

thmboutet **Theorem 1.1** *For $\epsilon > 0$ small enough the following holds true. The eigenfunctions e_j extends holomorphically to B_ϵ and the family $(e^{-\epsilon \omega_j} \langle \omega_j \rangle^{-(m-1)/4} e_j(z))_{j \geq 0}$ is a Riesz basis of $H(B_\epsilon)$. For $f \in H(B_\epsilon)$ and $a_j = \int_M f e_j d_g x$, one has*

$$f(z) = \sum a_j e_j(z) \quad (1.6) \quad \boxed{1.6}$$

where the sum is uniformly convergent on any compact subset of B_ϵ and convergent in $H(B_\epsilon)$. There exists a constant C_ϵ such that one has the equivalence of norms

$$\frac{1}{C_\epsilon} \|f\|_{H(B_\epsilon)}^2 \leq \sum_j |e^{\epsilon \omega_j} \langle \omega_j \rangle^{-(m-1)/4} a_j|^2 \leq C_\epsilon \|f\|_{H(B_\epsilon)}^2 \quad (1.7) \quad \boxed{1.7}$$

A detailed proof of this theorem has been given recently by S. Zelditch in [\[11\]](#)^{Zel1}, following the lines indicate in [\[1\]](#)^{Boutet1} and using the Hadamard parametrix for the wave equation, and also by M. Stenzel in [\[10\]](#)^{St1} which uses the asymptotic expansions of the heat kernel. Here, we will give a proof based on non-characteristic deformation techniques and a direct calculus of the Hadamard type parametrix for the Poisson Kernel.

The paper is organized as follows:

In section [2](#)^{sec2}, we just recall explicit formulas in the euclidian space \mathbb{R}^m and we give a proof of the Boutet theorem in the special case of the flat torus $(\mathbb{R}/2\pi\mathbb{Z})^m$.

In section [3](#)^{sec3}, we recall basic facts on symplectic geometry. We introduce the fundamental function Φ and we give some of his properties. We refer to [\[7\]](#)^{Sch1} for a detailed study of the relationships between real and complex symplectic geometry.

Section [4](#)^{sec4} is devoted to the proof of the Boutet theorem.

In section [5](#)^{sec5}, we state our "conjecture" on the ramification locus in the complex domain for general Poisson kernels, and we give some exemples.

Finally, the appendix contains some proofs of technical results.

There is no need to have any knowledge about analytic microlocal analysis to read these notes. The only "analytic" things that we will use are: Cauchy-Kowalewski, Zerner-lemma (see lemma [4.2](#)^{lemzerner} in section [4](#)^{sec4}), and the analytic regularity for solutions of elliptic linear differential operator with analytic coefficients.

Finally, let us recall that the representation of the analytic wave front set as the analytic singular support of boundary values of holomorphic functions defined inside a strictly pseudoconvex domain, which is one of the most fundamental results in microlocal analysis, (and which is closely related to the Boutet theorem) is due to M. Sato, T. Kawai and M. Kashiwara and is explicit in their foundation article of 1971 [\[8\]](#).

2 Explicit formulas in the flat case

sec2

In this section, we just recall what are the explicit formulas for the Poisson kernel, heat kernel, and FBI transform on the euclidean space \mathbb{R}^m . Replacing \mathbb{R}^m by the standard m -dimensional torus $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$, this will give a straightforward proof of the Boutet theorem in this special case.

First observe that on \mathbb{R}^m one has $d^2(x, y) = (x - y)^2$, and therefore the function $\Phi(z)$ given by [\(1.3\)](#) is defined on all \mathbb{C}^m by

$$\Phi(z) = \text{Im}(z)^2/2 \quad (2.1) \quad \boxed{2.0}$$

The heat kernel in \mathbb{R}^m is equal to $p_t(x, y) = (2\pi t)^{-m/2} e^{-(x-y)^2/2t}$. The solution of the heat equation

$$\partial_t f - \frac{1}{2} \Delta f = 0 \text{ (in } t > 0), \quad f|_{t=0} = g \in \mathcal{S}'(\mathbb{R}^m) \quad (2.2) \quad \boxed{2.1}$$

is given by the formula

$$f(t, x) = \int_{\mathbb{R}^m} p_t(x, y) g(y) dy \quad (2.3) \quad \boxed{2.2}$$

On the Fourier side, one has the obvious identity

$$\hat{f}(t, \xi) = e^{-t\xi^2/2} \hat{g}(\xi) \quad (2.4) \quad \boxed{2.3}$$

Observe that if we replace $x \in \mathbb{R}^m$ by $z \in \mathbb{C}^m$, and if we set $\lambda = 1/t > 0$, we get

$$f(t, z) = \left(\frac{\lambda}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} e^{-\lambda(z-y)^2/2} g(y) dy = T_\lambda(g)(z) \quad (2.5) \quad \boxed{2.4}$$

where T_λ is exactly the most usual FBI transform introduced by J. Sjöstrand in [\[9\]](#) (up to the factor $(\frac{\lambda}{2\pi})^{m/2}$ in front of it). Therefore, we get that this FBI transform is just a complexification of the usual heat kernel. One has the obvious bound

$$|f(t, z)| \leq \left(\frac{\lambda}{2\pi}\right)^{m/2} e^{\lambda\Phi(z)} \|g\|_{L^1} \quad (2.6) \quad \boxed{2.4\text{bis}}$$

Now we recall the formula for the Poisson kernel $\mathbb{P}_s(x, y)$. The solution of the elliptic boundary value problem, with $f(s, \cdot)$ bounded in $s \geq 0$ with values in $L^2(\mathbb{R}^m)$

$$\partial_s^2 f + \Delta f = 0 \text{ (in } s > 0), \quad f|_{s=0} = g \in L^2(\mathbb{R}^m) \quad (2.7) \quad \boxed{2.5}$$

is given by the formula

$$f(s, x) = \mathbb{P}_s(g)(x) = \int_{\mathbb{R}^m} \mathbb{P}_s(x, y) g(y) dy \quad (2.8) \quad \boxed{2.6}$$

One has the obvious identity

$$\mathbb{P}_s(g)(x) = (2\pi)^{-m} \int e^{ix\xi - s|\xi|} \hat{g}(\xi) d\xi \quad (2.9) \quad \boxed{2.7}$$

Fix now $s > 0$. Then (2.9) clearly implies that $\mathbb{P}_s(g)$, (with g in any Sobolev space $H^\mu(\mathbb{R}^m)$) extends holomorphically for $s > 0$ in the domain

$$B_s = \{|Im(z)| < s\} = \{\Phi(z) < s^2/2\}$$

For $z \in B_s$, set $z = a + ib$. Then the map $g \mapsto T_s(g) = \mathbb{P}_s(g)|_{\partial B_s}$ is given by

$$T_s(g)(a, b) = (2\pi)^{-m} \int e^{i(a-x)\xi - b \cdot \xi - s|\xi|} g(x) dx d\xi \quad (2.10) \quad \boxed{2.8}$$

Clearly, T_s extends for all real μ to a map defined on the Sobolev space $H^\mu(\mathbb{R}^m)$ with values in $\mathcal{D}'(\partial B_s)$. Let $d\sigma_s$ be the standard measure on the sphere of radius s in \mathbb{R}^m , and let c_m be the volume of the unit sphere S^{m-1} in \mathbb{R}^m . Let $d\mu_s$ be the volume form on ∂B_s

$$d\mu_s = c_m^{-1} s^{-(m-1)} da d\sigma_s(b) \quad (2.11) \quad \boxed{2.9}$$

Let T_s^* be adjoint of T_s with respect to $L^2(\partial B_s, d\mu_s)$. One has

$$T_s^*(f)(x) = (2\pi)^{-m} \int e^{i(x-a)\xi - b \cdot \xi - s|\xi|} f(a, b) d\mu_s d\xi \quad (2.12) \quad \boxed{2.10}$$

and therefore we get

$$\begin{aligned} T_s^* T_s(g)(x) &= (2\pi)^{-m} \int e^{ix\xi} \Gamma_m(s\xi) \hat{g}(\xi) d\xi \\ \Gamma_m(\eta) &= c_m^{-1} \int_{S^{m-1}} e^{-2(|\eta|+u \cdot \eta)} d\sigma(u) \end{aligned} \quad (2.13) \quad \boxed{2.11}$$

It is clear that Γ_m is a real strictly positive function and $\Gamma_m(0) = 1$. The function $\Gamma_m(\eta)$ depends only on $|\eta|$ and $e^{2|\eta|} \Gamma_m(\eta)$ is an entire function of $|\eta|^2$. Moreover, by stationary phase, we get that $\Gamma_m(\eta)$ is an elliptic symbol of degree $-(m-1)/2$ in η (and even an analytic symbol). Therefore, with $\langle \eta \rangle = (1 + |\eta|^2)^{1/2}$ there exists $c > 1$ such that

$$\frac{1}{c} \langle \eta \rangle^{-(m-1)/2} \leq \Gamma_m(\eta) \leq c \langle \eta \rangle^{-(m-1)/2}, \quad \forall \eta \in \mathbb{R}^m$$

Since $T_s^* T_s$ is the Fourier multiplier by $\Gamma_m(s\xi)$, this shows that $T_s^* T_s$ is a self adjoint, non negative, elliptic pseudodifferential operator of degree $-(m-1)/2$, and also an isomorphism of the Sobolev space $H^{\mu-(m-1)/2}(\mathbb{R}^m)$ onto $H^\mu(\mathbb{R}^m)$ for any real μ . From the identity

$$(T_s^* T_s(g)|_g)_{L^2(\mathbb{R}^m, dx)} = \|T_s(g)\|_{L^2(\partial B_s, d\mu_s)}^2$$

we get

$$T_s(g) \in L^2(\partial B_s) \quad \text{if and only if} \quad g \in H^{-(m-1)/4}(\mathbb{R}^m)$$

From the above formulas, it is easy to get the Boutet theorem for $M = \mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$. The standard L^2 orthonormal basis is in that case $e_k(x) = (2\pi)^{-m/2} e^{ik \cdot x}$, with $k \in \mathbb{Z}^m$, and associated eigenvalue $|k|^2$. The Poisson operator is given by

$$\mathbb{P}_s(\sum c_k e_k)(x) = \sum c_k e^{-s|k|} e_k(x)$$

which clearly extends to $B_s = \{z = a + ib \in (\mathbb{C}/2\pi\mathbb{Z})^m, |b| < s\}$. If T_s still denotes the map $g \mapsto T_s(g) = \mathbb{P}_s(g)|_{\partial B_s}$, one has (T_s^* is the adjoint for the volume form (2.11) on ∂B_s)

$$T_s^* T_s \left(\sum c_k e_k \right) = \sum c_k \Gamma_m(sk) e_k$$

thus $T_s(g) \in L^2(\partial B_s)$ if and only if $g \in H^{-(m-1)/4}(\mathbb{T}^m)$. One has

$$T_s \left(\sum c_k e_k \right) (a + ib) = (2\pi)^{-m/2} \sum c_k e^{-s|k|} e^{ik \cdot a - k \cdot b}$$

The functions $(2\pi)^{-m/2} e^{ik \cdot a - k \cdot b} = E_k(a, b)$ are trivially orthogonal in $L^2(\partial B_s, d\mu_s)$, and the computation we have done to get (2.13) shows that one has

$$\|E_k\|_{L^2(\partial B_s)}^2 = e^{2s|k|} \Gamma_m(sk)$$

It will be proven in section 4 that the family $(e_k(z))_k$ is dense in the Hardy space $H(B_s)$ (we leave this as an exercise in the special case of the flat torus). Thus, in the flat case, we get the more precise statement that the family

$$e^{-s|k|} \Gamma_m^{-1/2}(sk) e_k(z), \quad k \in \mathbb{Z}^m$$

is an orthonormal basis of the Hardy space $H(B_s)$. Thus the Boutet theorem holds true in the special case of the flat torus.

Remark 2.1 *As one can see, in the flat case, $T_s^* T_s$ is in fact a function of the Laplace operator, and the eigenfunctions $e_k(z)|_{\partial B_s}$ remains orthogonal for any s for a natural choice of the volume form on ∂B_s . This will not remain true in the general case. **Véifier !** Also, one has to notice that, with respect to s , view as a small parameter and not view as a fixed constant, formula (2.13) indicate that $T_s^* T_s$ is a s -pseudodifferential operator and not at all a usual pseudo-differential operator uniformly in $s \in]0, 1]$.*

Let us now recall how one can recover the Poisson kernel from the heat kernel. We start from the formula, valid for all $x \in [0, \infty[$.

$$e^{-x} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x^2/4u} e^{-u} \frac{du}{\sqrt{u}} \quad (2.14) \quad \boxed{2.12}$$

This formula is easy to prove, since both side are continuous functions of $x \geq 0$ and satisfy the equation $f'' - f = 0$ in $x > 0$ and $f(0) = 1, \lim_{x \rightarrow \infty} f(x) = 0$. From (2.14), we get for $s > 0, \omega \geq 0$ (change of variable $u = s^2/2t$)

$$e^{-s\omega} = \frac{s}{\sqrt{2\pi}} \int_0^\infty e^{-s^2/2t} e^{-t\omega^2/2} \frac{dt}{t^{3/2}} \quad (2.15) \quad \boxed{2.13}$$

Therefore, one has the following identity which allows to recover the Poisson kernel from the heat kernel, (and which remains obviously valid on any Riemannian compact manifold (M, g) by decomposition on the orthonormal basis $(e_j)_j$):

$$\mathbb{P}_s(x, y) = \frac{s}{\sqrt{2\pi}} \int_0^\infty e^{-s^2/2t} p_t(x, y) \frac{dt}{t^{3/2}} \quad (2.16) \quad \boxed{2.14}$$

This identity is used by M. Stenzel in [St1] in his proof of the Boutet theorem. If we express this in term of the FBI transform defined in (2.14), we get (recall $\lambda = 1/t$)

$$\mathbb{P}_s(z, y) = \frac{s}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s^2/2} T_\lambda(z, y) \lambda^{-1/2} d\lambda \quad (2.17) \quad \boxed{2.15}$$

From (2.6), we recover from (2.15) that in the flat case, $\mathbb{P}_s(z, y)$ extends holomorphically in the domain $|Im(z)| < s$. Therefore, the FBI transform (i.e the complexification of the heat kernel) contains at least as much information than the Poisson Kernel. In fact, the two points of view are essentially equivalent if the FBI transform acts on functions independent of λ . The use of the FBI transform is of course more relevant in semi-classical analysis, with small parameter $h = 1/\lambda = t$. We refer to the article by F.Golse, E.Leichtnam and M. Stenzel, [GLS] for a study of the FBI transform as a complexification of the heat kernel on compact Riemannian analytic manifolds.

3 Symplectic geometry

sec3

Let T^*X be the complex cotangent bundle to the complex manifold X . Let us recall that for $(z, \zeta) \in T^*X$, ζ is a \mathbb{C} -linear form on the complex vector space $T_z X$ with values in \mathbb{C} , i.e $\zeta(iu) = i\zeta(u)$ for all $u \in T_z X$. As usual, if f is a function defined on X with values in \mathbb{C} , we denote by ∂f (resp $\bar{\partial} f$) its holomorphic (resp. antiholomorphic) derivative, that is

$$\partial f(u) = \frac{1}{2}(df(u) - idf(iu)), \quad \bar{\partial} f(u) = \frac{1}{2}(df(u) + idf(iu))$$

Then ∂f is a section of T^*X and one has $d = \partial + \bar{\partial}$.

Let us denote by $X^{\mathbb{R}}$ the real analytic manifold X without its complex structure. In these notes, we shall identify the real cotangent bundle $T^*(X^{\mathbb{R}})$ with the complex cotangent bundle T^*X by the following rule

$$(z, \zeta) \in T^*X \text{ is identified with } (z, \xi) \in T^*X^{\mathbb{R}} : \quad \xi(u) = Re(\zeta(u)) \quad (3.1) \quad \boxed{3.1}$$

With this identification, for any smooth function $\varphi : X \rightarrow \mathbb{R}$,

$$d\varphi(z) \in T_z^*X^{\mathbb{R}} \text{ is identified with } 2\partial\varphi(z) \in T_z^*X \quad (3.2) \quad \boxed{3.2}$$

Let $\omega = d\zeta \wedge dz$ be the canonical complex symplectic 2-form on T^*X . Then $Re(\omega)$ and $Im(\omega)$ are real symplectic 2-forms on $T^*X^{\mathbb{R}}$, and moreover, $Re(\omega) = \omega^{\mathbb{R}}$ is the canonical symplectic 2-form on $T^*X^{\mathbb{R}}$. This facts are easy to verify in local coordinates. We shall say that a real submanifold Λ of T^*X is R-symplectic (resp I-lagrangian) iff Λ is symplectic for $Re(\omega) = \omega^{\mathbb{R}}$ (resp lagrangian for $Im(\omega)$). In other words, Λ is R-symplectic iff $dim_{\mathbb{R}}\Lambda = 2m$ and $Re(\omega)|_{\Lambda}$ is non degenerate, and Λ is I-lagrangian iff $dim_{\mathbb{R}}\Lambda = 2m$ and $Im(\omega)|_{\Lambda} = 0$.

lem3.1

Lemma 3.1 *Let $z \mapsto \zeta(z)$ be a smooth section of $T^*X \simeq T^*X^{\mathbb{R}}$ defined on an open contractible subset Ω of X and let $\Lambda = \{(z, \zeta(z)), z \in \Omega\}$. Then Λ is I-lagrangian iff there exists a smooth function $\varphi : \Omega \rightarrow \mathbb{R}$ such that $\zeta(z) = 2i\partial\varphi(z)$. Moreover, Λ is also R-symplectic iff the 2-form of type (1, 1) $2i\bar{\partial}\partial\varphi$ on $T^*X|_{\Omega}$ is non degenerate.*

Proof. If Λ is I-lagrangian, then $-i\Lambda = \{(z, -i\zeta(z)), z \in \Omega\}$ is R-lagrangian, $\omega^{\mathbb{R}}|_{-i\Lambda} = 0$. Since Ω is contractible, there exists a function $\varphi : \Omega \rightarrow \mathbb{R}$ such that $-i\Lambda$ view as a subset of $T^*X^{\mathbb{R}}$ is of the form $\{(z, d\varphi(z))\}$. With the identification $T^*X \simeq T^*X^{\mathbb{R}}$, and by (3.2), we get $-i\zeta(z) = 2\partial\varphi(z)$, i.e

$$\zeta(z) = 2i\partial\varphi(z)$$

Let $j : \Omega \rightarrow T^*X$ be defined by $j(z) = (z, 2i\partial\varphi(z))$. One has $j^*(Im(\omega)) = 0$. Moreover Λ is R-symplectic iff $j^*(\omega^{\mathbb{R}})$ is non degenerate and the result follows from

$$j^*(\omega^{\mathbb{R}}) = j^*(\omega) = j^*(d(\zeta dz)) = d(j^*(\zeta dz)) = d(2i\partial\varphi) = 2i\bar{\partial}\partial\varphi$$

□

The Levi form on $TX|_{\Omega}$, $\mathcal{L}_{\varphi}(u, v) = 2i\bar{\partial}\partial\varphi(u, v)$ is given in local complex coordinates (z_1, \dots, z_m) by the formula

$$\mathcal{L}_{\varphi}(u, v) = 2i \sum_{j,k} \frac{\partial^2 \varphi}{\partial \bar{z}_j \partial z_k}(z) (\bar{u}_j v_k - \bar{v}_j u_k)$$

One has obviously $\mathcal{L}_{\varphi}(u, v) \in \mathbb{R}$, and \mathcal{L}_{φ} is entirely determinate by the associated hermitian form $q_{\varphi}(u) = \mathcal{L}_{\varphi}(iu, u)$. In local coordinates, one has

$$q_{\varphi}(u) = 4 \sum_{j,k} \frac{\partial^2 \varphi}{\partial \bar{z}_j \partial z_k}(z) \bar{u}_j u_k \quad (3.3) \quad \boxed{3.3}$$

Therefore, Λ is I-lagrangian and R-symplectic iff the hermitian form q_{φ} is non degenerate, hence of signature (p, q) with $p + q = m$.

The real cotangent bundle T^*M is a subset of T^*X : for $x \in M$, any $u \in T_x X$ can be written in a unique way $u = a + ib$, $a, b \in T_x M$, and $(x, \xi) \in T^*M$ defines $(x, \zeta) \in T^*X$, $\zeta(u) = \xi(a) + i\xi(b)$. Then it is obvious that T^*M is both R-symplectic and I-lagrangian. Moreover, T^*M is a totally real submanifold of T^*X and the complex symplectic manifold T^*X is a complexification of the real symplectic manifold T^*M .

Let $p(z, \zeta)$ be the holomorphic extension of $p(x, \xi) = \frac{1}{2}|\xi|_x^2$. In local coordinates, one has

$$p(z, \zeta) = \frac{1}{2} \sum_{j,k} g^{j,k}(z) \zeta_j \zeta_k$$

and $p(z, \zeta)$ is well defined on $T^*X|_W$ if W is a small neighborhood of M in X . For $t \in \mathbb{C}$, let us denote by $exp(tH_p)(z, \zeta) = (Z(t, z, \zeta), \Xi(t, z, \zeta))$ the complex integral curve of the hamiltonian vector field of p starting at (z, ζ) . One has the Hamilton-Jacobi equations

$$\begin{aligned} \partial_t Z &= (\partial_{\zeta} p)(Z, \Xi), & Z(0, z, \zeta) &= z \\ \partial_t \Xi &= -(\partial_z p)(Z, \Xi), & \Xi(0, z, \zeta) &= \zeta \end{aligned} \quad (3.4) \quad \boxed{3.4}$$

Since $p(z, \zeta)$ is homogeneous of degree 2 in ζ , one has for $\lambda \neq 0$

$$Z(\lambda t, z, \zeta/\lambda) = z(t, z, \zeta), \quad \Xi(\lambda t, z, \zeta/\lambda) = \lambda^{-1}\Xi(t, z, \zeta) \quad (3.5) \quad \boxed{3.5}$$

Therefore, $\exp(tH_p)(z, \zeta)$ is well defined for $|t\zeta|$ small and $(z, \zeta) \in T^*X|_W$ if W is small enough, and one has the Taylor expansion

$$\begin{aligned} Z(t, z, \zeta) &= z + t(\partial_{\zeta}p)(z, \zeta) + 0((t\zeta)^2) \\ \Xi(t, z, \zeta) &= \zeta - t(\partial_z p)(z, \zeta) + 0(\zeta(t\zeta)^2) \end{aligned} \quad (3.6) \quad \boxed{3.6}$$

Let $\epsilon_0 > 0$ given and small. For $s \in]0, 1]$, set

$$\Lambda_s = \{(z, \zeta) = \exp(isH_p)(x, \xi) \in T^*X, (x, \xi) \in T^*M, |\xi|_x < \epsilon_0/s\} \quad (3.7) \quad \boxed{3.7}$$

Then for ϵ_0 small enough and all $s \in]0, 1]$, Λ_s is well defined and from (3.5), one has $\Lambda_s = s^{-1}\Lambda_1$. Moreover, since the map $\exp(tH_p)$ preserves the complex symplectic structure of T^*X for any $t \in \mathbb{C}$, Λ_s is both R-symplectic and I-lagrangian. By (3.6), the map $(x, \xi) \mapsto Z(is, x, \xi)$ is given in local coordinates by

$$(x, \xi) \mapsto Z(is, x, \xi), \quad Z_k(is, x, \xi) = x_k + is \sum_j g^{j,k}(x) \xi_j + 0((s\xi)^2) \quad (3.8) \quad \boxed{3.8}$$

hence is an isomorphism near $\xi = 0$. By lemma 3.1, near any point $x \in M$ there exists a unique function $\Phi_s(z) = s^{-1}\Phi(z)$ define in a neighborhood of x , with $\Phi_s(x) = 0$ such that one has

$$\Lambda_s = \{(z, \zeta), \zeta = 2i\partial\Phi_s(z) = 2is^{-1}\partial\Phi(z)\}$$

From (3.6) and (3.8), one has $\partial\Phi|_M = 0$, and therefore the function Φ is globally defined in a neighborhood of M in X and one has

$$\Phi|_M = 0, \quad d\Phi|_M = 0 \quad (3.9) \quad \boxed{3.9}$$

lem3.2 **Lemma 3.2** *The following identity holds true*

$$\Phi(Z(i, x, \xi)) = |\xi|_x^2/2 \quad (3.10) \quad \boxed{3.10}$$

Proof. For $s \in [0, 1]$, set $(\gamma(s), \eta(s)) = (Z(is, x, \xi), \Xi(is, x, \xi))$ and $\zeta(s) = 2i\partial\Phi(\gamma(s))$. One has, for $s > 0$, $(\gamma(s), \eta(s)) \in \Lambda_s = s^{-1}\Lambda_1$, and therefore $\eta(s) = s^{-1}2i\partial\Phi(\gamma(s)) = s^{-1}\zeta(s)$. Let

$$g(s) = \Phi(Z(is, x, \xi)) = \Phi(\gamma(s))$$

Then we get

$$\begin{aligned} g'(s) &= d\Phi(\gamma(s))(\gamma'(s)) = Re\left(2\partial\Phi(\gamma(s))(i\partial_t Z(is, x, \xi))\right) \\ &= Re\left(2i\partial\Phi(\gamma(s))(\partial_t Z(is, x, \xi))\right) = Re(\zeta(s)\partial_{\zeta}p(\gamma(s), \eta(s))) \\ &= sRe(2p(\gamma(s), \eta(s))) = sRe(2p(\gamma(0), \eta(0))) = s|\xi|_x^2 \end{aligned} \quad (3.11) \quad \boxed{3.11}$$

Here we have used that $\partial\Phi$ is \mathbb{C} -linear, the Hamilton-Jacobi equations (3.4), $\zeta(s) = s\eta(s)$, and the fact that $p(z, \zeta)$ is homogeneous of degree 2 in ζ and invariant by the flow of the hamiltonian vector field H_p . Since $g(0) = 0$, we thus get $g(s) = s^2|\xi|_x^2/2$. The proof of lemma 3.2 is complete.

□

As a byproduct of lemma [3.2](#), ^{lem3.2} the function Φ is strictly pluri-subharmonic, i.e the hermitian form q_Φ defined in [\(3.3\)](#) is strictly positive. Moreover the map

$$(x, \xi) \mapsto Z(i, x, \xi) \quad (3.12) \quad \boxed{3.49}$$

gives a real analytic identification between the neighborhood $\{|\xi|_x < \epsilon_0\}$ of the zero section in the symplectic manifold T^*M , and the neighborhood $B_{\epsilon_0} = \{\Phi(z) < \epsilon_0^2/2\}$ of M in the complex manifold X . With this identification, the symplectic structure on B_{ϵ_0} is defined by the *real and closed* 2-form $2i\bar{\partial}\partial\Phi$, and the associated hermitian metric q_Φ defines a Kahlerian structure on B_{ϵ_0} . Since Φ is an exhaustion strictly pluri-subharmonic function on B_{ϵ_0} , B_{ϵ_0} is a Stein manifold.

Moreover, this identification induces a complex structure J on $\{|\xi|_x < \epsilon_0\}$. We refer to the article of Lempert and Szöke ([\[5\]](#)) for more details on this complex structure J on T^*M , which is canonically defined by the metric g on M . In particular, it is shown in [\[5\]](#), theorem 4.3, that if this complex structure can be extended to $\{|\xi|_x < R\}$, then the sectional curvatures of g are bounded from below by $-\pi^2/(4R^2)$.

We denote by β_z (resp ζ_z) the *real* (resp *complex*) 1-form on the real (resp complex) analytic manifold B_{ϵ_0} defined by

$$\beta_z = \text{Re}(\zeta_z), \quad \zeta_z = \Xi(i, x, \xi), \quad z = Z(i, x, \xi), \quad (x, \xi) \in T^*M \quad (3.13) \quad \boxed{3.50}$$

By construction, one has

$$\zeta_z = 2i\bar{\partial}\Phi(z) \quad (3.14) \quad \boxed{3.50\text{bis}}$$

Let $q(x, \xi) = |\xi|_x$. Then the hamiltonian $\exp(tH_q)(z, \zeta) = (\tilde{Z}(t, z, \zeta), \tilde{\Xi}(t, s, \zeta))$ is well defined for $t \in \mathbb{C}$ close to 0 and $(z, \zeta) \in T^*X$ in a conic neighborhood of $T^*M \setminus M$. Since $p = q^2/2$, one has by homogeneity, with the notation $|\zeta|_z = (g^{-1}(z)(\zeta))^{1/2}$, which is preserved by the flow of H_q ,

$$\tilde{Z}(t, z, \zeta) = Z(t, z, \zeta/|\zeta|_z), \quad \tilde{\Xi}(t, s, \zeta) = |\zeta|_z \Xi(t, z, \zeta/|\zeta|_z) \quad (3.15) \quad \boxed{3.51}$$

For $s \in]0, \epsilon_0[$ let $\kappa(is) = \exp(isH_q)$. Then $\kappa(is)$ is an homogeneous canonical complex transformation of T^*X , defined in a conic neighborhood U of $T^*M \setminus M$. From [\(3.5\)](#), one has

$$\kappa(is)(z, \zeta) = (Z(i, z, s\zeta/|\zeta|_z), |\zeta|_z \Xi(i, z, s\zeta/|\zeta|_z)) \quad (3.16) \quad \boxed{3.51\text{bis}}$$

Since $\kappa(is)$ preserves the canonical 1-form ζdz on T^*X , one has

$$|\zeta|_z \Xi(i, z, s\zeta/|\zeta|_z) d_{z, \zeta}(Z(i, z, s\zeta/|\zeta|_z)) = \zeta dz \quad (3.17) \quad \boxed{3.51\text{ter}}$$

For $y \in M$, let $\Lambda_{s,y} = \kappa(is)(T_y^*M \setminus 0)$, and let $\Lambda_{s,y}^{\mathbb{C}} = \kappa(is)(U \cap T_y^*X \setminus 0)$ be its complexification. Then $\Lambda_{s,y}^{\mathbb{C}} \subset T^*X$ is a \mathbb{C} -lagrangian homogeneous submanifold of T^*X . One has by [\(3.5\)](#), [\(3.13\)](#), and [\(3.15\)](#):

$$\Lambda_{s,y} = \{(z = Z(i, y, \eta), \zeta = t\zeta_z), (y, \eta) \in T_y^*M, |\eta|_y = s, t > 0\} \quad (3.18) \quad \boxed{3.52}$$

Since $d^2(Z(t, y, \eta), y) = t^2|\eta|_y^2$, and these functions are analytic in t , we get

$$d^2(Z(i, y, \eta), y) = -|\eta|_y^2 = -2\Phi(Z(i, y, \eta)), \quad \forall \eta \in T_y^*M \quad (3.19) \quad \boxed{3.53}$$

and therefore the function $s^2 + d^2(z, y)$ vanishes on $\pi(\Lambda_{s,y})$, where π is the projection $T^*X \rightarrow X$. Since $\pi(\Lambda_{s,y}^{\mathbb{C}})$ is a complexification of $\pi(\Lambda_{s,y})$ (a real analytic manifold of real dimension $m - 1$), we get that $\Lambda_{s,y}^{\mathbb{C}}$ is the conormal bundle to the complex hypersurface $s^2 + d^2(z, y) = 0$ near the points $z = Z(i, y, \eta)$, $|\eta|_y = s$:

$$\Lambda_{s,y}^{\mathbb{C}} = T_{\Sigma_{s,y}}^* X \setminus 0, \quad \Sigma_{s,y} = \{z, s^2 + d^2(z, y) = 0\} \quad (3.20) \quad \boxed{3.54}$$

The following lemma (and [\(3.10\)](#)) gives in particular a proof for the definition [\(1.3\)](#) of the function Φ given in the introduction.

lem3.3

Lemma 3.3 *There exists $c > 0$ and a neighborhood U of $\text{Diag}(M)$ such that for all $s \in]0, \epsilon_0]$, all $(x, y) \in U$ and all $z = Z(i, x, \xi) \in \partial B_s$ (i.e. $|\xi|_x = s$), one has*

$$\partial_z d^2(z, y)|_{z=Z(i,y,\xi)} = 2i\zeta_z \quad (3.21) \quad \boxed{3.13\text{bis}}$$

and

$$\text{Re}(d^2(z, y) + s^2) \geq cd^2(x, y) \quad (3.22) \quad \boxed{3.13}$$

Proof. From [\(3.53\)](#) one has $d^2(Z(i, y, \eta), y) = -|\eta|_y^2$ and from [\(3.18\)](#) and [\(3.20\)](#), one has $\partial_z d^2(z, y)|_{z=Z(i,y,\eta)} = \lambda\zeta_z$ for some $\lambda \in \mathbb{C} \setminus 0$. Let $z(t) = Z(i, y, e^t\eta) = Z(ie^t, y, \eta)$. One has $z(0) = Z(i, y, e^t\eta) = z$ and $d^2(z(t), y) = -e^{2t}|\eta|_y^2$. By evaluation of the derivative at $t = 0$, we find:

$$-2|\eta|_y^2 = d_t(d^2(z(t), y))|_{t=0} = \lambda\zeta_z(d_t z(t)|_{t=0}) = i\lambda\zeta_z \frac{\partial p}{\partial \zeta}(z, \zeta_z) = 2i\lambda p(z, \zeta_z) = i\lambda|\eta|_y^2$$

This implies $\lambda = 2i$. Let us now verify [\(3.13\)](#). In geodesic coordinates $\exp_x(a)$ centered at x , set $d^2(a, b) = (a - b)^2 + R_x(a, b)$. The function $R_x(a, b)$ is symmetric in a, b . From $d^2(0, b) = b^2$, we get $R_x(0, b) = 0$, thus $R_x(a, 0) = 0$, and $R_x(a, b) = \sum_{j,l} a_j b_l Q_x^{j,l}(a, b)$. From $(\nabla_a d^2)(0, b) = -2b$, one gets $\sum_l b_l Q_x^{j,l}(0, b) = 0$, hence $d^2(a, b) = (a - b)^2 + O(a^2 b)$, and since $R_x(a, b)$ is symmetric

$$d^2(a, b) = (a - b)^2 + O(a^2 b^2) \quad (3.23) \quad \boxed{3.14}$$

Set $y = \exp_x(a)$ and $z = Z(i, x, \xi)$. In geodesic coordinates centered at x , one has $g(x) = \text{Id}$, $Z(t, x, \xi) = t\xi$, thus $z = i\xi$, and from $|\xi|_x = s$ and $d^2(x, y) = a^2$, we get

$$d^2(z, y) = d^2(x, y) - s^2 - 2ia\xi + O(s^2 d^2(x, y)) \quad (3.24) \quad \boxed{3.14}$$

Since s is small, [\(3.13\)](#) holds true. The proof of lemma [1.3](#) is complete. \square

4 A proof of the Boutet de Monvel theorem

sec4

Recall that for $\epsilon \in]0, \epsilon_0]$, B_ϵ is the tubular neighborhood of M in X

$$B_\epsilon = \{z, \Phi(z) < \epsilon^2/2\} = \{Z(i, x, \xi), (x, \xi) \in T^*M, |\xi|_x < \epsilon\} \quad (4.1) \quad \boxed{4.0\text{bis}}$$

As in [\(1.5\)](#), the Hardy space $H(B_\epsilon)$ is defined as the Hilbert space:

$$H(B_\epsilon) = \{f \in \mathcal{O}(B_\epsilon), f|_{\partial B_\epsilon} \in L^2(\partial B_\epsilon)\}, \quad \|f\|_{H(B_\epsilon)} = \|f|_{\partial B_\epsilon}\|_{L^2(\partial B_\epsilon)} \quad (4.2) \quad \boxed{4.0}$$

Recall that for $f \in \mathcal{O}(B_\epsilon)$, f satisfies the elliptic system of Cauchy Riemann equations $\bar{\partial}f = 0$. Hence the trace $f|_{\partial B_\epsilon}$ is well defined as an hyperfunction on ∂B_ϵ , and if this trace is analytic, then f is analytic up to the boundary. In particular, if the trace is equal to 0, the extension \tilde{f} of f by 0 outside B_ϵ still satisfy $\bar{\partial}\tilde{f} = 0$; therefore \tilde{f} is holomorphic, and since \tilde{f} vanishes outside B_ϵ , one gets $\tilde{f} = 0$. This shows that $\|f|_{\partial B_\epsilon}\|_{L^2(\partial B_\epsilon)}$ is a Hilbert norm, and $H(B_\epsilon)$ an Hilbert space.

The Poisson kernel $\mathbb{P}_s(x, y)$ on (M, g) is the smooth function on $]0, \infty[\times M \times M$ given by the formula

$$\mathbb{P}_s(x, y) = \sum_j e^{-s\omega_j} e_j(x) e_j(y) \quad (4.3) \quad \boxed{4.1}$$

For any $v \in L^2(M)$, the smooth function on $]0, \infty[\times M$ defined by $u(s, x) = \int_M \mathbb{P}_s(x, y) v(y) d_g y$ satisfies the elliptic boundary problem

$$(\partial_s^2 + \Delta_g)u = 0, \quad \lim_{s \rightarrow 0} u(s, x) = v(x) \text{ in } L^2(M) \quad (4.4) \quad \boxed{4.2}$$

We start with purely geometric lemmas about the holomorphic extension of the e_j , and more generally of solutions to the elliptic operator $\partial_s^2 + \Delta_g$.

lem4.1 **Lemma 4.1** *Let $u(s, x)$ be a solution of the elliptic equation $(\partial_s^2 + \Delta_g)u = 0$ on $]0, \infty[\times M$. Then u extends holomorphically in the open set*

$$\mathbb{D} = \{(s, z) \in \mathbb{C} \times X, \operatorname{Re}(s) > 0, z \in B_{\min(\epsilon_0, \operatorname{Re}(s))}\} \quad (4.5) \quad \boxed{4.3}$$

Proof. By translation invariance in s , it is sufficient to prove the following property: Let $a \in]0, \epsilon_0[$, and $u(s, x)$ a solution of the equation $(\partial_s^2 + \Delta_g)u = 0$ on $] - a, a[\times M$. Then u extends holomorphically in the open set

$$\mathbb{G}_a = \{(s, z) \in \mathbb{C} \times X, \operatorname{Re}(s) \in] - a, a[, z \in B_{a - |\operatorname{Re}(s)|}\} \quad (4.6) \quad \boxed{4.31}$$

The proof of this fact uses a classical non-characteristic deformation argument based on the following Zerner lemma (see [12]). This lemma is a consequence of the precise form of the Cauchy-Kowalewski theorem given by J. Leray (see [2], theorem 9.4.7 for a proof).

lemzerner **Lemma 4.2 (Zerner)** *Let $Q(z, \partial_z) = \sum_{\alpha, |\alpha| \leq m} q_\alpha(z) \partial_z^\alpha$ be a linear differential operator with holomorphic coefficients defined near 0 in \mathbb{C}^N and let $q(z, \zeta) = \sum_{|\alpha|=m} q_\alpha(z) \zeta^\alpha$ be its principal symbol. Let $f : \mathbb{C}^N \rightarrow \mathbb{R}$ be a C^1 function such that $f(0) = 0$ and such that, with $\zeta_0 = 2i\partial f(0)$, one has $q(0, \zeta_0) \neq 0$. Then, if $u(z)$ is an holomorphic function defined in a half-neighborhood of 0 in $f < 0$, such that $Q(u)$ extends holomorphically near 0, then u extends holomorphically near 0.*

For $\mu \in [0, a]$ let $\psi_\mu(t), t \in \mathbb{R}$, be the non negative Lipschitz function

$$\psi_\mu(t) = \max(a - (\mu^2 + t^2)^{1/2}, 0) \quad (4.7) \quad \boxed{4.32}$$

Let $\tau > 0$ be given. For $\mu \in [0, a]$, let $K_{\mu, \tau}$ be the set

$$K_{\mu, \tau} = \{(s, z) \in \mathbb{C} \times B_{\epsilon_0}, \Phi(z) + \tau \operatorname{Im}(s)^2 \leq \psi_\mu(\operatorname{Re}(s))^2 / 2, |\operatorname{Re}(s)| \leq (a^2 - \mu^2)^{1/2}\} \quad (4.8) \quad \boxed{4.34}$$

From $0 \leq \psi_\mu \leq a < \epsilon_0$, we get that $K_{\mu,\tau}$ is a compact set, and its interior, $Int(K_{\mu,\tau})$, is defined by the equation

$$Int(K_{\mu,\tau}) = \{(s, z), \Phi(z) + \tau Im(s)^2 < \psi_\mu(Re(s))^2/2, |Re(s)| < (a^2 - \mu^2)^{1/2}\} \quad (4.9) \quad \boxed{4.34bis}$$

One has $K_{\mu,\tau} \subset K_{\mu',\tau}$ for $\mu' \leq \mu$ and the closure of $Int(K_{\mu,\tau})$ is equal to $K_{\mu,\tau}$ for $\mu < a$. Since one has

$$\mathbb{G}_a \subset \cup_{\tau>0} Int(K_{0,\tau})$$

we have just to prove that u extends holomorphically to $Int(K_{0,\tau})$. Set

$$J = \{\mu, u \text{ extends holomorphically to } Int(K_{\mu,\tau})\}$$

Since $K_{a,\tau} = \{s = 0\} \times M$, J contains a neighborhood of a , and it remains to show that for $\mu > 0$ in J , u extends holomorphically to a neighborhood of $K_{\mu,\tau}$. Let $\mu > 0$ in J .

Let $(s_0, z_0) \in \partial K_{\mu,\tau} = K_{\mu,\tau} \setminus Int(K_{\mu,\tau})$. Set $s_0 = \alpha + i\beta$. If $\psi_\mu(\alpha) = 0$, then one has $z_0 \in M, \beta = 0$, and therefore u is holomorphic near (s_0, z_0) since u is analytic on $] - a, a[\times M$. We may thus assume $\psi_\mu(\alpha) \neq 0$, i.e $|\alpha| < (a^2 - \mu^2)^{1/2}$. The function

$$f(s, z) = \Phi(z) + \tau Im(s)^2 - \psi_\mu(Re(s))^2/2$$

is smooth for $|Re(s)| < (a^2 - \mu^2)^{1/2}$, one has $f(s_0, z_0) = 0$, and the differential of f is given by (with the identification of section [3](#))

$$2i\partial f = (\zeta_s, \zeta_z) = i(-\psi_\mu \psi'_\mu(Re(s)) - 2i\tau Im(s), 2\partial_z \Phi(z))$$

The differential of f at (s_0, z_0) , $(\zeta_s(s_0), \zeta_z(z_0))$ does not vanish. (Otherwise, we will have $z \in M$ and $Im(s_0) = 0$ and this contradict $f(s_0, z_0) = 0$ and $\psi_\mu(\alpha) \neq 0$) Moreover, u satisfies the equation $Qu = (\partial_s^2 + \Delta_g)u = 0$ in a half-neighborhood of (s_0, z_0) in $f < 0$. The principal symbol of Q is $q(s, z; \zeta_s, \zeta_z) = \zeta_s^2 + 2p(z, \zeta_z)$. Therefore, by the Zerner lemma, it remains to show

$$(\psi_\mu \psi'_\mu(\alpha) + 2i\tau\beta)^2 \neq 2p(z_0, \zeta_{z_0}) \quad (4.10) \quad \boxed{4.35}$$

Let $(x_0, \xi_0) \in T^*M$ such that $Z(i, x_0, \xi_0) = z_0$. Then one has $(z_0, \zeta_{z_0}) = exp(iH_p)(x_0, \xi_0)$, and since the function p is invariant by the hamiltonian flow H_p , one has by [\(3.10\)](#)

$$2p(z_0, \zeta_{z_0}) = |\xi_0|_{x_0}^2 = 2\Phi(z_0) = \psi_\mu(\alpha)^2 - 2\tau\beta^2 \in \mathbb{R}$$

We first verify that [\(4.10\)](#) holds true for $\beta \neq 0$. For $\beta \neq 0$, equality in [\(4.10\)](#) implies (take imaginary part) $\psi'_\mu(\alpha) = 0$, and equality of the real part gives $-4\tau^2\beta^2 = 2\Phi(z_0) \geq 0$ which is impossible. It remains to verify $\psi'_\mu(\alpha) \neq \pm 1$ for $\mu > 0$ and $|\alpha| < (a^2 - \mu^2)^{1/2}$, which is obvious since one has

$$\psi'_\mu(\alpha) = \frac{-\alpha}{\sqrt{\mu^2 + \alpha^2}}$$

The proof of lemma [3.1](#) is complete. \square

If one apply the above lemma to the function $u(s, x) = e^{-s\omega_j} e_j(x)$, we get that all the eigenfunctions $e_j(x)$ extends holomorphically to the neighborhood B_{ϵ_0} of M in X , which is independent of j . In fact, we can deduce easily from lemma [3.1](#) a more precise statement.

lem4.2 **Lemma 4.3** Let $a \in]0, \epsilon_0[$. For all $\delta > 0$ small, there exists C_δ such that

$$\forall j, \quad \sup_{z \in B_a} |e_j(z)| \leq C_\delta e^{(a+\delta)\omega_j} \quad (4.11) \quad \boxed{4.36}$$

Proof. Set $E = L^2(M, d_g x)$ and $F = \{f \in \mathcal{O}(B_a), \sup_{z \in B_a} |f(z)| < \infty\}$. These are Banach spaces, and the canonical injection $i : F \rightarrow E$, $i(f) = f|_M$ is continuous. Let $\delta > 0$ such that $a + \delta < \epsilon_0$ and let A_δ be the linear continuous map from E to E defined by

$$A_\delta \left(\sum_j c_j e_j(x) \right) = \sum_j e^{-(a+\delta)\omega_j} c_j e_j(x)$$

By lemma [3.1](#), one has $\text{Im}(A_\delta) \subset \mathcal{O}(B_{a+\delta}) \subset F$. By the closed graph theorem, the map A_δ from E to F is continuous, and therefore, there exists a constant C_δ such that

$$\|A_\delta(f)\|_F \leq C_\delta \|f\|_E, \quad \forall f \in E \quad (4.12) \quad \boxed{4.37}$$

If one applies [\(4.12\)](#) to $f = e_j$, we get that [\(4.11\)](#) holds true. The proof of lemma [4.3](#) is complete. \square

Remark 4.4 The estimate [\(4.11\)](#) on the sup-norm of the eigenfunctions in B_a is of course very weak. The exponential factor $e^{a\omega_j}$ is the correct one, but the sub-exponential factor $C_\delta e^{\delta\omega_j}$ (for any $\delta > 0$) is far to be optimal. To my knowledge, the best estimate is proven by S.Zelditch in [\[Zel\]](#), corollary 3: $\sup_{z \in B_a} |e_j(z)| \leq C \omega_j^{(m+1)/4} e^{a\omega_j}$.

Another interesting by-product of Zerner-lemma is the following characterisation of the space $\mathcal{O}(B_a)$ of holomorphic functions on B_a . This gives the "analytic" version of the Boutet theorem (i.e without any precise information on Sobolev spaces and polynomial growth of the Fourier coefficients). It implies in particular that the Poisson operator $\mathbb{P}_a(\sum c_j e_j(x)) = \sum c_j e^{-a\omega_j} e_j(z)$ is an isomorphism from the space $\mathcal{A}'(M)$ of Sato-hyperfunctions on M , onto the space $\mathcal{O}(B_s)$ of holomorphic functions in B_s .

lem4.3 **Lemma 4.5** Let $a \in]0, \epsilon_0[$ and let $f(x) = \sum c_j e_j(x)$ an analytic function on M . Then f extends holomorphically to B_a iff

$$\forall \delta > 0, \exists C_\delta, \text{ such that for all } j \text{ one has } |c_j| \leq C_\delta e^{-(a-\delta)\omega_j} \quad (4.13) \quad \boxed{4.38}$$

Moreover, for any function $f(z) \in \mathcal{O}(B_a)$, the Fourier coefficients $c_j = \int_M f(x) e_j(x) d_g x$ satisfy [\(4.13\)](#), and one has $f(z) = \sum_j c_j e_j(z)$ for all $z \in B_a$, where the sum is uniformly convergent on compact subsets of B_a .

Proof. If [\(4.13\)](#) is satisfied, then by lemma [4.3](#), formula [\(4.11\)](#), the sum $\sum c_j e_j(z)$ is uniformly convergent on $B_{a'}$ for all $a' < a$, (since by Weyl formula, $\#\{j, \omega_j \leq R\} \leq CR^m$) hence f extends holomorphically to B_a . It remains to show that for a function $f(z) \in \mathcal{O}(B_a)$, its Fourier coefficients $c_j = \int_M f(x) e_j(x) d_g x$ satisfy [\(4.13\)](#): with $g(z) = \sum c_j e_j(z)$, we will have $g \in \mathcal{O}(B_a)$ by the first part of the lemma, and since $(f - g)|_M = 0$, we will get $f = g$ by analytic continuation. The proof of the estimate [\(4.13\)](#) on the Fourier

coefficients c_j uses the Zerner Lemma. Let $F(s, z)$ be the Cauchy-Kowalewski solution of the analytic Cauchy problem:

$$(\partial_s^2 + \Delta_z)F = 0, \quad F(a, z) = f(z), \quad \partial_s F(a, z) = 0 \quad (4.14) \quad \boxed{4.39}$$

Zerner lemma implies that F extends holomorphically to the open set

$$\mathbb{F}_a = \{(s, z) \in \mathbb{C} \times X, |Re(s) - a| < a, z \in B_{a-|Re(s)-a|}\} \quad (4.15) \quad \boxed{4.31bis}$$

The proof of this point follows the same line as the proof of lemma ^{lem4.1}4.1. We first change s in $s + a$ so that the Cauchy data for ^{4.39}(4.14) are now on the set $\{s = 0\} \times B_a$, and we have to prove that F extends to the open set \mathbb{G}_a defined in ^{4.31}(4.6). We use the non-characteristic deformation associated to the function, with $\tau > 0$,

$$\tilde{f}_\tau(s, z) = \frac{1}{2}Re(s)^2 + \tau Im(s)^2 - \frac{1}{2} \left(\max(a - \sqrt{\mu^2 + 2\Phi(z)}, 0) \right)^2 \quad (4.16) \quad \boxed{4.40}$$

Observe that in comparison with the proof of lemma ^{lem4.1}4.1, we just exchange the role of $2Re(s)^2$ and $\Phi(z)$. For $\mu \in [0, a]$, we define $\tilde{K}_{\mu, \tau}$ by

$$\tilde{K}_{\mu, \tau} = \{(s, z) \in \mathbb{C} \times X, \tilde{f}_\tau(s, z) \leq 0, 2\Phi(z) \leq a^2 - \mu^2\} \quad (4.17) \quad \boxed{4.41}$$

The function F is holomorphic in a neighborhood of $\tilde{K}_{\mu, \tau} = \{s = 0\} \times M$, and as in the proof of lemma ^{lem4.1}4.1, we just have to verify that for $\mu \in]0, a[$, if F extends to $Int(\tilde{K}_{\mu, \tau})$, then F extends to a neighborhood of $\tilde{K}_{\mu, \tau}$. Let $(s_0, z_0) \in \partial K_{\mu, \tau} = K_{\mu, \tau} \setminus Int(K_{\mu, \tau})$. Set $s_0 = \alpha + i\beta$. If $2\Phi(z_0) = a^2 - \mu^2 < a$, then one has $z_0 \in B_a$ and $s = 0$, and therefore F is holomorphic near (s_0, z_0) by Cauchy-Kowalewski theorem. We may thus assume $2\Phi(z_0) < a^2 - \mu^2$. Then the function \tilde{f}_τ is smooth near (s_0, z_0) and its differential at $(s_0 = \alpha + i\beta, z_0)$ is equal to

$$2i\partial\tilde{f}_\tau = (\zeta_{s_0}, \zeta_{z_0}) = 2i(\alpha/2 - i\tau\beta, \frac{a - \sqrt{\mu^2 + 2\Phi(z_0)}}{\sqrt{\mu^2 + 2\Phi(z_0)}}\partial\Phi(z_0)) \quad (4.18) \quad \boxed{4.42}$$

By the Zerner lemma, it remains to show $\zeta_s^2 + 2p(z_0, \zeta_{z_0}) \neq 0$. Since $2p(z_0, \zeta_{z_0}) = 2\Phi(z_0)$, this is equivalent to

$$(\alpha - 2i\tau\beta)^2 \neq 2\Phi(z_0) \frac{(\sqrt{\mu^2 + 2\Phi(z_0)} - a)^2}{\mu^2 + 2\Phi(z_0)} \in [0, \infty[\quad (4.19) \quad \boxed{4.43}$$

We first verify that ^{4.43}(4.19) holds true for $\beta \neq 0$. For $\beta \neq 0$, equality in ^{4.43}(4.19) implies (take imaginary part) $\alpha\beta = 0$, hence $\alpha = 0$ and $-4\tau^2\beta^2 \geq 0$ which is impossible. For $\beta = 0$, from $\tilde{f}_\tau(s_0, z_0) = 0$ and $2\Phi(z_0) < a^2 - \mu^2$, we get $|\alpha| = a - \sqrt{\mu^2 + 2\Phi(z_0)} > 0$. It remains to verify

$$\alpha^2 \neq \frac{2\Phi(z_0)}{\mu^2 + 2\Phi(z_0)}\alpha^2$$

for $\mu \in]0, a[$ and $\alpha \neq 0$ which is obvious. Thus F extends holomorphically to $Int(\tilde{K}_{0, \tau})$ for all $\tau > 0$, and since one has $\cup_{\tau > 0} Int(\tilde{K}_{0, \tau}) = \mathbb{G}_a$, we get the desired result.

For $s \in]0, 2a[$, set now $F_j(s) = \int_M F(s, x)e_j(x)dx$. Then $F_j(s)$ is analytic on $]0, 2a[$ and satisfies the equation

$$\partial_s^2 F_j - \omega_j^2 F_j = 0, \quad F_j(a) = c_j, \quad \partial_s F_j(a) = 0$$

This gives $F_j(s) = a_j \text{ch}((a-s)\omega_j)$. Since for all $s \in]0, a]$, the function $x \mapsto F(s, x)$ is analytic on M , its Fourier coefficients are bounded, i.e

$$\forall s \in]0, a], \exists C_s \text{ such that } \sup_j |c_j \text{ch}((a-s)\omega_j)| \leq C_s$$

By taking $s = \delta$ small, this implies [\(4.13\)](#) ^{4.38}. The proof of lemma [4.5](#) ^{lem4.3} is complete. \square

We will now recall the classical construction of the Hadamard type parametrix for the Poisson kernel near $s = 0$ and $x = y$. Let $\delta(s, x, y)$ be defined by the formula

$$\delta(s, x, y) = s^2 + d^2(x, y) \tag{4.20} \quad \boxed{4.4}$$

The function δ is holomorphic in a small neighborhood W of $\{s = 0\} \times \text{Diag}_M$ in $\mathbb{C} \times X \times X$. Let $c_W = \sup_W |\delta|$. Clearly, we may assume c_W as small as we want by choosing W small enough. Set $\mu = -(m+1)/2$.

prop4.1

Proposition 4.6 *For W small enough, the following holds true.*

For all $j \in \mathbb{N}$, there exists holomorphic functions $a_j(s, x, y)$ defined on W , such that

$$\sum_j \sup_W |a_j| c_W^j < \infty \tag{4.21} \quad \boxed{4.5}$$

and such that if one defines $G(s, x, y)$ by the formula

$$\begin{aligned} G &= s\delta^\mu \sum_{j \geq 0} \delta^j a_j \quad \text{if } m \text{ is even} \\ G &= s\delta^\mu \sum_{j=0}^{|\mu|-1} \delta^j a_j + s \log(\delta) \sum_{j \geq |\mu|} \delta^j a_j \quad \text{if } m \text{ is odd} \end{aligned} \tag{4.22} \quad \boxed{4.6}$$

then the function $\mathbb{P}_s(x, y) - G(s, x, y)$ which is defined a priori for $s > 0$ small and $(x, y) \in M \times M$ close to Diag_M , extends holomorphically to W . Moreover, the functions a_j are even in s and one has

$$a_0(0, y, y) = d_m^{-1}, \quad d_m = \int_{\mathbb{R}^m} (1+x^2)^{-(m+1)/2} dx \tag{4.23} \quad \boxed{4.7}$$

Proof. Let us denote by ∇f the gradient of a function f , i.e the vector fields on M which is associated to the differential df via the identification of TM and T^*M . An easy computation shows that the following formula holds true:

$$\begin{aligned} (\partial_s^2 + \Delta)(f^l b) &= l(l-1)f^{l-2}((\partial_s f)^2 + |\nabla f|_g^2)b \\ &+ lf^{l-1} \left(2\partial_s f \partial_s b + 2(\nabla f | \nabla b)_g + (\partial_s^2 f + \Delta f)b \right) + f^l(\partial_s^2 b + \Delta b) \end{aligned} \tag{4.24} \quad \boxed{4.8}$$

For a given y , the function $f(s, x) = \delta(s, x, y)$ satisfies the identity $(\partial_s \delta)^2 + |\nabla_x \delta|_g^2 = 4\delta$ (the analog of the eiconal equation). Thus we get from [\(4.24\)](#) ^{4.8}

$$\begin{aligned} (\partial_s^2 + \Delta)(\delta^l b) &= l\delta^{l-1} \left(4s\partial_s b + 2(\nabla_x d^2 | \nabla b)_g + (\Delta_x(d^2) + 4l - 2)b \right) \\ &+ \delta^l(\partial_s^2 b + \Delta b) \end{aligned} \tag{4.25} \quad \boxed{4.9}$$

If we set $b = sa$, with a even in s , we thus get

$$\begin{aligned} (\partial_s^2 + \Delta)(s\delta^l a) &= s l \delta^{l-1} \left(4s \partial_s a + 2(\nabla_x d^2 | \nabla a)_g + (\Delta_x(d^2) + 4l + 2)a \right) \\ &\quad + s \delta^l (\partial_s^2 a + 2s^{-1} \partial_s a + \Delta a) \end{aligned} \quad (4.26) \quad \boxed{4.9\text{bis}}$$

Let us first assume that m is even. We will apply the identity $\boxed{4.26}$ with $l = \mu + j, j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, one has $l \neq 0$. Let us denote by Z_l the first order operator

$$Z_l(a) = 4s \partial_s a + 2(\nabla_x d^2 | \nabla a)_g + (\Delta_x(d^2) + 4l + 2)a \quad (4.27) \quad \boxed{4.10}$$

Then the function G defines by the first line of $\boxed{4.22}$ will be formally a solution of the equation $(\partial_s^2 + \Delta)G = 0$ if one choose the functions a_j solutions of the transport equations:

$$\begin{aligned} Z_\mu(a_0) &= 0 \\ Z_{\mu+j}(a_j) &= -\frac{1}{\mu+j} (\partial_s^2 + 2s^{-1} \partial_s a + \Delta_x) a_{j-1} \quad \forall j \geq 1 \end{aligned} \quad (4.28) \quad \boxed{4.11}$$

The key point here is that the equation $Z_\mu(a_0) = 0$ admits a unique even in s holomorphic solution in W for any given data $a(0, y, y)$, and the equation $Z_{\mu+j}(a) = b$ with $j \geq 1$ and $b(s, x, y)$ even in s and holomorphic in W , admits a unique solution $a(s, x, y)$, even in s and holomorphic in W . Therefore, the system of transport equations $\boxed{4.28}$ admits a unique solution such that formula $\boxed{4.23}$ holds true. We refer to the appendix for a proof of these affirmations, and also for a proof of the estimate $\boxed{4.21}$ for small enough W . From the estimate $\boxed{4.21}$, the function $\sum_{j \geq 0} \delta^j a_j$ is a holomorphic function on W , and therefore

$$G = s \delta^\mu \sum_{j \geq 0} \delta^j a_j$$

is an holomorphic function on the set $W \cap \{Re(\delta) > 0\}$. In this set, which clearly contains $W \cap \{s > 0, x, y \in M\}$, G satisfies by construction the equation $(\partial_s^2 + \Delta_x)G = 0$, and extends as a holomorphic function on the two sheets covering of the set $W \setminus \{\delta = 0\}$. Now we claim that with the choice $\boxed{4.23}$ of the initial data for the solution a_0 of the transport equation $Z_\mu(a_0) = 0$, one has

$$\lim_{s \rightarrow 0} G(s, x, y) = \delta_{x=y} \quad (4.29) \quad \boxed{4.45}$$

Here, we identify a measure on M with a distribution by factorization of the volume form $d_g x$. In other words, $\boxed{4.29}$ means

$$\lim_{s \rightarrow 0} \int_M G(s, x, y) \varphi(x) d_g x = \varphi(y) \quad (4.30) \quad \boxed{4.45\text{bis}}$$

for any smooth test function φ with support close to y . The verification of $\boxed{4.30}$ is easy: take near y , the geodesic coordinate system $v \mapsto \exp_y(v), v \in T_y M$. Then one has $d^2(x, y) = v^2$ and $d_g x = (1 + O(v^2)) dv$. For f smooth with support near 0 one has

$$\begin{aligned} \lim_{s \rightarrow 0} \int_{\mathbb{R}^m} s(s^2 + v^2)^{-(m+1)/2} a_0(s, \exp_y v, y) f(v) (1 + O(v^2)) dv \\ = a_0(0, y, y) f(0) \int_{\mathbb{R}^m} (1 + w^2)^{-(m+1)/2} dw = f(0) \end{aligned} \quad (4.31) \quad \boxed{4.46}$$

by the choice ^{4.7}(4.23) of $a_0(0, y, y)$ (use the change of variables $v = sw$ and Lebesgue dominated convergence theorem). The same argument shows that the other terms in the development of G in powers of δ do not contribute to the limit in ^{4.45}(4.29).

Therefore, $H(s, x, y) = \mathbb{P}_s(x, y) - G(s, x, y)$ satisfies the elliptic boundary value problem in variables (s, x) close to $(0, y)$

$$(\partial_s^2 + \Delta_x)H = 0 \text{ in } s > 0, \quad \lim_{s \rightarrow 0} H = 0 \quad (4.32) \quad \boxed{4.47}$$

Hence $H(s, x, y)$ is analytic in (s, x) near $(0, y)$. This is a classical result for this kind of elliptic boundary problem with analytic coefficients, but here, one can use a most elementary reflection argument: near $(0, y)$ in $\mathbb{R} \times M$, the function $u(s, x) = \text{sign}(s)H(|s|)(x, y)$ satisfies the elliptic equation $(\partial_s^2 + \Delta_x)u = 0$, hence is analytic. The proof of the fact that $H(s, x, y)$ is analytic in (s, x, y) near $\{s = 0\} \times \text{Diag}_M$ is of the same kind: One has the symmetry $\mathbb{P}_s(x, y) = \mathbb{P}_s(y, x)$ and from the uniqueness in the construction of the coefficients a_j (see the appendix), one has also $G(s, x, y) = G(s, y, x)$. Hence, $H(s, x, y)$ satisfies the elliptic boundary value problem in variables (s, x, y) close to $\{s = 0\} \times \text{Diag}_M$

$$(2\partial_s^2 + \Delta_x + \Delta_y)H = 0 \text{ in } s > 0, \quad \lim_{s \rightarrow 0} H = 0 \quad (4.33) \quad \boxed{4.47}$$

Therefore, we conclude that $H(s, x, y)$ is analytic near $\{s = 0\} \times \text{Diag}_M$.

In the case m odd, the proof follows the same lines. In addition to formulas ^{4.8}(4.24) and ^{4.9bis}(4.26), one also use the formulas with $n \in \mathbb{N}$

$$\begin{aligned} (\partial_s^2 + \Delta)(f^n \log(f)b) &= nf^{n-2}(2 + (n-1)\log(f))((\partial_s f)^2 + |\nabla f|_g^2)b \\ &+ f^{n-1}(1 + n\log(f))\left(2\partial_s f \partial_s b + 2(\nabla f | \nabla b)_g + (\partial_s^2 f + \Delta f)b\right) + f^n \log(f)(\partial_s^2 b + \Delta b) \end{aligned} \quad (4.34) \quad \boxed{4.8bis}$$

which gives since $(\partial_s \delta)^2 + |\nabla \delta|_g^2 = 4\delta$

$$\begin{aligned} (\partial_s^2 + \Delta)(\delta^n \log(\delta)b) &= n\delta^{n-1} \log(\delta) \left(4s\partial_s b + 2(\nabla_x d^2 | \nabla b)_g + (\Delta_x(d^2) + 4n - 2)b\right) \\ &+ \delta^n \log(\delta)(\partial_s^2 b + \Delta b) + \delta^{n-1} \left(4s\partial_s b + 2(\nabla_x d^2 | \nabla b)_g + (\Delta_x(d^2) + 8n + 2)b\right) \end{aligned} \quad (4.35) \quad \boxed{4.9ter}$$

If we set $b = sa$, with a even in s , we thus get

$$\begin{aligned} (\partial_s^2 + \Delta)(s\delta^n \log(\delta)a) &= sn\delta^{n-1} \log(\delta) \left(4s\partial_s a + 2(\nabla_x d^2 | \nabla a)_g + (\Delta_x(d^2) + 4n + 2)a\right) \\ &+ s\delta^n \log(\delta)(\partial_s^2 a + 2s^{-1}\partial_s a + \Delta a) \\ &+ s\delta^{n-1} \left(4s\partial_s a + 2(\nabla_x d^2 | \nabla a)_g + (\Delta_x(d^2) + 8n + 6)a\right) \end{aligned} \quad (4.36) \quad \boxed{4.9quad}$$

Then one find that the second line of ^{4.6}(4.22) holds true with an additional term of the form $sh(s, x, y)$ with h holomorphic near $s = 0, x = y$, and this term plays no role in the verification of the boundary condition at $s = 0$ nor in the fact that $\mathbb{P}_s(x, y) - G(s, x, y)$ is analytic near $\{s = 0\} \times \text{Diag}_M$ (see the appendix for the details).

The proof of proposition ^{prop 4.1}4.6 is complete. \square

lem4.4

Lemma 4.7 *There exists $\epsilon_0 > 0$, such that for all $s \in]0, \epsilon_0[$ the following holds true.*

i) *The function $\mathbb{P}_s(z, y)$ is holomorphic in (z, y) near any point $(z, y) \in B_s \times M$.*

ii) *The function $\mathbb{P}_s(z, y)$ extends holomorphically near any point $(z, y) \in \partial B_s \times M$ such that $z \notin \{Z(i, y, \eta), |\eta|_y = s\}$.*

Proof. Point i) follows directly from the identity (4.1) and the bound (4.36) of lemma 4.3. Point ii) is also easy to prove: the function $(s, x) \in]0, \infty[\times M \mapsto \mathbb{P}_s(x, y)$ satisfies the elliptic boundary value problem

$$(\partial_s^2 + \Delta_x)\mathbb{P}_s(x, y) = 0 \text{ in } s > 0, \quad \mathbb{P}_0(x, y) = \delta_{x=y}$$

Therefore, as in the proof of proposition 4.6, we get that $\mathbb{P}_s(x, y)$ is analytic in (s, x) near any point $(0, x)$ with $x \neq y$. By choosing $\epsilon_0 > 0$ small enough, we may thus assume that $z = Z(i, x, \xi)$, $|\xi|_x = s$ and x close to $y \in M$. Then by proposition 4.6, the singularities of $\mathbb{P}_s(z, y)$ near such points are on the subcomplex manifold $\{(z, y), s^2 + d^2(z, y) = 0\}$ and the result follows from the formula (3.22) of lemma 3.3. The proof of lemma 4.7 is complete. \square

Recall that we use the identification of $\{(x, \xi) \in T^*M, |\xi|_x = s\}$ with ∂B_s given by the map $(x, \xi) \mapsto Z(i, x, \xi)$, and that c_m is the volume of the unit sphere in \mathbb{R}^m , so c_m/m is the volume of the unit ball in \mathbb{R}^m . Let $dx d\xi$ be the canonical Liouville measure on T^*M . We define the measure $d\mu_s$ on ∂B_s by the formula

$$\int_{\partial B_s} f d\mu_s = \frac{m}{c_m} \int_{|\xi|_x \leq s} f(x, \frac{s\xi}{|\xi|_x}) dx d\xi = \int_M \left(\int_{S^{m-1}} f(x, s g_x^{1/2}(u)) \frac{d\sigma(u)}{c_m} \right) d_g x \quad (4.37) \quad 4.68$$

This is compatible with the definition of $d\mu_s$ that we have used in the flat case in section 2, and if $f(z)$ is a smooth function on X defined near M , one has

$$\lim_{s \rightarrow 0} \int_{\partial B_s} f d\mu_s = \int_M f(x) d_g x \quad (4.38) \quad 4.69$$

The real 1-form β_z introduced in (3.13) defines by restriction to ∂B_s a 1-form that we still denote by β_z . This defines a canonical half line bundle $L^- \subset T^*(\partial B_s)$

$$L^- = \{(z, \zeta) \in T^*(\partial B_s), \zeta = t\beta_z, t < 0\} \quad (4.39) \quad 4.62$$

For $s \in]0, \epsilon_0[$, we denote by T_s the map from $\mathcal{D}'(M)$ into $\mathcal{D}'(\partial B_s)$

$$\sum c_j e_j = f \mapsto T_s(f) = \mathbb{P}_s(f)|_{\partial B_s} = \sum_j e^{-s\omega_j} c_j e_j|_{\partial B_s} \quad (4.40) \quad 4.60$$

lem4.5

Lemma 4.8 *For all $s \in]0, \epsilon_0[$, T_s is well defined and injective map. The Hörmander wave front set of its distribution kernel $T_s(z, y)$ is given by*

$$WF(T_s) = \{(z, \zeta; y, \eta) \in T^*(\partial B_s) \times T^*(M) \setminus M, z = Z(i, y, s\eta/|\eta|), \zeta = -\beta_z|\eta|/s\} \quad (4.41) \quad 4.63$$

In particular, $WF(T_s)$ is parametrized by $(y, \eta) \in T^(M) \setminus M$.*

Moreover, for any $f \in \mathcal{D}'(M)$, one has

$$T_s(f) = \lim_{\mathcal{D}', r \rightarrow 0^+} \int_M \mathbb{P}_{s+r}(z, y) f(y) d_g y = \lim_{\mathcal{D}', r \rightarrow 0^+} \int_M T_s(e^{-r|\Delta_g|^{1/2}} f) \quad (4.42) \quad 4.64$$

Proof. One has $\mathbb{P}_s(f) = \int_M \mathbb{P}_s(z, y)f(y)d_g y \in \mathcal{O}(B_s)$, thus the injectivity of T_s is obvious. The fact that $T_s(f) \in \mathcal{D}'(\partial B_s)$ for any $f \in \mathcal{D}'(M)$ follows easily from proposition 4.6 and point ii) of lemma 4.7. By lemma 4.7, the singular support of the Kernel $T_s(z, y)$ is contained in $\{(z, y), \exists \eta \in T^*M, |\eta|_y = s, \text{ and } z = Z(i, y, \eta)\}$. Then to compute $WF(T_s)$, we may use proposition 4.6, and this reduce to the computation of $WF(s^2 + d^2(z, y))^\mu$, which is easy if one uses lemma 3.3 and gives formula (4.41). Finally, the assertion (4.42) is obvious. The proof of lemma 4.21 is complete. \square

In the following proposition, T_s^* is the adjoint of T_s for the measures $d_g x$ on M and $d\mu_s$ on ∂B_s .

prop4.2

Proposition 4.9 *Let $I = [c, d] \subset]0, \epsilon_0[$. Then $T_s^*T_s$ is a smooth family in $s \in I$ of elliptic pseudodifferential operators of degree $-(m-1)/2$. Moreover, there exists a constant $C(I) > 1$ such that one has the equivalence of norms*

$$\frac{1}{C(I)} \|T_s g\|_{L^2(\partial B_s, d\mu_s)} \leq \|g\|_{H^{-(m-1)/4}(M)} \leq C(I) \|T_s g\|_{L^2(\partial B_s, d\mu_s)} \quad (4.43) \quad 4.70$$

Proof. The proof of this lemma is suggested in [Boutet1]: essentially, we use the fact that T_s is a "Fourier Integral Operator with complex phase", which is a direct consequence of proposition 4.3 and lemma 4.7 and then we apply the general machinery. (this is the proof given in [Zel]). Here, to avoid invocation of a general machinery, and also to get the principal symbol, we shall directly verify that $T_s^*T_s$ is an elliptic pseudodifferential operator of degree $-(m-1)/2$, by computing its distribution kernel. This will just involve the knowledge of the stationary phase theorem in the case of complex phase, but with phase and symbol analytic in the parameters, which is not so difficult.

We start with the following lemma. For his proof, which is elementary, and basic definitions on analytic symbols, we refer to the appendix. For the computation of $\sigma_0(0, y, y)$ given in 4.46, we use the identity, with d_m defined in (4.23)

$$d_m \Gamma((m+1)/2) = \int_{\mathbb{R}^m} \int_0^\infty e^{-t(1+x^2)} t^{(m+1)/2} \frac{dt dx}{t} = \pi^{m/2} \int_0^\infty e^{-t} \frac{dt}{\sqrt{t}} = \pi^{(m+1)/2}$$

Here, $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$ is the usual Gamma function.

lem4.6

Lemma 4.10 *There exists a classical analytic symbol of degree 0, $\sigma(\lambda; s, x, y)$, with holomorphic dependance on $(s, x, y) \in W$ such that the function defined for $s > 0$ and (x, y) close to $Diag_M$*

$$G(s, x, y) - s \int_0^\infty e^{-\lambda(s^2 + d^2(x, y))} \lambda^{(m+1)/2} \sigma(\lambda; s, x, y) \frac{d\lambda}{\lambda} \quad (4.44) \quad 4.71$$

extends holomorphically in W . One has for some constants A, B

$$\sigma(\lambda; s, x, y) \simeq \sum_{j \geq 0} \lambda^{-j} \sigma_j(s, x, y), \quad \sup_W |\sigma_j| \leq AB^j j! \quad (4.45) \quad 4.72$$

and

$$\sigma_0(0, y, y) = \pi^{-(m+1)/2} \quad (4.46) \quad 4.73$$

Let us verify that $T_s^*T_s$ is an elliptic pseudodifferential operator of degree $-(m-1)/2$. For $f \in C^\infty(M)$, one has $T_s^*T_s(f)(x) = \int_M K_s(x, y)f(y)d_g y$ with the distribution kernel $K_s \in \mathcal{D}'(M \times M)$ defined by the formula

$$K_s(x, y) = \int_{\partial B_s} \overline{\mathbb{P}_s(z, x)} \mathbb{P}_s(z, y) d\mu_s(z) \quad (4.47) \quad \boxed{4.74}$$

One has to take care of the fact that the integral in (4.47) is not an "usual" integral, but the distribution product $\overline{\mathbb{P}_s(z, x)} \mathbb{P}_s(z, y) \in \mathcal{D}'(M \times M \times \partial B_s)$ is well defined. This is a consequence of lemma 4.5, formula (4.41), and of general results on wave front set of tensor product, non characteristic trace, and proper direct image (see [2]). Moreover, one gets from this general results and (4.41) the inclusion

$$WF(K_s) \subset \{(x, y, \xi, \eta), x = y, \xi + \eta = 0\} = T_{Diag(M)}^*M$$

Therefore, to compute the kernel $K_s(x, y)$ modulo a smooth function, we may assume that (x, y) is close to $Diag(M)$. We will choose the coordinate system $(p, w) \in TM$, w small and $x = \exp_p(w/2)$, $y = \exp_p(-w/2)$ so that p is the middle point of the geodesic connecting y to x , and in these geodesic coordinates centered at p , one has $w = x - y$. By lemma 4.20 we may localize the integral in 4.47 for $z = Z(i, u, \xi)$ with u close to p . Let $n_{j,p}$, $1 \leq j \leq m$ be an orthonormal basis of T_pM . In geodesic coordinates we write $u = \exp_p(\sum a_j n_{j,p})$, and we denote by $\xi = (\xi_1, \dots, \xi_m)$ the dual coordinates of the (a_j) . Recall that in geodesic coordinates, one has $g(a) = Id + O(a^2)$ and we define new coordinates b by the formula

$$b = b(a, \xi) = (g^{-1}(a))^{1/2}(\xi) = \xi + O(a^2\xi) \quad (4.48) \quad \boxed{4.75}$$

Then one has $b^2 = |\xi|_a^2$, and we shall parametrize the set of points $z = Z(i, u, \xi)$, u close to p and $|\xi|_a \leq \epsilon_0$ by the coordinates $(a, b) \in \mathbb{R}^{2m}$ close to $(0, 0)$. From lemma 4.6, proposition 4.1, and formulas 4.42 and 4.37, one find that near $Diag(M)$, the kernel $K_s(x, y)$ is equal to (modulo a smooth function) **préciser l'argument**

$$\begin{aligned} & \lim_{\mathcal{D}', r \rightarrow 0^+} \int_0^\infty \int_{S^{m-1}} E_{s+r}(s, x, y; \rho, u) s^2 \rho^m d\rho \frac{d\sigma(u)}{c_m}, \quad (\rho \in]0, \infty[, u \in S^{m-1}) \\ E_{s+r}(s, x, y; \rho, u) &= \int_0^{\pi/2} \int_{\mathbb{R}^m} e^{-\rho \Psi_{s+r}} \Sigma_{s+r}(\sin \theta \cos \theta)^{(m-1)/2} \chi(a) \sqrt{\det(g(a))} d\theta da \\ \Psi_{s+r}(s, x, y, u; a, \theta) &= \sin \theta ((s+r)^2 + \bar{d}^2(z, x)) + \cos \theta ((s+r)^2 + d^2(z, y)) \\ \Sigma_{s+r}(s, x, y, u, \rho; a, \theta) &= \sigma(\rho \cos \theta, s+r, z, y) \bar{\sigma}(\rho \sin \theta, s+r, z, x) \\ z &= Z(i, \exp_p(\sum a_j n_{j,p}), sg^{1/2}(a)(u)) \end{aligned} \quad (4.49) \quad \boxed{4.75}$$

Here, $\chi \in C_0^\infty(|a| \leq 2c_0)$ is a smooth cutoff function, equal to 1 in the ball $|a| \leq c_0$, with c_0 such that one has $|w| \ll c_0 \ll \inf(s \in I)$ (recall $x = \exp_p(w/2)$, $y = \exp_p(-w/2)$). By lemma 3.3, one has

$$Re(\Psi_{s+r}) \geq (\sin \theta + \cos \theta)((s+r)^2 - s^2) + c_I(\sin \theta d^2(a, x) + \cos \theta d^2(a, y))$$

and in particular, for $r > 0$, the integral in (4.49) is absolutely convergent. The key technical point is to verify that the analytic function

$$(a, \theta) \mapsto \Psi_{s+r}(s, x, y, u; a, \theta)$$

admits a unique non degenerate critical point $(a_c(r, s, x, y, u), \theta_c(r, s, x, y, u))$ close to $(0, \pi/4)$ for $s \in I$, r close to 0, x, y close to p and any $u \in S^{m-1}$, and that the Hessian of Ψ_{s+r} at the critical point is non degenerate. To this end, we have just to verify that it is true for $r = 0$, $x = y = p$, and since s is small, we may even assume that the metric is flat. But in that case, we get easily

$$\Psi_s(s, p, p, u; a, \theta) = a^2(\sin \theta + \cos \theta) - 2is(\sin \theta - \cos \theta)a.u \quad (4.50) \quad \boxed{4.77}$$

which admits a unique critical point $(a_c, \theta_c) = (0, \pi/4)$. From the Taylor expansion

$$\Psi_s(s, p, p, u; a, \pi/4 + \varphi) = \sqrt{2}(a^2 - 2is\varphi a.u) \quad (4.51) \quad \boxed{4.77bis}$$

we get that this critical point is non degenerate. Observe also that the Hessian of $Re(\Psi_s)$ is strictly positive in the a directions. Therefore, for $s \in I$, r close to 0, x, y close to p and any $u \in S^{m-1}$, one has a unique non degenerate critical point, and the Hessian of Ψ_{s+r} is strictly positive in the a directions. Let

$$\psi_s(r, x, y, u) = \Psi_{s+r}(s, x, y, u; a_c(r, s, x, y, u), \theta_c(r, s, x, y, u))$$

be the critical value, which depends analytically on all parameters. In the flat case, one verifies easily that one has $(a_c, \theta_c) = (0, \pi/4)$ independently of $(x, y) = (w/2, -w/2)$. By lemma [3.3](#) and Taylor expansion in $w = x - y$, one gets $(a_c, \theta_c) = (0, \pi/4) + O(w^2)$, and **vérifier détailler**

$$\psi_s(r, x, y, u) = \sqrt{2}\left((s+r)^2 - s^2 + is(x-y).u + Q(p, s, u; r, x-y)\right) \quad (4.52) \quad \boxed{4.78}$$

where $Q(p, s, u; r, w)$ is analytic in $(p, s, u; r, w)$ and satisfies

$$Q(p, s, u; r, 0) = 0, \quad \nabla_w Q(p, s, u; r, 0) = 0, \quad Re(\partial_w^2 Q(p, s, u; 0, 0)) \gg 0 \quad (4.53) \quad \boxed{4.78bis}$$

To compute the integral in [\(4.49\)](#), one has also to take care of the end points $\theta = 0, \pi/2$. Let $1 = \chi_0(\theta) + \chi_c(\theta) + \chi_{\pi/2}(\theta)$ with $\chi_0(\theta)$ supported near 0, $\chi_{\pi/2}(\theta)$ supported near $\pi/2$ and $\chi_c(\theta) \in C_0^\infty(]0, \pi/2[)$ equal to 1 near $\pi/4$. Then the contributions of $\chi_0, \chi_{\pi/2}$ to the kernel $K_s(x, y)$ are smooth functions near $Diag(M)$ (see the appendix for a proof of this point). Now, we can apply the phase stationary theorem to the contribution of χ_c , and we get

$$E_{c,s+r}(s, x, y; \rho, u) = e^{-\rho\psi_s(r, x, y, u)} \rho^{-(m+1)/2} \tilde{\sigma}_s(r, x, y, u; \rho) \quad (4.54) \quad \boxed{4.79}$$

where $\tilde{\sigma}_s(r, x, y, u; \rho)$ is a classical symbol of degree 0 in ρ , $\tilde{\sigma}_s \simeq \sum_{j \geq 0} \tilde{\sigma}_{s,j}(r, x, y, u) \rho^{-j}$ with $\tilde{\sigma}_{s,j}$ analytic in (r, x, y, u) . Then it is easy to pass to the limite $r \rightarrow 0^+$, and we get for (x, y) near $Diag(M)$, the equality, modulo a smooth function near $Diag(M)$:

$$K_s(x, y) = \int_0^\infty \int_{S^{m-1}} e^{i((x-y)s\sqrt{2}\rho u + i\rho Q(p, s, u; 0, x-y))} \rho^{-(m-1)/2} \tilde{\sigma}_s(0, x, y, u; \rho) \frac{\rho^{m-1} d\rho d\sigma(u)}{c_m} \quad (4.55) \quad \boxed{4.80}$$

Then from [\(4.53\)](#) and [\(4.55\)](#), we get that $T_s^* T_s$ is a pseudodifferential operator of degree $-(m-1)/2$ (set $\xi = s\sqrt{2}\rho u$). The ellipticity follows easily from the definition of Σ_s given in [\(4.49\)](#) and formula [\(4.46\)](#).

Finally, from the identity

$$(T_s^* T_s(g)|g)_{L^2(M, d_g x)} = \|T_s(g)\|_{L^2(\partial B_s, d\mu_s)}^2$$

and the injectivity of T_s , we get that (4.70) holds true. The proof of proposition 4.9 is complete. \square

Remark 4.11 *It is also true that $T_s^* T_s$ is an analytic elliptic pseudodifferential operator, but we will not use this fact. If one wants to prove it, one has to modify the above arguments. First prove that $K_s(x, y)$ is analytic outside $\text{Diag}(M)$. Second, use the analytic version of the stationary phase theorem. Third, do not use a cutoff in the θ variable; instead, integrate in θ along a suitable complex path from 0 to $\pi/2$ and passing through the critical point θ_c . The cutoff function $\chi(a)$ is harmless since the Hessian of the real part is positively defined at the critical point.*

End of proof of the Boutet theorem.

Take $s \in]0, \epsilon_0[$. From proposition 4.9, the map

$$g \in H^{-(m-1)/4}(M) \mapsto \mathbb{P}_s(g)(z) = \int_M \mathbb{P}_s(z, y) d_g y \in H(B_s) \quad (4.56) \quad \boxed{4.100}$$

is well defined, continuous, injective, and has closed range. Let us prove that \mathbb{P}_s is surjective, hence an isomorphism of Hilbert space. Let $f \in H(B_s) \subset \mathcal{O}(B_s)$. From lemma 4.3, one has

$$f(z) = \sum c_j e_j(z), \quad c_j = \int_M f(x) e_j(x) d_g x \quad (4.57) \quad \boxed{4.101}$$

where the sum is uniformly convergent on compact subset of B_s and the Fourier coefficients c_j satisfy the bounds $|c_j| \leq C_\delta e^{-(s-\delta)\omega_j}$ for all $\delta > 0$. For $0 < s' < s$, one has

$$f(z)|_{B_{s'}} = \sum c_j e_j(z) = \mathbb{P}_{s'}(g_{s'}), \quad g_{s'} = \sum e^{s'\omega_j} c_j e_j \quad (4.58) \quad \boxed{4.102}$$

From the bounds on the c_j , the function $g_{s'}$ is smooth (and in fact analytic) on M , and from (4.43), we get with a constant C independent of $s' \in [s/2, s[$

$$\begin{aligned} \left(\sum \langle \omega_j \rangle^{-(m-1)/2} e^{2s'\omega_j} |c_j|^2 \right)^{1/2} &= \|g_{s'}\|_{H^{-(m-1)/4}(M)} \\ &\leq C \|T_{s'} g_{s'}\|_{L^2(\partial B_{s'}, d\mu_{s'})} = C \|f\|_{L^2(\partial B_{s'}, d\mu_{s'})} \end{aligned} \quad (4.59) \quad \boxed{4.103}$$

Since one has

$$\lim_{s' \rightarrow s} \|f\|_{L^2(\partial B_{s'}, d\mu_{s'})} = \|f\|_{L^2(\partial B_s, d\mu_s)} = \|f\|_{H(B_s)}$$

we get the "optimal" bound on the c_j :

$$\sum \langle \omega_j \rangle^{-(m-1)/2} e^{2s\omega_j} |c_j|^2 < \infty$$

and therefore,

$$f(z) = \mathbb{P}_s(g_s), \quad g_s = \sum e^{s\omega_j} c_j e_j \in H^{-(m-1)/4}(M)$$

Finally, the family $\langle \omega_j \rangle^{(m-1)/4} e_j$ an orthonormal basis of $H^{-(m-1)/4}(M)$, and \mathbb{P}_s is an isomorphism of Hilbert spaces. Therefore, the family

$$\mathbb{P}_s(\langle \omega_j \rangle^{(m-1)/4} e_j) = e^{-s\omega_j} \langle \omega_j \rangle^{(m-1)/4} e_j$$

is a Riesz basis of $H(B_s)$. The proof of the Boutet theorem [thmboutet I.1](#) is complete.

Let us end these section by some results about the principal symbol of $T_s^*T_s$. The calculus we have done gives the principal symbol A of $T_s^*T_s$ equal to

$$\begin{aligned} A(s, x, \xi) &= C^{-1/2}(s, x, \xi/|\xi|_x) \Gamma_m(s|\xi|_x), \quad (\text{mod } |\xi|_x^{-(m+1)/2}) \\ C(s, x, u) &= s^{-2} (2\sqrt{2})^{-(m+1)} \det(\text{Hess}(\Psi_s(s, x, x, u; \dots)))_{a_c=0, \theta_c=\pi/4} \end{aligned} \quad (4.60) \quad \boxed{4.104}$$

where the function Γ_m is defined in formula [\(2.11\)](#). To prove this point, we use formula [\(4.80\)](#) [\(4.55\)](#) which gives

$$A(s, x, \xi) = (2\pi)^m (|\xi|_x/s\sqrt{2})^{-(m-1)/2} \tilde{\sigma}_{s,0}(0, x, x, \xi/|\xi|_x) (s\sqrt{2})^{-m} c_m^{-1}$$

Now we use stationary phase expansion to compute $\tilde{\sigma}_{s,0}(0, x, x, \xi/|\xi|_x)$. One has

$$\begin{aligned} \Psi_s(s, x, x, u; a, \theta) &= \sin \theta (\bar{d}^2(z, x) + s^2) + \cos \theta (d^2(z, x) + s^2) \\ z &= Z(i, \exp_x(a), sg^{1/2}(a)(u)) \end{aligned}$$

From lemma [3.3](#), we get that the critical point is $(a_c, \theta_c) = (0, \pi/4)$. Thus the function $A(s, x, \xi)$ is equal to (here we use [\(4.46\)](#) and the formula [\(4.49\)](#) for Σ_s)

$$A(s, x, \xi) = (2\pi)^m (|\xi|_x/s\sqrt{2})^{-(m-1)/2} s^2 \pi^{-(m+1)} \left(\frac{1}{2}\right)^{(m-1)/2} (\det^{-1/2}(2\pi)^{(m+1)/2}) (s\sqrt{2})^{-m} c_m^{-1}$$

where \det is the value of the Hessian determinant of $\Psi_s(s, x, x, u; a, \theta)$ at the critical point $(a_c, \theta_c) = (0, \pi/4)$ which is equal to $s^2(2\sqrt{2})^{m+1} C(s, x, \xi/|\xi|_x)$. Hence we get

$$A(s, x, \xi) = C(s, x, \xi/|\xi|_x) \frac{\pi^{(m-1)/2} (s|\xi|_x)^{-(m-1)/2}}{c_m}$$

and the result follows from the fact that the principal symbol of $\Gamma_m(\eta)$ is equal to $(\pi/|\eta|)^{(m-1)/2} c_m^{-1}$.

Remark 4.12 *If one replace the measure $d\mu_s$ on ∂B_s by $d\tilde{\mu}_s = J(z)d\mu_s$, then the new principal symbol of $T_s^*T_s$ will be*

$$|J(Z(i, x, s, s\xi/|\xi|_x))|^2 C^{-1/2}(s, x, \xi/|\xi|_x) \Gamma_m(s|\xi|_x), \quad (\text{mod } |\xi|_x^{-(m+1)/2})$$

and therefore, with the choice $J(Z(i, x, s, su)) = C^{1/4}(s, x, u)$, we will recover the same principal symbol as the one of the flat case.

The function C involves the second derivative in z of $d^2(z, x)$ at $z = Z(i, x, su)$, hence the curvature tensor of M . When $M = S_R^m = \{x \in \mathbb{R}^{m+1}, x^2 = R^2\}$ is the sphere of radius R in \mathbb{R}^{m+1} , one has

$$d^2(x, y) = R^2 \psi\left(\frac{x \cdot y}{R^2}\right), \quad \psi(u) = \theta^2 \Leftrightarrow \cos \theta = u$$

and

$$Z(i, x, su) = x \cosh(s/R) + iRu \sinh(s/R), \quad x \in S_R^m, \quad u \in S_1^m, \quad x \cdot u = 0$$

which gives

$$d^2(Z(i, x, su), y) = R^2 \psi\left(\frac{x \cdot y \cosh(s/R) + iRu \cdot y \sinh(s/R)}{R^2}\right)$$

These formulas allows to find the Taylor expansion at order 2 of Ψ_s at the critical point $(a_c, \theta_c) = (0, \pi/4)$, $(\theta = \pi/4 + \varphi)$:

$$\Psi_s \simeq \sqrt{2} \left(|a|^2 L(s/R) + (1 - L(s/R))(a \cdot u)^2 - 2is\varphi a \cdot u \right), \quad L(u) = u \frac{\cosh(u)}{\sinh(u)}$$

Observe that $L(0) = 1$, thus when $R \rightarrow \infty$, this is compatible with the formula (4.77bis) of the flat case. Therefore, in the case of S_R^m , we get

$$C(s, x, u) = C(s) = (L(s/R))^{m-1}$$

which depends effectively on the parameter s .

5 A conjecture on the ramification locus for general Poisson kernels

sec5

The reader has to observe that the construction of the Hadamard parametrix for the Poisson kernel (in the analytic category) given in proposition 4.6 ^{prop4.1} uses strongly the fact that on the Riemannian manifold with boundary $[0, \infty[\times M$, with metric $ds^2 + g$, the boundary $s = 0$ satisfies the following property:

Every null-complex characteristic curve $u \mapsto (s(u), z(u); \sigma(u), \zeta(u))$ of the hamiltonian function $\sigma^2 + g^{-1}(z, \zeta)$ such that $s(0) = 0, \sigma(0) = 0$, satisfies $s(u) = 0$. (5.1) 6.0

This fact explains why the ramification locus in the complex domain of the kernel of the Poisson operator associated to the operator $\partial_s^2 + \Delta_g$ is simply given by the equation $s^2 + d^2(x, y) = 0$. To my knowledge, the description of the ramification locus in the complex domain of the kernel of the Poisson operator in a general analytic Riemannian manifold with boundary is an open problem. In this section, we will state a "conjecture" about this ramification locus, and we will give some examples in favor of it.

Let $g(x)$ be an analytic metric defined in a neighborhood of 0 in \mathbb{R}^n , and Ω a half space near 0 defined by an analytic equation $f > 0$ with $f(0) = 0$ and $df(0) \neq 0$. We are interested by local solutions near 0 in Ω of the Cauchy problem

$$\Delta_g(v) = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = \delta_0 \quad (5.2) \quad \text{6.1}$$

Any solution of (5.2)^{6.1} is analytic in Ω . Our problem is to determine the "maximal holomorphic extension" of a solution u in a neighborhood of $0 \in \mathbb{C}^n$. This does not depend on the particular solution v of (5.2)^{6.1} since if v_1, v_2 are two solutions of (5.2)^{6.1}, then $v_1 - v_2$ extends analytically in a neighborhood of 0.

One has

$$\Delta_g = \det(g)^{-1/2} \sum_i \partial_i \left(\sum_j \det(g)^{1/2} g^{i,j} \partial_j \right)$$

In the system of coordinates (x', x_n) "geodesic normal to the boundary" with $f = x_n = \text{dist}((x', x_n), \partial\Omega)$, let us denote by $\tilde{g}(a, \cdot)$ the metric on the hypersurface $x_n = a$. Then $u = \det(g)^{1/4} v$ satisfies an equation of the form

$$P(u) = (\partial_{x_n}^2 + R(x_n, x', \partial_{x'}))u = 0 \quad \text{in } x_n > 0, \quad u|_{x_n=0} = \delta_0 \quad (5.3) \quad \boxed{6.2}$$

where $R(x_n, x', \partial_{x'})$ is a second order differential operator with analytic coefficients and real principal symbol equal to

$$r(x_n, x', \xi') = \sum_{i,j < n} \tilde{g}^{i,j}(x_n, x') \xi_i \xi_j$$

Without loss of generality, we may assume $r(0, 0, \xi') = \xi'^2$. We will state more generally a conjecture on the holomorphic extension of a solution u of an equation of type (5.3)^{6.2}, with the assumption that $r(x_n, \cdot)$ is an analytic family of analytic metrics. For $(z, \zeta) \in \mathbb{C}^{2n}$ and z close to 0, we define $p(z, \zeta)$ by

$$p(z, \zeta) = \zeta_n^2 + r(z_n, z', \zeta') \quad (5.4) \quad \boxed{6.3}$$

Let $B_\epsilon = \{z \in \mathbb{C}^n, |z| < \epsilon\}$. We denote by $F = F_\epsilon$ the *smallest* closed subset of $(T^*B_\epsilon \setminus B_\epsilon) \cap p^{-1}(0)$, \mathbb{C}^* homogeneous (i.e for $(z, \zeta) \in F_\epsilon$, one has $|z| < \epsilon$, $\zeta \neq 0$, $p(z, \zeta) = 0$, and $(z, s\zeta) \in F_\epsilon$ for any $s \in \mathbb{C}^*$) which satisfies the following 3 conditions

$$\begin{aligned} a) & \quad (z, \zeta) \in F_\epsilon \Rightarrow \exp(sH_p)(z, \zeta) \in F_\epsilon \quad \text{for } |s| \text{ small} \\ b) & \quad (z', 0, \zeta', \zeta_n) \in F_\epsilon \Rightarrow (z', 0, \zeta', -\zeta_n) \in F_\epsilon \\ c) & \quad \{(0, \zeta), \zeta \neq 0, p(0, \zeta) = 0\} \subset F_\epsilon \end{aligned} \quad (5.5) \quad \boxed{6.4}$$

Observe that a) is a propagation assumption, b) stands for the reflection of singularities at the boundary $z_n = 0$, and c) takes care of the Cauchy data δ_0 .

With $\pi(z, \zeta) = z$, we set $Z = Z_\epsilon = \pi(F_\epsilon)$. Then Z_ϵ is a closed subset of B_ϵ . We denote by $\dim_{\mathcal{H}}(A)$ the real Hausdorff dimension of a set A . We assume $\epsilon > 0$ small enough and we drop the indices ϵ . Our first conjecture is the following.

Conjecture A

- 1. $\dim_{\mathcal{H}}(Z) = 2(n-1)$, $B \setminus Z$ is connected, and u can be holomorphically extended along any path $t \geq 0 \rightarrow q(t)$ such that $q(0) \in \Omega$ and $q(t) \in B \setminus Z$ for $t > 0$.

- 2. Let

$$Z_{reg} = \{z \in Z, Z \text{ is a complex smooth hypersurface near } z\} \quad (5.6) \quad \boxed{6.5}$$

Then $\dim_{\mathcal{H}}(Z) = 2(n-1)$, $\overline{Z_{reg}} = Z$, and near any point of the universal covering which projects on Z_{reg} , the holomorphic extension of u is regular holonomic.

Recall that u is regular holonomic near a point z_0 of a smooth hypersurface defined by an equation $g(z) = 0$ with $g(z_0) = 0, dg(z_0) \neq 0$ iff u is near z_0 a finite linear combination with coefficients in \mathcal{O}_{z_0} of functions of the form $g^\mu, g^\mu \log(g)$.

We define a closed subset Z_{bad} of Z by

$$Z_{bad} = \{z \in Z, Z \text{ is not a constructible set near } z\} \quad (5.7) \quad \boxed{6.6}$$

Here, by "constructible near z ", we just mean that Z is defined by an holomorphic equation near z . Then one has by definition

$$\begin{aligned} Z &= Z_{reg} \cup Z_{sing} \cup Z_{bad} \\ Z_{sing} &= \{z \in Z, Z \text{ is constructible but not a smooth hypersurface near } z\} \end{aligned} \quad (5.8) \quad \boxed{6.7}$$

Remark 5.1 *One has to take care that in general, Z is not the set of zeros of an holomorphic function and we may have $\dim_{\mathcal{H}}(Z_{bad}) = 2(n-1)$. The behavior of u near points of Z_{bad} is most probably not describable (at least not by me).*

Observe that if conjecture **A** is true, this will be a substitute in the complex domain for the celebrated *propagation at the boundary of singularities* theorem of R. Melrose and J. Sjöstrand (see [6]).

The second conjecture is almost the same, but is probably weaker since it is just about the normal derivative

$$\partial_{x_n} u(0, x') = w(x')$$

which is a well defined distribution on the boundary, and which is analytic in $x' \neq 0$. Set $B^0 = B \cap z_n = 0$ and $Z^0 = Z \cap z_n = 0$. Let Z_{reg}^0 be defined by

$$Z_{reg}^0 = \{z' \in Z^0, Z^0 \text{ is a smooth complex hypersurface of the boundary near } z'\}$$

Conjecture B

- 1. $\dim_{\mathcal{H}}(Z^0) = 2(n-2)$, $B^0 \setminus Z^0$ is connected, and w can be holomorphically extends along any path $t \geq 0 \rightarrow q(t)$ such that $q(0) \in \{x' \neq 0\}$ and $q(t) \in B^0 \setminus Z^0$ for $t > 0$.
- 2. One has $\overline{Z_{reg}^0} = Z^0$, and near any point of the universal covering which projects on Z_{reg}^0 , the holomorphic extension of w is regular holonomic.

Let us study a very particular case, where $Z_{bad} = \emptyset$ and $Z = Z_{reg} \cup \{z = 0\}$. Let $r_0(z', \zeta') = r(0, z', \zeta')$. Let us assume that the following hypothesis holds true:

$$\begin{aligned} & \text{for } \eta' \in \mathbb{C}^{n-1} \setminus 0 \text{ such that } r_0(0, \eta') = \eta'^2 = 0 \text{ one has} \\ & (z', \zeta') = \exp(sH_{r_0})(0, \eta') \Rightarrow \frac{\partial r}{\partial z_n}(0, z', \zeta') = 0 \end{aligned} \quad (5.9) \quad \boxed{6.6}$$

In that case, we claim that the set F is equal to the union of null bicaracteristics of p starting at a point of $T_0^*\mathbb{C}^n$, i.e

$$F = \Lambda_0 = \cup_{\zeta \neq 0, p(0, \zeta) = 0} \{\exp(sH_p)(0, \zeta)\} \quad (5.10) \quad \boxed{6.7}$$

and therefore

$$Z = \{z \in \mathbb{C}^n, d^2(z, 0) = 0\} \quad (5.11) \quad \boxed{6.8}$$

Z is a complex cone, and $Z_{reg} = Z \setminus \{z = 0\}$. (by a complex cone, we mean that it is defined by an equation ${}^t z M(z) z = 0$ with a matrix $M(z)$ such that $\det(M(0)) \neq 0$) One has just to verify that the closed \mathbb{C}^* -homogeneous set Λ_0 given by (5.10) satisfies the 3 conditions of (5.5), since any closed \mathbb{C}^* -homogeneous set F which satisfies these conditions contains Λ_0 . Conditions a) and c) are obvious. If $n = 2$, the equation $p(0, \zeta) = 0$ is equivalent to $\zeta_2 = \pm i\zeta_1$, thus there is only 2 null bicaracteristics starting at $z = 0$, and they are transversal to the boundary, thus condition b) is obvious. If $n \geq 3$, then $C = \{d^2(z, 0) = 0\} \cap \{z_n = 0\}$ is still a complex cone in the boundary. By (5.9) C contains the complex cone $\tilde{C} = \{\tilde{d}^2(z', 0) = 0\}$ where \tilde{d} is the distance in the boundary, and therefore, one has $C = \tilde{C}$. Thus if a point $((z', 0), \zeta)$ is in Λ_0 , one has $d^2(z', 0) = 0$ and therefore condition b) holds true.

Observe in particular that (5.9) is obviously satisfied when r is independent of z_n , which was the case for the operator $\partial_s^2 + \Delta_g$ with $s = x_n$ of section 4, and of course proposition 4.3 shows that conjecture **A** holds true in that case.

Observe also that the hypothesis (5.9) always holds true in dimension $n = 2$ (since in that case, the hypothesis is void). Of course, $n = 2$ is very particular since the principal symbol of an operator of the form (5.3) factorize in the product of two complex linear forms.

Observe finally that for any n , when Δ is the flat laplacian on \mathbb{R}^n and when Ω is ever the interior or the exterior of a ball of radius $R > 0$, then condition (5.9) holds true. This is a consequence of the trivial fact that if $x \in \mathbb{R}^n$ is such that $x^2 = R^2$, then for any vector $z \in \mathbb{C}^n$ such that $z^2 = 0$ and $z \cdot x = 0$, one has $(x + z)^2 = R^2$. Observe that one has an explicit formula for the Poisson kernel inside or outside a ball, and these formulas are compatible with conjecture **A**.

We will end this section with the special case of the operator

$$\partial_x^2 + (1 + x)\partial_y^2 + \partial_z^2 \quad \text{with} \quad (x, y, z) \in \mathbb{R}^3, \quad \Omega = \{x > 0\}$$

In that case, we will indicate briefly why conjecture **B** is true, the idea of the proof being the one of [3]. This example shows that the set of bad points Z_{bad} do exists, and they are really obstruction in the holomorphic extension of u . Let us first describe what are the sets F and Z in that case.

It is easy to compute in coordinates $(x, y, z; \xi, \eta, \zeta)$ the null bicaracteristic $\exp(sH_p)(0, y_0, z_0; \xi_0, \eta_0, \zeta_0)$. One finds

$$\begin{aligned} x(s) &= 2s\xi_0 - s^2\eta_0^2, & \xi(s) &= \xi_0 - s\eta_0^2 \\ y(s) &= y_0 + 2s\eta_0 + 2s^2\xi_0\eta_0 - 2s^3\eta_0^3/3, & \eta(s) &= \eta_0 \\ z(s) &= z_0 + 2s\zeta_0, & \zeta(s) &= \zeta_0 \end{aligned} \quad (5.12) \quad \boxed{6.10}$$

Let Λ_0 be defined by (5.7). For $N \geq 1$, we denote by Λ_N the set of points in $T^*\mathbb{C}^3 \setminus 0$, with (x, y, z) close to 0, which are connected to a point in Λ_0 by N reflections on the boundary $x = 0$. From (5.12), it is easy to compute Λ_N , and we find that it is parametrized by the complex curve $\{\alpha^2 + \beta^2 + 1 = 0\} \subset \mathbb{C}^2$, with $|\alpha| \leq \epsilon/N$, with the following formula

$$\begin{aligned} x(t) &= 2t\alpha - t^2 \\ y(t) &= 4N(\alpha + 2\alpha^3/3) + 2t + 2t^2\alpha - 2t^3/3 \\ z(t) &= 4N\alpha\beta + 2t\beta \\ (\xi(t), \eta(t), \zeta(t)) &= \lambda(\alpha - t, 1, \beta), \quad \lambda \in \mathbb{C}^* \end{aligned} \quad (5.13) \quad \boxed{6.11}$$

Since α is close to 0, this gives $\Lambda_N = \Lambda_N^+ \cup \Lambda_N^-$ with $\beta = \pm i\sqrt{1 + \alpha^2}$ on Λ_N^\pm . The Λ_N^\pm are \mathbb{C} -lagrangian in $T^*\mathbb{C}^3 \setminus 0$. Obviously, the set $\cup_{k \geq 0} \Lambda_k$ satisfies conditions a), b), c) of (5.5), and thus we get

$$F = \text{closure}(\cup_{k \geq 0} \Lambda_k) \quad (5.14) \quad \boxed{6.12}$$

If we define Λ_∞^\pm by the parameterization with $(u, t) \in \mathcal{C}^2$ close to $(0, 0)$

$$\begin{aligned} x(t) &= -t^2 \\ y(t) &= 4u + 2t - 2t^3/3 \\ z(t) &= \pm i(4u + 2t) \\ (\xi(t), \eta(t), \zeta(t)) &= \lambda(-t, 1, \pm i), \quad \lambda \in \mathbb{C}^* \end{aligned} \quad (5.15) \quad \boxed{6.13}$$

we find with $\Lambda_\infty = \Lambda_\infty^+ \cup \Lambda_\infty^-$

$$F = (\cup_{k \geq 0} \Lambda_k) \cup \Lambda_\infty \quad (5.16) \quad \boxed{6.14}$$

Set $Z_0 = \{d^2((x, y, z), (0, 0, 0)) = 0\}$, $Z_N^\pm = \pi(\Lambda_N^\pm)$, $Z_N = Z_N^+ \cup Z_N^-$, $Z_\infty^\pm = \pi(\Lambda_\infty^\pm)$ and $Z_\infty = Z_\infty^+ \cup Z_\infty^-$. Then one gets

$$\begin{aligned} Z &= (\cup_{k \geq 0} Z_k) \cup Z_\infty \\ Z_{bad} &= Z_\infty \end{aligned} \quad (5.17) \quad \boxed{6.15}$$

Observe that Z_{bad}^\pm is the union of two complex sets of codimension 1 with a cusp singularity on the boundary $x = 0$. In particular, $\dim_{\mathcal{H}}(Z_{bad}) = 4 = 2(n - 1)$. The intersection Z^0 of Z with the boundary $x = 0$ is easy to compute. One find

$$\begin{aligned} Z_0^0 &= \{(0, 0)\} \cup \Sigma_1, & Z_N^0 &= \Sigma_N \cup \Sigma_{N+1} \\ \Sigma_N &= \Sigma_N^+ \cup \Sigma_N^-, & \Sigma_N^\pm &= \{(y, z) = (4u(1 + 2u^2/3N^2), \pm 4iu\sqrt{1 + u^2/N^2})\} \\ Z_\infty^0 &= \Sigma_\infty = \Sigma_\infty^+ \cup \Sigma_\infty^-, & \Sigma_\infty^\pm &= \{z = \pm iy\} \end{aligned} \quad (5.18) \quad \boxed{6.16}$$

This gives $Z_{reg}^0 = Z^0 \setminus (\Sigma_\infty)$ and therefore $\overline{Z_{reg}^0} = Z^0$.

The solution $u(x, y, z), x > 0$ of the boundary value problem (6.2) is easy to compute via Fourier transform in (y, z) . One finds

$$\begin{aligned} u(x, y, z) &= \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} e^{i(y\eta+z\zeta)} \frac{Ai(x\eta^{2/3} + \eta^{-4/3}(\eta^2 + \zeta^2))}{Ai(\eta^{-4/3}(\eta^2 + \zeta^2))} d\eta d\zeta \\ w(y, z) &= \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} e^{i(y\eta+z\zeta)} \eta^{2/3} \frac{Ai'(\eta^{-4/3}(\eta^2 + \zeta^2))}{Ai(\eta^{-4/3}(\eta^2 + \zeta^2))} d\eta d\zeta \end{aligned} \quad (5.19) \quad \boxed{6.17}$$

where $Ai(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(su+s^3/3)} ds$ is the Airy function. Recall that $\frac{Ai'}{Ai}(w)$ is an analytic symbol on any angular sector $-\pi + \epsilon < \arg(w) < \pi + \epsilon, |w| \geq 1$, and

$$\frac{Ai'}{Ai}(w) \simeq -w^{1/2} \left(1 + \sum_{j \geq 1} c_j w^{-3j/2}\right), \quad |c_j| \leq AB^j j! \quad (5.20) \quad \boxed{6.17bis}$$

Thus we get from (6.17), with $D = -i\partial$, $|\Delta_0| = -\partial_y^2 - \partial_z^2$, and where N denotes the Dirichlet to Neumann operator (an analytic pseudodifferential operator of degree 1)

$$w = N(\delta_0), \quad N(D_y, D_z) \simeq -|\Delta_0|^{1/2} \left(1 + \sum_{j \geq 1} c_j D_y^{2j} |\Delta_0|^{-3j/2}\right) \quad (5.21) \quad \boxed{6.17ter}$$

From (6.17) and (6.17bis), one gets for $(y, z) \in \mathbb{R}^2$ and $\varphi \in]-\pi/2, \pi/2[$

$$w(e^{-i\varphi}y, e^{-i\varphi}z) = \left(\frac{1}{2\pi}\right)^2 e^{8i\varphi/3} \int_{\mathbb{R}^2} e^{i(y\eta+z\zeta)} \eta^{2/3} \frac{Ai'}{Ai}(e^{2i\varphi/3} \eta^{-4/3}(\eta^2 + \zeta^2)) d\eta d\zeta \quad (5.22) \quad \boxed{6.18}$$

$\frac{Ai'}{Ai}(e^{i\pi/3}u)$ is a tempered distribution on \mathbb{R} and one has

$$\frac{Ai'}{Ai}(e^{i\pi/3}u) = ie^{i\pi/3} \int_{\mathbb{R}} e^{-isu} \mathcal{G}(s) ds \quad (5.23) \quad \boxed{6.19}$$

where $\mathcal{G}(s)$ is a tempered distribution with support in $s \geq 0$, analytic in $s > 0$, with

$$\forall s > 0, \quad \mathcal{G}(s) = \sum_{k \geq 0} e^{-e^{i\pi/6} s \omega_k} \quad (5.24) \quad \boxed{6.18}$$

where $\omega_1 < \omega_2 < \dots$ are such that $-\omega_k$ is the k -th zero of the Airy function. Recall $\omega_k = (3\pi(k-1/4)/2)^{2/3} f(k)$ where $f(\lambda) = 1 + O(\lambda^{-2})$ is a classical analytic symbol in λ . Near $s = 0$, one has $\mathcal{G}(s) \in O(s^{-3/2})$. Set $\tilde{w}(y, z) = w(e^{-i\pi/2}y, e^{-i\pi/2}z)$. From (6.18) and (5.25) we get

$$\begin{aligned} \tilde{w}(y, z) &= \int_0^\infty s^5 \mathcal{G}(s) e^{is^3 \phi(y, z)} \sigma(y, z; s^3) ds \\ e^{is^3 \phi(y, z)} \sigma(y, z; s^3) &= e^{i7\pi/9} \frac{is^3}{4\pi^2} \int_{\mathbb{R}^2} e^{is^3(ya+zb-a^{-4/3}(a^2+b^2))} a^{2/3} da db \end{aligned} \quad (5.25) \quad \boxed{6.19}$$

where $\phi(y, z)$ is the critical value of the phase

$$(a, b) \mapsto ya + zb - a^{-4/3}(a^2 + b^2)$$

and $\sigma(y, z; \lambda)$ is a classical analytic symbol of degree 0. Let $G(u)$ be a function of the type

$$G(u) = \int_0^\infty \mathcal{G}(\lambda^{1/3}) e^{i\lambda u} \sigma(u; \lambda) \lambda d\lambda \quad (5.26) \quad \boxed{6.20}$$

As in [3], one finds that near 0, G is ramified on the complement of $u = 0$ and the set of points $u_n = \frac{-1}{12n^2}$, and it remains to check that the set of equation $\phi(-iy, -iz) = \frac{-1}{12n^2}$ are the set Z_n given in 5.18, and $\phi(-iy, -iz) = 0$ iff $z = \pm iy$.

6 Appendix

appendix

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