Spectral inequalities for Schrödinger operators on manifolds with constant curvature

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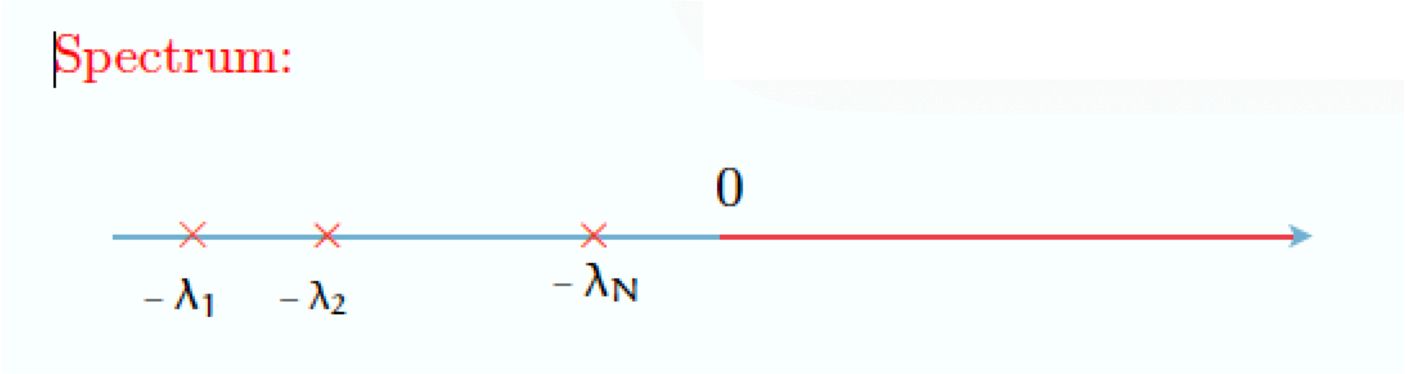
• CLR and Lieb-Thirring inequalities

Consider a Schrödinger operator

$$\mathcal{H} = -\Delta - V(x), \qquad \text{in}\, L^2(\mathbb{R}^d),$$

where $V \to 0$ as $|x| \to \infty$ and $V \geq 0$ and let $\{-\lambda_k\}$ be negative eigenvalues of $\mathcal{H}$.



$$\sum_j \lambda_j^\gamma = \sum_j \lambda_j^\gamma(V) \leq C_{d,\gamma} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, dx d\xi = L_{d,\gamma} \int V(x)_+^{\gamma+d/2}\, dx.$$

This inequality holds true for $d = 1$, $\gamma \geq 1/2$; $d = 2$, $\gamma > 0$; $d \geq 3$, $\gamma \geq 0$.

Compare it with Weyl's asymptotic formula:

$$\sum_j \lambda_j(\alpha V)^\gamma \sim_{\alpha\to\infty} L^{cl}_{d,\gamma} \int (\alpha V_+)^{\gamma+d/2}\, dx = (2\pi)^{-d} \iint (|\xi|^2 - \alpha V)_-^\gamma \, d\xi dx,$$

which implies $L^{cl}_{d,\gamma} \leq L_{d,\gamma}$.

Applications.

- Weyl's asymptotics.
- Stability of matter.
- Study of properties of continuous spectrum of Schrödinger operators.
- Estimate of dimensions of attractors in theory of Navier-Stokes equations.
- Bounds on the maximum ionization of atoms.

Theorem. Let $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d \geq 2$, $\gamma \geq 0$ if $d \geq 3$ and let $0 \leq V \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then the negative eigenvalues $\{-\lambda_k\}$ of the operator $-\Delta - V$ satisfy

$$\sum_k \lambda_k^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V^{\gamma+d/2}\, dx.$$

M.S.Biman, M.Z.Solomyak, G.Rosenblum, M.Cwikel, J.G. Conlon, E.H.Lieb, W.Thirring, M.Aizenmann, D.Hundertmark, L.Thomas, AL & T.Weidl, R.Frank, M. Jex, P. T. Nam, R.Benguria & M.Loss, J.Dolbeault, M.Rumin, J.P. Solovej, R. Seiringer.

Sharp constant were obtained in the following cases:

Theorem. It is known that $L_{1,1/2} = 1/2$ ($L^{cl}_{1,1/2} = 1/4$) and $L_{d,\gamma} = L^{cl}_{d,\gamma}$ if $\gamma \geq 3/2$, $d \geq 1$.

In other cases the sharp constants are unknown.

- Sharp multidimensional Lieb-Thirring inequalities, $\gamma \geq 3/2$.

The main argument is based on a Lieb-Thirring inequality for Schrödinger operators with matrix-valued potentials.

Theorem. (AL & T.Weidl)
Let $M \geq 0$ be a Hermitian $m \times m$ matrix-function and let $\mathcal{H} = -d^2/dx^2 - M$ in $L^2(\mathbb{R})$. Then

$$\sum_n \lambda_n^{3/2}(\mathcal{H}) \leq \frac{3}{16} \int \operatorname{Tr} M^2(x)\, dx.$$

Our proof was based on the matrix valued version of the so-called second trace (BFZ) formula related to the integral of motion for the KdV equation

$$\sum_k \lambda_k^{3/2} + \frac{3}{2\pi} \int_{\mathbb{R}} k^2 \log |a(k)|\, dk = \frac{3}{16} \int V^2\, dx$$

and the fact that $|a(k)| \geq 1$.

Using the L-Th inequality for the matrix-valued potentials we prove the following

Theorem. (AL & T.Weidl)

Let $V \geq 0$ with $\gamma \geq 3/2$. Then for the negative eigenvalues $\{-\lambda_k\}$ of the Schrödinger operator $\mathcal{H} = -\Delta - V$ in $L^2(\mathbb{R}^d)$ we have

$$\sum \lambda_n^\gamma \leq L_{d,\gamma}^{cl} \int_{\mathbb{R}^d} V^{\gamma+d/2}(x)\, dx.$$

For the proof we use the "lifting argument" with respect to dimension.

Let for simplicity $d = 2$, $V \in C_0^\infty(\mathbb{R}^2)$, $V \geq 0$, $x = (x_1, x_2)$ and $\gamma = 3/2$. Then

$$H = -\Delta - V = -\partial^2_{x_1 x_1} - \underbrace{(\partial^2_{x_2 x_2} + V)}_{\tilde{H}(x_1)}.$$

Spectrum $\sigma(\tilde{H})$ of $\tilde{H}(x_1)$ has a finite number of positive eigenvalues $\mu_l(x_1)$. Thus $\tilde{H}_+(x_1)$ has a finite rank. We find

$$\sum_j \lambda_j^{3/2}(H) \leq \sum_j \lambda_j^{3/2}(-\partial^2_{x_1 x_1} - \tilde{H}_+)$$

$$\leq \frac{3}{16} \int \operatorname{Tr} \tilde{H}_+^2(x_1)\, dx_1 \leq \underbrace{\frac{3}{16} L_{1,2}^{\mathrm{cl}}}_{L_{2,3/2}^{cl}} \iint V^{3/2+1}(x)\, dx.$$

The operator-valued approach was fruitful and used in the cases:

- $\gamma = 1/2$ - D.Hundertmark, AL, T.Weidl $\left(2 \times L^{\mathrm{cl}}_{d,1/2} - \text{sharp if } d = 1\right)$.

- $\gamma = 1$ - J. Dolbeault, AL, M. Loss $\left(1.8 \times L^{\mathrm{cl}}_{d,1}\right)$

  - and lately by R. Frank, M. Jex, P. T. Nam $\left(1.4 \times L^{\mathrm{cl}}_{d,1}\right)$.

$$\sum_j \lambda_j^\gamma = \sum_j \lambda_j^\gamma(V) \le C_{d,\gamma} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, dx d\xi.$$

Remark.
Let $\Omega \subset \mathbb{R}^d$ be a domain of finite measure and for $\lambda > 0$ let

$$V(x) = \begin{cases} \lambda, & x \in \Omega, \\ -\infty, & x \notin \Omega. \end{cases}$$

In this case we reduce the problem to the eigenvalues $\{\lambda_k\}$ of the Dirichlet Laplacian in Ω and the L-Th inequalities become

$$\sum_k (\lambda - \lambda_k)_+^\gamma \le C_{d,\gamma} |\Omega| \int (|\xi|^2 - \lambda)_-^\gamma \, d\xi.$$

- $\gamma = 0$, $d \ge 1$.

This inequality was proved for bounded domains by Birman & Solomyak ('70) and Ciesielski ('70) with some finite constant.
For domains of finite measure it was proved by G.Rosenblum ('71) and E.Lieb ('80).
The sharp constant $C_{d,\gamma} = (2\pi)^{-d}$ was obtained by Pólya for tiling domains.
Recently N. Filonov, M. Levitin, I. Polterovich & D.A. Sher proved this inequality for discs.
AL proved for some product type domains.

- $\gamma \ge 1$, $d \ge 1$.

The best constant $C_{d,\gamma} = (2\pi)^{-d}$ is due to Berezin and Li & Yau.

• The Berezin–Li–Yau inequalities.

Theorem. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure and $\lambda > 0$. Then

$$\sum_k (\lambda - \lambda_k)_+ \leq (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \, d\xi$$

$$= \lambda^{1+d/2} |\Omega| \, (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+ \, d\xi.$$

Proof. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure. Then the spectrum of $-\Delta_\Omega^D$ is discrete. We denote by λ_k the non-decreasing sequence of eigenvalues (counting with multiplicities) and let φ_k be an associated othonormal system of Dirichlet eigenfunctions. Note that these eigenfunctions can be continued by zero outside of Ω to H^1-functions on $\mathbb{R}^d$.

For any $\lambda > 0$ we have

$$\sum_k (\lambda - \lambda_k)_+ = \sum_k \left(\int_\Omega (\lambda |\varphi_k|^2 - |\nabla \varphi_k|^2) \, dx \right)_+$$

$$= \sum_k \left((2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2) \, |\widehat{\varphi_k}|^2) \, d\xi \right)_+ \leq (2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \sum_k |\widehat{\varphi_k}|^2 \, d\xi.$$

Clearly

$$\sum_k |\widehat{\varphi_k}|^2 = \sum_k \left| (\varphi_k, e^{-ix\xi})_{L^2(\Omega)} \right|^2 = \| e^{-ix\xi} \|^2 = |\Omega|.$$

Corollary. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure. Then for all $\lambda > 0$, we have

$$N(\lambda, -\Delta_\Omega^D) \leq \left(1 + \frac{2}{d}\right)^{d/2} L_{0,d}^{cl} \, |\Omega| \, \lambda^{d/2}.$$

Proof. Consider

Here $\rho = \eta - \lambda$. Then by the BLY inequality we have

$$N(\lambda, -\Delta_\Omega^D) \leq \frac{1}{\eta - \lambda} \sum_k (\eta - \lambda_k)_+ \leq \frac{1}{\eta - \lambda} \, |\Omega| \, (2\pi)^{-d} \int_{\mathbb{R}^d} (\eta - |\xi|^2)_+ \, d\xi$$

$$= \frac{\eta^{1+d/2}}{\eta - \lambda} \, |\Omega| \, (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+ \, d\xi.$$

Minimising wrt η we find $\eta = \lambda \, \frac{1+d/2}{d/2}$ and obtain the proof.

- Berezin-Li & Yau type inequalities on sphere.

Let now $\Omega \subset \mathbb{S}^{d-1}$ with $(d-1)$-dimensional surface area $|\Omega| \le |\mathbb{S}^{d-1}| =: \sigma_d$. Denote by $\{\lambda_k\}_{k=1}^{\infty}$ the eigenvalues of the Laplace–Beltrami operator $-\Delta$ with the Dirichlet boundary conditions in $L^2(\Omega)$. Then

Theorem. (A.Ilyin, AL) *We have*

$$\sum (\lambda - \lambda_k)_+^{\gamma} \le \frac{|\Omega|}{\sigma_d} \sum_{n=0}^{\infty} (\lambda - \Lambda_n)^{\gamma} k_d(n),$$

where $\gamma \ge 1$, Λ_n *and* k_d *are the eigenvalues and their multiplicities of the Laplace-Beltrami operator on the whole sphere* $\mathbb{S}^{d-1}$.

- Lieb–Thirring inequalities on manifolds with negative constant curvature.

Let $\mathbb{H}^d$ be the upper half-space

$$\mathbb{H}^d = \{(x,y) : \, x \in \mathbb{R}^{d-1}, \, y \in \mathbb{R}_+\}$$

with the Poincare metric $ds^2 = y^{-2}(\sum_{n=1}^{d-1} dx_n^2 + dy^2)$ assuming that the negative curvature equals -1.
We consider the self-adjoint Laplace-Beltrami operator in $L^2\left(\mathbb{R}^{d-1} \times \mathbb{R}_+, \frac{dx\,dy}{y^d}\right)$

$$-\Delta_h = -y^d \frac{\partial}{\partial y} \, y^{2-d} \, \frac{\partial}{\partial y} - y^2 \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2} .$$

The spectrum of the standard Laplacian $-\Delta$ in $L^2(\mathbb{R}^d, dx)$ is continuous and covers the whole left half-line $[0, \infty)$.

It is well-known that the spectrum of the hyperbolic Laplace operator $-\Delta_h$ covers the interval $[(d-1)^2/4, \infty)$.

Proposition. Let $-\Delta_h$ be the Laplacian in $L^2\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$. Then the continuous spectrum coincides with $\sigma_c = [(d-1)^2/4, \infty)$.

Proof. Let us consider the quadratic form of the operator $-\Delta_h$

$$(-\Delta_h u, u) = \int_{\mathbb{H}^d} y^{2-d} \left(|\partial_y u|^2 + \sum_{n=1}^{d-1} |\partial_{x_n} u|^2 \right) dx\,dy.$$

The substitution

$$y = e^t, \qquad u = e^{\frac{d-1}{2}\,t}\, v,$$

implies

$$\iint_{\mathbb{R}^{d-1}\times\mathbb{R}_+} |u|^2 \,\frac{dx\,dy}{y^d} = \int_{\mathbb{R}^d} |v|^2 \,dx\,dt.$$

and

$$(-\Delta_h u, u) = \int_{\mathbb{R}^d} \left(|v_t'|^2 + \frac{(d-1)^2}{4}\,|v|^2 + e^{2t} \sum_{n=1}^{d-1} |\partial_{x_n} v|^2 \right) dxdt.$$

Thus we reduce the hyperbolic Laplacian to the operator in $L^2(\mathbb{R}^d)$

$$-\frac{\partial^2}{\partial t^2} - e^{2t}\,\Delta_x + \frac{(d-1)^2}{4} \geq \frac{(d-1)^2}{4}.$$

Let $V \geq 0$ and let us consider the spectrum problem for the Schrödinger operator in $L^2(\mathbb{H}^d, y^{-d}\, dxdy)$

$$\mathcal{H}_h u = (-\Delta_h - V)\, u = \left(\frac{(d-1)^2}{4} - \mu\right) u.$$

Theorem. (A. Ilyin, AL, T. Weinmann) Let $\gamma \geq 1/2$, $V \geq 0$ and let $V \in L^{\gamma+d/2}(\mathbb{H}^d, y^{-d}\, dxdy)$. Then

$$\sum_k \mu_k^\gamma \leq L_{d,\gamma} \int_{\mathbb{H}^d} V^{\gamma+d/2}\, \frac{dxdy}{y^d}.$$

Proof. Applying the exponential change of variables we reduce the problem to the study of the spectrum defined by the form

$$\int_{\mathbb{R}^d} \left(|v_t'|^2 + e^{2t} \sum_{n=1}^{d-1} |\partial_{x_n} v|^2 - V(x, e^t)|v|^2 \right) dxdt = -\mu \int_{\mathbb{R}^d} |v|^2\, dxdt.$$

Using the L-Th inequalities for 1D operators with operator-valued symbols we obtain

$$\sum \mu_k^\gamma \leq L_{1,\gamma} \int_{\mathbb{R}} \mathrm{Tr} \left(-e^{2t} \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2} - V(x, e^t) \right)_-^{\gamma+1/2} dt$$

$$\leq L_{1,\gamma} L_{d-1,\gamma+1/2} \int_{\mathbb{R}^d} e^{(1-d)t}\, V(x, e^t)^{\gamma+d/2}\, dxdt = L_{d,\gamma} \int_{\mathbb{H}^d} V(x,y)^{\gamma+d/2}\, \frac{dxdy}{y^d}.$$

- Dual inequalities.

Consider an orthonormal set of function $\{u_m\}_{m=1}^{M}$ in $L^2(\mathbb{H}^d, y^{-d}\, dxdy)$. Then using the exponential change

$$y = e^t, \qquad u_m = e^{(d-1)t/2}\, v_m$$

we find

$$\delta_{m,l} = \int_{\mathbb{H}} u_m \overline{u}_l \frac{dxdy}{y^d} = \int_{\mathbb{R}^n} v_m \overline{v}_l \, dxdt.$$

Assuming $\gamma = 1$ we obtain

$$\sum_{m=1}^{M} \int_{\mathbb{R}^d} \left(|\partial_t v_m|^2 + e^{2t} \sum_{n=1}^{d-1} |\partial_{x_n} v_m|^2 - V(x, e^t)|v_m|^2 \right) dxdt$$
$$\geq - \sum_m \mu_m \geq -L_{d,1} \int_{\mathbb{R}^d} V(x, e^t)^{1+d/2} \, dxdt.$$

Thus

$$\sum_{m=1}^{M}\int_{\mathbb{R}^d}\left(|\partial_t v_m|^2+e^{2t}\sum_{n=1}^{d-1}|\partial_{x_n}v_m|^2\right)dxdt$$
$$\geq \int_{\mathbb{R}^d}\left(V(x,e^t)\sum_{m=1}^{M}|v_m|^2-L_{d,1}\,V(x,e^t)^{1+d/2}\right)dxdt$$
$$=\int_{\mathbb{R}^d}\Big(V(x,e^t)\,\widetilde{\rho}-L_{d,1}\,V(x,e^t)^{1+d/2}\Big)\,dxdt,$$

where $\widetilde{\rho}=\sum_{m=1}^{N}|v_m|^2$. We choose now

$$V=\left(\frac{\widetilde{\rho}}{L_{d,1}(1+d/2)}\right)^{2/d}$$

and obtain

$$\sum_{m=1}^{M}\int_{\mathbb{R}^d}\left(|\partial_t v_m|^2+\sum_{n=1}^{d-1}|\partial_{x_n}v_m|^2\right)dxdt\geq K_{d,1}\,\int_{\mathbb{R}^d}(\widetilde{\rho})^{1+2/d}\,dxdt,$$

where

$$K_{d,1}^{-1}=\frac{2}{d}\left(1+\frac{d}{2}\right)^{1+2/d}L_{d,1}^{2/d}.$$

Returning back to the orthonormal system of functions $\{u_m\}$ and denoting by $\rho = \sum_{m=1}^{M} |u_m|^2$ we obtain

Theorem. Let $\gamma = 1$ and let $\{u_m\}_{m=1}^{M}$ be an orthonormal system of function in $L^2(\mathbb{H}^d, y^{-d} dxdy)$. Then

$$\sum_{m=1}^{M} (-\Delta_h u_m, u_m) \geq K_{d,1} \int_{\mathbb{H}^d} \rho^{1+2/d} \, \frac{dxdy}{y^d} + M \, \frac{(d-1)^2}{4}.$$

Assume that $M = 1$ and let $u \in H^1 \left(\mathbb{H}^d, y^{-d} \, dxdy\right)$. Denote

$$\|u\|_h^2 = \int_{\mathbb{H}^d} |u|^2 \frac{dxdy}{y^d}.$$

Theorem. We have the following Sobolev type inequality

$$\|u\|_h^{2/d} \, \|y^{2-d} |\nabla_{y,x} u|\|_{L^2(\mathbb{H})} \geq \sqrt{K_{d,1}} \, \|u^{1+2/d}\|_h + \frac{(d-1)^2}{4} \|u\|_h^{2/d}.$$

- Hyperbolic Dirichlet Laplacian.

Let $\Omega \subset \overline{\Omega} \subset \mathbb{H}^d$ satisfies the inequality

$$|\Omega|_h = \int_\Omega \frac{dxdy}{y^d} < \infty.$$

We consider the Dirichlet eigenvalue problem for the Laplace–Beltrami operator $-\Delta_h$ in $L^2(\Omega, y^{-d}dxdy)$

$$-\Delta_h u = \lambda u, \qquad u\big|_{(x,y)\in\partial\Omega} = 0.$$

The spectrum of this operator is discrete and we denote by $\{\lambda_k\}$ its eigenvalues. Such eigenvalues satisfy the inequality

$$\lambda_k > \frac{(d-1)^2}{4}.$$

It is convenient to introduce number ν_k such that

$$\lambda_k = \frac{(d-1)^2}{4} + \nu_k$$

and study the counting function $\mathcal{N}(\Lambda)$ of the spectrum

$$\mathcal{N}(\Lambda) = \#\{k : \nu_k < \Lambda\}, \qquad \Lambda > 0.$$

Theorem. Let $|\Omega|_h < \infty$. Then the counting function $\mathcal{N}(\Lambda)$ of the eigenvalues of the hyperbolic Dirichlet Laplacian satisfies the following inequality

$$\mathcal{N}(\Lambda) \le \left(1 + \frac{2}{d}\right)^{d/2} \left(1 + \frac{d}{2}\right) L_{d,1}\, \Lambda^{d/2}\, |\Omega|_h,$$

where $L_{d,1}$ is the best known constant in L-Th inequality. so that

$$(1 + d/2) L_{d,1} \le R_{1,1} (1 + d/2)\, L^{\mathrm{cl}}_{d,1} = R_{1,1}\, L^{\mathrm{cl}}_{d,0}.$$

This inequality is a Pólya type inequality for manifolds with constant negative curvature, where, we believe, the constant is not sharp.

• *Conjecture 1.* For the counting function $\mathcal{N}(\Lambda)$ of the eigenvalues $\lambda_k = (d-1)^2/4 + \nu_k$ of the hyperbolic Dirichlet Laplacian we have

$$\mathcal{N}(\Lambda) \le L^{\mathrm{cl}}_{d,0}\, \Lambda^{d/2}\, |\Omega|_h.$$

• *Conjecture 2.* Prove the Berezin - Li & Yau type inequality

$$\sum_k (\Lambda - \nu_k)_+ \le L^{\mathrm{cl}}_{d,1}\, \Lambda^{d/2+1}\, |\Omega|_h,$$

(Kontrovich-Lebedev's inversion formula).

Remark. At the moment we do not have any examples of Ω for which such inequalities hold.

Happy birthday Anders