On the spectrum of certain semiregular global systems

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(Joint work with Marcello Malagutti)

To Anders

Outline.

1. Semiregular systems.
2. The class of systems.
3. Examples:
   1. Jaynes-Cummings;
   2. Geometric extensions.
4. The results:
   1. Weyl Law;
   2. Refined Weyl Law.
5. Quasi-clustering (time permitting).

Weyl asymptotics: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots \to +\infty$ repeated eigenvalues (according to multiplicity) of (semibdd) operator A with a discrete spectrum,

$$\mathsf{N}_A(\lambda) = \sum_{\lambda_j \leq \lambda} 1.$$

1. Weyl, Hörmander, Ivrii, Shubin, Duistermaat-Guillemin (elliptic operators on a closed manifold), Safarov-Vassiliev;
2. For global operators, Hörmander, Shubin, Chazarain, Helffer-Robert, Ivrii, recently Doll-Gannot-Wunsch, Doll-Zelditch(semiregular scalar symbol), Coriasco-Doll (SG classes);
3. For systems: Ivrii, myself for NCHOs.

I will give here a survey of some recent results of M. Malagutti and myself on Weyl asymptotics and quasi-clustering for semiregular systems.

The symbols.

Hörmander's class of symbols: for $\mu \in \mathbb{R}$, $X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, $\langle X \rangle = (1 + |X|^2)^{1/2}$,

$$S^\mu = S(\langle X \rangle^\mu, |dX|^2/\langle X \rangle^2; \mathrm{M}_N),$$

hence $A \in S^\mu$ iff

$$|\partial_X^\alpha A(X)|_{\mathrm{M}_N} \lesssim \langle X \rangle^{\mu - |\alpha|}, \quad \forall \alpha \in \mathbb{Z}_+^n.$$

Weyl-Hörmander's quantization: given $A \in S^\mu$,

$$A^{\mathrm{w}}(x, D)u = (2\pi)^{-n} \iint e^{i\langle x-y, \xi \rangle} A(\frac{x+y}{2}, \xi) u(y) dy d\xi, \ \ u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N).$$

Always:

$$p_2(X) = |X|^2/2 \in S^2_{\mathrm{reg}}, \quad \psi_j(X) = \frac{x_j + i\xi_j}{\sqrt{2}} \in S^1_{\mathrm{reg}}, \ 1 \le j \le n.$$

Hence $\psi_j^{\mathrm{w}}(x, D)$ is the annihilation operator and $\psi_j^{\mathrm{w}}(x, D)^* = \bar{\psi}_j^{\mathrm{w}}(x, D)$ is the creation operator in the (x_j, ξ_j) variables.

Regular and semiregular symbols.

$A \in S^{\mu}_{\mathrm{reg}}$ when there exists $(A_{\mu-2j})_{j\geq 0} \subset C^{\infty}(\dot{\mathbb{R}}^{2n} := \mathbb{R}^{2n} \setminus \{0\}; \mathrm{M}_N)$ positively homogeneous of degree $\mu - 2j$ (in X) such that $A \sim \sum_{j\geq 0} A_{\mu-2j}$ i.e. (with χ an excision function)

$$A - \chi \sum_{j=0}^{N} A_{\mu-2j} \in S^{\mu-2(N+1)}, \ \ \forall N \in \mathbb{Z}_+.$$

$A \in S^{\mu}_{\mathrm{sreg}}$ when $A = A^0 + A^1$ where $A^j \in S^{\mu-j}_{\mathrm{reg}}$ for $j = 0, 1$.

The principal, semiprincipal and subprincipal parts of A:

are, respectively, given by

$$A^0_{\mu}, \quad A^1_{\mu-1}, \quad A^0_{\mu-2}.$$

Globally elliptic when

$$|\det A^0_{\mu}(X)| \approx |X|^{N\mu}, \quad \forall X \in \dot{\mathbb{R}}^{2n}.$$

Our class of $N \times N$ systems: $A \in S^{\mu}_{\mathrm{sreg}}$, $\mu > 0$ globally elliptic,

where

$$A(X) = A(X)^* = p_\mu(X) I_N + A_{\mu-1}(X) + A_{\mu-2}(X) + S^{\mu-3}_{\mathrm{sreg}}, \; X \neq 0,$$

where $p_\mu \gtrsim |X|^\mu$, $A_{\mu-1}$ is such that there exists $e_0 \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathrm{M}_N)$ unitary and positively homogeneous of degree 0 such that

$$e_0^* A_{\mu-1} e_0 = \mathrm{diag}(\lambda_{j,\mu-1} I_{N_j}; 1 \leq j \leq r),$$

where $N = N_1 + N_2 + \ldots + N_r$, and $\lambda_{j,\mu-1} \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ are positively homogeneous of degree 1 such that

$$j < k \Longrightarrow \lambda_{j,\mu-1}(X) < \lambda_{k,\mu-1}(X), \; \forall X \neq 0.$$

Hence: constant multiplicity.

The Jaynes-Cummings model (interaction of a two-level atom with photon).

Consider the Pauli matrices

$$\boldsymbol{\sigma}_0 = I_2, \;\; \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \;\; \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \;\; \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\boldsymbol{\sigma}_\pm = (\boldsymbol{\sigma}_1 \pm i\boldsymbol{\sigma}_2)/2$, and the operator in $n = 1$

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_2 + \alpha(\boldsymbol{\sigma}_- \psi^{\mathrm{w}}(x, D)^* + \boldsymbol{\sigma}_+ \psi^{\mathrm{w}}(x, D)) + \gamma \boldsymbol{\sigma}_3,$$

where $\alpha, \gamma \in \mathbb{R}$. Hence:

$$A_1(X) = \alpha \begin{bmatrix} 0 & \psi(X) \\ \overline{\psi(X)} & 0 \end{bmatrix},$$

with eigenvalues

$$\lambda_{\pm,1}(X) = \pm|\alpha||X|/\sqrt{2}.$$

$A^{\mathrm{w}}(x, D)$ is a Laplacian of a complex.

Geometric models. $(d_k^* = (-1)^{nk+1} \star d_k \star)$

$$D = D_k = \frac{1}{\sqrt{2}}\Big(d_k + \sum_{j=1}^{n} x_j\, dx_j \wedge\Big) \colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$$

$$D^* = D_k^* = \frac{1}{\sqrt{2}}\Big(d_k^* + \sum_{j=1}^{n} x_j\, \mathrm{i}_{\partial/\partial_j}\Big) \colon \Omega^{k+1}(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n)$$

$$\Box_k := D_k^* D_k + D_{k-1} D_{k-1}^* = \Big(p_2^{\mathrm{w}}(x, D) + k - \frac{n}{2}\Big)\mathbf{1}_k \colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n)$$

Geometric models (cont'd.): Connection on $E = \mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$

$$\boldsymbol{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\,) \otimes E_{j,j+1}, \quad E_{j,k} w = \langle w, e_k \rangle e_j,$$

$n = N - 1$, $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, then extended

$$\boldsymbol{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\,) \otimes E_{j,j+1} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \to \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N)$$

$$\boldsymbol{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j \mathrm{i}_{\partial/\partial x_j} \otimes E_{j+1,j} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \to \Omega^k(\mathbb{R}^n; \mathbb{C}^N)$$

with curvature (we put $E_{j,N+1} = 0$ for all j)

$$\boldsymbol{F_D} = \sum_{j=1}^{N-2} \alpha_j \alpha_{j+2} (dx_j \wedge dx_{j+1} \wedge\,) \otimes E_{j,j+2} \in \Omega^2(\mathbb{R}^n; \mathrm{M}_N)$$

Geometric models (cont'd.): associated Laplacian

$$\Box_k^{(N)} = \boldsymbol{D}_k^*\boldsymbol{D}_k + \boldsymbol{D}_{k-1}\boldsymbol{D}_{k-1}^* \colon \Omega^k(\mathbb{R}^n;\mathbb{C}^N) \to \Omega^k(\mathbb{R}^n;\mathbb{C}^N)$$

$$\Box_k^{(N)} = (p_2^{\mathrm{w}}(x,D) + k - \frac{n}{2})\mathbf{1}_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\mathbf{1}_k \otimes E_{j,j+1} + \psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k \otimes E_{j+1,j}\Big)$$
$$+ \sum_{j=1}^{N-1} \alpha_j^2 \mathbf{1}_k \otimes E_{j+1,j+1} + \sum_{j=1}^{N-1} \alpha_j^2 dx_j \wedge \mathrm{i}_{\partial/\partial x_j}\mathbf{1}_k \otimes (E_{j,j} - E_{j+1,j+1}).$$

Hence, relation between the JC model and $\Box_0^{(2)}$ $(n=1)$, and $\Box_k^{(N)}$ is a generalization.

Decoupling Thm. (M-P)

$\mu > 0$, $A = A^* = A^0 + A^1 \in S^{\mu}_{\mathrm{sreg}}$, $A \sim \sum_{j\geq 0} A_{\mu-j}$, $A_\mu = p_\mu I_N$, p_μ smooth and real valued, $\exists e_0 \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathrm{M}_N)$ unitary and positively homogeneous of degree 0 such that

$$e_0^* A_{\mu-1} e_0 = \left[\begin{array}{c|c} \Lambda_{1,\mu-1} & 0 \\ \hline 0 & \Lambda_{2,\mu-1} \end{array}\right], \quad X \neq 0,$$

$\Lambda_{j,\mu-1} \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathrm{M}_{\mathrm{N_j}})$, $N = N_1 + N_2$, pos. homog. deg. $\mu - 1$ such that

$$\mathrm{Spec}(\Lambda_{1,\mu-1}(X)) \cap \mathrm{Spec}(\Lambda_{2,\mu-1}(X)) = \emptyset, \ \forall X \neq 0.$$

Then $\exists E \in S^0_{\mathrm{sreg}}$ such that $E^{\mathrm{w}}(x,D)E^{\mathrm{w}}(x,D)^* \equiv I \equiv E^{\mathrm{w}}(x,D)^* E^{\mathrm{w}}(x,D)$,

$$E^{\mathrm{w}}(x,D)^* A^{\mathrm{w}}(x,D) E^{\mathrm{w}}(x,D) \equiv B^{\mathrm{w}}(x,D), \ \ B \sim \sum_{j\geq 0} \left[\begin{array}{c|c} B_{1,\mu-j} & 0 \\ \hline 0 & B_{2,\mu-j} \end{array}\right]$$

$$B_\mu = A_\mu, \quad B_{\mu-1} = \left[\begin{array}{c|c} \Lambda_{1,\mu-1} & 0 \\ \hline 0 & \Lambda_{2,\mu-1} \end{array}\right].$$

Remark.

There are formulas for the subprincipal term B_0 in terms of A and its transformation-law when changing gauge e_0.

Principal, semiprincipal, subprincipal:

With $N_1 + \ldots + N_r = N$,

$$A_\mu = B_\mu = p_\mu I_N, \;\; B_{\mu-1} = \mathrm{diag}(\lambda_{j,\mu-1} I_{N_j};\; 1 \le j \le r),$$

and if

$$\pi_j \colon \mathbb{C}^N \longrightarrow \mathbb{C}^N = \bigoplus_{j=1}^{r} \mathbb{C}^{N_j},$$

is the orthogonal projection onto the jth $\mathbb{C}^{N_j} \subset \mathbb{C}^N$,

$$B_{\mu-2} = \mathrm{diag}(B^{(j)}_{\mu-2};\; 1 \le j \le r),$$

$$B^{(j)}_{\mu-2} = \pi_j e_0^* A_{\mu-2} e_0 \pi_j^* - i \pi_j e_0^* \{p_\mu, e_0\} \pi_j^*, \;\; j = 1, \ldots, r.$$

The Weyl spectral counting function $N_A(\lambda) = \sum_{\lambda_j \le \lambda} 1$, $\mu = 2$

Give meaning to

$$\mathscr{F}_{\lambda \to t}\Big(\sum_{j \ge 0} \delta(\lambda - \lambda_j)\Big) = \sum_{j \ge 0} e^{-it\lambda_j},$$

by considering the Schrödinger equation: for $\phi \in \mathscr{S}(\mathbb{R})$ let

$$U_\phi := \int \phi(t) e^{-itA^{\mathrm{w}}} dt = \int (-i)^k (\partial_t^k \phi)(t) (A^{\mathrm{w}})^{-k} e^{-itA^{\mathrm{w}}} dt,$$

hence $S_A(t) := \mathrm{Tr}(e^{-itA^{\mathrm{w}}}) \colon \mathscr{S}(\mathbb{R}) \ni \phi \longmapsto \mathrm{Tr}(U_\phi) \in \mathbb{C}$, is a tempered distribution. (Tr is the matrix trace of the trace of the Schwartz kernel.) If $\varphi \in C_c^\infty(\mathbb{R})$ One has to construct a parametrix of the Schrödinger group $t \longmapsto e^{-itA^{\mathrm{w}}}$ in the form of an FIO.

Crucial step: construction for the blockwise-reduced system and further use of Mehler formulas. Use of WKB. Transport equations are scalar in the principal part, but not scalar in the zeroth order coefficient, due to the subprincipal part, but no problem, since characteristics are existing for all times.

Reduced propagator:

Suppose $\Lambda_{j,1} = \lambda_{j,1} I_{N_j}$ and consider (Egorov's Thm.)

$$t \mapsto P(t) := e^{itp_2^{\mathrm{w}}} (B^{\mathrm{w}} - p_2^{\mathrm{w}}) e^{-itp_2^{\mathrm{w}}}.$$

With $\phi_{1,j}(t, X) = -\int_0^t \lambda_{j,1} \circ \exp(sH_{p_2})(X) ds$, then $\exists$ $N_j \times N_j$ matrix $\alpha_j \in C^\infty(\mathbb{R}_t; S^0_{\mathrm{sreg}})$ such that $t \longmapsto F(t) = \mathrm{diag}\Big((e^{i\phi_{1,j}(t)} \alpha_j(t))^{\mathrm{w}}; 1 \le j \le r\Big)$ solves

$$(i\partial_t - P)F \in C^\infty(\mathbb{R}_t; \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathrm{M}_N), \quad F\big|_{t=0} = I_N + \text{smoothing}.$$

The parametrix for the Schrödinger propagator $t \mapsto e^{-itB^{\mathrm{w}}}$:

$$U_B(t) = U_0(t)F(t) = \operatorname{diag}\Big((e^{i(\phi_2(t)+\tilde{\phi}_{1,j}(t))}\tilde{\alpha}_j(t))^{\mathrm{w}}; 1 \leq j \leq r \Big),$$

$t \in (2k\pi - \varepsilon, 2k\pi + \varepsilon), \quad k \in \mathbb{Z}$, where

$$U_0(t) = \cos(t/2)^{-n}(e^{i\phi_2(t)})^{\mathrm{w}}, \quad \phi_2(t,X) = -2\tan(t/2)p_2(X),$$

$$\tilde{\phi}_{1,j}(t,X) = \phi_{1,j}(X - 2\tan(t/2)JX).$$

Remark:

Weyl composition formula to compute $U_0(t)F(t)$.

The parametrix for the Schrödinger propagator $t \mapsto e^{-itA^{\mathrm{w}}}$:

$$U_A(t) \equiv E^{\mathrm{w}} U_B(t) (E^{\mathrm{w}})^*$$

where now we have the Weyl calculus and have symbol composition formulas.

Thm. (M-P, Weyl asymptotics).

$A \in S^2_{\mathrm{sreg}}$, $N \times N$ system of the class considered here. If $\rho \in \mathscr{S}(\mathbb{R})$ is s.t. $\hat{\rho} \in C_c^\infty$, supp $\hat{\rho} \subset (-\varepsilon, \varepsilon)$, $\hat{\rho} \equiv 1$ near 0,

$$(\mathrm{N}_A \star \rho)(\lambda) = \sum_{j=1}^{r} \frac{N_j}{(2\pi)^n} \int_{p_2 + \lambda_{j,1} \leq \lambda} dX - \frac{1}{(2\pi)^n} \int_{p_2 = \lambda} \mathrm{Tr}(a_0) \frac{ds}{|\nabla p_2|} + O(\lambda^{n-3/2}),$$

as $\lambda \to +\infty$. Hence

$$\mathrm{N}_A(\lambda) = \frac{N}{(2\pi)^n} \int_{p_2 \leq 1} dX \ \lambda^n - \Big((2\pi)^{-n} \int_{p_2 = 1} \mathrm{Tr}(a_1) \frac{ds}{|\nabla p_2|} \Big) \lambda^{n-1/2} + O(\lambda^{n-1}),$$

as $\lambda \to +\infty$.

In particular

$$N'_A \star \rho = O(\lambda^{n-1}), \ \ \lambda \to +\infty.$$

Tauberian Thm.

ρ as above. If $\exists\ \gamma \in \mathbb{R}$ such that $(N'_A \star \rho)(\lambda) = O(\lambda^\gamma)$ and $(N'_A \star \chi)(\lambda) = o(\lambda^\gamma)$ for all χ s.t. $\hat{\chi} \in C_c^\infty(0, +\infty)$, then

$$N_A(\lambda) = (N_A \star \rho)(\lambda) + o(\lambda^\gamma), \ \ \lambda \to +\infty.$$

Thm. (M-P, Refined Weyl asymptotics).

$A \in S^2_{\mathrm{sreg}}$, $N \times N$ system of the class considered here. Suppose that

$$Z_j := \{\omega \in \mathbb{S}^{2n-1};\ \partial_\omega^\alpha \int_0^{2\pi} (\lambda_{j,1} \circ \exp(tH_{p_2}))(\omega)dt = 0,\ \forall|\alpha| \geq 1\}$$

has measure 0, $1 \leq j \leq r$. Then

$$\mathrm{N}_A(\lambda) = (2\pi)^{-n}\Big(\sum_{j=1}^{r} N_j \int_{p_2+\lambda_{j,1}\leq\lambda} dX - \int_{p_2=\lambda} \mathrm{Tr}(a_0)\frac{ds}{|\nabla p_2|}\Big) + o(\lambda^{n-1}),$$

as $\lambda \to +\infty$. In particular,

$$\mathrm{N}_A(\lambda) = (2\pi)^{-n}\Big(N\lambda^n \int_{p_2\leq 1} dX - \lambda^{n-1/2}\int_{p_2=1} \mathrm{Tr}(a_1)\frac{ds}{|\nabla p_2|}$$

$$+\lambda^{n-1}\int_{p_2=1}(\frac{n}{2}\mathrm{Tr}(a_1^2) - \mathrm{Tr}(a_0))\frac{ds}{|\nabla p_2|}\Big) + o(\lambda^{n-1}),$$

as $\lambda \to +\infty$.

Example.

The JC system satisfies the assumptions of the refined Weyl law. In fact, (with $\alpha \neq 0$) $\lambda_{\pm,1}(X) = \pm|\alpha||X|/\sqrt{2}$, and $Z_{\pm}$ has full measure. ($n = 1$, $N = 2$.)

Asymptotics:

$$\mathrm{N}(\lambda) = (2\pi)^{-1} 2\lambda \int_{p_2 \leq 1} dX + O(1) = 2\lambda + O(1), \quad \lambda \to +\infty.$$

Recall: $p_2(X) = |X|^2/2$.

Example.

The system $\square_0^{(3)}\colon C^\infty(\mathbb{R}^2;\mathbb{C}^3)\to C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ (recall that $\alpha_1,\alpha_2\neq 0$) is such that the eigenvalues are

$$\lambda_{\pm,1}(X)=\pm(\alpha_1^2|\psi_1(X)|^2+\alpha_2^2|\psi_2(X)|^2)^{1/2},\quad \text{and } 0,$$

constant multiplicity 1. Here $n=2$, $N=3$. Consider then $\mu_\kappa(X):=\kappa\lambda_+(X)+(1-\kappa)\lambda_-(X)$, $\kappa\in(0,1)$, and the deformation of $\square_0^{(3)}$ obtained by considering the semiprincipal term whence the assumptions for the refined Weyl law are fulfilled when $\kappa\neq 1/2$ and $\alpha_1^2\neq\alpha_2^2$:

$$A(X)=p_2(X)I_3+A_{1,\kappa}(X)+A_0(X),$$

$$A_{1,\kappa}(X)=\begin{bmatrix}\mu_\kappa|f_2|^2 & \lambda_+\bar f_1 & -\mu_\kappa\bar f_1\bar f_2\\ \lambda_+ f_1 & 0 & \lambda_+\bar f_2\\ -\mu_\kappa f_1 f_2 & \lambda_+ f_2 & \mu_\kappa|f_1|^2\end{bmatrix},\quad f_j=\alpha_j\psi_j/|\alpha\psi|,$$

$$A_0(X)=\sum_{j=1}^{2}\alpha_j^2E_{j+1,j+1}+\sum_{j=1}^{2}(E_{j,j}-E_{j+1,j+1})\Rightarrow \mathrm{Tr}(A_0)=\alpha_1^2+\alpha_2^2.$$

$$\mathrm{N}(\lambda)=(2\pi)^{-2}\Big(3|\mathbb{S}^3|\lambda^2-\lambda^{3/2}\int_{p_2=1}\mu_\kappa\frac{ds}{|\nabla p_2|}+\lambda\int_{p_2=1}(2\lambda_+^2+\mu_\kappa^2-(\alpha_1^2+\alpha_2^2))\frac{ds}{|\nabla p_2|}\Big)+o(1),$$

as $\lambda\to+\infty$.