* Non CM p-adic analytic families of modular forms

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Abstract: A p-adic analytic family is a family of modular forms f_P (eigenforms of all Hecke operators) parameterized by geometric points P of an integral scheme Spec($\mathbb{I}$) finite flat over Spec($\mathbb{Z}_p[[T]]$) whose geometric points is an open unit disk of $\overline{\mathbb{Q}}_p$. The Hecke eigenvalues of $T(l)$ of f_P is given by $a_l(P)$ for $a_l \in \mathbb{I}$ (so analytic). Such a family has corresponding Galois representation $\rho_{\mathbb{I}}$: Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) $\to GL_2(\mathbb{I})$. The family is said to have complex multiplication if $\rho_{\mathbb{I}}$ has abelian image over an open subgroup of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$). In this talk, we emphasize by examples importance of characterizing CM and non CM families.

§0. Notation:

Fix an odd prime p, a positive integer N prime to p and field embeddings $\mathbb{C} \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Consider the space of cusp forms $S_{k+1}(\Gamma_0(Np^r), \psi)$ $(r \geq 1)$.

Let the ring $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively.

The Hecke algebra over $\mathbb{Z}$ is $H = \mathbb{Z}[\psi][T(n)|n = 1, 2, \cdots] \subset \mathrm{End}(S_{k+1}(\Gamma_0(Np^r), \psi))$. We put $H_{k+1,\psi} = H \otimes_{\mathbb{Z}[\psi]} \mathbb{Z}_p[\psi]$. Sometimes our $T(p)$ is written as $U(p)$ as the level is divisible by p. The ordinary part $h_{k+1,\psi} \subset H_{k+1,\psi}$ is the maximal ring direct summand on which $U(p)$ is invertible. Let $\psi_1 = \psi_N \times$ the tame p-part of ψ.

We call a Galois representation ρ *CM* if there exists an open subgroup $G \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the semi-simplification $(\rho|_G)^{ss}$ has abelian image over G.

§1. Big Hecke algebra

We have a unique 'big' Hecke algebra $\mathbf{h} = \mathbf{h}_{\psi_1}$ characterized by the following two properties:

1. $\mathbf{h}$ is free of finite rank over $\Lambda := \mathbb{Z}_p[[T]]$ equipped with $T(n) \in \mathbf{h}$ for $n = 1, 2, \dots,$

2. if $k \geq 1$ and $\varepsilon : \mathbb{Z}_p^\times/\mu_{p-1} \to \mu_{p^\infty}$ is a character,
$$\mathbf{h} \otimes_{\Lambda, t \mapsto \varepsilon(\gamma)\gamma^k} \mathbb{Z}_p[\psi_k \varepsilon] \cong h_{k+1, \varepsilon\psi_k}$$
$(\gamma = 1 + p)$ for $\psi_k := \psi_1 \omega^{1-k}$, sending $T(n) \otimes 1$ to $T(n)$, where ω is the Teichmüller character.

Hereafter, we put $t = 1 + T \in \Lambda$.

§2. Analytic family

A point P of Spec($\mathbb{I}$)($\overline{\mathbb{Q}}_p$) is called **arithmetic** if it contains $(1+T-\varepsilon(\gamma)\gamma^k)$ with $k \geq 1$. If P is arithmetic, we have a Hecke eigenform

$$f_P \in S_{k+1}(\Gamma_0(Np^{r(P)}), \varepsilon\psi_k)$$

such that $f_P|T(n) = a_P(n)f_P \ \ (n = 1, 2, \dots)$ for $a_P(n) := P(a(n)) = (a(n) \mod P) \in \overline{\mathbb{Q}}_p$. We write $\varepsilon_P = \varepsilon$ and $k(P) = k$ for such a P. Thus $\mathbb{I}$ gives rise to an **analytic family**

$$\mathcal{F}_{\mathbb{I}} = \{f_P | \text{arithemtic } P \in \text{Spec}(\mathbb{I})\}.$$

The Hecke field $K(f_P)$ is the field generated over a number field K by all Hecke eigenvalues of f_P.

Note that $[\mathbb{Q}(\psi_k, f_P) : \mathbb{Q}(\psi_k, a_P(p))]$ is bounded by $\text{rank}_{\mathbb{Z}_p[\varepsilon\psi_k]} h_{k+1,\varepsilon\psi_k}$ equal to $\text{rank}_\Lambda \mathbf{h}$ independent of P as f_P^σ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixeing $\mathbb{Q}(\varepsilon\psi_k, a_P(p))$ has slope 0 still comeing from $h_{k+1,\varepsilon\psi_k}$.

§3. Galois representation

Each **irreducible component** Spec($\mathbb{I}$) $\subset$ Spec($\mathbf{h}$) has a **Galois representation**

$$\rho_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(2)$$

with **coefficients** in the quotient field of $\mathbb{I}$ such that

$$\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) = a(l)$$

(for the image $a(l)$ in $\mathbb{I}$ of $T(l)$) for almost all primes ℓ. Usually $\rho_{\mathbb{I}}$ has values in $GL_2(\mathbb{I})$, and we suppose this for simplicity. The component $\mathbb{I}$ is called *CM* if $\rho_{\mathbb{I}}$ has abelian image over an open subgroup.

We regard $P \in$ Spec($\mathbb{I}$)($\overline{\mathbb{Q}}_p$) as an algebra homomorphism $P : \mathbb{Z}_p[[T]] \to \overline{\mathbb{Q}}_p$, and we put $\rho_P = P \circ \rho_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$.

If $\mathbb{I}$ is a CM component, it is known that for an imaginary quadratic field M in which p splits, there exists a Galois character $\varphi_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^{\times}$ such that $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \varphi_{\mathbb{I}}$.

§4. p-adic L

One expects to have a two variable p-adic L-function $L_p(s, \rho_{\mathbb{I}}^{sym\otimes n})$ for each symmetric power interpolating $L(s, f_P^{sym\otimes n})$. In particular, we have $L_p(s, Ad(\rho_{\mathbb{I}}))$. We have $L_p := L_p(1, Ad(\rho_{\mathbb{I}})) = L_p(1, \rho_{\mathbb{I}}^{sym\otimes 2} \otimes \det(\rho_{\mathbb{I}})^{-1}) \in \mathbb{I}$ and

$$(L_p \mod P) = L(1, Ad(f_P))/\text{period} \quad \text{for all arithemtic } P.$$

We also know that if $\text{Spec}(\mathbf{h}) = \text{Spec}(\mathbb{I}) \cup \text{Spec}(\mathbb{X})$ for the complement $\mathbb{X}$, we have a congruence criterion

$$\text{Spec}(\mathbb{I}) \cap \text{Spec}(\mathbb{X}) = \text{Spec}(\mathbb{I} \otimes_{\mathbf{h}} \mathbb{X}) \cong \text{Spec}(\mathbb{I}/(L_p)).$$

Adding the cyclotomic variable, because of the modifying Euler p-factor, $L_p(s, Ad(\rho_{\mathbb{I}}))$ has an exceptional zero at $s = 1$, and for an $\mathcal{L}$-invariant $0 \neq \mathcal{L}(Ad(\rho_{\mathbb{I}})) \in \mathbb{I}$, we have

$$\frac{dL_p}{ds}(s, Ad(\rho_{\mathbb{I}}))|_{s=1} = \mathcal{L}(Ad(\rho_{\mathbb{I}}))L_p.$$

§5. Is characterizing CM components important?

• (Well known) $\mathbb{I}$ is CM $\Leftrightarrow$ there exist an imaginary quadratic field $M = \mathbb{Q}[\sqrt{-D}]$ in which p splits into $\mathfrak{p}\overline{\mathfrak{p}}$ and a character $\varphi : G_M = \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^\times$ of conductor $\mathfrak{c}\mathfrak{p}^\infty$ for an ideal $\mathfrak{c} \nmid p$ with $\rho_{\mathbb{I}} = \mathrm{Ind}_M^{\mathbb{Q}} \varphi$. Moreover we have

$$L_p = L_p(\varphi^-) L(0, \left(\frac{M/\mathbb{Q}}{}\right)),$$

where $\varphi^-(\sigma) = \varphi(c\sigma c^{-1}\sigma^{-1})$ for complex conjugation c, and $L_p(\varphi^-)$ is the Katz p-adic L-function associated to φ^-. This is a base of the proof by Mazur/Tilouine of the anticyclotomic main conjecture.

• (Known) $\mathbb{I}$ is CM $\Leftrightarrow$ $\rho_{\mathbb{I}}$ modulo an arithmetic prime is CM.

• (Almost known) $\mathbb{I}$ CM $\Leftrightarrow$ $\rho_{\mathbb{I}} \mod p$ is CM. This is almost equivalent to the vanishing of the Iwasawa μ-invariant for $L_p(\varphi^-)$.

§6. Examples continue.

• (Known for Greenberg's $\mathcal{L}$-invariant) $\mathcal{L}(Ad(\rho_{\mathbb{I}}))$ is constant in $\overline{\mathbb{Q}}_p$ if and only if $\mathbb{I}$ is CM.
• (Likely) $[\mathbb{Q}(a_P(l)) : \mathbb{Q}]$ is bounded for P running over infinitely many arithmetic points with fixed level $N \Leftrightarrow \mathbb{I}$ is CM? Maeda's conjecture implies this if the level is $N = 1$.
• (Mostly true) Fix $k \geq 1$ and a prime $l \nmid N$. Then $[K(a_P(l)) : K]$ ($K = \mathbb{Q}(\mu_{p^\infty})$) is bounded for P running over infinitely many arithmetic points with fixed weight $k \Leftrightarrow \mathbb{I}$ is CM. This is true for a prime l in a density 1 set (including p).
• (Mostly true) Let $L := \mathbb{Z}_{(p)}[t^{\log_p(\alpha)/\log_p(\gamma)}] \subset \Lambda$, where α runs all algebraic integers in $\overline{\mathbb{Q}}_p$. Then $a(l)$ is transcendental over L. True for density one set of primes l. Note that $P(L) \subset \overline{\mathbb{Q}}$.
• (Mostly known) $\mathbb{I}$ non CM $\Leftrightarrow$ $\mathrm{Im}(\rho_{\mathbb{I}}) \cap \mathrm{SL}_2(\Lambda) \supset \Gamma_\Lambda(\mathfrak{c}) = \mathrm{Ker}(\mathrm{SL}_2(\Lambda) \to \mathrm{SL}_2(\Lambda/\mathfrak{c}))$ for a non-zero ideal $\mathfrak{c}$. What is $\mathfrak{c}$? Often a p-adic L.
• (Known) $\mathbb{I}$ non CM $\Leftrightarrow$ $\rho_{\mathbb{I}}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is upper triangular indecomposable. This is due first to Ghate–Vatsal and later Hida–Zhao (in a different way).

§7. Wild Guesses.

• (?) $a(l) \mod p$ is transcendental over L/pL in $\mathbb{I} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ if $p \nmid N$. This implies Iwasawa's conjecture of vanishing of μ-invariant of Kubota–Leopoldt p-adic L.

• (Wild guess) Is $\mathbb{I}$ is CM if and only if $\mathcal{L}(Ad(f_P)) = \log_p(\mathfrak{p}/\overline{\mathfrak{p}})$ for one arithmetic P up to algebraic numbers? Here taking a high power $(\mathfrak{p}/\overline{\mathfrak{p}})^h = (\alpha)$, $\log_p(\mathfrak{p}/\overline{\mathfrak{p}}) = \frac{1}{h}\log_p(\alpha)$.

• (Wild guess) Then $\mathbb{I}$ is CM if the family contains a rational Hecke eigenform f_P of weight $k \geq 27$??

All interesting outcome; so, perhaps, characterization is important!

§8. Theorem:

For any infinite set $\mathcal{A}$ of arithmetic points in Spec$(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *of a fixed weight (varying p-power level), there exists a density one subset Ξ of primes including p such that*

$$\lim_{P \in \mathcal{A}} [\mathbb{Q}(\mu_{p^\infty}, a_P(l)) : \mathbb{Q}(\mu_{p^\infty})] = \infty$$

for each $l \in \Xi$.

There is a Hilbert modular version, which is somehow more interesting as Spec$(\mathbb{I})$ has dimension equal to the degree d of the base totally real field, but still this is valid for any infinite set $\mathcal{A}$ of fixed weight (whose closure could be very small).

If one allows to move all Hecke eigenforms of fixed weight $k \geq 2$ (not just those in the family), there is an analytic method invented by Serre to show infinite growth of Hecke fields. This method is now generalized to automorphic forms on unitary Shimura varieties by Shin and Templier.

We might give a sketch of a proof for $l = p$ later.

§9. p-Adic L-function

Recall of adjoint L-functions and $\mathcal{L}$-invariant to state an application.

We have one variable $L_p \in \mathbb{I}$ characterized by

$$L_p := L_p(1, Ad(\rho_{\mathbb{I}})) = L_p(1, \rho_{\mathbb{I}}^{sym\otimes 2} \otimes \det(\rho_{\mathbb{I}})^{-1})$$

and

$$L_p(P) := P(L_p) = \frac{L(1, Ad(f_P))}{\text{period}}$$

for all arithmetic P.

§10. Congruence criterion and $\mathcal{L}$-invariant

If $\mathrm{Spec}(\mathbf{h}) = \mathrm{Spec}(\mathbb{I}) \cup \mathrm{Spec}(\mathbb{X})$ for the complement $\mathbb{X}$, (under a mild assumption)

$$\mathrm{Spec}(\mathbb{I}) \cap \mathrm{Spec}(\mathbb{X}) = \mathrm{Spec}(\mathbb{I} \otimes_{\mathbf{h}} \mathbb{X}) \cong \mathrm{Spec}(\frac{\mathbb{I}}{(L_p)})$$

Adding the cyclotomic variable, because of the modifying Euler p-factor,

$L_p(s, Ad(\rho_{\mathbb{I}}))$ has **exceptional zero** at 1,

and for an analytic $\mathcal{L}$-**invariant** $0 \neq \mathcal{L}^{an}(Ad(\rho_{\mathbb{I}}))$ in $\mathbb{I}[\frac{1}{p}]$, we expect to have

$$L_p'(s, Ad(\rho_{\mathbb{I}})) \stackrel{?}{=} \mathcal{L}^{an}(Ad(\rho_{\mathbb{I}}))L_p.$$

§11. An application

The p-adic L-function $L_p(s, Ad(f_P))$ has an exceptional zero at $s = 1$ coming from modifying Euler p-factor. Greenberg proposed Galois cohomological definition of an $\mathcal{L}$-invariant $\mathcal{L}(Ad(f_P))$, and we have the following formula in IMRN **59** (2004) 3177–3189: for $c = -2\log_p(\gamma)$ and $a = a(p)$,

$$\mathcal{L}(Ad(f_P)) = c \cdot a^{-1} t \frac{da}{dt}|_{t=\gamma^{k(P)}\varepsilon_P(\gamma)}.$$

Thus $P \to \mathcal{L}(Ad(f_P))$ is interpolated over Spec($\mathbb{I}$) as an analytic function. If $\mathbb{I}$ has CM, as we have seen, $a(p) = ct^s$ for some $s \in \mathbb{Z}_p$. Thus we get

Theorem 1 (Constancy). $P \to \mathcal{L}(Ad(f_P))$ *is* **constant** *if and only if* $\rho_{\mathbb{I}}$ *has CM.*

§12. Finiteness proposition, start of the proof of Theorem

Two nonzero numbers a and b **equivalent** if a/b is a root of unity. Let $\mathcal{K}_d$ be the set of all extensions of $\mathbb{Q}[\mu_{p^\infty}]$ of **degree** $d < \infty$ inside $\overline{\mathbb{Q}}$ whose ramification at l is **tame**. Here is a slight improvement:

Proposition 1 (Finiteness Proposition)**.** *We have only* **finitely many** *Weil l-numbers of a given weight in the set-theoretic union* $\bigcup_{K\in\mathcal{K}_d} K$ **up to equivalence***.*

The tameness assumption is not necessary if $l \neq p$ actually. The proof is an elementary but subtle analysis of prime decomposition of the Weil number. Tameness is assumed since in that case, there are only **finitely many** isomorphism classes of $K \otimes_{\mathbb{Q}} \mathbb{Q}_l$ for $K \in \mathcal{K}_d$, and one can consider the prime factorization in a fixed algebra $K \otimes_{\mathbb{Q}} \mathbb{Q}_l$ picking one isomorphism class.

§13. A rigidity lemma

Let W be a p-adic valuation ring finite flat over $\mathbb{Z}_p$ and $\Phi(T) \in W[[T]]$. Regard Φ as a function of $t = 1 + T$; so, $\Phi(1) = \Phi|_{T=0}$. We start with a lemma:

Lemma 1 (Rigidity)**.** *Suppose that there is an infinite subset $\Omega \subset \mu_{p^\infty}(\overline{K})$ such that $\Phi(\Omega) \subset \mu_{p^\infty}$. Then there exist $\zeta_0 \in \mu_{p^\infty}$ and $s \in \mathbb{Z}_p$ such that $\zeta_0^{-1}\Phi(t) = t^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n$.*

Note here that if

$$Z \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m = \mathrm{Spf}(W[t, \widehat{t^{-1}, t'}, t'^{-1}])$$

is a formal subtorus, it is defined by the equation $t = t'^s$ for $s \in \mathbb{Q}_p$. Thus we need to prove that the graph of the function $t \mapsto \Phi(t)$ in $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ is a **formal subtorus**.

This is an exercise (or see Section §3.1.4 of my recent Springer book).

§14. Frobenius eigenvalues.

Let A be an eigenvalue of $\rho_{\mathbb{I}}(Frob_l)$ if $l \nmid Np$, and if $l = p$, we put $A = a(p)$.

Suppose $[\mathbb{Q}(\mu_{p^\infty}, a_P(l)) : \mathbb{Q}(\mu_{p^\infty})]$ is bounded over $\mathcal{A}$ (so, we want to show that $\mathbb{I}$ is CM). This implies $K_P = \mathbb{Q}(\mu_{p^\infty})(A_P)$ has bounded degree over $\mathbb{Q}(\mu_{p^\infty})$; so, for primes $l \gg 0$, K_P is **tamely** ramified in K_P (the tameness assumption in Finiteness Proposition). Indeed, $[\mathbb{Q}(\psi_k, f_P) : \mathbb{Q}(\psi_p)] \leq \mathrm{rank}_\Lambda \mathbf{h}$.

Proposition 2 (Eigenvalue formula)**.** *For a root of unity ζ_0,*

$$A(T) = \zeta_0 t^s = \zeta_0 \sum_{n=0}^{\infty} \binom{s}{n} T^n.$$

Since $a(l)_P$ hits the same Weil number α up to p-power root of unity, $a(l)/\alpha$ satisfies the assumption of the rigidity lemma (after some variable change); so, the above formula follows (a difficult point is to show ζ_0 is a root of unity, which we do not discuss here).

§15. Abelian image lemma

Consider the endomorphism $\sigma_s : t \mapsto t^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n$ of a power series ring $W[[T]]$ for $s \in \mathbb{Z}_p$. Let R be an integral domain over $W[[T]]$ of characteristic different from 2. Assume that the endomorphism σ_2 on $W[[T]]$ extends to an endomorphism σ of R.

Lemma 2 (Abelian image)**.** *Take a continuous representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(R)$ for a field $F \subset \overline{\mathbb{Q}}$, and put $\rho^\sigma := \sigma \circ \rho$. If $\mathrm{Tr}(\rho^\sigma) = \mathrm{Tr}(\rho^2)$. Then ρ is absolutely reducible over the quotient field Q of R.*

Heuristically, the assumption implies that the **square map**: $\sigma \mapsto \rho^2(\sigma)$ is still a representation ρ^σ; so, it has to have an abelian image. Since any automorphism of the quotient field Q of $\mathbb{Z}_p[[T]]$ extends to its algebraic closure $\overline{Q} \supset \mathbb{I}$, we can apply the above lemma to $\rho_{\mathbb{I}}$.

§**15. Proof of the theorem for $l = p$.** Suppose

$$\lim_{P\in\mathcal{A}}[\mathbb{Q}(\mu_{p^\infty}, a_P(p)) : \mathbb{Q}(\mu_{p^\infty})] < \infty.$$

Step 1: We have $[K_P : \mathbb{Q}(\mu_{p^\infty})]$ bounded independent of l; so, if $l \gg 0$, K_P is at most tamely ramified.
Step 2: We have $\mathrm{Tr}(\rho(Frob_l)) = \zeta(1+T)^a + \zeta'(1+T)^{a'}$ for two roots of unity ζ, ζ' and $a, a' \in \mathbb{Q}_p$.
Step 3: Not too difficult to show that the order of ζ, ζ' is bounded independent of l.
Step 4: Let $\mathfrak{m}_N = \mathfrak{m}_{\mathbb{I}}^N + (T)$ and $\overline{\rho} = \rho_{\mathbb{I}} \mod \mathfrak{m}_N$ for $N \gg 0$ and F be the splitting field of $\overline{\rho}$; so, taking $N \gg 0$, we may assume

$$\mathrm{Tr}(\rho(Frob_l^f)) = (1+T)^{fa} + (1+T)^{fa'}$$

for all $l \gg 0$ as long as $Frob_l^f \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.
Step 5: This shows

$$\mathrm{Tr}(\sigma_s \circ \rho) = \mathrm{Tr}(\rho^s)$$

over $G = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then by the above lemma, $\rho^{ss}|_G$ is abelian, and hence $\mathbb{I}$ is CM. □