$\mathcal{L}$-INVARIANT OF p-ADIC L-FUNCTIONS

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Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the field of all algebraic numbers. We fix a prime $p > 2$ and a p-adic absolute value $|\cdot|_p$ on $\overline{\mathbb{Q}}$. Then $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}$ under $|\cdot|_p$. We write $W = \left\{x \in K \middle| |x|_p < 1\right\}$ for the p-adic integer ring of sufficiently large extension $K/\mathbb{Q}_p$ inside $\mathbb{C}_p$. We write $\overline{\mathbb{Q}}_p$ for the field of all numbers in $\mathbb{C}_p$ algebraic over $\mathbb{Q}_p$.

We consider a Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_{\ell} \left(\sum_{n=0}^{\infty} a_{\ell^n} \ell^{-ns} \right) = \prod_{\ell} \Big((1-\alpha_\ell^{(1)}\ell^{-s})(1-\alpha_\ell^{(2)}\ell^{-s}) \cdots (1-\alpha_\ell^{(d)}\ell^{-s}) \Big)^{-1}$$

with an Euler product over primes ℓ. Heuristically, if $|n^m - n^{m'}|_p < \varepsilon$ and $|a_n - b_n|_p < \varepsilon$ for all integers n, we would expect that

$$|\sum_{n=1}^{\infty} a_n n^m - \sum_{n=1}^{\infty} b_n n^m|_p < \varepsilon$$

up to some constant (the transcendental factor of the L-values) at good integers $s = m$. Even if $s \equiv s' \mod p^M(p-1)$ for $g \gg 0$ ($\Leftrightarrow |s - s'|_p \leq p^{-M}$), p^s and $p^{s'}$ may not be very close p-adically, while $|\ell^s - \ell^{s'}|_p \leq p^{-1-M}$ if $\ell \neq p$. If m is negative, we will have further trouble interpolating the L-values if we do not remove Euler p-factors. Thus mod p class of L-values is better represented by the L-value with a certain **Euler p-factor removed**.

Here is an example. Start with a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \overline{\mathbb{Q}}^{\times}$ with $\chi(-1) = -1$. Here the word "Dirichlet character" means that it is multiplicative and $\chi(n) = 0$ if n has a nontrivial common factor with N. For positive integer m and m', as long as $|n^m - n^{m'}|_p < 1$ (that is, $m \equiv m' \mod (p-1)$), it is known from the time of Euler

$$|(1-\chi(p)p^{m-1})L(1-m,\chi) - (1-\chi(p)p^{m'-1})L(1-m',\chi)|_p < 1.$$

By a work of Kubota–Leopoldt and Iwasawa, we have a p-adic analytic L-function $L_\chi(s) = \Phi(\gamma^{1-s} - 1)$ for a power series $\Phi(X) \in \Lambda = W[[X]]$ and $\gamma = 1 + p$ such that

$$L_\chi(m) = \Phi(\gamma^{1-m} - 1) = (1-\chi(p)p^{m-1})L(1-m,\chi)$$

for all positive integer m as long as $|n^m - n|_p < 1$ for all n prime to p. The Iwasawa's analyticity $L_\chi(s) = \Phi(\gamma^{1-s} - 1)$ guarantees that there are only finitely many zeros (counting with multiplicity) of $L_\chi(s)$ in W.

If we suppose $\chi = \left(\frac{-D}{\cdot}\right)$ for a square free positive integer D, the modifying Euler factor vanishes at $s = 1$ if the Legendre symbol $\left(\frac{-D}{p}\right) = 1 \Leftrightarrow (p) = \mathfrak{p}\overline{\mathfrak{p}}$ in $\mathbb{Z}[\sqrt{-D}]$

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with $\mathfrak{p} = \{x \in \mathbb{Z}[\sqrt{-D}] \big| |x|_p < 1\}$. In other words, $L_\chi(1) = 0$ and $\lambda \geq 1$. This type of zeros of a p-adic L-function is called an *exceptional zero*. We may regard χ as a Galois character $\mathrm{Gal}(\mathbb{Q}[\mu_N]/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \{\pm 1\}$, and we remark that $\chi(Frob_p) = 1$ to have the exceptional zero. For a given p-adic L-function, we write e for the number of (linear) Euler factors producing the exceptional zero.

Here is another example. Start with an elliptic curve $E_{/\mathbb{Q}}$ defined by the equation $y^2 = 4x^3 - g_2 x - g_3$ with $g_j \in \mathbb{Z}$. If $4x^3 - g_2 x - g_3 \equiv 0 \mod \ell$ has three distinct roots in $\overline{\mathbb{F}}_\ell$, the reduced curve $E_\ell = E \mod \ell$ over $\mathbb{F}_\ell$ defined by $y^2 \equiv 4x^3 - g_2 x - g_3 \mod \ell$ remains to be an elliptic curve. Counting the number of points of $E_\ell(\mathbb{F}_\ell)$, we define $a_\ell = |\mathbf{P}^1(\mathbb{F}_\ell)| - |E_\ell(\mathbb{F}_\ell)| = 1 + \ell - |E_\ell(\mathbb{F}_\ell)|$. Then the Hasse-Weil L-function of E twisted by χ is given by

$$L(s, E, \chi) = \prod_\ell (1 - a_\ell \chi(\ell)\ell^{-s} + \chi(\ell)^2 \ell^{1-2s})^{-1} \qquad \text{(Hasse)}.$$

Take the Galois representation ρ_E of E on $\varprojlim_n \mathrm{Ker}(p^n : E \to E) \cong \mathbb{Z}_p^2$. Then

$$L(s, E) = \prod_\ell \det(1 - \rho_E(Frob_\ell)|_{V_{I_\ell}} \ell^{-s})^{-1} \qquad \text{(Weil)}.$$

Split the Euler factor as a product of linear factors

$$(1 - a_p p^{-s} + p^{1-2s}) = (1 - \alpha p^{-s})(1 - \beta p^{-s}),$$

if one of α and β, say α, is a p-adic unit (so, $|\alpha|_p = 1$), E has either ordinary or multiplicative reduction modulo p. We suppose this *ordinarity condition*. Then by the solution of Shimura-Taniyama conjecture by Wiles et al, this L-function has p-adic analogue constructed by Mazur such that we have $\Phi_E(X) \in \Lambda$ with $\Phi_E(\varepsilon(\gamma) - 1) = (1 - \alpha^{-1}\varepsilon(p))\frac{G(\varepsilon^{-1})L(1,E,\varepsilon)}{\Omega_E}$ for all p-power order character $\varepsilon : \mathbb{Z}_p^\times \to W^\times$; in other words, $L_p(s, E) = \Phi_E(\gamma^{1-s} - 1)$. Here Ω_E is the period of the Néron differential of E. The $\rho_E(Frob_p)$ has eigenvalue 1 if and only if E has multiplicative reduction mod p if and only if $E(\mathbb{C}_p) \cong \mathbb{C}_p^\times / q_E^{\mathbb{Z}}$ as Galois modules.

1. $\mathcal{L}$-INVARIANT

For a p-adic Galois representation ρ acting on $V \cong W^d$, we define $L(s, \rho) = \prod_\ell \det(1 - \rho(Frob_\ell)|_{V_{I_p}} p^{-s})^{-1}$, assuming that $\det(1 - \rho(Frob_\ell)|_{V_{I_\ell}} X) \in T[X]$ for a number field $T \subset \overline{\mathbb{Q}}$ independent of ℓ. We suppose that ρ is *p-ordinary* in the sense that ρ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is upper triangular with diagonal characters $\mathcal{N}^{a_j}$ on the inertia I_p for the p-adic cyclotomic character $\mathcal{N}$ ordered from top to bottom as $a_1 \geq a_2 \geq \cdots \geq 0 \geq \cdots \geq a_d$. Thus

$$\rho|_{I_p} = \begin{pmatrix} \mathcal{N}^{a_1} & * & \cdots & * \\ 0 & \mathcal{N}^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{N}^{a_d} \end{pmatrix} \qquad \text{(Ordinarity)}.$$

Under the existence of a good analytic p-adic L-function $L_p(s, \rho)$ of Iwasawa type for ρ, we make

Conjecture 1.1. *If the eigenvalue of $Frob_p$ contains 1 with multiplicity e, then $L_p(s,\rho)$ has zero of order $e+\mathrm{ord}_{s=1} L(s,\rho)$ and for a nonzero constant $\mathcal{L}(\rho)\in\mathbb{C}_p^\times$,*

$$\lim_{s\to1}\frac{L_p(s,\rho)}{(s-1)^e}=\mathcal{L}(\rho)\mathcal{E}^+(\rho)\frac{L(1,\rho)}{c^+(\rho(1))},$$

where $c^+(\rho(1))$ is the transcendental factor of the critical complex L-value $L(1,\rho)$ and $\mathcal{E}^+(\rho)$ is the product of nonvanishing modifying p-factors.

The problem of $\mathcal{L}$-invariant is to compute explicitly the $\mathcal{L}$-invariant $\mathcal{L}(\rho)$. The $\mathcal{L}$-invariant in the cases where $\rho=\chi=\left(\frac{-D}{\cdot}\right)$ as above and $\rho=\rho_E$ for E with split multiplicative reduction is computed in the 1970s to 90s, and the results are

Theorem 1.2. *Let the notation and the assumption be as above.*

(1) $\mathcal{L}(\chi)=\frac{\log_p(q)}{\mathrm{ord}_p(q)}=\frac{\log_p(q)}{h}$ *for $q\in\mathbb{C}_p$ given by $q=\varpi/\overline{\varpi}$, where h is the class number h of $\mathbb{Q}[\sqrt{-D}]$ and $\mathfrak{p}^h=(\varpi)$ (Gross–Koblitz and Ferrero–Greenberg);*

(2) *For E split multiplicative at p, writing $E(\mathbb{C}_p)=\mathbb{C}_p^\times/q^{\mathbb{Z}}$ for the Tate period $q\in\mathbb{Q}_p^\times$, we have $\mathcal{L}(\rho_E)=\frac{\log_p(q)}{\mathrm{ord}_p(q)}$. This was conjectured by Mazur–Tate–Teitelbaum and later proven by Greenberg–Stevens.*

Here $\log_p$ is the Iwasawa logarithm and $|x|_p=p^{-\mathrm{ord}_p(x)}$.

Starting with a 2-dim p-adic Galois representation for a number field F, there is a systematic way to create many Galois representations whose eigenvalues of $Frob_p$ contain 1. Take a symmetric n-th tensor of ρ_E twisted by m times $\det(\rho)^{-1}$. Then $\rho_{n,m}=\rho_E^{\otimes n}\otimes\det(\rho_E)^{-m}$ has exceptional zero at $s=1$ if $n=2m$. If m is odd and F is totally real, $\rho_{2m,m}$ is critical at $s=1$, and $e=|\{\mathfrak{p}|p\}|$.

There is an arithmetic way of constructing p-adic L-function due to Iwasawa and others. We can define Galois cohomologically the Selmer group

$$\mathrm{Sel}_M(\rho)\subset H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/M),\rho\otimes\mathbb{Q}_p/\mathbb{Z}_p)\ (M=F,F_\infty)$$

for the $\mathbb{Z}_p$-extension F_∞/F inside $F(\mu_{p^\infty})$. The Galois group $\Gamma=\mathrm{Gal}(F_\infty/F)$ acts on $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F_\infty),\rho\otimes\mathbb{Q}_p/\mathbb{Z}_p)$ and hence on $\mathrm{Sel}_{F_\infty}(\rho)$, making it as a discrete module over the group algebra $W[[\Gamma]]=\varprojlim_n W[\Gamma/\Gamma^{p^n}]$. Identifying Γ with (a subgroup of) $1+p\mathbb{Z}_p$ by the cyclotomic character, we may regard $\gamma\in\Gamma$. Then $W[[\Gamma]]\cong\Lambda$ by $\gamma\mapsto1+X$. By the classification theory of compact Λ-modules, the Pontryagin dual $\mathrm{Sel}^*(\rho)$ is pseudo-isomorphic to $\prod_{f\in\Omega}\Lambda/f\Lambda$ for a finite set $\Omega\subset\Lambda$. The power series $\Phi_\rho=\prod_{f\in\Omega}f(X)$ is uniquely determined up to unit multiple. We then define $L_p^{arith}(s,\rho)=\Phi_\rho(\gamma^{1-s}-1)$. Greenberg verified in 1994 the conjecture for this $L_p(s,\rho)$ except for the nonvanishing of $\mathcal{L}(\rho)$ (under some restrictive conditions). I also did verify in my book from Oxford university press his conjecture under milder assumptions by an automorphic way. If there exists a good analytic way of making the p-adic L-function $L_p^{an}(s,\rho)=\Phi_\rho^{an}(\gamma^{1-s}-1)$ interpolating complex L-values, the main conjecture of Iwasawa's theory confirms $\Phi_\rho=\Phi_\rho^{an}$ up to unit multiple.

An important point is to describe $\mathcal{L}$ without recourse to the above formula involving L-functions. Greenberg's computation of the $\mathcal{L}$-invariant is via Galois cohomology groups; for example, for $Ad(\rho)=\rho_{2,1}$: He found a unique subspace $\mathrm{Sel}_F^{cyc}(\rho)\subset H^1(F,Ad(\rho))$ of dimension $e=|\{\mathfrak{p}|p\}|$ responsible to the order e zero at $s=1$. This space is represented by cocycles $c_{\mathfrak{p}}:G_F=\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to Ad(\rho)$ such that

(1) $c|_{D_\mathfrak{p}} \sim \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ for the decomposition group $D_\mathfrak{p}$ at each $\mathfrak{p}|p$;
(2) c is unramified outside p and c modulo nilpotent matrices is unramified over $F_\mathfrak{p}[\mu_{p^\infty}]$ at all $\mathfrak{p}|p$ (automatic if $F_\mathfrak{p} = \mathbb{Q}_p$).

Take a basis $\{c_\mathfrak{p}\}_{\mathfrak{p}|p}$ of $\mathcal{L}$ over K. Write

$$c_\mathfrak{p}(\sigma) \sim \begin{pmatrix} -a_\mathfrak{p}(\sigma) & * \\ 0 & a_\mathfrak{p}(\sigma) \end{pmatrix} \quad \text{for } \sigma \in D_{\mathfrak{p}'}.$$

Then $a_\mathfrak{p} : D_{\mathfrak{p}'} \to K$ is a homomorphism. His $\mathcal{L}$-invariant is defined by

$$\mathcal{L}_{2,1} = \det\left((a_\mathfrak{p}([p, F_{\mathfrak{p}'}])_{\mathfrak{p},\mathfrak{p}'} \left(\log_p(\gamma) a_\mathfrak{p}([\gamma, F_{\mathfrak{p}'}]))_{\mathfrak{p},\mathfrak{p}'}\right)^{-1} \right).$$

The goal of this talk is to relate Greenberg's $\mathcal{L}$-invariant with Galois deformation theory, and give a couple of conjectures on $\mathcal{L}_{n,m}$ and the deformation ring.

2. A conjecture

Take an elliptic curve $E_{/F}$ for a totally real field F with split multiplicative reduction modulo at every prime $\mathfrak{p}|p$; so, $E(\overline{F}_\mathfrak{p}) \cong \overline{F}_\mathfrak{p}^\times / q_\mathfrak{p}^\mathbb{Z}$ for $q_\mathfrak{p} \in F_\mathfrak{p}^\times$. Let $Q_\mathfrak{p} = N_{F_\mathfrak{p}/\mathbb{Q}_p}(q_\mathfrak{p})$.

Conjecture 2.1 ($\mathcal{L}$-invariant)**.** *Suppose that the motive $Sym^{\otimes n}(H_1(E))(m)$ for $m \in \mathbb{Z}$ with $0 \le m < n$ is critical at 1 ($\Leftrightarrow$ either n is odd or $n = 2m$ with odd m). Then if the arithmetic or analytic $L_p(s, \rho_{n,m})$ has an exceptional zero at $s = 1$, we have*

$$\mathcal{L}_{n,m} = \prod_{\mathfrak{p}|p} \frac{\log_p(Q_\mathfrak{p})}{\mathrm{ord}_p(Q_\mathfrak{p})}.$$

3. Galois deformation

The Greenberg's Selmer group $\mathrm{Sel}_F^{cyc}(\rho)$ can be identified with the tangent space at the origin of the universal deformation space of $\rho_n = \rho_{n,0}$. Consider $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$and put $J_n = J_1^{\otimes n}$. We have ${}^t\rho_n(\sigma) J_n \rho_n(\sigma) = \mathcal{N}^n(\sigma) J_n$. Define an algebraic group G_n over $\mathbb{Z}_p$ by

$$G_n(A) = \left\{ \alpha \in GL_{n+1}(A) \middle| {}^t\alpha J_n \alpha = \nu(\alpha) J_n \right\}$$

for the similitude homomorphism $\nu : G_n \to \mathbb{G}_m$ (note that $G_1 = GL(2)$). The Galois representation ρ_n has values in $G_n(\mathbb{Z}_p)$. Consider the p-adic Lie algebra $Ad(\rho_n)$ of the derived group of G_n. Then $\sigma \in G_F$ acts on $Ad(\rho_n)$ by $X \mapsto \rho_n(\sigma) X \rho_n(\sigma)^{-1}$. Then

$$Ad(\rho_n) \cong \bigoplus_{j:\text{odd},\ 1 \le j \le n} \rho_{2j,j}. \tag{3.1}$$

Start with ρ_n, and consider the deformation ring $(R_n, \boldsymbol{\rho}_n)$ which is universal among Galois representations: $\rho_A : G_F \to G_n(A) \equiv \rho_n \mod \mathfrak{m}_A$ for local artinian $\mathbb{Q}_p$-algebras A with residue field $\mathbb{Q}_p$ such that

(Q_n1) unramified outside bad primes for E, ∞ and p;

(Q_n2) $\rho_A|_{D_\mathfrak{p}} \cong \begin{pmatrix} \alpha_{0,A,\mathfrak{p}} & * & \cdots & * \\ 0 & \alpha_{1,A,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,A,\mathfrak{p}} \end{pmatrix}$ with $\alpha_{i,A,\mathfrak{p}} \equiv \mathcal{N}_\mathfrak{p}^{n-i} \mod \mathfrak{m}_A$ with $\alpha_{i,A,\mathfrak{p}}|_{I_\mathfrak{p}}$ ($i = 0, 1, \ldots, n$) factoring through the cyclotomic inertia group $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$ for all $\mathfrak{p}|p$;

(Q_n3) $\nu \circ \rho_A = \mathcal{N}^n$ for the global p-adic cyclotomic character $\mathcal{N}$.

The universal couple $(R_n, \boldsymbol{\rho}_n)$ exists under (Q_n1–3). Each diagonal character, say, j-th $\boldsymbol{\delta}_j$ from the top, of $\boldsymbol{\rho}|_{D_\mathfrak{p}}$ induces a $\mathbb{Z}_p$-algebra homomorphism of $\mathbb{Z}_p[[X_{j,\mathfrak{p}}]]$ sending $1+X_{j,\mathfrak{p}}$ to the value of $\boldsymbol{\delta}_j$ at the generator of the p-primary part of $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$.

Conjecture 3.1. *We have $R_n \cong \mathbb{Q}_p[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p,\ j:odd,\ 1\le j\le n}$ for variables $X_{j,\mathfrak{p}}$ induced by j-th diagonal character of $\boldsymbol{\rho}_n|_{D_\mathfrak{p}}$; in particular,* $\dim R_n = e\cdot \mathrm{rank}\, G_n = e\lceil \frac{n}{2}\rceil$.

When $F=\mathbb{Q}$ and $n=1$, the conjecture holds, and for general totally real F, it holds if $n=1$ and $\rho_E \mod p$ has nonsoluble image (Wiles/Taylor, Skinner/Wiles, Fujiwara, Kisin, Khare/Wintenberger, Lin Chen). Since $G_3 \cong GSp(4)$ is the spin cover of $G_4 = GO(2,3)$. Some progress has been made by A. Genestier and J. Tilouine towards the identification of Galois deformation rings and $GSp(4)$-Hecke algebras (for $F=\mathbb{Q}$), there is a good prospect to get a proof of Conjecture 3.1 when $n=3$ and 4. Further, when $F=\mathbb{Q}$, in view of the recent results of Clozel–Harris–Taylor and Taylor (in the paper proving the Sato–Tate conjecture for Tate curves), one would be able to treat general n in future not so far away. Conjecture 3.1 implies the $\mathcal{L}$-invariant conjecture for Greenberg's $\mathcal{L}$-invariant:

Theorem 3.2. *Suppose* Conjecture 3.1 *and that n is odd. Then we have*

$$\prod_{j:odd, 0<j\le n} \mathcal{L}(\rho_{2j,j}) = \mathcal{L}(Ad(\rho_n)) = \prod_{\mathfrak{p}|p}\left(\frac{\log_p(Q_\mathfrak{p})}{\mathrm{ord}_p(Q_\mathfrak{p})}\right)^{(n+1)/2}.$$

This follows from the fact that $\left(\frac{\partial\boldsymbol{\rho}_n}{\partial X_{j,\mathfrak{p}}}\boldsymbol{\rho}_n^{-1}\right)\Big|_{X=0}$ gives a canonical basis of $\mathrm{Sel}^{cyc}(Ad(\rho_n))$; so, we can compute Greenberg's $\mathcal{L}$-invariant explicitly.

For the abelian case, if $\chi = \left(\frac{M/F}{}\right)$ for a CM field in which all $\mathfrak{p}|p$ in F splits into $\mathfrak{P}\overline{\mathfrak{P}}$ with $\mathfrak{P}^h = (\varpi(\mathfrak{P}))$, we also get

Corollary 3.3. *Up to a simple constant, for a half subset $\Sigma \sqcup \Sigma^c = \{\mathfrak{P}|p\}$, we have*

$$\mathcal{L}(\chi) = \frac{\det\left(\log_p(N_{\mathfrak{P}'}(\varpi(\mathfrak{P})^{(1-c)}))\right)_{\mathfrak{P},\mathfrak{P}'\in\overline{\Sigma}}}{\prod_{\mathfrak{P}\in\overline{\Sigma}}\mathrm{ord}_p(N_\mathfrak{p}(\varpi(\mathfrak{P})^{(1-c)})},$$

where $N_\mathfrak{P}$ is the local norm $N_{M_\mathfrak{p}/\mathbb{Q}_p}$ and c is a complex conjugation.

The above two results are obtained by explicitly computing the universal representation $\boldsymbol{\rho}$. As for Corollary 3.3, we take a CM Hecke eigenform so that $\rho = \mathrm{Ind}_M^F\psi$ for a CM Hecke character ψ of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Writing $\boldsymbol{\kappa}$ for the universal character deforming ψ whose restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})$ factors through $\mathrm{Gal}(F[\mu_{p^\infty}]/F_\mathfrak{p})$, we have $\boldsymbol{\kappa}([u,M_\mathfrak{P}]) = (1+X_\mathfrak{p})^{\log_p(N_\mathfrak{p}(u)/\log_p(\gamma)}\psi([u,M_\mathfrak{P}])$ for $\mathfrak{P}$-adic unit u, and we get $\boldsymbol{\rho} = \mathrm{Ind}_M^F\boldsymbol{\kappa}$. By this fact, we can compute $\mathcal{L}(\mathrm{Ind}_F^\mathbb{Q}\chi) = \mathcal{L}(\mathrm{Ind}_F^\mathbb{Q} Ad(\rho))$.

The general case for all $n>0$ is treated in my paper appeared in IMRN 2007 Vol. 2007, Article ID rnm102, 49 pages. doi:10.1093/imrn/rnm102, and the proof of the case: $n=1$ is given in my book "Hilbert modular forms and Iwasawa theory" from Oxford University Press published in 2006.