$\mathcal{L}$-INVARIANTS OF TATE CURVES

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1. Lecture 3

1.1. **Extensions of $\mathbb{Q}_p$ by its Tate twist.** Let $K \subset \mathbb{C}_p$ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{W}$ be a two dimensional vector space over K with a K-linear action of $D := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We start with an extension of local Galois modules

$$0 \to K(1) \to \mathcal{W} \to K \to 0$$

over D. This type of extensions (for $K = \mathbb{Q}_p$) can be obtained by the p-adic Tate module $\mathcal{W} = T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of an elliptic curve $E_{/\mathbb{Q}_p}$ with multiplicative reduction.

We prepare some general facts. The following is a description of a result in [GS1] Section 2 (see also [H07]). We write $H^i(?)$ for $H^i(D, ?)$. By definition, $H^1(M) = \mathrm{Ext}^1_{K[D]}(K, M)$ for a D-module M, and hence, there is a one-to-one correspondence:

$$\left\{ \begin{matrix} \text{nontrivial extensions} \\ \text{of } K \text{ by } M \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{1-dimensional subspaces} \\ \text{of } H^1(M) \end{matrix} \right\}.$$

From the left to the right, the map is given by $(M \hookrightarrow X \twoheadrightarrow K) \mapsto \delta_X(1)$ for the connecting map $K = H^0(K) \xrightarrow{\delta_X} H^1(M)$ of the long exact sequence attached to $(M \hookrightarrow X \twoheadrightarrow K)$. Out of a 1-cocycle $c : D \to M$, one can easily construct an extension $(M \hookrightarrow X \twoheadrightarrow K)$ taking $X = M \oplus K$ and letting D acts on X by $g(v, t) = (gv + t \cdot c(g), t)$, and $[c] \mapsto (M \hookrightarrow X \twoheadrightarrow K)$ gives the inverse map.

By Kummer's theory, we have a canonical isomorphism:

$$H^1(K(1)) \cong \left(\varprojlim_n \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n} \right) \otimes_{\mathbb{Z}_p} K.$$

We write $\gamma_q \in H^1(K(1))$ for the cohomology class associated to $q \otimes 1$ for $q \in \mathbb{Q}_p^\times$. The class γ_q is called the Kummer class of q. A canonical cocycle ξ_q in the class γ_q is given as follows. Then $\xi_q(\sigma) = \varprojlim_n (q^{1/p^n})^{\sigma-1}$ having values in $\mathbb{Z}_p(1) \subset K(1)$.

In summar, we have

Proposition 1.1. *If $\mathcal{W}$ is isomorphic to the representation $\sigma \mapsto \left(\begin{smallmatrix} \mathcal{N}(\sigma) & \xi_q(\sigma) \\ 0 & 1 \end{smallmatrix} \right)$ with $0 < |q|_p < 1$, then for the extension class of $[\mathcal{W}] \in H^1(K(1))$, we have $K[\mathcal{W}] = K\gamma_q$. In particular, $K\gamma_q$ is in the image of the connecting homomorphism $H^0(K) \xrightarrow{\delta_0} H^1(K(1))$ coming from the extension $K(1) \hookrightarrow \mathcal{W} \twoheadrightarrow K$.*

Corollary 1.2. *Let $E_{/\mathbb{Q}_p}$ be an elliptic curve. If E has split multiplicative reduction over W, the extension class of $[T_p E \otimes \mathbb{Q}]$ is in $\mathbb{Q}_p \gamma_{q_E}$ for the Tate period $q_E \in \mathbb{Q}_p^\times$.*

Let $\widetilde{K} := K[\varepsilon] = K[t]/(t^2)$ with $\varepsilon \leftrightarrow (t \mod t^2)$. A $\widetilde{K}[D]$-module $\widetilde{\mathcal{W}}$ is called an infinitesimal deformation of $\mathcal{W}$ if $\widetilde{\mathcal{W}}$ is $\widetilde{K}$-free of rank 2 and $\widetilde{\mathcal{W}}/\varepsilon\widetilde{\mathcal{W}} \cong \mathcal{W}$ as $K[D]$-modules. Since the map $\varepsilon : \widetilde{\mathcal{W}} \twoheadrightarrow \mathcal{W} \subset \widetilde{\mathcal{W}}$ given by $v \mapsto \varepsilon v$ is Galois equivariant, we have an exact sequence of D-modules

$$0 \to \mathcal{W} \to \widetilde{\mathcal{W}} \to \mathcal{W} \to 0.$$

Each infinitesimal deformation gives rise to an infinitesimal character $\psi : D \to \widetilde{K}^\times$ with $\psi \mod (\varepsilon) = 1$. Define $\widetilde{K}(\psi)$ for the space of the character ψ. Obviously, $\frac{d\psi}{dt} : D \to K$ is a homomorphism; so, $\frac{d\psi}{dt} \in \mathrm{Hom}(D, K) = H^1(K)$. Since the extension $\widetilde{K}(\psi)$ is split if and only if $\frac{d\psi}{dt} = 0$, we get

Proposition 1.3. *The correspondence* $\widetilde{K}(\psi) \leftrightarrow \frac{d\psi}{dt} \in H^1(K)$ *gives a one-to-one correspondence:*

$$\left\{ \begin{array}{c} \textit{Nontrivial infinitesimal} \\ \textit{deformations of } K \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{1-dimensional} \\ \textit{subspaces of } H^1(K) \end{array} \right\}.$$

Note that

$$H^1(D, K) \cong \mathrm{Hom}(D, K) = \mathrm{Hom}(D^{ab}, K) \cong K^2,$$

where, as we have seen in Lecture 2, the last isomorphism is given by

$$\mathrm{Hom}(D^{ab}, K) \ni \phi \mapsto (\frac{\phi([\gamma, \mathbb{Q}_p])}{\log_p(\gamma)}, \phi([p, \mathbb{Q}_p])) \in K^2.$$

This follows from local class field theory. Since the Tate duality $\langle \cdot, \cdot \rangle$ is perfect, for any line ℓ in $H^1(D, K)$, one can assign its orthogonal complement $\ell^\perp$ in $H^1(D, K(1))$.

Proposition 1.4. *The correspondence of a line in $H^1(D, K)$ and its orthogonal complement in $H^1(D, K(1))$ gives a one-to-one correspondence:*

$$\left\{ \begin{array}{c} \textit{Nontrivial extensions} \\ \textit{of } K \textit{ by } K(1) \textit{ as } K[D]\textit{-modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{nontrivial infinitesimal} \\ \textit{deformations of } K \textit{ over } D \end{array} \right\}.$$

Theorem 1.5. *Let $E_{/L}$ be an elliptic curve with split multiplicative reduction defined over a finite extension $L/\mathbb{Q}_p$, and let $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widetilde{\mathbb{Q}}_p^\times$ be a nontrivial character which is congruent to 1 modulo ε. Let $\mathcal{W} = T_p E \otimes \mathbb{Q}_p$ for the p-adic Tate module $T_p E$ of E and $q_E \in L^\times$ be the Tate period of E. Then the following statements are equivalent:*

(a) $\frac{d\psi}{dt}(\sigma_{N_{L/\mathbb{Q}_p}(q_E)}) = 0$ *for* $\sigma_q = [q, \mathbb{Q}_p]^{-1}$;

(b) $\mathcal{W}$ *corresponds to* $\widetilde{\mathbb{Q}}_p(\psi)$ *under the correspondence of* Proposition 1.4;

(c) *There is a deformation $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ and a commutative diagram of $\widetilde{\mathbb{Q}}_p[D]$-modules with exact row:*

$$
\begin{array}{ccccc}
\widetilde{\mathbb{Q}}_p(1) & \overset{\hookrightarrow}{\longrightarrow} & \widetilde{\mathcal{W}} & \overset{\twoheadrightarrow}{\longrightarrow} & \widetilde{\mathbb{Q}}_p(\psi) \\
{\scriptstyle \mathrm{mod}\ \varepsilon} \downarrow & & {\scriptstyle \mathrm{mod}\ \varepsilon} \downarrow & & \downarrow {\scriptstyle \mathrm{mod}\ \varepsilon} \\
\mathbb{Q}_p(1) & \underset{\hookrightarrow}{\longrightarrow} & \mathcal{W} & \underset{\twoheadrightarrow}{\longrightarrow} & \mathbb{Q}_p.
\end{array}
$$

Normalize the Artin symbol $[x, \mathbb{Q}_p]$ so that

- $\mathcal{N}([u, \mathbb{Q}_p]) = u^{-1}$ for $u \in \mathbb{Z}_p^\times$,

- $[p, \mathbb{Q}_p]$ is the arithmetic Frobenius element.

By an explicit form of Tate duality, we have $\langle \gamma_q, \phi \rangle = \phi(\sigma_q)$ for $\gamma_q \in H^1(D, \mathbb{Q}_p(1))$ and $\phi \in \mathrm{Hom}(D, \mathbb{Q}_p) = H^1(D, \mathbb{Q}_p)$.

Proof. The case $L = \mathbb{Q}_p$ is treated in [GS1] 2.3.4 and the gneral fact is in [H07]. For simplicity, we assume $L = \mathbb{Q}_p$. Since $\langle \gamma_q, \phi \rangle = \phi(\sigma_q)$ for $\phi \in H^1(D, \mathbb{Q}_p) = \mathrm{Hom}(D, \mathbb{Q}_p)$ and $\gamma_q \in H^1(D, \mathbb{Q}_p(1))$, applying these formulas to $\phi = \frac{d\psi}{dt}$, we get (a) $\Leftrightarrow$ (b) by the definition of the correspondence in Proposition 1.4.

Here is the argument proving (b) $\Rightarrow$ (c). Let ξ_Q be a 1-cocycle representing γ_Q for $Q = q_E$. Then $D \times D \ni (\sigma, \tau) \mapsto c(\sigma) \frac{d\psi}{dt}(\tau) \in \mathbb{Q}_p(1)$ is the 2-cocycle representing the cup product $\gamma_Q \cup [\widetilde{\mathbb{Q}_p}(\psi)]$ (another expression of the Tate pairing), which vanishes by (b) ($\Leftrightarrow$ (a)). Thus it is a 2-coboundary:

$$(1.1) \quad \xi_Q(\sigma)\frac{d\psi}{dt}(\tau) = \partial\Xi(\sigma, \tau) = \Xi(\sigma\tau) - \mathcal{N}(\sigma)\Xi(\tau) - \Xi(\sigma)$$

$$(\Leftrightarrow \Xi(\sigma\tau) = \xi_Q(\sigma)\frac{d\psi}{dt}(\tau) + \mathcal{N}(\sigma)\Xi(\tau) + \Xi(\sigma))$$

for a 1-chain $\Xi : D \to \mathbb{Q}_p(1)$. Then defining an action of $\sigma \in D$ on $\widetilde{\mathbb{Q}_p}^2$ via the matrix multiplication by $\widetilde{\rho}(\sigma) := \begin{pmatrix} \mathcal{N}(\sigma) & \xi_Q(\sigma) + \Xi(\sigma)\varepsilon \\ 0 & \psi(\sigma) \end{pmatrix}$. One checks that this is well defined by computation (the relation (1.1) shows up in the ε-term of $\widetilde{\rho}(\sigma\tau) \overset{?}{=} \widetilde{\rho}(\sigma)\widetilde{\rho}(\tau)$ at the shoulder). The resulting $\widetilde{\mathbb{Q}_p}[D]$-module $\widetilde{\mathcal{W}}$ fits well in the diagram in (c).

Conversely suppose we have the commutative diagram as in (c), which can be written as the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{\mathbb{Q}_p}(1) & \longrightarrow & \widetilde{\mathcal{W}} & \longrightarrow & \widetilde{\mathbb{Q}_p}(\psi) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The connecting homomorphism $d : H^1(D, \mathbb{Q}_p(1)) \to H^2(D, \mathbb{Q}_p(1))$ vanishes because the leftmost vertical sequence splits. On the other hand, letting $\delta_\psi : H^0(D, \mathbb{Q}_p) \to H^1(D, \mathbb{Q}_p)$ stand for the connecting homomorphism of degree 0 coming from the rightmost vertical sequence, and letting $\delta_i : H^i(D, \mathbb{Q}_p) \to H^{i+1}(D, \mathbb{Q}_p(1))$ be the connecting homomorphism of degree i associated to the bottom row and the top row. By the commutativity of the diagram, we get the following commutative square:

$$
\begin{array}{ccc}
H^0(D, \mathbb{Q}_p) = \mathbb{Q}_p & \xrightarrow{\delta_0} & H^1(D, \mathbb{Q}_p(1)) \\
\delta_\psi \downarrow & & \downarrow d = 0 \\
H^1(D, \mathbb{Q}_p) & \xrightarrow[\delta_1]{} & H^2(D, \mathbb{Q}_p(1)).
\end{array}
$$

Since $\delta_\psi(1) = \frac{d\psi}{dt}$, we confirm $\frac{d\psi}{dt} \in \mathrm{Ker}(\delta_1)$. By Proposition 1.1, γ_Q is the in the image of δ_0. Thus (a)/(b) follows if we can show that $\mathrm{Ker}(\delta_1)$ is orthogonal to $\mathrm{Im}(\delta_0)$.

Since $\mathcal{W} = T_p E \otimes \mathbb{Q}$, $\mathcal{W}$ is self dual under the canonical polarization pairing, which induces a self duality of $\mathcal{W}$ and also the self (Cartier) duality of the exact sequence $0 \to \mathbb{Q}_p(1) \xrightarrow{\iota} \mathcal{W} \xrightarrow{\pi} \mathbb{Q}_p \to 1$. In particular the inclusion ι and the projection π are mutually adjoint under the pairing. Thus the connecting maps $\delta_0 : H^0(D, \mathbb{Q}_p) \to H^1(D, \mathbb{Q}_p(1))$ and $\delta_1 : H^1(D, \mathbb{Q}_p) \to H^2(D, \mathbb{Q}_p(1))$ are mutually adjoint each other under the Tate duality pairing. In particular, $\mathrm{Im}(\delta_0)$ is orthogonal to $\mathrm{Ker}(\delta_1)$. $\square$

1.2. **How to relate the $\mathcal{L}$-invariant with the logarithm of Tate period.** Take an elliptic curve E with multiplicative reduction over the finite extension $L_{/\mathbb{Q}_p}$. Let $\mathcal{W} = T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Write $Q_i = N_{F_{\mathfrak{p}_i}/\mathbb{Q}_p}(q_{\mathfrak{p}_i})$.

Theorem 1.6 ($\mathcal{L}$-invariant). *If Deformation conjecture holds for ρ_E, then $\mathrm{Sel}_F(V) = 0$ and we have*

$$\mathcal{L}(Ad(\rho_E)) = \left(\prod_{i=1}^{b} \frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)} \right) \mathcal{L}(1).$$

We have $\mathcal{L}(m) = 1$ if $b = e$, and the value $\mathcal{L}(1)$ when $b < e$ is given by

$$\mathcal{L}(1) = \det \left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}_i}([p, F_{\mathfrak{p}_i}])}{\partial X_j} \right)_{i>b, j>b} \Big|_{X_1 = X_2 = \cdots = X_e = 0} \prod_{i>b} \frac{\log_p(\gamma_{\mathfrak{p}_i})}{[F_{\mathfrak{p}_i} : \mathbb{Q}_p] \alpha_{\mathfrak{p}_i}([p, F_{\mathfrak{p}_i}])}$$

for the local Artin symbol $[p, F_{\mathfrak{p}_i}]$.

Proof. In the proof, we continue to assume $F_{\mathfrak{p}_j} = \mathbb{Q}_p$ for all $j = 1, \ldots, e$. Fix an index j. Write $D_j = \mathrm{Gal}(\overline{F}_{\mathfrak{p}_j}/F_{\mathfrak{p}_j}) \cong D$. We consider the universal couple $(R, \boldsymbol{\rho})$ of ρ_E under the conditions (K1–4). Put $\mathfrak{m}_j := (X_1, \ldots, X_{j-1}, X_j^2, X_{j+1}, \ldots, X_e) \subset R = K[[X_j]]_{j=1,\ldots,e}$ for $X_j = X_{\mathfrak{p}_j}$. Consider $\widetilde{\mathcal{W}}_j = \mathbf{W}/\mathfrak{m}_j \mathbf{W}$ for $\mathbf{W} = \boldsymbol{\rho}$.

Suppose $j \leq b$. We have a D_j-stable filtration $0 = \mathcal{F}^2 \widetilde{\mathcal{W}}_j \subset \mathcal{F}^1 \widetilde{\mathcal{W}}_j \subset \widetilde{\mathcal{W}}_j = \mathcal{F}^0 \widetilde{\mathcal{W}}_j$. Let δ_j be the nearly ordinary character

$$\delta_j := (\boldsymbol{\delta}_j \mod (X_1, \ldots, X_{j-1}, X_j^2, X_{j+1}, \ldots, X_e)).$$

The character δ_j satisfies $\delta_j \equiv \alpha_{\mathfrak{p}_j} = \mathbf{1} \mod (X_j)$ for the trivial character $\mathbf{1}$ of D. Since $\det(\boldsymbol{\rho}) = \mathcal{N}$ (K3), we have

$$\frac{\widetilde{\mathcal{W}}_j}{\mathcal{F}^1 \widetilde{\mathcal{W}}_j} = \delta_j \cong \widetilde{\mathbb{Q}}_p \quad \text{and} \quad \mathcal{F}^1 \widetilde{\mathcal{W}}_j = \widetilde{\mathbb{Q}}_p(\delta_j^{-1} \mathcal{N}).$$

The matrix form of the D_j-representation $\widetilde{\mathcal{W}}_j$ is $\begin{pmatrix} \delta_j^{-1} \mathcal{N} & * \\ 0 & \delta_j \end{pmatrix}$. Twist $\widetilde{\mathcal{W}}_j$ by δ_j; then, $\widetilde{\mathcal{W}}_j \otimes \delta_j$ has the matrix form $\begin{pmatrix} \mathcal{N} & * \\ 0 & \psi_j \end{pmatrix}$ for $\psi_j = \delta_j^2$. Then $\widetilde{\mathcal{W}}_j \otimes \psi_j$ is an infinitesimal extension making the following diagram commutative:

$$\begin{array}{ccccc}
\widetilde{\mathbb{Q}}_p(1) & \xrightarrow{\hookrightarrow} & \widetilde{\mathcal{W}}_j \otimes \psi_j & \xrightarrow{\twoheadrightarrow} & \widetilde{\mathbb{Q}}_p(\psi_j) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}_p(1) & \xrightarrow{\hookrightarrow} & \mathcal{W} & \xrightarrow{\twoheadrightarrow} & \mathbb{Q}_p.
\end{array}$$

This diagram satisfies the condition (c) of Theorem 1.5, for $Q_j = q_{E/F_{\mathfrak{p}_j}}$,

$$\left.\frac{\partial \psi_j}{\partial X_j}\right|_{X_j=0}([Q_j, \mathbb{Q}_p]) = 2\left(\delta_j \left.\frac{\partial \psi_j}{\partial X_j}\right|_{X_j=0}\right)([Q_j, \mathbb{Q}_p]) = 0 \Rightarrow \left.\frac{\partial \delta_j}{\partial X_j}\right|_{X_j=0}([Q_j, \mathbb{Q}_p]) = 0.$$

Write $Q_j = p^a u$ for $a = \mathrm{ord}_p(Q_j)$ and $u \in \mathbb{Z}_p^\times$. Then $\log_p(u) = \log_p(Q_j)$. We have

$$\begin{aligned}
\delta_j([Q_j, F_{\mathfrak{p}_j}]) &= \delta_j([p, F_{\mathfrak{p}_j}])^a \delta_j([u, F_{\mathfrak{p}_j}]) \\
&= \delta_j([p, F_{\mathfrak{p}_j}])^a (1 + X_j)^{-\log_p(\mathcal{N}([u, F_{\mathfrak{p}_j}]))/\log_p(\gamma_{\mathfrak{p}_j})} \\
&= \delta_j([p, F_{\mathfrak{p}_j}])^a (1 + X_j)^{-\log_p(u)/\log_p(\gamma_{\mathfrak{p}_j})}
\end{aligned}$$

(because $\mathcal{N}([u, F_{\mathfrak{p}_j}]) = u^{-1}$). Differentiating this identity with respect to X_j, we get from $\delta_j([u, F_{\mathfrak{p}_j}])|_{X_j=0} = \delta_j([p, F_{\mathfrak{p}_j}])|_{X_j=0} = \alpha_j([p, F_{\mathfrak{p}_j}]) = 1$

$$a \left.\frac{\partial \boldsymbol{\delta}_j}{\partial X_j}\right|_{X_j=0}([p, F_{\mathfrak{p}_j}]) - \frac{\log_p(u)}{\log_p(\gamma_j)} = 0.$$

From this we conclude

$$(1.2) \qquad \left.\frac{\partial \boldsymbol{\delta}_j([p, F_{\mathfrak{p}_j}])}{\partial X_j}\right|_{X_j=0} \log_p(\gamma_j)\alpha_{\mathfrak{p}_j}([p, F_{\mathfrak{p}_j}])^{-1} = \frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)},$$

since $\alpha_j([p, F_{\mathfrak{p}_j}]) = 1$ (by split multiplicative reduction of E at $\mathfrak{p}_j$ with $j \le b$).

As already seen, $\mathrm{Sel}_F(Ad(\rho_E)) = 0$, assuming that $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. We will prove the following factorization in the fourth lecture:

$$(1.3) \quad \mathcal{L}(Ad(\rho_E)) = \prod_{i=1}^b \left.\frac{\partial \boldsymbol{\delta}_i([p, F_{\mathfrak{p}_i}])}{\partial X_i}\right|_{X_i=0} \log_p(\gamma_{\mathfrak{p}_i})\alpha_{\mathfrak{p}_i}([p, F_{\mathfrak{p}_i}])^{-1}$$

$$\times \det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_{\mathfrak{p}_i}])}{\partial X_j}\right)_{i>b, j>b} \Big|_{X=0} \prod_{j>b} \log_p(\gamma_{\mathfrak{p}_j})\alpha_{\mathfrak{p}_j}([p, F_{\mathfrak{p}_j}])^{-1}.$$

From this and (1.2), the desired formula follows. $\qquad \square$

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