GALOIS DEFORMATION AND $\mathcal{L}$-INVARIANT

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1. Lecture 2

The notation is as in the first lecture (F: a totally real field, $p > 2$ is a fixed prime). For simplicity, we assume that p **splits completely** in $F/\mathbb{Q}$. We start with a Galois representation $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(W)$ associated to a Hilbert modular form (on $GL(2)_{/F}$) with coefficients in W. We assume the ordinarity of ρ_F:

$$\rho_F|_{D_\mathfrak{p}} \cong \begin{pmatrix} \beta_\mathfrak{p} & * \\ 0 & \alpha_\mathfrak{p} \end{pmatrix} \quad \text{with} \quad \beta_\mathfrak{p} \neq \alpha_\mathfrak{p},\ \beta_\mathfrak{p}|_{I_\mathfrak{p}} = \mathcal{N}^{k-1} \quad \text{and} \quad \alpha_\mathfrak{p}(I_\mathfrak{p}) = 1$$

on the decomposition group and the inertia group $I_\mathfrak{p} \subset D_\mathfrak{p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for all prime factor $\mathfrak{p}$ of p in F. Here $\mathcal{N}(\sigma) \in \mathbb{Z}_p^\times$ is the p-adic cyclotomic character with $\exp(\frac{2\pi i}{p^n})^\sigma = \exp(\frac{\mathcal{N}(\sigma)2\pi i}{p^n})$ for all $n > 0$ and $k > 1$ is an integer. Again for simplicity, we assume that ρ is **unramified outside** p.

We consider the **universal** nearly ordinary couple $(R, \boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(R))$ considered in the first lecture where R is a pro-Artinian local K-algebra. The couple $(R, \boldsymbol{\rho})$ is universal among Galois deformations $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ (for Artinian local K-algebras A with $A/\mathfrak{m}_A = K$) such that

- (K1) unramified outside p;
- (K2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$ (and the local cyclotomy condition if p does not split completely in F);
- (K3) $\det(\rho_A) = \det \rho_F$;
- (K4) $\rho_A \equiv \rho_F \mod \mathfrak{m}_A$.

Recall $\Gamma_\mathfrak{p} = 1 + p\mathbb{Z}_p = \gamma_\mathfrak{p}^{\mathbb{Z}_p} \overset{\mathcal{N}^{-1}}{\hookrightarrow} \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$. Identify $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$ by $\gamma_\mathfrak{p} \leftrightarrow 1 + X_\mathfrak{p}$. Since $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_\mathfrak{p} \end{smallmatrix}\right)$, $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1} : \Gamma_\mathfrak{p} \to R$ induces an algebra structure on R over $W[[X_\mathfrak{p}]]$. Thus R is an algebra over $K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$.

Here is the theorem we have seen in the first lecture:

Theorem 1.1 (Derivative)**.** *Suppose $R \cong K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. Then, if $\varphi \circ \boldsymbol{\rho} \cong \rho_F$, for the local Artin symbol $[p, F_\mathfrak{p}] = Frob_\mathfrak{p}$, we have*

$$\mathcal{L}(\mathrm{Ind}_F^\mathbb{Q} Ad(\rho_F)) = \mathcal{L}(Ad(\rho_F)) = \det\left(\frac{\partial \boldsymbol{\delta}_\mathfrak{p}([p, F_\mathfrak{p}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'} \Big|_{X=0} \prod_\mathfrak{p} \log_p(\gamma_\mathfrak{p})\alpha_\mathfrak{p}([p, F_\mathfrak{p}])^{-1}.$$

Greenberg proposed a conjectural recipe of computing the $\mathcal{L}$–invariant. When $V = Ad(\rho_F)$, his definition goes as follows. Under some hypothesis, he found a

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unique subspace $\mathbb{H} \subset H^1(F, Ad(\rho_F))$ of dimension $e = |\{\mathfrak{p}|p\}|$ represented by cocycles $c : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to Ad(\rho_F)$ such that

(1) c is unramified outside p;
(2) c restricted to $D_{\mathfrak{p}}$ is upper triangular after conjugation for all $\mathfrak{p}|p$.

By the condition (2), $c|_{I_{\mathfrak{p}}}$ modulo upper nilpotent matrices factors through the cyclotomic Galois group $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p)$ because $F_{\mathfrak{p}} = \mathbb{Q}_p$, and hence $c|_{D_{\mathfrak{p}}}$ modulo upper nilpotent matrices becomes unramified everywhere over the cyclotomic $\mathbb{Z}_p$-extension F_∞/F; so, the cohomology class $[c]$ is in $\mathrm{Sel}_{F_\infty}(Ad(\rho_F))$ but not in $\mathrm{Sel}_F(Ad(\rho_F))$. Take a basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$ of $\mathbb{H}$ over K. Write

$$c_{\mathfrak{p}}(\sigma) \sim \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \quad \text{for } \sigma \in D_{\mathfrak{p}'} \text{ with any } \mathfrak{p}'|p.$$

Then $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \to K$ is a homomorphism. His $\mathcal{L}$-invariant is defined by

$$\mathcal{L}(Ad(\rho_F)) = \det\left((a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}])_{\mathfrak{p},\mathfrak{p}'|p} \left(\log_p(\gamma_{\mathfrak{p}'})^{-1} a_{\mathfrak{p}}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}\right)^{-1} \right).$$

The above value is independent of the choice of the basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$. As we remarked in the first lecture, assuming the following condition:

(ns) $\overline{\rho} = (\rho \mod \mathfrak{m}_W)$ has nonsoluble image,

by using basically a result of Fujiwara and potential modularity of Taylor (plus a very recent work of Lin Chen), we have $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. The following conjecture for the arithmetic L-function is almost a theorem except for the nonvanishing $\mathcal{L}(Ad(\rho_F)) \neq 0$ (see [HMI] Theorem 5.27 combined with (5.2.6) there):

Conjecture 1.2 (Greenberg)**.** *Suppose* (ns)*. Let* $? = arith, an$*. For* $L_p^?(s, Ad(\rho_F)) = \Phi_\rho^{arith}(\gamma^{1-s} - 1)$*, then* $L_p^?(s, Ad(\rho_F))$ *has zero of order equal to* $e = |\{\mathfrak{p}|p\}|$ *and for the constant* $\mathcal{L}(Ad(\rho_F)) \in K^\times$ *specified by the determinant as in the theorem, we have*

$$\lim_{s\to 1} \frac{L_p^?(s, Ad(\rho_F))}{(s-1)^e} = \mathcal{L}(Ad(\rho_F)) \left| |\mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)^*)| \right|_p^{-1/[K:\mathbb{Q}_p]}.$$

If $? = arith$*, the identity is up to units.*

The factor $\mathcal{E}^+(Ad(\rho))$ does not show up in the above formula. If ρ_F is crystalline at p, writing $S_F(Ad(\rho_F)^*)$ for the Bloch-Kato Selmer group $H^1_f(F, Ad(\rho)^*)$, we have

$$\left| |\mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)^*)| \right|_p^{-1/[K:\mathbb{Q}_p]} = \mathcal{E}^+(Ad(\rho_F)) \left| |S_F(Ad(\rho_F)^*)| \right|_p^{-1/[K:\mathbb{Q}_p]} \text{ up to units,}$$

and the value $\left| |S_F(Ad(\rho_F)^*)| \right|_p^{-1/[K:\mathbb{Q}_p]}$ is directly related to the primitive complex L-value $L(1, Ad(\rho_F))$ up to a period (see [MFG] page 284). In the following section, we describe the Selmer group and how to specify $\mathbb{H}$.

1.1. **Greenberg's Selmer Groups.** Write $F^{(p)}/F$ for the maximal extension unramified outside p and ∞. Put $\mathfrak{G} = \mathrm{Gal}(F^{(p)}/F)$ and $\mathfrak{G}_M = \mathrm{Gal}(F^{(p)}/M)$. Let $V = Ad(\rho_F)$. We fix a W-lattice T in V stable under $\mathfrak{G}$.

Write $D = D_{\mathfrak{p}} \subset \mathfrak{G}$ for the decomposition group of each prime factor $\mathfrak{p}|p$. Choosing a basis of ρ_F so that $\rho_F|_D$ is upper triangular, we have a 3-step filtration:

$$\text{(ord)} \qquad V \supset \mathcal{F}_{\mathfrak{p}}^- V \supset \mathcal{F}_{\mathfrak{p}}^+ V \supset \{0\},$$

where $\mathcal{F}_{\mathfrak{p}}^{-}V$ is made up of upper triangular matrices and $\mathcal{F}_{\mathfrak{p}}^{+}V$ is made up of upper nilpotent matrices, and on $\mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V$, D acts trivially (getting eigenvalue 1 for $Frob_{\mathfrak{p}}$). Since V is self-dual, its dual $V^*(1) = \mathrm{Hom}_K(V,K)\otimes\mathcal{N}$ again satisfies (ord).

Let M/F be a subfield of $F^{(p)}$, and put $\mathfrak{G}_M = \mathrm{Gal}(F^{(p)}/M)$. We write $\mathfrak{p}$ for a prime of M over p and $\mathfrak{q}$ for general primes of M. We put

$$L_{\mathfrak{p}}(V) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{p}}, V) \to H^1(I_{\mathfrak{p}}, \frac{V}{\mathcal{F}_{\mathfrak{p}}^{+}(V)})).$$

Define for the image $L_{\mathfrak{p}}(V/T)$ of $L_{\mathfrak{p}}(V)$ in $H^1(M_{\mathfrak{p}}, V/T)$

$$\mathrm{Sel}_M(A) = \mathrm{Ker}(H^1(\mathfrak{G}_M, A) \to \prod_{\mathfrak{p}} \frac{H^1(M_{\mathfrak{p}}, A)}{L_{\mathfrak{p}}(A)})) \text{ for } A = V, V/T. \tag{1.1}$$

The classical Selmer group of V is given by $\mathrm{Sel}_M(V/T)$. We define the "$-$" Selmer group replacing $L_{\mathfrak{p}}(A)$ in the above definition by

$$L_{\mathfrak{p}}^{-}(V) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{p}}, V) \to H^1(I_{\mathfrak{p}}, \frac{V}{\mathcal{F}_{\mathfrak{p}}^{-}(V)})).$$

Lemma 1.3 (Vanishing)**.** *Suppose* $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. *Then* $\mathrm{Sel}_F^{-}(V) \cong \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$ *and* $\mathrm{Sel}_F(V) = 0$.

Proof. We consider the space $Der_K(R,K)$ of continuous K-derivations. Let $K[\varepsilon] = K[t]/(t^2)$ for the dual number $\varepsilon = (t \mod t^2)$. Then writing K-algebra homomorphism $\phi : R \to K[\varepsilon]$ as $\phi(r) = \phi_0(r) + \phi_1(r)\varepsilon$ and sending ϕ to $\phi_1 \in Der_K(R,K)$, we have $\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong Der_K(R,K) = \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$. Note here that $\phi_1 = \frac{\partial\phi}{\partial t}$. By the universality of $(R, \boldsymbol{\rho})$, we have

$$\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong \frac{\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K[\varepsilon]) | \rho \text{ satisfies the condtions (K1–4)}\}}{\cong}.$$

Pick ρ as above. Write $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$. Note here again $\rho_1 = \frac{\partial\rho}{\partial t}$. Then $c_\rho = \rho_1\rho_F^{-1}$ can be easily checked to be a 1-cocycle having values in $M_2(K) \supset V$. Since $\det(\rho) = \det(\rho_F) \Rightarrow \mathrm{Tr}(c_\rho) = 0$, c_ρ has values in V. By the reducibility condition (K2), $[c_\rho] \in \mathrm{Sel}_F^{-}(V)$. We see easily that $\rho \cong \rho' \Leftrightarrow [c_\rho] = [c_{\rho'}]$. We can reverse the above argument starting with a cocycle c giving an element of $\mathrm{Sel}_F^{-}(V)$ to construct a deformation ρ_c with values in $K[\varepsilon]$. Thus we have

$$\frac{\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K[\varepsilon]) | \rho \text{ satisfies the condtions (K1–4)}\}}{\cong} \cong \mathrm{Sel}_F^{-}(V).$$

Since the algebra structure of R over $W[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ is given by $\boldsymbol{\delta}_{\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1}$, the K-derivation $\delta : R \to K$ corresponding to a $K[\varepsilon]$-deformation ρ is a $W[[X_{\mathfrak{p}}]]$-derivation if and only if $\rho_1|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \sim \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$, which is equivalent to $[c_\rho] \in \mathrm{Sel}_F(V)$, because we already knew that $\mathrm{Tr}(c_\rho) = 0$. Thus we have $\mathrm{Sel}_F(V) \cong Der_{W[[X_{\mathfrak{p}}]]}(R, K) = 0$. $\square$

If $\rho|_{D_{\mathfrak{p}}}$ is isomorphic to $\left(\begin{smallmatrix} \mathcal{N} & \xi \\ 0 & 1 \end{smallmatrix}\right) \otimes \eta$ for a finite order character η of $D_{\mathfrak{p}}$ and a cocycle $\xi : D_{\mathfrak{p}} \to K(1)$ of the form $\xi(\sigma) = \lim_{n\to\infty}(\sqrt[p^n]{q_{\mathfrak{p}}})^{\sigma-1}$ for a non-unit $q_{\mathfrak{p}} \in F_{\mathfrak{p}}^{\times}$, we call ρ *multiplicative* at $\mathfrak{p}$. If ρ comes from an elliptic curve $E_{/F}$, E has multiplicative reduction modulo $\mathfrak{p}$ if and only if it is multiplicative at $\mathfrak{p}$. We order primes $\mathfrak{p}|p$ so that ρ is multiplicative at $\mathfrak{p}_i$ if and only if $i \leq b$. The number b can be zero.

We need to have a slightly different definition of Selmer groups behaving well under Tate duality. For each prime $\mathfrak{q} \in \{\mathfrak{p}|p\}$, we put

(1.2)
$$\overline{L}_{\mathfrak{q}}(V) = \begin{cases} \mathrm{Ker}(H^1(F_{\mathfrak{p}_j}, V) \to H^1(F_{\mathfrak{p}_j}, \frac{V}{\mathcal{F}^+_{\mathfrak{p}_j}(V)})) \subset L_{\mathfrak{p}_j}(V) & \text{if } \mathfrak{q} = \mathfrak{p}_j \text{ with } j \leq b, \\ L_{\mathfrak{q}}(V) & \text{otherwise} \end{cases}$$

Once $\overline{L}_{\mathfrak{q}}(V)$ is defined, we define $\overline{L}_{\mathfrak{q}}(V^*(1)) = \overline{L}_{\mathfrak{q}}(V)^{\perp}$ under the local Tate duality between $H^1(F_{\mathfrak{q}}, V)$ and $H^1(F_{\mathfrak{q}}, V^*(1))$, where $V^*(1) = \mathrm{Hom}_K(V, \mathbb{Q}_p(1))$ as Galois modules. Then we define the balanced Selmer group $\overline{\mathrm{Sel}}_F(V)$ (resp. $\overline{\mathrm{Sel}}_F(V^*(1))$) by the same formula as in (1.1) replacing $L_{\mathfrak{p}}(V)$ (resp. $L_{\mathfrak{p}}(V^*(1))$) by $\overline{L}_{\mathfrak{p}}(V)$ (resp. $\overline{L}_{\mathfrak{p}}(V^*(1))$). By definition, $\overline{\mathrm{Sel}}_F(V) \subset \mathrm{Sel}_F(V)$.

Lemma 1.4 (Isomorphism)**.** *Let V be $Ad(\rho_E)$. We have*

$$\mathrm{Sel}_F(V) = 0 \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_{\mathfrak{p}|p} \frac{H^1(F_{\mathfrak{p}}, V)}{\overline{L}_{\mathfrak{p}}(V)}. \tag{V}$$

Proof. Since $\overline{\mathrm{Sel}}_F(V) \subset \mathrm{Sel}_F(V)$, the assumption implies $\overline{\mathrm{Sel}}_F(V) = 0$. Then the Poitou-Tate exact sequence tells us the exactness of the following sequence:

$$\overline{\mathrm{Sel}}_F(V) \to H^1(\mathfrak{G}, V) \to \prod_{\mathfrak{l} \in \{\mathfrak{p}|p\}} \frac{H^1(F_{\mathfrak{l}}, V)}{\overline{L}_{\mathfrak{l}}(V)} \to \overline{\mathrm{Sel}}_F(V^*(1))^*.$$

It is an old theorem of Greenberg (which assumes criticality at $s = 1$) that

$$\dim \overline{\mathrm{Sel}}_F(V) = \dim \overline{\mathrm{Sel}}_F(V^*(1))^*$$

(see [G] Proposition 2 or [HMI] Proposition 3.82); so, we have the assertion (V). In [HMI], Proposition 3.82 is formulated in terms of $\overline{\mathrm{Sel}}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} V)$ and $\overline{\mathrm{Sel}}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} V^*(1))$ defined in [HMI] (3.4.11), but this does not matter because we can easily verify $\overline{\mathrm{Sel}}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} ?) \cong \overline{\mathrm{Sel}}_F(?)$ (similarly to [HMI] Corollary 3.81). $\square$

Actually, for the Selmer group with coefficients in a Galois representation of adjoint type in characteristic 0, we will later prove (in the fourth lecture) that

$$\overline{\mathrm{Sel}}_F(V) = \mathrm{Sel}_F(V).$$

2. Greenberg's $\mathcal{L}$–invariant

Here is Greenberg's definition of $\mathcal{L}(V)$: The long exact sequence of $\mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V \hookrightarrow V/\mathcal{F}^+_{\mathfrak{p}} V \twoheadrightarrow V/\mathcal{F}^-_{\mathfrak{p}} V$ gives a homomorphism, noting $F_{\mathfrak{p}} = \mathbb{Q}_p$ and writing $\mathfrak{G}_{F_{\mathfrak{p}}} = \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$,

$$\begin{aligned} &H^1(F_{\mathfrak{p}}, \mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V) = \mathrm{Hom}(\mathfrak{G}^{ab}_{\mathbb{Q}_p}, \mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V) \\ &\xrightarrow{\iota_{\mathfrak{p}}} H^1(F_{\mathfrak{p}}, V)/L_{\mathfrak{p}}(V) = \mathrm{Im}(H^1(F_{\mathfrak{p}}, V) \to H^1(F_{\mathfrak{p}}, V/\mathcal{F}^+_{\mathfrak{p}} V) \xrightarrow{\mathrm{Res}} H^1(I_{\mathfrak{p}}, V/\mathcal{F}^+_{\mathfrak{p}} V)). \end{aligned}$$

Note that $\mathrm{Hom}(\mathfrak{G}^{ab}_{\mathbb{Q}_p}, \mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V) \cong (\mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V)^2 \cong K^2$ canonically by

$$\phi \mapsto \left(\frac{\phi([\gamma, F_{\mathfrak{p}}])}{\log_p(\gamma)}, \phi([p, F_{\mathfrak{p}}])\right).$$

Here $[x, F_{\mathfrak{p}}] = [x, \mathbb{Q}_p]$ is the local Artin symbol (suitably normalized). Restricting to $I_{\mathfrak{p}}$, we lose one coordinate: $\phi([p, F_{\mathfrak{p}}]$ (the Frobenius coordinate). Since

$$L_{\mathfrak{p}}(\mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V) = \mathrm{Ker}(H^1(F_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V) \xrightarrow{\mathrm{Res}} H^1(I_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V)),$$

the image of $\iota_{\mathfrak{p}}$ is isomorphic to $\mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V \cong K$. By (V), we have a unique subspace $\mathbb{H}$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\prod_{\mathfrak{p}} \mathrm{Im}(\iota_{\mathfrak{p}}) \hookrightarrow \prod_{\mathfrak{p}} \frac{H^1(F_{\mathfrak{p}}, V)}{L_{\mathfrak{p}}(V)}.$$

Then by the restriction, $\mathbb{H}$ gives rise to a subspace L of

$$\prod_{\mathfrak{p}} \mathrm{Hom}(\mathfrak{G}_{F_{\mathfrak{p}}}^{ab}, \mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V) \cong \prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V)^2$$

isomorphic to $\prod_{\mathfrak{p}}(\mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V)$. If a cocycle c representing an element in $\mathbb{H}$ is unramified, it gives rise to an element in $\mathrm{Sel}_F(V)$. By the vanishing of $\mathrm{Sel}_F(V)$ (Lemma 1.3), this implies $c = 0$; so, the projection of L to the first factor $\prod_{\mathfrak{p}}(\mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V)$ (via $\phi \mapsto (\phi([\gamma, F_{\mathfrak{p}}])/\log_p(\gamma))_{\mathfrak{p}}$) is surjective. Thus this subspace L is a graph of a K–linear map $\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{-}V/\mathcal{F}_{\mathfrak{p}}^{+}V$. We then define $\mathcal{L}(V) = \det(\mathcal{L}) \in K$.

Let $\boldsymbol{\rho} : \mathfrak{G}_F \to GL_2(R)$ be the universal nearly ordinary deformation with $\boldsymbol{\rho}\big|_D = \begin{pmatrix} * & * \\ 0 & \boldsymbol{\delta} \end{pmatrix}$. Then $c_{\mathfrak{p}} = \frac{\partial \boldsymbol{\rho}}{\partial X_{\mathfrak{p}}}|_{X=0}\rho_F^{-1}$ is a 1-cocycle (by the argument proving Lemma 1.3) giving rise to a class of $\mathbb{H}$. By Lemma 1.3, $\mathbb{H} = \mathrm{Sel}_F^{-}(V)$, and $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ gives a basis of $\mathbb{H}$ over K. We have $\boldsymbol{\delta}([u, F_{\mathfrak{p}}]) = (1 + X_{\mathfrak{p}})^{\log_p(u)/\log_p(\gamma)}$ for $u \in O_{\mathfrak{p}}^{\times} = \mathbb{Z}_p^{\times}$. Writing

$$c_{\mathfrak{p}}(\sigma) = \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \rho_F(\sigma)^{-1},$$

we have $a_{\mathfrak{p}} = \boldsymbol{\delta}^{-1}\frac{d\boldsymbol{\delta}}{dX_{\mathfrak{p}}}|_{X=0}$, and from this we get the desired formula of $\mathcal{L}(Ad(\rho_F))$.

Write F_∞ for the cyclotomic $\mathbb{Z}_p$–extension of F. If one restricts $c \in \mathbb{H}$ to $\mathfrak{G}_\infty = \mathrm{Gal}(F^{(p)}/F_\infty)$, its ramification is exhausted by $\Gamma = \mathrm{Gal}(F_\infty/F)$ (because $F_{\mathfrak{p}} = \mathbb{Q}_p$) giving rise to a class $[c] \in \mathrm{Sel}_{F_\infty}(V)$. The kernel of the restriction map: $H^1(\mathfrak{G}, V) \to H^1(\mathfrak{G}_\infty, V)$ is given by $H^1(\Gamma, H^0(\mathfrak{G}_\infty, V)) = 0$ because $H^0(\mathfrak{G}_\infty, V) = 0$. Thus the image of $\mathbb{H}$ in $\mathrm{Sel}_{F_\infty}(V/T)$ gives rise to the order d exceptional zero of $L^{arith}(s, Ad(\rho_F))$ at $s = 1$. We have proved

Proposition 2.1. *For the number of prime factors $d = [F : \mathbb{Q}]$ of p in F, we have*

$$\mathrm{ord}_{s=1} L_p^{arith}(s, Ad(\rho_F)) \geq d.$$

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