Lecture 9: We discuss the material in Section 6.1. The Fourier expansion theorem of theta descent. Since the proof is easier for a definite quaternion algebra, we assume that $D/\mathbb{Q}$ is indefinite. So $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$. 
§0. Quaternion subalgebras of $B$. For each $\alpha \in D_\sigma \cap B^\times$, define the $\alpha$-twist $\sigma_\alpha$ of $\sigma$ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^\sigma_\alpha$. Then $\sigma_\alpha$ is another action of $\text{Gal}(E/Q)$ on $B$, and $D_\alpha = H^0(E/F, B)$ under this twisted action is a quaternion subalgebra of $B$.

- All quaternion $\mathbb{Q}$-subalgebras of $B$ are realized as $D_\alpha$ for some $\alpha \in D_\sigma$, and $D_z = D \Leftrightarrow z \in Z$;
- $\alpha = \xi^{-1} \beta \xi^{-\iota_\sigma}$ for $\xi \in B^\times \Leftrightarrow D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\alpha \cong D_\beta$ by an inner automorphism of $B$ if $N(\alpha) = N(\beta)$ and $B_\infty \cong M_2(E_\infty)$ (strong approximation);
- The even Clifford group $G_\alpha$ of $D_{\alpha,0} = \{v \in D_{\sigma_\alpha} | \text{Tr}(v) = 0\}$ is $D_\alpha^\times$ and $B^\times$ is a covering of the similitude group $GO_{D_\sigma}$ of $D_\sigma$.

Let $\hat{\Gamma}_\phi = \{h \in B_\hat{A}(\infty) | \phi(\infty)(h^{-1}vh^{-\iota_\sigma}) = \phi(\infty)(v), \forall v \in D_{\sigma,\hat{A}(\infty)}\}$ for each Schwartz-Bruhat function $\phi$ on $D_{\sigma,\hat{A}(\infty)}$.

Let $Sh_B = B^\times \backslash B_\hat{A}^\times / E_\hat{A}^\times \hat{\Gamma}_\phi C_\infty$ be the Shimura variety for $B^\times$ of level $\hat{\Gamma}_\phi$, and $Sh_\alpha$ be the image of $D_{\alpha,\hat{A}}^\times$ in $Sh_B$ for $\alpha \in D_\sigma$. 

Regard $Sh_\alpha \in H_2(Sh_B, \mathbb{Z})$ and write $(\cdot, \cdot) : H^2 \times H_2 \to \mathbb{C}$ for the Poincaré duality.
§1. **Theta differential form.** As in Lecture 4, for $\tau \in \mathfrak{h}$ and $(z, w) \in \mathfrak{h} \times \mathfrak{h}$, consider the Schwartz function

$$
\Psi(v; \tau, z, w) = \text{Im}(\tau) \frac{[v; z, w]^2}{(z - \bar{z})^2(w - \bar{w})^2} e(N(v)\tau + i\frac{\text{Im}(\tau)}{2|\text{Im}(z)\text{Im}(w)|}[v; z, w]^2).
$$

Here recall $[v; z, w] = s(v, p(z, w))$ for $p(z, w) = (\begin{matrix} 1 & \bar{z} \\ \bar{z} & 1 \end{matrix})(w, 1)J$. For $\gamma \in \text{GL}_2(\mathbb{R})$, $\alpha(\begin{matrix} \bar{z} \\ 1 \end{matrix}) = (\begin{matrix} \alpha(z) \\ 1 \end{matrix}) j(\alpha, z)$ and $\alpha J^t \alpha = \det(\alpha)J \Leftrightarrow ^t \alpha J = J \alpha^t$. From this, we have for $\alpha, \beta \in \text{GL}_2(\mathbb{R})$

$$
(*) \quad \alpha p(z, w) \beta^t = p(\alpha(z), \beta(w)) j(\alpha, z) j(\beta, w),
$$

$$
\text{det}(\alpha)[\alpha^{-1}v\beta; z, w] = j(\alpha, z) j(\beta, w)[v; \alpha(z), \beta(w)].
$$

Put $\phi = \phi^{(\infty)}\Psi$ for a Bruhat function $\phi^{(\infty)} \in \mathcal{S}(D_{\sigma, A})$. Then the theta kernel is given by

$$
\theta(\phi) = \theta(\phi)(\tau; z, w) = \sum_{v \in D_\sigma} \phi(v),
$$

$$
\Theta(\phi)(\tau; z, w) = \theta(\phi)(\tau; z, w)d\bar{z} \wedge dw.
$$

By $(*)$, $\theta$ is of weight $(2, 2)$ in $(z, w)$ and of weight $2$ in $\tau$. See §3.1.4 for the weight in $\tau$ and §6.1.3 for higher weight analog.
§2. Automorphic differential form and period. Let \( \iota : E \to \mathbb{R} \) be the identity embedding; so, \( \sigma \) gives rise to another embedding. For \( a \in F \), we just write \( a \) for \( a^\iota \). Write \( S^+_{2^-}(\Gamma) \) for space of cusp forms \( F : \mathcal{Z}_B \to \mathbb{C} \) holomorphic in \( z \) and antiholomorphic in \( w \) such that \( F(\gamma(z, w)) = j(\gamma, z, \overline{w})^2 F(z, w) \) for \( z, w \in \mathcal{H} \times \mathcal{H} \), where \( \gamma(z, w) = \left( \frac{az+b}{cz+d}, \frac{a^\sigma w+b^\sigma}{c^\sigma w+d^\sigma} \right) \) for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \). Define

\[
\omega(F) := F(z, w)dz \wedge d\overline{w}
\]

as a harmonic differential form.

Let \( SO_\alpha := SO_{D_\alpha,0} \) and \( \Gamma_\alpha = \Gamma菲 \cap SO_\alpha(\mathbb{R}) \). Then \( Sh_\alpha \cong \Gamma_\alpha \backslash \mathcal{H} \). Recall the period of \( \omega(F) \) over \( Sh_\alpha \) given by

\[
(Sh_\alpha, [\omega(F)]) := \int_{Sh_\alpha} \omega(F)|_{Sh_\alpha},
\]

where \([\omega(F)] \in H^2_{DR}(Sh_B, \mathbb{C})\) is the de Rham class of \( \omega(F) \).
3. **Theta descent.** The identity connected component of $SO_{D\sigma}(E_R)$ is

$$SO^+_{D\sigma}(R) := \{ (x, y) \in SO_{D\sigma}(E_R) | \det(x) = 1 \}$$

and $SO^+_{D\sigma}(Q) = SO^+_{D\sigma}(R) \cap SO_{D\sigma}(Q)$. Let

$$\Gamma_\phi := \{ \gamma \in SO^+_{D\sigma}(Q) = G^+_{D\sigma}/Z_{G^+}(Q) | \phi(\infty)(\gamma^{-1}v\gamma^\sigma) = \phi(\infty)(v) \},$$

$$\hat{\Gamma}_\phi := \{ u \in SO_{D\sigma}(A(\infty)) | \phi(\infty)(u^{-1}vu^\sigma) = \phi(\infty)(v) \},$$

$$U_\phi := \{ u \in B^\times_{A(\infty)} | \phi(\infty)(u^tvu^\sigma) = \phi(\infty)(v) \}.$$

Then $\Gamma_\phi = \hat{\Gamma}_\phi \cap SO^+_{D\sigma}(Q)$. Define for $g \in SL_2(\mathbb{A})$

$$\theta_{SL,*}(\phi)(F)(g) := \int_{SO^+_{D\sigma}(Q) \backslash SO^+_{D\sigma}(A)} \theta(\phi)(g, h)F(h) d\mu_h.$$ 

The Haar measure $d\mu_h$ satisfies $\int_{\hat{\Gamma}_\phi C_i} d\mu_h = 1$ for the stabilizer $C_i$ in $SO^+_{D\sigma}(R)$ of $i := (\sqrt{-1}, \sqrt{-1}) \in \mathbb{H}^2$, $d\mu_h = d\mu_{z,w} = y^{-2}dx dy v^{-2}du dv$ writing $z = x + y\sqrt{-1}$ and $w = u + v\sqrt{-1}$ and induces the Dirac measure at each point of $SO^+_{D\sigma}(Q)$. 
§4. Descent $q$-expansion.

**Theorem 1.** Let the notation and the assumption be as above. Then $\theta_{SL,*}(\phi)(\mathcal{F}) = \int_{\Gamma_{\phi}} \theta(\phi)(\tau; z, w)\mathcal{F}(z, w)(z - \overline{z})^2(w - \overline{w})^2\omega_{inv}$ is equal to

$$(4\sqrt{-1})^{-1} \sum_{\alpha \in D_{\sigma}/\Gamma_{\phi}; N(\alpha) > 0} \phi(\infty)(\alpha)(Sh_{\alpha}, [\omega(\mathcal{F})])e(N(\alpha)\tau),$$

where $\omega_{inv} := (z - \overline{z})^{-2}(w - \overline{w})^{-2}dz \wedge d\overline{z} \wedge dw \wedge d\overline{w}$.

We make some preparation for the proof. By ($\ast$), for $\gamma \in \Gamma_{\alpha}$ and $h \in SO_{\alpha}(\mathbb{R})$,

$$[\alpha; \gamma h(z), \gamma^\sigma h^\sigma(\overline{w})]^2\mathcal{F}(\gamma h(z), \gamma^\sigma h^\sigma(w)) = [\gamma^{-1} \alpha \gamma^\sigma; h(z), h(\overline{w})]^2 \times j(\gamma, z)^{-2}j(\gamma^\sigma, \overline{w})^{-2}j(\gamma, z)^2j(\gamma^\sigma, \overline{w})^2\mathcal{F}(h(z), h(w)) = [\alpha; h(z), h(\overline{w})]^2\mathcal{F}(h(z), h(w)).$$

The function $SO_{\alpha}(\mathbb{R}) \ni h \mapsto [\alpha; h(z), h(\overline{w})]^2\mathcal{F}(h(z), h(w))$ if left $\Gamma_{\alpha}$-invariant.
§5. Period as an integral over $SO_{\alpha}(\mathbb{R})$. For a right invariant measure $dh$ on $\Gamma_{\alpha}\backslash SO_{\alpha}(\mathbb{R})$, the integral

$$\varphi(z, w) = \int_{\Gamma_{\alpha}\backslash SO_{\alpha}(\mathbb{R})} [\alpha; h(z), h(w)]^2 F(h(z), h(w)) dh,$$

is well defined. Since $z \mapsto h(z)$ is a holomorphic action, $\varphi(z, w)$ is holomorphic in $z$ and anti-holomorphic in $w$.

**Lemma 1.** Let the notation and the assumption be as above. In particular, $F \in S_{k}^{++}(\hat{\Gamma}_\phi)$. Then

1. $\varphi(z, w)$ is a constant independent of $(z, w)$,
2. If $k > 0$ and $N(\alpha) < 0$, we have $\varphi(z, w) = 0$ for all $(z, w)$,
3. If $k > 0$ and $N(\alpha) = 0$, we have $\varphi(z, w) = 0$ for all $(z, w)$,
4. $\varphi(z, w)$ is a nonzero constant multiple of $(Sh_{\alpha}, [\omega(F)])$. 
6. **Proof of the assertion (1), Step 1.** We replace complex structure of $\mathcal{Z}_B = \mathcal{H}_z \times \mathcal{H}_w$ by the anti-holomorphic automorphism $\epsilon = (1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}) : (z, w) \mapsto (z, -\bar{w})$. We write $\mathcal{Z}_B'$ for this space with the new complex structure. Thus we regard $\mathcal{F}(z, w)$ as a function $\mathcal{F}' : \mathcal{Z}_B' \to \mathbb{C}$ by $\mathcal{F}'(z, w) = \mathcal{F}|\epsilon(z, w) = \mathcal{F}(z, \bar{w})$. Thus $\mathcal{F}'$ is a holomorphic modular form on $\mathcal{Z}_B'$. Then for $\Gamma'_\alpha := \epsilon \Gamma_\alpha \epsilon^{-1}$ and $\Gamma'_\phi = \epsilon \Gamma_\phi \epsilon^{-1}$,

$$\text{Sh}'_\alpha = \Gamma'_\alpha \backslash \text{SO}_\alpha(\mathbb{R})(\sqrt{-1}, \sqrt{-1}) \subset \Gamma'_\phi \backslash \mathcal{Z}_B' =: \text{Sh}'_B$$

is a real analytic submanifold of real dimension 2 but is not a complex manifold. Note $\alpha' := \epsilon^{-1} \alpha \epsilon^\sigma = (\alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, (\alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})^\sigma) \in B_\infty^\times$ and $N(\alpha') = -N(\alpha)$. Let

$$\phi(z, w) = \int_{\Gamma'_\alpha \backslash \text{SO}_\alpha(\mathbb{R})} [\alpha'; h(z), h(w)]^2 \mathcal{F}'(h(z), h(w)) dh,$$

which is a holomorphic function of $(z, w) \in \mathcal{Z}_B'$. For the assertion (1), we need to prove that $\phi$ is constant.
§7. Step 2, Use of holomorphy. Since $\phi(h(z), h(w)) = \phi(z, w)$ for all $h \in SO_{\alpha}(\mathbb{R})$, it is constant on $Sh'_{\alpha} \subset Sh'_{E}$, and it is constant on $\mathcal{Z}'_{B}$ as it is a constant on a real analytic submanifold of real dimension 2 which is not a complex manifold. This proves the assertion (1).

As for (4), after bringing $Sh_{\alpha}$ to the diagonal embedding by a variable change $(z - \overline{z})^{-2}(w - \overline{w})^{-2}|_{z=w} = (z - \overline{z})^{-4}$ and the invariant measure is proportional to $(z - \overline{z})^{-2}|dz \wedge d\overline{z}|$; so, the difference of the exponent 2 of the factor $(z - \overline{z})^{-2}(w - \overline{w})^{-2}|_{z=w}$ in $\omega_{inv}$ covers the discrepancy of the exponent $k = 2$ and $n = 0$ in the definition of $\varphi(z, w)$ and the definition of $(Sh_{\alpha}, [\omega(\mathcal{F})])$. Then it is just a simple computation. The nonzero constant can be taken to be 1 if we choose $dh$ matching $(z - \overline{z})^{-2}|dz \wedge d\overline{z}|$.

For the assertion (2) and (3), see Lemma 6.17. Hereafter we assume $N(\alpha) > 0$. 
§8. Reduction to the case where $\alpha = \delta \in \mathbb{R}_+^\times$, Step 1.
Consider a differential 4-form given by

$$\Omega_{\alpha}(F) := [\alpha; z, \bar{w}]^2 \exp(-\pi \frac{\eta [\alpha; z, \bar{w}]^2}{|\text{Im}(z) \text{Im}(w)|}) F(z, w)\omega_{\text{inv}}.$$

Since $\theta(\phi)$ is an average over $D_\sigma$ under the action $\alpha \mapsto \gamma^{-1} \alpha \gamma^s$,

$$\int_{\Gamma \phi \setminus \mathbb{Z}_B} \sum_{\alpha \in D_\sigma/\Gamma \phi} \sum_{\gamma \in \Gamma \phi/\Gamma \alpha} \Omega_{\gamma^{-1} \alpha \gamma} = \sum_{\alpha \in D_\sigma/\Gamma \phi} \int_{\Gamma \alpha \setminus \mathbb{Z}_B} \Omega_{\alpha}.$$

We pick $h_L, h_R \in \text{SL}_2(\mathbb{R})$ so that $\alpha = h_L^{-1} \delta h_R$ for $\delta \in \mathbb{R}_+^\times$ as $N(\alpha) > 0$. Put $h = (h_L, h_R) \in \text{SO}^+_\sigma(\mathbb{R})$. Then

$$h \alpha h^{-\sigma} = (h_L, h_R)(\alpha, \alpha^\sigma)(h_R^{-1}, h_L^{-1}) = (h_L \alpha h_R^{-1}, \ast) = (\delta, \ast).$$

Since $(h \alpha h^{-\sigma})^\sigma = h^\sigma \alpha^\sigma h^{-1} = h^\sigma \alpha^t h^{-1} = (h \alpha h^{-\sigma})^t$, we find that $\ast = \delta = \delta^\sigma$ in the $\sigma$-component of $E_\mathbb{R}$. Thus in $B_\infty$, we have $\alpha = h^{-1} \delta h^\sigma$ writing $(\delta, \delta)$ as $\delta \in E_\mathbb{R}$.

By computation, the function $\exp(-\pi \frac{\eta [\alpha; z, \bar{w}]^2}{|\text{Im}(z) \text{Im}(w)|})$ factors through $\text{SL}_2^\Delta(\mathbb{R}) \setminus \mathfrak{g}^2$ for $\text{SL}_2^\Delta(\mathbb{R}) = \{(g, g) | g \in \text{SL}_2(\mathbb{R})\}$ in $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. 
§9. Step 2. Noting that $\omega_{\text{inv}}$ is invariant under the action of holomorphic automorphisms of $\mathcal{H}^2$,
$$\Omega_\alpha(\mathcal{F}) = [h^{-1}\delta h^\sigma; z, \overline{w}]^2 \exp(-\pi \frac{\eta[|h^{-1}\delta h^\sigma; z, \overline{w}|]^2}{\Im(z) \Im(w)}) \mathcal{F}(z, w) \omega_{\text{inv}}$$
$$h = (h_L, h_R) \mapsto [\delta, h_L(z), h_R(\overline{w})]^2 j(h_L, z)^2 j(h_R, \overline{w})^2 \times \exp(-\pi \frac{\eta[|\delta; h_L(z), h_R(\overline{w})|^2}{\Im(h_L(z)) \Im(h_R(\overline{w}))}) \mathcal{F}(z, w) \omega_{\text{inv}}$$
$$h_L(z) \mapsto z, h_R(w) \mapsto w \Rightarrow [\delta; z, \overline{w}]^2 \exp(-\pi \frac{\eta[|\delta; z, \overline{w}|]^2}{\Im(z) \Im(w)}) \mathcal{F}|h^{-1}(z, w) \omega_{\text{inv}},$$
where $\mathcal{F}|h^{-1}(z, w) = j(h_L, z)^2 j(h_R, \overline{w})^2 \mathcal{F}(h_L^{-1}(z), h_R^{-1}(w))$.

We have $Sh_\alpha = \Gamma_\alpha \backslash SO^+_\alpha(\mathbb{R})(i) \cong h\Gamma_\alpha h^{-1} \backslash SO^+_\alpha(\mathbb{R})(i) =: Sh^h_\alpha$ for $h\Gamma_\alpha h^{-1} \subset D^\times_\infty = GL_2(\mathbb{R}) \xrightarrow{\text{diag}} (B_\infty)^\times = GL_2(\mathbb{R})^2$ diagonally embedded. Noting $[\delta; z, \overline{z}] = \delta(z - \overline{z})$, $\omega^h_\alpha(\mathcal{F})(z) := \mathcal{F}|h^{-1}(z, z)dz \wedge d\overline{z}$ is a closed harmonic 2-form on $Sh^h_\alpha$, and pulling back to $Sh_\alpha$ by $h : Sh_\alpha \cong Sh^h_\alpha$, we rediscover
$$\int_{Sh_\alpha} \omega(\mathcal{F})(z, w; \mathbf{x})|_{Sh_\alpha} = c \cdot \varphi(z, w).$$

The constant $c \neq 0$ depending on the choice of $dh$ in the lemma.
§10. Cartan decomposition. Since $\varphi(z, w)$ is the constant $(Sh_\alpha, \omega(F))$, we need to compute $\int_{SL_2^\Delta(\mathbb{R})\backslash \mathfrak{h}^2} \exp(-\pi \eta |[\delta; z, w]|^2 / |\text{Im}(z)\text{Im}(w)|)$. We analyze the quotient $SL_2^\Delta(\mathbb{R})\backslash \mathfrak{h}^2$.

**Lemma 2** (Cartan decomposition). We have a real analytic isomorphism $SL_2^\Delta(\mathbb{R})\backslash \mathfrak{h}^2 \cong [1, \infty)$ given by $(z, w) \mapsto v := \text{Im}(w)$.

For $(z, w) \in \mathfrak{h}_z \times \mathfrak{h}_w$, we have $(g_z^{-1}, g_z^{-1})(z, w) = (\sqrt{-1}, g_z^{-1}(w))$. Thus the quotient $SL_2^\Delta(\mathbb{R})\backslash \mathfrak{h}^2$ is a quotient of $\mathfrak{h}_w$. Replacing $g_z$ by $g_z u$, we find $(u^{-1}g_z^{-1}, u^{-1}g_z^{-1})(z, w) = (\sqrt{-1}, u^{-1}(w))$. The morphism $SL_2^\Delta(\mathbb{R})\backslash \mathfrak{h}^2 \to SO_2(\mathbb{R})\backslash \mathfrak{h}$ is an isomorphism as $w$ is an arbitrary element in $\mathfrak{h} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$.

Consider $i : w \mapsto \frac{w - \sqrt{-1}}{w + \sqrt{-1}}$ which induces an isomorphism $\mathfrak{h} \cong \mathcal{D} := \{ \bar{z} \in \mathbb{C} | |\bar{z}| < 1 \}$ whose inverse is $\bar{z} \mapsto \sqrt{-1} \frac{1 + z}{1 - z}$. Since $SO_2(\mathbb{R})$ acts on $\mathcal{D}$ by rotation, we find $SO_2(\mathbb{R})\backslash \mathfrak{h} \cong SO_2(\mathbb{R})\backslash \mathcal{D} \cong [0, 1) \xrightarrow{i^{-1}} \{ v\sqrt{-1} | v \in [1, \infty) \}$ as stated in the lemma.
§11. Proof of the theorem. Let $\Phi(v)$ be a function on $[1, \infty) \cong SO_2(\mathbb{R}) \backslash \mathfrak{h} \cong SL_2^+(\mathbb{R}) \backslash \mathfrak{h}^2 \to \mathbb{R}$ induced by $(z, w) \mapsto \exp(-\pi \frac{\eta[\delta; z, \overline{w}]^2}{|\text{Im}(z)\text{Im}(w)|})$. Then by the lemma, we get

**Corollary 1.** For $v \in [1, \infty)$, we have

$$
\Phi(v) = \exp(-\pi \eta[\delta; z, \overline{w}]^2 |\text{Im}(z)\text{Im}(w)|) = \exp(-\pi \eta\delta^2(1 + v)^2)
$$

Then by computation, we can evaluate $\int_1^\infty \Phi(v) d\mu_v$ easily, and we reach the theorem. See §6.1.7 full details of the computation, including an explicit form of the measure $d\mu_v$. 