# Descent Theorem for $F = \mathbb{Q}$

Haruzo Hida
Department of Mathematics, UCLA,

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**Lecture 9:** We discuss the material in Section 6.1. The Fourier expansion theorem of theta descent. Since the proof is easier for a definite quaternion algebra, we assume that $D_{/\mathbb{Q}}$ is indefinite. So $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$.

§**0. Quaternion subalgebras of $B$.** For each $\alpha \in D_\sigma \cap B^\times$, define the $\alpha$-twist $\sigma_\alpha$ of $\sigma$ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^{\sigma_\alpha}$. Then $\sigma_\alpha$ is another action of $\mathrm{Gal}(E/\mathbb{Q})$ on $B$, and $D_\alpha = H^0(E/F, B)$ under this twisted action is a quaternion subalgebra of $B$.

- All quaternion $\mathbb{Q}$-subalgebras of $B$ are realized as $D_\alpha$ for some $\alpha \in D_\sigma$, and $D_z = D \Leftrightarrow z \in Z$;
- $\alpha = \xi^{-1}\beta\xi^{-\iota\sigma}$ for $\xi \in B^\times \Leftrightarrow D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\alpha \cong D_\beta$ by an inner automorphism of $B$ if $N(\alpha) = N(\beta)$ and $B_\infty \cong M_2(E_\infty)$ (strong approximation);
- The even Clifford group $G_\alpha$ of $D_{\alpha,0} = \{v \in D_{\sigma_\alpha} | \mathrm{Tr}(v) = 0\}$ is $D_\alpha^\times$ and $B^\times$ is a covering of the similitude group $\mathrm{GO}_{D_\sigma}$ of $D_\sigma$.

Let $\widehat{\Gamma}_\phi = \{h \in B^\times_{\mathbb{A}(\infty)} | \phi^{(\infty)}(h^{-1}vh^{-\iota\sigma}) = \phi^{(\infty)}(v), \forall v \in D_{\sigma,\mathbb{A}(\infty)}\}$ for each Schwartz-Bruhat function $\phi$ on $D_{\sigma,\mathbb{A}(\infty)}$.

Let $Sh_B = B^\times \backslash B^\times_{\mathbb{A}} / E^\times_{\mathbb{A}} \widehat{\Gamma}_\phi C_\infty$ be the Shimura variety for $B^\times$ of level $\widehat{\Gamma}_\phi$, and $Sh_\alpha$ be the image of $D^\times_{\alpha,\mathbb{A}}$ in $Sh_B$ for $\alpha \in D_\sigma$. Regard $Sh_\alpha \in H_2(Sh_B, \mathbb{Z})$ and write $(\cdot,\cdot) : H^2 \times H_2 \to \mathbb{C}$ for the Poincaré duality.

§**1. Theta differential form.** As in Lecture 4, for $\tau \in \mathfrak{H}$ and $(z,w) \in \mathfrak{Z}_B = \mathfrak{H} \times \mathfrak{H}$, consider the Schwartz function $\Psi(v;\tau,z,w)$

$$\mathrm{Im}(\tau)\frac{[v;z,\overline{w}]^2}{(z-\overline{z})^2(w-\overline{w})^2}\mathbf{e}(N(v)\overline{\tau} + i\frac{\mathrm{Im}(\tau)}{2|\,\mathrm{Im}(z)\,\mathrm{Im}(w)|}|[v;z,\overline{w}]|^2).$$

Here recall $[v;z,w] = s(v,p(z,w))$ for $p(z,w) = \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)(w,1)J$. For $\gamma \in \mathrm{GL}_2(\mathbb{R})$, $\alpha\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha(z) \\ 1 \end{smallmatrix}\right) j(\alpha,z)$ and $\alpha J {}^t\alpha = \det(\alpha)J \Leftrightarrow {}^t\alpha J = J\alpha^\iota$. From this, we have for $\alpha,\beta \in \mathrm{GL}_2(\mathbb{R})$

$$(*)\ \alpha p(z,w)\beta^\iota = p(\alpha(z),\beta(w))j(\alpha,z)j(\beta,w)$$

$$\det(\alpha)[\alpha^{-1}v\beta;z,w] = j(\alpha,z)j(\beta,w)[v;\alpha(z),\beta(w)].$$

Put $\phi = \phi^{(\infty)}\Psi$ for a Bruhat function $\phi^{(\infty)} \in \mathcal{S}(D_{\sigma,\mathbb{A}})$. Then the theta kernel is given by

$$\theta(\phi) = \theta(\phi)(\tau;z,w) = \textstyle\sum_{v\in D_\sigma}\phi(v),$$

$$\Theta(\phi)(\tau;z,w) = \theta(\phi)(\tau;z,w)d\overline{z}\wedge dw.$$

By $(*)$, $\theta$ is of weight $(2,2)$ in $(z,w)$ and of weight 2 in $\tau$. See §3.1.4 for the weight in $\tau$ and §6.1.3 for higher weight analog.

§2. **Automorphic differential form and period.** Let $\iota : E \to \mathbb{R}$ be the identity embedding; so, $\sigma$ gives rise to another embedding. For $a \in F$, we just write $a$ for $a^\iota$. Write $S_2^{+-}(\Gamma)$ for space of cusp forms $\mathcal{F} : \mathfrak{Z}_B \to \mathbb{C}$ holomorphic in $z$ and antiholomorphic in $w$ such that $\mathcal{F}(\gamma(z,w)) = j(\gamma, z, \overline{w})^2 \mathcal{F}(z,w)$ for $z, w \in \mathfrak{H} \times \mathfrak{H}$, where $\gamma(z,w) = \left(\frac{az+b}{cz+d}, \frac{a^\sigma w + b^\sigma}{c^\sigma w + d^\sigma}\right)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Define

$$\omega(\mathcal{F}) := \mathcal{F}(z,w) dz \wedge d\overline{w}$$

as a harmonic differential form.

Let $\mathsf{SO}_\alpha := \mathsf{SO}_{D_{\alpha,0}}$ and $\Gamma_\alpha = \Gamma_\phi \cap \mathsf{SO}_\alpha(\mathbb{R})$. Then $Sh_\alpha \cong \Gamma_\alpha \backslash \mathfrak{H}$. Recall the period of $\omega(\mathcal{F})$ over $Sh_\alpha$ given by

$$(Sh_\alpha, [\omega(\mathcal{F})]) := \int_{Sh_\alpha} \omega(\mathcal{F})|_{Sh_\alpha},$$

where $[\omega(\mathcal{F})] \in H^2_{DR}(Sh_B, \mathbb{C})$ is the de Rham class of $\omega(\mathcal{F})$.

§**3. Theta descent.** The identity connected component of $\mathsf{SO}_{D_\sigma}(E_{\mathbb{R}})$ is

$$\mathsf{SO}^+_{D_\sigma}(\mathbb{R}) := \{(x,y) \in \mathsf{SO}_{D_\sigma}(E_{\mathbb{R}}) | \det(x) = 1\}$$

and $\mathsf{SO}^+_{D_\sigma}(\mathbb{Q}) = \mathsf{SO}^+_{D_\sigma}(\mathbb{R}) \cap \mathsf{SO}_{D_\sigma}(\mathbb{Q})$. Let
$\Gamma_\phi := \{\gamma \in \mathsf{SO}^+_{D_\sigma}(\mathbb{Q}) = G^+_{D_\sigma}/Z_{G^+}(\mathbb{Q}) | \phi^{(\infty)}(\gamma^{-1} v \gamma^\sigma) = \phi^{(\infty)}(v)\}$,
$\widehat{\Gamma}_\phi := \{u \in \mathsf{SO}_{D_\sigma}(\mathbb{A}^{(\infty)}) | \phi^{(\infty)}(u^{-1} v u^\sigma) = \phi^{(\infty)}(v)\}$,
$U_\phi := \{u \in B^\times_{\mathbb{A}^{(\infty)}} | \phi^{(\infty)}(u^\iota v u^\sigma) = \phi^{(\infty)}(v)\}$.
Then $\Gamma_\phi = \widehat{\Gamma}_\phi \cap \mathsf{SO}^+_{D_\sigma}(\mathbb{Q})$. Define for $g \in \mathsf{SL}_2(\mathbb{A})$

$$\theta_{SL,*}(\phi)(\mathcal{F})(g) := \int_{\mathsf{SO}^+_{D_\sigma}(\mathbb{Q})\backslash \mathsf{SO}^+_{D_\sigma}(\mathbb{A})} \theta(\phi)(g,h)\mathcal{F}(h) d\mu_h.$$

The Haar measure $d\mu_h$ satisfies $\int_{\widehat{\Gamma}_\phi C_{\mathbf{i}}} d\mu_h = 1$ for the stabilizer $C_{\mathbf{i}}$ in $\mathsf{SO}^+_{D_\sigma}(\mathbb{R})$ of $\mathbf{i} := (\sqrt{-1}, \sqrt{-1}) \in \mathfrak{H}^2$, $d\mu_h = d\mu_{z,w} = y^{-2} dx dy v^{-2} du dv$ writing $z = x + y\sqrt{-1}$ and $w = u + v\sqrt{-1}$ and induces the Dirac measure at each point of $\mathsf{SO}^+_{D_\sigma}(\mathbb{Q})$.

## §4. Descent $q$-expansion.

**Theorem 1.** *Let the notation and the assumption be as above. Then* $\theta_{SL,*}(\phi)(\mathcal{F}) = \int_{\Gamma_\phi\backslash\mathfrak{H}^2} \theta(\phi)(\tau; z, w)\mathcal{F}(z, w)(z-\overline{z})^2(w-\overline{w})^2\omega_{inv}$ *is equal to*

$$(4\sqrt{-1})^{-1} \sum_{\alpha\in D_\sigma/\Gamma_\phi; N(\alpha)>0} \phi^{(\infty)}(\alpha)(Sh_\alpha, [\omega(\mathcal{F})])\mathbf{e}(N(\alpha)\tau),$$

*where* $\omega_{inv} := (z-\overline{z})^{-2}(w-\overline{w})^{-2}dz \wedge d\overline{z} \wedge dw \wedge d\overline{w}$.

We make some preparation for the proof. By $(*)$, for $\gamma \in \Gamma_\alpha$ and $h \in \mathrm{SO}_\alpha(\mathbb{R})$,

$$\begin{aligned}
&[\alpha; \gamma h(z), \gamma^\sigma h^\sigma(\overline{w})]^2\mathcal{F}(\gamma h(z), \gamma^\sigma h^\sigma(w)) \\
&= [\gamma^{-1}\alpha\gamma^\sigma; h(z), h(\overline{w})]^2 \\
&\qquad \times j(\gamma, z)^{-2} j(\gamma^\sigma, \overline{w})^{-2} j(\gamma, z)^2 j(\gamma^\sigma, \overline{w})^2 \mathcal{F}(h(z), h(w)) \\
&= [\alpha; h(z), h(\overline{w})]^2\mathcal{F}(h(z), h(w)).
\end{aligned}$$

The function $\mathrm{SO}_\alpha(\mathbb{R}) \ni h \mapsto [\alpha; h(z), h(\overline{w})]^2\mathcal{F}(h(z), h(w))$ if left $\Gamma_\alpha$-invariant.

## §5. Peirod as an integral over $\mathrm{SO}_\alpha(\mathbb{R})$.

For a right invariant measure $dh$ on $\Gamma_\alpha\backslash\mathrm{SO}_\alpha(\mathbb{R})$, the integral

$$\varphi(z,w) = \int_{\Gamma_\alpha\backslash\mathrm{SO}_\alpha(\mathbb{R})} [\alpha; h(z), h(\overline{w})]^2 \mathcal{F}(h(z), h(w)) dh,$$

is well defined. Since $z \mapsto h(z)$ is a holomorphic action, $\varphi(z,w)$ is holomorphic in $z$ and anti-holomorphic in $w$.

**Lemma 1.** *Let the notation and the assumption be as above. In particular, $\mathcal{F} \in S_k^{+-}(\widehat{\Gamma}_\phi)$. Then*

(1) *$\varphi(z,w)$ is a constant independent of $(z,w)$,*

(2) *If $k > 0$ and $N(\alpha) < 0$, we have $\varphi(z,w) = 0$ for all $(z,w)$,*

(3) *If $k > 0$ and $N(\alpha) = 0$, we have $\varphi(z,w) = 0$ for all $(z,w)$,*

(4) *$\varphi(z,w)$ is a nonzero constant multiple of $(Sh_\alpha, [\omega(\mathcal{F})])$.*

§**6. Proof of the assertion (1), Step 1.** We replace complex structure of $\mathfrak{Z}_B = \mathfrak{H}_z \times \mathfrak{H}_w$ by the anti-holomorphic automorphism $\epsilon = (1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) : (z, w) \mapsto (z, -\overline{w})$. We write $\mathfrak{Z}'_B$ for this space with the new complex structure. Thus we regard $\mathcal{F}(z, w)$ as a function $\mathcal{F}' : \mathfrak{Z}'_B \to \mathbb{C}$ by $\mathcal{F}'(z, w) = \mathcal{F}|\epsilon(z, w) = \mathcal{F}(z, \overline{w})$. Thus $\mathcal{F}'$ is a holomorphic modular form on $\mathfrak{Z}'_B$. Then for $\Gamma'_\alpha := \epsilon\Gamma_\alpha\epsilon^{-1}$ and $\Gamma'_\phi = \epsilon\Gamma_\phi\epsilon^{-1}$,

$$Sh'_\alpha = \Gamma'_\alpha \backslash \mathrm{SO}_\alpha(\mathbb{R})(\sqrt{-1}, \sqrt{-1}) \subset \Gamma'_\phi \backslash \mathfrak{Z}'_B =: Sh'_B$$

is a real analytic submanifold of real dimension 2 but is not a complex manifold. Note $\alpha' := \epsilon^{-1}\alpha\epsilon^\sigma = (\alpha \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \alpha^\sigma) \in B_\infty^\times$ and $N(\alpha') = -N(\alpha)$. Let

$$\phi(z, w) = \int_{\Gamma'_\alpha \backslash \mathrm{SO}_\alpha(\mathbb{R})} [\alpha'; h(z), h(w)]^2 \mathcal{F}'(h(z), h(w)) dh,$$

which is a holomorphic function of $(z, w) \in \mathfrak{Z}'_B$. For the assertion (1), we need to prove that $\phi$ is constant.

§**7. Step 2, Use of holomorphy.** Since $\phi(h(z), h(w)) = \phi(z, w)$ for all $h \in \mathrm{SO}_\alpha(\mathbb{R})$, it is constant on $Sh'_\alpha \subset Sh'_E$, and it is constant on $\mathfrak{Z}'_B$ as it is a constant on a real analytic submanifold of real dimension 2 which is not a complex manifold. This proves the assertion (1).

As for (4), after bringing $Sh_\alpha$ to the diagonal embedding by a variable change $(z - \overline{z})^{-2}(w - \overline{w})^{-2}|_{z=w} = (z - \overline{z})^{-4}$ and the invariant measure is proportional to $(z - \overline{z})^{-2}|dz \wedge d\overline{z}|$; so, the difference of the exponent 2 of the factor $(z - \overline{z})^{-2}(w - \overline{w})^{-2}|_{z=w}$ in $\omega_{inv}$ covers the discrepancy of the exponent $k = 2$ and $n = 0$ in the definition of $\varphi(z, w)$ and the definition of $(Sh_\alpha, [\omega(\mathcal{F})])$. Then it is just a simple computation. The nonzero constant can be taken to be 1 if we choose $dh$ matching $(z - \overline{z})^{-2}|dz \wedge d\overline{z}|$.

For the assertion (2) and (3), see Lemma 6.17. Hereafter we assume $N(\alpha) > 0$.

## §8. Reduction to the case where $\alpha = \delta \in \mathbb{R}_+^\times$, Step 1.

Consider a differential 4-form given by

$$\Omega_\alpha(\mathcal{F}) := [\alpha; z, \overline{w}]^2 \exp(-\pi \frac{\eta|[\alpha; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})\mathcal{F}(z,w)\omega_{inv}.$$

Since $\theta(\phi)$ is an average over $D_\sigma$ under the action $\alpha \mapsto \gamma^{-1}\alpha\gamma^s$,

$$\int_{\Gamma_\phi\backslash\mathfrak{Z}_B} \sum_{\alpha\in D_\sigma/\Gamma_\phi} \sum_{\gamma\in\Gamma_\phi/\Gamma_\alpha} \Omega_{\gamma^{-1}\alpha\gamma} = \sum_{\alpha\in D_\sigma/\Gamma_\phi} \int_{\Gamma_\alpha\backslash\mathfrak{Z}_B} \Omega_\alpha.$$

We pick $h_L, h_R \in \mathrm{SL}_2(\mathbb{R})$ so that $\alpha = h_L^{-1}\delta h_R$ for $\delta \in \mathbb{R}^\times$ as $N(\alpha) > 0$. Put $h = (h_L, h_R) \in \mathrm{SO}_{D_\sigma}^+(\mathbb{R})$. Then

$$h\alpha h^{-\sigma} = (h_L, h_R)(\alpha, \alpha^\sigma)(h_R^{-1}, h_L^{-1}) = (h_L\alpha h_R^{-1}, *) = (\delta, *).$$

Since $(h\alpha h^{-\sigma})^\sigma = h^\sigma\alpha^\sigma h^{-1} = h^\sigma\alpha^\iota h^{-1} = (h\alpha h^{-\sigma})^\iota$, we find that $* = \delta = \delta^\sigma$ in the $\sigma$-component of $E_\mathbb{R}$. Thus in $B_\infty$, we have $\alpha = h^{-1}\delta h^\sigma$ writing $(\delta, \delta)$ as $\delta \in E_\mathbb{R}$.

By computation, the function $\exp(-\pi\frac{\eta|[\alpha;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})$ factors through $\mathrm{SL}_2^\Delta(\mathbb{R})\backslash\mathfrak{H}^2$ for $\mathrm{SL}_2^\Delta(\mathbb{R}) = \{(g,g)|g \in \mathrm{SL}_2(\mathbb{R})\}$ in $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$.

§**9. Step 2.** Noting that $\omega_{inv}$ is invariant under the action of holomorphic automorphisms of $\mathfrak{H}^2$,

$$\begin{aligned}\Omega_\alpha(\mathcal{F}) &= [h^{-1}\delta h^\sigma; z, \overline{w}]^2 \exp(-\pi\tfrac{\eta|[h^{-1}\delta h^\sigma; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})\mathcal{F}(z,w)\omega_{inv} \\ &\overset{h=(h_L,h_R)}{=} [\delta, h_L(z), h_R(\overline{w})]^2 j(h_L,z)^2 j(h_R,\overline{w})^2 \\ &\quad\times \exp(-\pi\tfrac{\eta|[\delta; h_L(z), h_R(\overline{w})]|^2}{|\operatorname{Im}(h_L(z))\operatorname{Im}(h_R(w))|})\mathcal{F}(z,w)\omega_{inv} \\ &\overset{h_L(z)\mapsto z, h_R(w)\mapsto w}{=} [\delta; z, \overline{w}]^2 \exp(-\pi\tfrac{\eta|[\delta; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})\mathcal{F}|h^{-1}(z,w)\omega_{inv},\end{aligned}$$

where $\mathcal{F}|h^{-1}(z,w) = j(h_L,z)^2 j(h_R,\overline{w})^2 \mathcal{F}(h_L^{-1}(z), h_R^{-1}(w))$.

We have $Sh_\alpha = \Gamma_\alpha\backslash \mathrm{SO}_\alpha^+(\mathbb{R})(\mathbf{i}) \cong h\Gamma_\alpha h^{-1}\backslash \mathrm{SO}_\alpha^+(\mathbb{R})(\mathbf{i}) =: Sh_\alpha^h$ for $h\Gamma_\alpha h^{-1} \subset D_\infty^\times = \mathrm{GL}_2(\mathbb{R}) \overset{\text{diag}}{\hookrightarrow} (B_\infty)^\times = \mathrm{GL}_2(\mathbb{R})^2$ diagonally embedded. Noting $[\delta; z, \overline{z}] = \delta(z - \overline{z})$, $\omega_\alpha^h(\mathcal{F})(z) := \mathcal{F}|h^{-1}(z,z)dz \wedge d\overline{z}$ is a closed harmonic 2-form on $Sh_\alpha^h$, and pulling back to $Sh_\alpha$ by $h : Sh_\alpha \cong Sh_\alpha^h$, we rediscover

$$\int_{Sh_\alpha} \omega(\mathcal{F})(z,w;\mathbf{x})|_{Sh_\alpha} = c \cdot \varphi(z,w).$$

The constant $c \neq 0$ depending on the choice of $dh$ in the lemma.

§**10. Cartan decomposition.** Since $\varphi(z,w)$ is the constant $(Sh_\alpha, \omega(\mathcal{F}))$, we need to compute $\int_{\mathrm{SL}_2^\Delta(\mathbb{R})\backslash\mathfrak{H}^2} \exp(-\pi\frac{\eta|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})$. We analyze the quotient $\mathrm{SL}_2^\Delta(\mathbb{R})\backslash\mathfrak{H}^2$.

**Lemma 2** (Cartan decomposition)**.** *We have a real analytic isomorphism* $\mathrm{SL}_2^\Delta(\mathbb{R})\backslash\mathfrak{H}^2 \cong [1,\infty)$ *given by* $(z,w) \mapsto v := \operatorname{Im}(w)$.

For $(z,w) \in \mathfrak{H}_z \times \mathfrak{H}_w$, we have $(g_z^{-1}, g_z^{-1})(z,w) = (\sqrt{-1}, g_z^{-1}(w))$. Thus the quotient $\mathrm{SL}_2^\Delta(\mathbb{R})\backslash\mathfrak{H}^2$ is a quotient of $\mathfrak{H}_w$. Replacing $g_z$ by $g_z u$, we find $(u^{-1}g_z^{-1}, u^{-1}g_z^{-1})(z,w) = (\sqrt{-1}, u^{-1}(w))$. The morphism $\mathrm{SL}_2^\Delta(\mathbb{R})\backslash\mathfrak{H}^2 \to \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H}$ is an isomorphism as $w$ is an arbitrary element in $\mathfrak{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

Consider $i : w \mapsto \frac{w-\sqrt{-1}}{w+\sqrt{-1}}$ which induces an isomorphism $\mathfrak{H} \cong \mathfrak{D} := \left\{\mathfrak{z} \in \mathbb{C} \middle| |\mathfrak{z}| < 1\right\}$ whose inverse is $\mathfrak{z} \mapsto \sqrt{-1}\frac{1+\mathfrak{z}}{1-\mathfrak{z}}$. Since $\mathrm{SO}_2(\mathbb{R})$ acts on $\mathfrak{D}$ by rotation, we find $\mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H} \cong \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{D} \cong [0,1) \xrightarrow[\sim]{i^{-1}} \{v\sqrt{-1} | v \in [1,\infty)\}$ as stated in the lemma.

§**11. Proof of the theorem.** Let $\Phi(v)$ be a function on $[1,\infty) \cong \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H} \cong \mathrm{SL}_2^{\Delta}(\mathbb{R})\backslash\mathfrak{H}^2 \to \mathbb{R}$ induced by $(z,w) \mapsto \exp(-\pi\frac{\eta|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})$. Then by the lemma, we get

**Corollary 1.** *For $v \in [1,\infty)$, we have*

$$\Phi(v) = \exp(-\pi\frac{\eta|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|}) = \exp(-\pi\frac{\eta\delta^2(1+v)^2}{v}).$$

Then by computation, we can evaluate $\int_1^\infty \Phi(v)d\mu_v$ easily, and we reach the theorem. See §6.1.7 full details of the computation, including an explicit form of the measure $d\mu_v$.