Lecture 8: We describe how to extend the Weil representation $w$ from $\text{SL}_2(F_{\mathbb{A}}) \times \mathcal{O}_{D_{\sigma}}(\mathbb{A})$ to $(g, h) \in \text{GL}_2(F_{\mathbb{A}}) \times B_{\mathbb{A}}^\times$ so that $\theta(\phi)(g, h) = \sum_{\alpha \in D_{\sigma}} (L(h)w(g)\phi)(\alpha)$ is kept automorphic. The idea is that $g \in \text{GL}_2(F_{p_{F}})$ for primes $p_{F}$ splitting in $E$ together with $\text{GL}_2(F)$ generates almost the entire group $\text{GL}_2(F_{\mathbb{A}})$. We prove that $\theta(\phi)(g, h)|_{T_p^\ast} = \theta(\phi)(g, h)|_{T_{p_{F}}^\perp}$ for a prime $p$ split in $E/F$ as long as $\phi = \phi_{p_{F}}\phi^{(p_{F})}$ with $\phi_{p_{F}}$ given by characteristic function of $M_2(O_{F_{p}})$. Details are in §4.7.5.
§0. A set of density 1 split primes. Let $S$ be a set of split primes $p|p_F$ of $E$ for each prime $p_F$ of $F$ with $D_{\sigma,E_p,F} \cong M_2(E_p)$. Suppose

(S) The set $S$ has Chebotarev density 1 over $E$.

Choose an inert prime $p'$ with $B_{p'} \cong M_2(E'_p)$ and put $S' = S \cup \{p'\}$. Let $G \subset GL_2(F_\mathbb{A})$ (resp. $G' \subset GL_2(F_\mathbb{A})$) generated by $SL_2(F_\mathbb{A})$, $GL_2^+(F_\infty)$ and $\{GL_2(E_p)\}_{p \in S}$ (resp. $\{GL_2(E_p)\}_{p \in S'}$). Chebotarev density 1 means that for each ray class modulo $N$ of $E$ for every integer $N > 0$, there is a prime $p|p_F \in S$ in the class (cf. [CFN,V.6]).

For each $p_F$, we fix a choice of a prime ideal $p$ of $E$ such that $p|p_F$ and identify $F_{p_F} = E_p$. We identify $S$ with $\{p|p_F \in S\}$. Since $\theta(\phi)(g, h) = \theta(g, \sigma \cdot h)$ by definition as $\sigma \in O_{D_\sigma}(\mathbb{Q})$, the choice of $p$ does not matter. For $G = G$ and $G'$, we have the representation $w$ extended to subgroup $G$ of $GL_2(F_\mathbb{A})$. 
§1. $B_{\mathbb{A}}^\times$ action on $S(D_{\sigma,\mathbb{A}})$. Define a representation $L$ of $B_{\mathbb{A}}^\times$ on $S(D_{\sigma,\mathbb{A}})$ by $(L(h)\phi)(v) = \phi(h^t v h^\sigma)$. Then the action $L(h)$ of $B_{\mathbb{A}}^\times$ on the center factors through $N_{E/F}$. This is because the algebraic group $B^\times$ is sent into $GO_{D_{\sigma}}$ by the action $v \mapsto h^t v h^\sigma$ and the similitude map $N_{D_{\sigma}} : GO \rightarrow \mathbb{G}_m/F$ is given by $N_{D_{\sigma}}(h) = N(h h^\sigma) = N_{E/F}(N(h))$. For the Weil representation $w_b$ with respect to $x \mapsto e_v(b x)$, Jacquet–Langlands [AFG, Chapter I] extended $w$ to a representation $w_p$ of $GL_2(F_p)$ by $w_p(g \text{ diag}[b,1])\phi(v) = |b|^p w_{b^{-1}}(g)$ for $g \in SL_2(F_p)$.

Consider $\theta(\phi)(g, h) : GL_2(F) \backslash GL_2(F)SL_2(F_{\mathbb{A}}) \times B^\times \backslash B_{\mathbb{A}}^\times \rightarrow \mathbb{C}$ so that

$$\theta(\phi)(\gamma g, \delta h) = \theta(L(h)w(g)\phi) = \theta(\phi)(g, h)$$

for $\gamma \in GL_2(F)$, $g \in SL_2(F_{\mathbb{A}})$, $\delta \in B^\times$ and $h \in B_{\mathbb{A}}^\times$ with $L(h)\phi(v) = \phi(h^t v h^\sigma)$.
§2. Legitimacy of extension in §1. If $\gamma g = \gamma' g'$ for $g, g' \in \text{SL}_2(F_\mathbb{A})$ and $\gamma, \gamma' \in \text{GL}_2(F)$, then $g = \gamma^{-1} \gamma' g'$ and $\theta(\phi)(g, h) = \theta(\phi)(g', h')$ because of $\gamma^{-1} \gamma' \in \text{SL}_2(F)$, and hence the value $\theta(\phi)(\gamma g, \delta h)$ is independent of the choice of $g, g' \in \text{SL}_2(F_\mathbb{A})$ and $\gamma, \gamma' \in \text{GL}_2(F)$ as long as $\gamma g = \gamma' g'$. For an element $\delta \in B^\times$, $\theta(\phi)(g, \delta h) = \theta(\phi)(g, \delta h')$ as the theta series is an average of the terms over $D_\sigma$ and $\alpha \mapsto \delta' \alpha \delta$ preserves $D_\sigma$. For a pair $\psi_\infty$ and $\phi_\infty$ as in Lemma 4 in Lecture 7, we may define $w(zg)\phi(v) = \psi_\infty(z)w(g)\phi(v)$ for $g \in \text{SL}_2(F_\infty)$.

For $G = G$ or $G'$, we extend further $\theta$ to $\text{GL}_2(F)\text{SL}_2(F_\mathbb{A}) \cdot G \times B^\times_\mathbb{A}$ by

$$(E) \quad \theta(\phi)(\gamma gg, h) := \theta(\phi)(gg, h)$$

for $\gamma \in \text{GL}_2(F), \ g \in \text{SL}_2(F_\mathbb{A}), \ g \in G$. 
§3. Legitimacy of extension adding $G$

Lemma 1. The extension $(\mathcal{E})$ of $\theta(\phi)$ to $\text{GL}_2(F)\text{SL}_2(F_{\mathbb{A}}) \cdot G \times B_{\mathbb{A}}^\times$ is well defined and is left invariant under $\text{GL}_2(F') \times B^\times$.

Proof. If $gg = g'g'$ with $g, g' \in \text{SL}_2(F_{\mathbb{A}})$ and $g, g' \in G$, then

$$w(g'^{-1})w(g) = w(g'g^{-1}) = w(g')w(g)^{-1}$$

as the extension of $w$ to $\text{SL}_2(F_{\mathbb{A}})G$ coincides with the original Weil representation $w$ of $\text{SL}_2(F_{\mathbb{A}})$. Thus $w(gg) = w(g'g')$ and hence $\theta(\phi)(gg, h) = \theta(\phi)(g'g', h)$.

For $\gamma, \gamma' \in \text{GL}_2(F)$, if $\gamma gg = \gamma'g'g'$, $\gamma'^{-1}\gamma = g'g'^{-1}g^{-1}$. Taking a prime $l$ of $F$ non-split in $E$ outside $S'$, we find $\gamma'^{-1}\gamma = g'^{-1}g'^{-1}l^{-1}$, and hence $\det(\gamma'^{-1}\gamma) = 1 \Rightarrow \det(\gamma'^{-1}\gamma) = 1$. This implies $\theta(gg, h) = \theta(g'g', h)$, and therefore, $\theta(gg, h)$ factors through $\text{GL}_2(F) \backslash \text{GL}_2(F')\text{SL}_2(F_{\mathbb{A}})G \times B_{\mathbb{A}}^\times$. Since $L(\xi)$ for $\xi \in B^\times$ permutes elements of $D_\sigma$, plainly as a function of $h \in B_{\mathbb{A}}^\times$, $\theta(\phi)(g, h)$ factors through $B^\times \backslash B_{\mathbb{A}}^\times$. \qed

Lemma 2. For a quadratic field extension $E/F$, inside the idele group $F\times_{\mathbb{A}}$, the closed subgroup $H$ topologically generated by $\{F_{\mathfrak{l}}\}_{\mathfrak{l} \in S}$, $F\times$, $F_{\infty}^\times$, and $N_{E/F}(E_{\mathbb{A}}^\times)$ has index 2 and is open, and Artin symbol induces $F_{\mathbb{A}}^\times/H \cong \text{Gal}(E/F)$. Here $F_{\infty}^\times_+$ is the identity connected component of $F_{\mathbb{R}}^\times$.

Proof: We show that $E_{\mathbb{A}}^\times$ is generated topologically by

$$E^\times, \{E_p^\times\}_{p|p_F \in S} \text{ and } E_{\infty}^\times_+,$$

where $S$ is identified with subset of prime ideals of $E$. Let $U(N) := \{x \in \hat{O}_E^\times | x \equiv 1 \mod N\}$ for a positive integer $N$. Then the finite group $Cl_N := E_{\mathbb{A}}^\times/E^\times U(N)E_{\infty}^\times_+$ is the strict ray class group modulo $N$ of $E$. Thus it is canonically isomorphic to the Galois group $H_N$ of ray class field modulo $N$ over $E$ by class field theory.
§5. Proof continues.
Since $E_A^\times /E^\times E_\infty^\times \rightarrow \lim_{\leftarrow N} Cl_N$, we need to prove that $Cl_N$ is
generated by totally split primes in $E$. Since totally split primes
of $E/Q$ has density one among primes of $E$, by Chebotarev density
theorem, $E_A^\times$ is generated topologically by $E^\times$, $\{E_p^\times\}_{p|F \in S}$ and
$E_\infty^\times$.

Now by global class field theory,
$$\text{Coker}(N_{E/F} : E_A^\times /E^\times E_\infty^\times \rightarrow F_A^\times /F^\times F_\infty^\times) \cong \text{Gal}(E/F).$$
Since $N_{E/F} : E_A^\times \rightarrow F_A^\times$ is an open map, the assertion follows. $\square$
§6. GL(2)-version.

**Corollary 1.** Write $\chi_E^G$ for the composite of $\det : \operatorname{GL}_2(F_{\mathbb{A}}) \to F_{\mathbb{A}}^\times$ with $\chi_E = \left(\frac{E}{F}\right) : F_{\mathbb{A}}^\times \to \{\pm 1\}$, and define $G_{E/F}(\mathbb{A}) := \ker(\chi_E^G)$. Then $G_{E/F}(\mathbb{A})$ is an open-closed subgroup of $\operatorname{GL}_2(F_{\mathbb{A}})$ of index 2 containing $\operatorname{GL}_2(F)$, $\operatorname{GL}_2(F_{\infty})$ and the center $F_{\mathbb{A}}^\times$.

**Proof.** Since $\chi_E^G$ is a continuous open character of order 2, its kernel $G_{E/F}(\mathbb{A})$ is an open subgroup of index 2 containing $(F_{\mathbb{A}}^\times)^2$ which is the image of the center under $\det$. Since $\chi_E$ factors through the idele class group $F_{\mathbb{A}}^\times/F^\times N_{E/F}(E_{\mathbb{A}}^\times)$, $G_{E/F}(\mathbb{A})$ contains $\operatorname{GL}_2(F)$ and $\operatorname{GL}_2(F_{\infty})$. □

Since $G_{E/F}(\mathbb{A})$ contains $\operatorname{GL}_2(F_{\infty})$ and projects surjectively to $\operatorname{GL}_2(F_{\infty})$, writing $G_{E/F}(\mathbb{A}^{(\infty)})$ for $\ker(G_{E/F}(\mathbb{A}) \to \operatorname{GL}_2(F_{\infty}))$,

$$G_{E/F}(\mathbb{A}) = G_{E/F}(\mathbb{A}^{(\infty)}) \times \operatorname{GL}_2(F_{\infty}).$$
§7. Extension Theorem.

Theorem 1. The theta function

$$\theta(\phi)(g, h) : \text{GL}_2(F) \backslash \text{GL}_2(F) \text{SL}_2(F_\mathbb{A}) \cdot G \times B^\times \backslash B^\times_\mathbb{A} \to \mathbb{C}$$

extends to an automorphic form on $\text{GL}_2(F) \backslash G_{E/F}(\mathbb{A}) \times B^\times \backslash B^\times_\mathbb{A}$ independent of the choice of $S$.

Here “automorphic form” means analytic on the infinite component and right invariant under an open subgroup of $G_{E/F}(\mathbb{A}^{(\infty)}) \times B^\times_\mathbb{A}(\mathbb{A})$. Since $\theta'(\phi)(g, h) = \theta(\phi)(g, h^{-\iota})$, the twisted one $\theta'(\phi)$ is also well defined over $\text{GL}_2(F) \backslash G_{E/F}(\mathbb{A}) \times B^\times \backslash B^\times_\mathbb{A}$. Since $G_{E/F}(\mathbb{A})$ is independent of the choice of $S$, the theta function is also independent of the choice of $S$. 
§8. Proof. We need to show that $\theta(g, h)$ extends to $G_{E/F}(\mathbb{A})$ from the subset $\text{GL}_2(F) \cap \text{GL}_2^+(F_\infty)$. For $\phi = \phi(\infty) \phi_\infty \in S(D_\sigma, F_\mathbb{A})$, $\phi(\infty)$ is a finite linear combination of factorizable Bruhat functions in $S(D_\sigma, F_\mathbb{A}(\infty))$. Thus we may assume that $\phi(\infty)$ is factorizable to prove the extension property of $\theta(g, h) = \theta(\phi)(g, h)$. By Chebotarev density Lemma 2, for $F_S^\times := \prod_{p \in S} F_p^\times \cap F_\mathbb{A}(\infty)$, $F^\times F_S^\times F_\infty^\times$ is dense in $F^\times N_{E/F}(E_\mathbb{A}^\times)$, and hence its pull-back $\text{GL}_2(F) \cap \text{GL}_2^+(F_\infty)$ in $\text{GL}_2(F_\mathbb{A})$ by the determinant map is dense in $G_{E/F}(\mathbb{A})$. Thus for any open subgroup $U$ of $G_{E/F}(\mathbb{A})$, we have $\text{GL}_2(F) \cap U \text{GL}_2^+(F_\infty) = G_{E/F}(\mathbb{A})$. As we saw, $U^\phi = \prod_{p \in F} U_p^\phi$ is an open subgroup of the finite part $G_{E/F}(\mathbb{A}(\infty))$ and $\theta(\phi)(g, h)$ extends to $\text{GL}_2(F) \cap U^\phi \text{GL}_2^+(F_\infty) = G_{E/F}(\mathbb{A})$ left invariant under $\text{GL}_2(F)$ and right invariant under $U^\phi$. \qed

By the same proof using $G'$, we have an extension of $\theta(g, h)$ to $\text{GL}_2(F) \backslash \text{GL}_2(F_\mathbb{A}) \times B^\times \backslash B_\mathbb{A}^\times$. 
§9. Induction from $G_{E/F}(\mathbb{A})$ to $GL_2(F_{\mathbb{A}})$. Choose a pair $(\psi_\infty, \phi_\infty)$ with $\psi_\infty|_{F_\infty^+} = 1$ as in §10 of the last lecture. Then by all the extension lemmas combined, this newly extended $\theta(\phi)(g, h)$ is an automorphic form on $GL_2(F') \backslash GL_2(F_{\mathbb{A}}) \times B^\times \backslash B_{\mathbb{A}}^\times$. Since $\psi_\infty|_{F_\infty^+} = 1$, we have

$$
\theta(\phi)(zg, h) = \theta(\phi)(g, h) \text{ for } z \in F_\infty^+.
$$

Since $GL_2(\mathbb{Q}) \backslash GL_2(F_{\mathbb{A}})/F_\infty^\times$ has finite volume, we can project $\theta(\phi)$ to the cuspidal subspace, which is written as $\theta_{\text{cusp}}(\phi)$. Here is an obvious corollary

**Corollary 2.** Writing $\Theta$ (resp. $\Theta_0$) for the representation of $GL_2(F_{\mathbb{A}}) \times B_{\mathbb{A}}^\times$ (resp. $G_{E/F}(\mathbb{A}) \times B_{\mathbb{A}}^\times$) generated by $\theta_{\text{cusp}}(\phi)$ (i.e., $\Theta(g, h)(\theta_{\text{cusp}}(\phi)(g', h')) = \theta_{\text{cusp}}(\phi)(g'g, h'h)$), $\Theta$ is isomorphic to the induction $\text{Ind}_{G_{E/F}(\mathbb{A})}^{GL_2(F_{\mathbb{A}})} \Theta_0$. 
§ 10. Quaternion subalgebras of $B$. For each $\alpha \in D_\sigma \cap B^\times$, define the $\alpha$-twist $\sigma_\alpha$ of $\sigma$ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^{\sigma_\alpha}$. Then $\sigma_\alpha$ is another action of $\text{Gal}(E/Q)$ on $B$, and $D_\alpha = H^0(E/F, B)$ under this twisted action is a quaternion subalgebra of $B$.

- All quaternion $\mathbb{Q}$-subalgebras of $B$ are realized as $D_\alpha$ for some $\alpha \in D_\sigma$, and $D_z = D \iff z \in Z$;
- $\alpha = \xi^{-1} \beta \xi^{-i} \sigma$ for $\xi \in B^\times \iff D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\alpha \cong D_\beta$ by an inner automorphism of $B$ if $N(\alpha) = N(\beta)$ and $B_\infty \cong M_2(E_\infty)$ (strong approximation);
- The even Clifford group $G_\alpha$ of $D_{\alpha,0} = \{v \in D_{\sigma_\alpha} | \text{Tr}(v) = 0\}$ is $D_{\alpha}^\times$ and $B^\times$ is a covering of the similitude group $\text{GO}_{D_\sigma}$ of $D_\sigma$.

Let $\hat{\Gamma}_\phi = \{h \in B_{\mathbb{A}(\infty)}^\times | \phi(\infty)(h^{-1}v h^{-i} \sigma) = \phi(\infty)(v), \forall v \in D_{\sigma, \mathbb{A}(\infty)}\}$ for each Schwartz-Bruhat function $\phi$ on $D_{\sigma, \mathbb{A}(\infty)}$.

Let $Sh_B = B^\times \backslash B_{\mathbb{A}(\infty)}^\times / E_{\mathbb{A}}^\times \hat{\Gamma}_\phi C_\infty$ be the Shimura variety for $B^\times$ of level $\hat{\Gamma}_\phi$, and $Sh_\alpha$ be the image of $D_{\alpha, \mathbb{A}}^\times$ in $Sh_B$ for $\alpha \in D_\sigma$.

Regard $Sh_\alpha \in H_2(Sh_B, \mathbb{Z})$ and write $(\cdot, \cdot) : H^2 \times H_2 \to \mathbb{C}$ for the Poincaré duality.