Local Hecke equivariance

Haruzo Hida
Department of Mathematics, UCLA,

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Lecture 7: We describe how to extend the Weil representation $\mathbf{w}$ from $\mathrm{SL}_2(F_{\mathfrak{p}_F}) \times \mathrm{O}_{D_\sigma}(\mathbb{A})$ to $\mathrm{GL}_2(F_{\mathfrak{p}_F}) \times B^\times_{\mathfrak{p}_F}$ for each prime $\mathfrak{p}_F$ of F splitting in E ($\mathfrak{p}_F = \mathfrak{p}\mathfrak{p}^\sigma$). Then we define the local (dual) Hecke operator action $\mathbb{T}^*_{\mathfrak{p}}$ through $B^\times_{\mathfrak{p}}$-action and $\mathbb{T}^+_{\mathfrak{p}_F}$ action through the metaplectic $\mathrm{GL}_2(F_{\mathfrak{p}_F})$-action on $\mathcal{S}(D_{\sigma,\mathfrak{p}_F})$. For split prime $\mathfrak{p}_F$, note that $D_{\sigma,\mathfrak{p}_F} \subset B_{\mathfrak{p}_F} = D_{\mathfrak{p}_F} \times D_{\mathfrak{p}_F}$ and by the left projection $D_{\sigma,\mathfrak{p}_F} \cong D_{\mathfrak{p}_F}$. Then assuming $D_{\mathfrak{p}_F} \cong M_2(F_{\mathfrak{p}_F})$, for the characteristic function $\mathbf{1}$ of $M_2(O_{F_{\mathfrak{p}_F}})$, we show $\mathbf{1}|\mathbb{T}^*_{\mathfrak{p}} = \mathbf{1}|\mathbb{T}^+_{\mathfrak{p}_F}$ (local Hecke equivariance). We simply write F for $F_{\mathfrak{p}_F}$ and E for $E_{\mathfrak{p}_F} = F \times F$. Write $O = O_{E_{\mathfrak{p}}} = O_{F_{\mathfrak{p}_F}}$ (integer rings) with uniformizer ϖ. Details are in Section 4.7.

§**0. Double coset decomposition.** For $S = \mathrm{SL}_2(O)$ or $\mathrm{GL}_2(O)$, we have for a complete representative set U_j of $O/(\varpi)^j$

$$S\,\mathrm{diag}[\varpi,1]S = \bigsqcup_{\xi\in\mathbb{T}_{\mathfrak{p}}} \xi S \quad \text{with} \quad \mathbb{T}_{\mathfrak{p}} = \left\{\left(\begin{smallmatrix}\varpi & u\\ 0 & 1\end{smallmatrix}\right)\middle|\, u\in U_1\right\}\sqcup\left\{\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)\right\}.$$

We write $X^* = \{\xi^{-1}|\xi\in X\}$ for a subset X in $\mathrm{GL}_2(F)$. We let act $\mathrm{GL}_2(F)$ on the column vector space $V := M_{2,n}(F)$ by $v\mapsto \xi v$ and $g\in\mathrm{GL}_2(O)$ act on $\phi\in\mathcal{S}(V)$ by $g\cdot\phi(v)=\phi(g^{-1}v)$. Particularly, if ϕ is S-invariant, we write $\phi|\mathfrak{p}=\varpi\cdot\phi$. Let $\mathbf{1}$ be the characteristic function of $M_{2,n}(O)\subset V$. Then

Lemma 1. *We have for* $\mathbf{p} := |\varpi|_{\mathfrak{p}}^{-1} = |O_F/\mathfrak{p}_F|$

$$\mathbf{1}|\mathbb{T}_{\mathfrak{p}} = \mathbf{1}+\mathbf{p}\mathbf{1}|\mathfrak{p}, \mathbf{1}|\mathbb{T}^*_{\mathfrak{p}} = \mathbf{1}|\mathfrak{p}^{-1}+\mathbf{p}\mathbf{1}$$

and $\mathbf{1}|T_{\mathfrak{p}^2} = \mathbf{1}|T^*_{\mathfrak{p}^2} = \mathbf{1}|\mathfrak{p}^{-1}+(\mathbf{p}-1)\mathbf{1}+\mathbf{p}^2\mathbf{1}|\mathfrak{p}.$

§1. Proof, Case $n = 1$:

We only prove the formula for $\mathbb{T}_\mathfrak{p}$. The formula for $\mathbb{T}_\mathfrak{p}^*$ follows from $\mathbb{T}_\mathfrak{p}^* = \mathbb{T}_\mathfrak{p} \circ \mathfrak{p}^{-1}$. We want to compute

$$\mathbf{1}|\mathbb{T}_\mathfrak{p} = \sum_{\xi \in \mathbb{T}_\mathfrak{p}} \mathbf{1}(\xi^{-1} v) = \sum_{\xi \in \mathbb{T}_\mathfrak{p}} \mathbf{1}_{\xi O_{E_\mathfrak{p}}^2}.$$

For $v \in pO_{E_\mathfrak{p}}^2$, $v \in \mathrm{Supp}(\mathbf{1}_{\xi O_{E_\mathfrak{p}}^2})$ for all $\xi \in \mathbb{T}_\mathfrak{p}$. Thus $\mathbf{1}|\mathbb{T}_\mathfrak{p}(v) = \mathbf{p}+1$ if $v \in \mathfrak{p}O_{E_\mathfrak{p}}^2$ as $|\mathbb{T}_\mathfrak{p}| = \mathbf{p}+1$. If $v \in O_{E_\mathfrak{p}}^2 - \mathfrak{p}O_{E_\mathfrak{p}}^2$, then $vO_{E_\mathfrak{p}} + \mathfrak{p}O_{E_\mathfrak{p}}^2 = \xi O_{E_\mathfrak{p}}^2$ for a unique $\xi \in \mathbb{T}_\mathfrak{p}$. Thus

$$\mathbf{1}|\mathbb{T}_\mathfrak{p} = \mathbf{1} + \mathbf{p}\mathbf{1}|\mathfrak{p}.$$

The first remark to deal with the case $n > 1$ is:

The vector space $M_{2,n}(E_\mathfrak{p})$ is a left module over $D_{E_\mathfrak{p}} = M_2(E_\mathfrak{p})$ and a right module over $M_n(E_\mathfrak{p})$ via left and right matrix multiplication.

§**2. Case** $n > 1$. Write $L := M_{2,n}(O) \subset V$. Pick a representative set $\mathbb{T}$ for $S\,\mathrm{diag}[\varpi, 1]S/S$ So, $\mathbb{T} \cong \mathbb{T}_{\mathfrak{p}}$, and elements of $\mathbb{T}$ act from the left on L. Write $\mathbb{F} := O/\mathfrak{p}$. Then for $\xi \in \mathbb{T}$, consider the set $\mathcal{M} := \{(\xi L)/\mathfrak{p}L \subset M_{2,n}(\mathbb{F})\}$ of $M_n(O_{E_{\mathfrak{p}}})$-right submodules of $M_{2,n}(\mathbb{F}) = L \otimes_{O_{E_{\mathfrak{p}}}} \mathbb{F} = L/\mathfrak{p}L$. Each element of $\mathcal{M}$ is a simple (and irreducible) right $M_n(O_{E_{\mathfrak{p}}})$-module, and each such module appears once in $\mathcal{M}$; so, $\mathcal{M}$ is independent of the choice of $\mathbb{T}$ (so, we may assume $\mathbb{T} = \mathbb{T}_{\mathfrak{p}}$). The set $\mathcal{M} \cup \{0\} \cup \{L/\mathfrak{p}L\}$ exhausts all right $M_n(O_{E_{\mathfrak{p}}})$-submodules in $M_{2,n}(\mathbb{F}) = L/\mathfrak{p}L$. Therefore $\bigcap_{\xi\in\mathbb{T}} \xi L = \mathfrak{p}L$. Writing $\phi|\xi(v) = \phi(\xi^{-1}v)$ for $v \in M_{2,n}(E_{\mathfrak{p}})$ and a left-S-invariant function ϕ on $M_{2,n}(E_{\mathfrak{p}})$, define $\mathbf{1}|\mathbb{T} = \sum_{\xi\in\mathbb{T}} \mathbf{1}|\xi$ for $v \in L$, which is independent of the choice of $\mathbb{T}$. Since $\mathbf{1}|\xi = \mathbf{1}_{\xi L}$,

$$\mathbf{1}|\mathbb{T}_{\mathfrak{p}} = \begin{cases} |\mathbb{T}| = \mathbf{p} + 1 & \text{if } v \in \bigcap_{\xi\in\mathbb{T}} \xi L = \mathfrak{p}L, \\ 1 & \text{if } v \in L \text{ but } v \notin \mathfrak{p}L. \end{cases}$$

The formula is the same as in the case of $n = 1$, and writing $X_1 := \{v \in M_2(O_{E_{\mathfrak{p}}}) | N(v) \in \mathfrak{p}O_{E_{\mathfrak{p}}}\}$ with its characteristic function $\mathbf{1}_{X_1}$,

$$\mathbf{1}_{X_1} = \mathbf{1}|\mathbb{T}_{\mathfrak{p}} - \mathbf{p}\mathbf{1}|\mathfrak{p}.$$

§**3.** $\widetilde{\mathrm{GL}(2)} \twoheadrightarrow \mathrm{GL}(2)$ **covering** $\widetilde{\mathrm{SL}}(2) \twoheadrightarrow \mathrm{SL}(2)$**.** Let K be a local field. For $y \in K^\times$ and $s = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(K)$, we define

$$v(y,s) = \begin{cases} 1 & \text{if } c \neq 0, \\ (y,d) & \text{if } c = 0, \end{cases}$$

where $(\cdot,\cdot)$ is the quadratic Hilbert symbol for K. Then write $s^y := \mathrm{diag}[1,y]^{-1} s\, \mathrm{diag}[1,y]$ and $T := \{\mathrm{diag}[1,y] | y \in K^\times\}$. By a tedious computation of Kubota's cocycle, Kubota verified the following fact [WRS, Proposition 2.6] essentially:

Proposition 1. *The association* $\widetilde{\mathrm{SL}}_2(K) \ni (s,\zeta) \mapsto (s^y, \zeta v(y,s)) \in \widetilde{\mathrm{SL}}_2(K)$ *induces an automorphism of* $\widetilde{\mathrm{SL}}_2(K)$*, and hence defining a semi-direct product* $\widetilde{\mathrm{GL}}_2(K) := T \ltimes \widetilde{\mathrm{SL}}_2(K)$ *under this action of* T*, we get an extension* $\mu_2 \hookrightarrow \widetilde{\mathrm{GL}}_2(K) \twoheadrightarrow \mathrm{GL}_2(K)$*.*

§**4. Extension to** GL(2) **from** SL(2)**.** The Weil representation depends on the identification of $X \cong X^*$ for $X = D_{\sigma,F_v}, D_{\sigma,F_{\mathbb{A}}}$. So far we have used the standard additive character $\mathbf{e}_F$ and its local factor to do this by using the pairing $\langle x, x^* \rangle = \mathbf{e}_F(s(x,y))$ or its local version. We can replace $\mathbf{e}_F$ by $\mathbf{e}_{F,\beta} := \mathbf{e}_F \circ \beta$ composing any element $\beta \in \mathrm{Aut}(X)$. Then we write $\mathbf{w}_\beta$ for the Weil representation associated to $\mathbf{e}_{F,\beta}$ or its local factor; so, $\mathbf{w}_1$ is the original representation with respect to $\mathbf{e}_F$.

By [AFG, Proposition 1.3] and $\mathrm{GL}_2 = T \ltimes \mathrm{SL}_2$, we can descend $\mathbf{w}_\beta$ to $\mathrm{SL}_2(F_\mathfrak{p})$ for $V = D_{\sigma,\mathfrak{p}}$ for all prime $\mathfrak{p}$: The formula is $\mathbf{w}_\beta(\mathrm{diag}[a, a^{-1}])\phi(v) = \chi_E(a)|a|_\mathfrak{p}^2\phi(av)$ and $\mathbf{w}_\beta(J)\phi(v) = \gamma\widehat{\phi}(v^\iota)$ for an 8-th root of unity γ and $\gamma = 1$ if $\mathfrak{p} \nmid \boldsymbol{\partial}\boldsymbol{\Delta}_{E/F}$. If $\mathfrak{p}_F = \mathfrak{p}\mathfrak{p}^\sigma$ ($\mathfrak{p} \neq \mathfrak{p}^\sigma$) in E with $D_{\sigma,E_{\mathfrak{p}_F}} \cong M_2(E_\mathfrak{p})$ by the projection to the $\mathfrak{p}$-component, we identify $D_{\sigma,E_{\mathfrak{p}_F}}$ and $M_2(E_\mathfrak{p})$.

Since $\mathrm{diag}[\beta, 1]v(u) = v(\beta u)\,\mathrm{diag}[\beta, 1]$, we add the action of $\mathrm{diag}[\beta, 1]$ intertwining $\mathbf{w}_1$ to $\mathbf{w}_{\beta^{-1}}$. Namely we extend $\mathbf{w}$ to GL_2 combining all $\{\mathbf{w}_\beta\}_\beta$.

§5. An explicit extension to $\mathsf{GL}(2)$ **from** $\mathsf{SL}(2)$**.** We define locally $\mathbf{w}(g\,\mathsf{diag}[b,1]) := |b|_{\mathfrak{p}}\mathbf{w}_{b^{-1}}(g)$ as operators; in other words,

$$\mathbf{w}(g\,\mathsf{diag}[b,1])\phi(v) = |b|_{\mathfrak{p}}\mathbf{w}_{b^{-1}}(g)\phi(v)$$

for $g \in \mathsf{SL}_2(F_{\mathfrak{p}_F})$ and $b \in F_{\mathfrak{p}_F}^{\times}$. The following is [AFG, 1.3 and 1.4] and [WRS, Proposition 2.27]:

Lemma 2. *Assume that* $\mathfrak{p}_F = \mathfrak{p}\mathfrak{p}^{\sigma}$ $(\mathfrak{p} \neq \mathfrak{p}^{\sigma})$ *in* E *with* $D_{\sigma,\mathfrak{p}_F} \cong M_2(E_{\mathfrak{p}})$ *by the projection to the* $\mathfrak{p}$*-component. Identify* $D_{\sigma,\mathfrak{p}_F}$ *and* $M_2(E_{\mathfrak{p}})$*. The above extension* $\mathbf{w}$ *is a well defined representation, and for a given* $\phi \in \mathcal{S}(D_{\sigma,\mathfrak{p}_F})$ *(for* $D_{\sigma,\mathfrak{p}_F} := D_{\sigma,F_{\mathfrak{p}_F}}$*),*

(1) the stabilizer $U_{\mathfrak{p}_F}^{\phi}$ *of* ϕ *in* $\mathsf{GL}_2(F_{\mathfrak{p}_F})$ *under the extended action is an open subgroup of* $\mathsf{GL}_2(F_{\mathfrak{p}_F})$*;*

(2) $U_{\mathfrak{p}_F}^{\phi}$ *contains* $\mathsf{GL}_2(O_{\mathfrak{p}_F})$ *if* ϕ *is a characteristic function of* $D_{\sigma,\mathfrak{p}_F} \cap R_{\mathfrak{p}_F}$ *for a maximal order* $R_{\mathfrak{p}_F}$ *of* $D_{E_{\mathfrak{p}_F}}$ *and* $\mathfrak{p}_F$ *is prime to* $2\boldsymbol{\partial}$*.*

§**6. A corollary.** Define $L(h)\phi(v) := \phi(h^{\iota}vh^{\sigma})$ and $L'(h)\phi(v) := \phi(h^{-1}vh^{\sigma})$ for $h \in B^{\times}_{\mathfrak{p}_F}$ and $\phi \in \mathcal{S}(D_{\sigma,F_{\mathfrak{p}_F}})$.

Corollary 1. *Let the notation and assumption be as in* Lemma 2*. Then we have* $\mathbf{1}|\mathbb{T}^{+}_{\mathfrak{p}_F} = \mathbf{1}|\mathbb{T}_{\mathfrak{p}}$ *and* $\mathbf{1}|\mathbb{T}^{+}_{\mathfrak{p}_F} = \mathbf{1}|\mathbb{T}_{\mathfrak{p}}$ *under the action* L' *of* $D^{\times}_{E_{\mathfrak{p}}}$ *and* $\mathbf{1}|\mathbb{T}^{+}_{\mathfrak{p}_F} = \mathbf{1}|\mathbb{T}^{*}_{\mathfrak{p}}$ *and* $\mathbf{1}|\mathbb{T}^{+,*}_{\mathfrak{p}_F} = \mathbf{1}|\mathbb{T}_{\mathfrak{p}}$ *under the action* L *of* $D^{\times}_{E_{\mathfrak{p}}}$*, where we let* $\mathrm{GL}_2(E_{\mathfrak{p}})$ *act on* $\mathcal{S}(D_{\sigma,E_{\mathbb{A}}})$ *as in* Lemma 2*.*

Proof. Chnaging L' to L brings $\mathbb{T}_{\mathfrak{p}}$ to $\mathbb{T}^{*}_{\mathfrak{p}}$, we prove the assertion for $\mathbb{T} = \mathbb{T}_{\mathfrak{p}}$. Recall $\mathbf{1}|\mathbb{T}_{\mathfrak{p}} = \mathbf{1} + \mathbf{p}\mathbf{1}|\mathfrak{p}$. By the way of extending the Weil representation to $\mathrm{GL}_2(F_{\mathfrak{p}_F})$ in Lemma 2, writing $\left(\begin{smallmatrix}\varpi & u\\ 0 & 1\end{smallmatrix}\right) = \upsilon(u)\,\mathrm{diag}[\varpi,1]$ and $\mathrm{diag}[1,\varpi] = \mathrm{diag}[\varpi^{-1},\varpi]\,\mathrm{diag}[\varpi,1]$, by the extension of the Weil representation $\mathbf{w}$ in Lemma 2, we have

$$\mathbf{1}|\mathbb{T}^{+}_{\mathfrak{p}_F}(v) = T_1(v) + \mathbf{p}\mathbf{1}(\varpi^{-1}v)$$

for

$$T_1(v) := \mathbf{p}^{-1} \sum_{u \bmod \mathfrak{p}} \mathbf{e}(u\varpi^{-1}N(v))\mathbf{1}(v).$$

§7. Proof continues.

Note

$$T_1(v) = \begin{cases} 0 & \text{if } v \notin M_2(O_{E_\mathfrak{p}}) \text{ or } N(v) \in O_{E_\mathfrak{p}}^\times, \\ 1 & \text{if } v \in M_2(O_{E_\mathfrak{p}}) \text{ and } N(v) \in \mathfrak{p}O_{E_\mathfrak{p}}. \end{cases}$$

Recall $X_1 := \{v \in M_2(O_{E_\mathfrak{p}}) | N(v) \in \mathfrak{p}O_{E_\mathfrak{p}}\}$. Thus, $T_1(v) = \mathbf{1}_{X_1}$. Since $\mathbf{1}_{X_1} = \mathbf{1}|\mathbb{T}_\mathfrak{p} - \mathbf{p}\mathbf{1}|\mathfrak{p}$ as in §2, and $\mathbf{1}|\mathfrak{p}(v) = \mathbf{1}(\varpi^{-1}v)$,

$$\mathbf{1}|\mathbb{T}^+_{\mathfrak{p}_F}(v) = (\mathbf{1}|\mathbb{T}_\mathfrak{p})(v) - \mathbf{p}\mathbf{1}(\varpi^{-1}v) + \mathbf{p}\mathbf{1}(\varpi^{-1}v) = \mathbf{1}|\mathbb{T}_\mathfrak{p}(v)$$

as desired. □

§**8. Non-split primes.** Here is a version of Lemma 2 for non-split primes:

Lemma 3. *Let* $\mathfrak{p}_F$ *be non-split in* E *or ramified in* D*, and write* $\boldsymbol{\Delta}_{E/F}$ *for the discriminant of* $E_{/F}$*. Then for* $\phi \in \mathcal{S}(D_{\sigma,\mathfrak{p}_F})$*, there exists an open subgroup* $U^{\phi}_{\mathfrak{p}_F}$ *of* $\mathrm{GL}_2(F_{\mathfrak{p}_F})$ *such that*

0. $\chi_E \circ \det$ *is trivial on* U^{ϕ}*;*

1. the action of $\mathrm{SL}_2(F_{\mathfrak{p}_F})$ *via* $\mathbf{w}$ *extends to* $\mathrm{SL}_2(F_{\mathfrak{p}_F})U^{\phi}_{\mathfrak{p}_F}$ *and* $\mathbf{w}(u)\phi = \phi$ *for all* $u \in U^{\phi}_{\mathfrak{p}_F}$*;*

2. If $\mathfrak{p}_F$ *is prime to* $2\boldsymbol{\partial}\boldsymbol{\Delta}_{E/F}$ *and* ϕ *is a characteristic function of* $D_{\sigma,\mathfrak{p}_F} \cap R_{\mathfrak{p}_F}$ *for a maximal order* $R_{\mathfrak{p}_F}$ *of* $D_{E_{\mathfrak{p}_F}}$*, then* $U^{\phi}_{\mathfrak{p}_F}$ *contains* $\mathrm{GL}_2(O_{\mathfrak{p}_F})$*.*

Consider $\mathcal{U}^{\phi} := \{u \in O^{\times}_{\mathfrak{p}_F} | \phi(uv) = \phi(v) \text{ for all } v \in D_{\sigma,F_{\mathfrak{p}_F}}\}$, and let S^{ϕ} be an open subgroup of $\{s \in \mathrm{SL}_2(O_{\mathfrak{p}_F}) | \mathbf{w}(s)\phi = \phi\}$. By the smoothness of $\mathbf{w}$, we may assume that S^{ϕ} is normalized by $\delta_u := \mathrm{diag}[1, u]$ for $u \in O^{\times}_{\mathfrak{p}_F}$ (i.e., a principal congruence subgroup).

§**9. Proof continues.** Then

$$U = U^{\phi} := \{s\delta_u | u \in \mathcal{U}^{\phi}, s \in S^{\phi}\}$$

is an open subgroup of $\mathrm{GL}_2(O_{\mathfrak{p}_F})$ and can be taken to contain $\mathrm{GL}_2(O_{\mathfrak{p}_F})$ if $\mathfrak{p}_F$ is prime to $2\boldsymbol{\partial}\boldsymbol{\Delta}_{E/F}$ and ϕ is a characteristic function of $D_{\sigma,\mathfrak{p}_F} \cap R_{\mathfrak{p}_F}$ for a maximal order $R_{\mathfrak{p}_F}$ of $D_{E_{\mathfrak{p}_F}}$. Let $\mathbf{w} = \mathbf{w}_{D_{\sigma,\mathfrak{p}_F}}$ (the local Weil representation on $\mathcal{S}(D_{\sigma,F_{\mathfrak{p}_F}})$). We extend $\mathbf{w}$ to U by $\mathbf{w}(\delta_u)\phi = \phi$ for $u \in U^{\phi}$. For this identity, we need $\chi_E(u) = 1$ as $r(\mathrm{diag}[u, u^{-1}])\phi(v) = \chi_E(u)\phi(uv)$.

By the computation in the books [AFG, Lemma 1.4] and [WRS, Proposition 2.27], the conjugation by δ_u induces an automorphism of $\widetilde{\mathrm{SL}}_2(F_{\mathfrak{p}_F})$ without changing the center if $\chi_E(u) = 1$ and $\mathfrak{p}_F$ is odd (if $\mathfrak{p}_F | 2$, we need to assume that u is square), and therefore, this extension is consistent.

§10. Archimedean primes.

Lemma 4. *Write* $\mathrm{GL}_2^+(F_\infty)$ *for the identity connected component of* $\mathrm{GL}_2(F_\infty)$*. For a character* $\psi_\infty : F_\infty^\times \to \mathbb{C}^\times$ *and* $\phi_\infty \in \mathcal{S}(D_{\sigma,F_\infty})$ *with* $\phi_\infty(\epsilon v) = \psi_\infty(\epsilon)\phi_\infty(v)$ *for all* $v \in D_{\sigma,F_\infty}$ *and* $\epsilon \in \mu_2(F_\infty)$*, we can extend* $\mathbf{w}$ *defined on* $\mathrm{SL}_2(F_\infty)$ *to* $\mathrm{GL}_2^+(F_\infty)$ *so that the central character of* $\mathbf{w}$ *at infinity is given by* ψ_∞*.*

Proof. By $\mathrm{GL}_2(F_\infty) = \mathrm{SL}_2(F_\infty)F_\infty^\times$ with $\mathrm{SL}_2(F_\infty) \cap F_\infty^\times = \mu_2(F_\infty)$, we require for ϕ_∞ to satisfy $\mathbf{w}(gz)\phi_\infty = \psi_\infty^{-1}(z)\phi_\infty$ for $z \in F_\infty^\times$. This gives the extension. □