Lecture 7: We describe how to extend the Weil representation \( w \) from \( \text{SL}_2(F_{p_F}) \times \mathcal{O}_{D_\sigma}(\mathbb{A}) \) to \( \text{GL}_2(F_{p_F}) \times B_{p_F}^\times \) for each prime \( p_F \) of \( F \) splitting in \( E \) (\( p_F = pp^{\sigma} \)). Then we define the local (dual) Hecke operator action \( T^*_p \) through \( B_{p_F}^\times \)-action and \( T^+_p \) action through the metaplectic \( \text{GL}_2(F_{p_F}) \)-action on \( S(D_{\sigma,p_F}) \). For split prime \( p_F \), note that \( D_{\sigma,p_F} \subset B_{p_F} = D_{p_F} \times D_{p_F} \) and by the left projection \( D_{\sigma,p_F} \cong D_{p_F} \). Then assuming \( D_{p_F} \cong M_2(F_{p_F}) \), for the characteristic function \( 1 \) of \( M_2(O_{F_{p_F}}) \), we show \( 1|T^*_p = 1|T^+_p \) (local Hecke equivariance). We simply write \( F \) for \( F_{p_F} \) and \( E \) for \( E_{p_F} = F \times F \). Write \( O = O_{E_p} = O_{F_{p_F}} \) (integer rings) with uniformizer \( \varpi \). Details are in Section 4.7.
§0. **Double coset decomposition.** For \( S = \text{SL}_2(O) \) or \( \text{GL}_2(O) \), we have for a complete representative set \( U_j \) of \( O/(\varpi)^j \)

\[
S \text{diag} [\varpi, 1] S = \bigsqcup_{\xi \in T_p} \xi S \quad \text{with} \quad T_p = \left\{ \left( \begin{array}{cc} \varpi & u \\ 0 & 1 \end{array} \right) \mid u \in U_1 \right\} \cup \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi \end{array} \right) \right\}.
\]

We write \( X^* = \{ \xi^{-1} \mid \xi \in X \} \) for a subset \( X \) in \( \text{GL}_2(F) \). We let act \( \text{GL}_2(F) \) on the column vector space \( V := M_{2,n}(F) \) by \( v \mapsto \xi v \) and \( g \in \text{GL}_2(O) \) act on \( \phi \in \mathcal{S}(V) \) by \( g \cdot \phi(v) = \phi(g^{-1}v) \). Particularly, if \( \phi \) is \( S \)-invariant, we write \( \phi \mid p = \varpi \cdot \phi \). Let \( 1 \) be the characteristic function of \( M_{2,n}(O) \subset V \). Then

**Lemma 1.** We have for \( p := |\varpi|^{-1} = |O_F/p_F| \)

\[
1 | T_p = 1 + p 1 | p, \quad 1 | T_p^* = 1 | p^{-1} + p 1
\]

and \( 1 | T_{p^2} = 1 | T_{p^2}^* = 1 | p^{-1} + (p - 1) 1 + p^2 1 | p. \)
§1. Proof, Case $n = 1$:

We only prove the formula for $T_p$. The formula for $T_p^*$ follows from $T_p^* = T_p \circ p^{-1}$. We want to compute

$$1|_{T_p} = \sum_{\xi \in T_p} 1(\xi^{-1}v) = \sum_{\xi \in T_p} 1_{\xi O^2_{E_p}}.$$  

For $v \in pO^2_{E_p}$, $v \in \text{Supp}(1_{\xi O^2_{E_p}})$ for all $\xi \in T_p$. Thus $1|_{T_p}(v) = p + 1$ if $v \in pO^2_{E_p}$ as $|T_p| = p + 1$. If $v \in O^2_{E_p} - pO^2_{E_p}$, then $vO_{E_p} + pO^2_{E_p} = \xi O^2_{E_p}$ for a unique $\xi \in T_p$. Thus

$$1|_{T_p} = 1 + p1|_p.$$  

The first remark to deal with the case $n > 1$ is:

The vector space $M_{2,n}(E_p)$ is a left module over $D_{E_p} = M_2(E_p)$ and a right module over $M_n(E_p)$ via left and right matrix multiplication.
§2. Case $n > 1$. Write $L := M_{2,n}(O) \subset V$. Pick a representative set $T$ for $S \, \text{diag}[\varpi, 1] S/S$ So, $T \cong T_p$, and elements of $T$ act from the left on $L$. Write $F := O/p$. Then for $\xi \in T$, consider the set $\mathcal{M} := \{ (\xi L)/pL \subset M_{2,n}(F) \}$ of $M_n(O_{Ep})$-right submodules of $M_{2,n}(F) = L \otimes O_{Ep} F = L/pL$. Each element of $\mathcal{M}$ is a simple (and irreducible) right $M_n(O_{Ep})$-module, and each such module appears once in $\mathcal{M}$; so, $\mathcal{M}$ is independent of the choice of $T$ (so, we may assume $T = T_p$). The set $\mathcal{M} \cup \{ 0 \} \cup \{ L/pL \}$ exhausts all right $M_n(O_{Ep})$-submodules in $M_{2,n}(F) = L/pL$. Therefore $\bigcap_{\xi \in T} \xi L = pL$. Writing $\phi|\xi(v) = \phi(\xi^{-1}v)$ for $v \in M_{2,n}(E_p)$ and a left-$S$-invariant function $\phi$ on $M_{2,n}(E_p)$, define $1|T = \sum_{\xi \in T} 1|\xi$ for $v \in L$, which is independent of the choice of $T$. Since $1|\xi = 1_{\xi L}$,

$$1|T_p = \begin{cases} |T| = p + 1 & \text{if } v \in \bigcap_{\xi \in T} \xi L = pL, \\ 1 & \text{if } v \in L \text{ but } v \notin pL. \end{cases}$$

The formula is the same as in the case of $n = 1$, and writing $X_1 := \{ v \in M_2(O_{Ep}) | N(v) \in pO_{Ep} \}$ with its characteristic function $1_{X_1}$,

$$1_{X_1} = 1|T_p - p1|p.$$
§3. \( \widetilde{\text{GL}(2)} \to \text{GL}(2) \) covering \( \widetilde{\text{SL}(2)} \to \text{SL}(2) \). Let \( K \) be a local field. For \( y \in K^\times \) and \( s = (c^* d^*) \in \text{SL}_2(K) \), we define

\[
v(y, s) = \begin{cases} 
1 & \text{if } c \neq 0, \\
(y, d) & \text{if } c = 0,
\end{cases}
\]

where \((\cdot, \cdot)\) is the quadratic Hilbert symbol for \( K \). Then write \( s^y := \text{diag}[1, y]^{-1}s \text{diag}[1, y] \) and \( T := \{ \text{diag}[1, y] | y \in K^\times \} \). By a tedious computation of Kubota’s cocycle, Kubota verified the following fact [WRS, Proposition 2.6] essentially:

**Proposition 1.** *The association* \( \widetilde{\text{SL}}_2(K) \ni (s, \zeta) \mapsto (s^y, \zeta v(y, s)) \in \widetilde{\text{SL}}_2(K) \) *induces an automorphism of* \( \widetilde{\text{SL}}_2(K) \), *and hence defining a semi-direct product* \( \widetilde{\text{GL}}_2(K) := T \ltimes \widetilde{\text{SL}}_2(K) \) *under this action of* \( T \), *we get an extension* \( \mu_2 \hookrightarrow \widetilde{\text{GL}}_2(K) \to \text{GL}_2(K) \).
§4. Extension to $\text{GL}(2)$ from $\text{SL}(2)$. The Weil representation depends on the identification of $X \cong X^*$ for $X = D_{\sigma,F_v}, D_{\sigma,F_{\Delta}}$. So far we have used the standard additive character $e_F$ and its local factor to do this by using the pairing $\langle x, x^* \rangle = e_F(s(x,y))$ or its local version. We can replace $e_F$ by $e_{F,\beta} := e_F \circ \beta$ composing any element $\beta \in \text{Aut}(X)$. Then we write $w_\beta$ for the Weil representation associated to $e_{F,\beta}$ or its local factor; so, $w_1$ is the original representation with respect to $e_F$.

By [AFG, Proposition 1.3] and $\text{GL}_2 = T \ltimes \text{SL}_2$, we can descend $w_\beta$ to $\text{SL}_2(F_p)$ for $V = D_{\sigma,p}$ for all prime $p$:

The formula is

$$w_\beta(\text{diag}[a,a^{-1}])\phi(v) = \chi_E(a)|a|^{\frac{2}{p}}\phi(av)$$

and

$$w_\beta(J)\phi(v) = \gamma \hat{\phi}(v^\epsilon)$$

for an 8-th root of unity $\gamma$ and $\gamma = 1$ if $p \nmid \partial \Delta_{E/F}$. If $p_F = pp^\sigma (p \neq p^\sigma)$ in $E$ with $D_{\sigma,E_{p_F}} \cong M_2(E_p)$ by the projection to the $p$-component, we identify $D_{\sigma,E_{p_F}}$ and $M_2(E_p)$.

Since $\text{diag}[\beta,1]v(u) = v(\beta u)\text{diag}[\beta,1]$, we add the action of $\text{diag}[\beta,1]$ intertwining $w_1$ to $w_{\beta^{-1}}$. Namely we extend $w$ to $\text{GL}_2$ combining all $\{w_\beta\}_\beta$. 
§5. An explicit extension to $GL(2)$ from $SL(2)$. We define locally $w(g \text{ diag}[b,1]) := |b|_p w_{b^{-1}}(g)$ as operators; in other words,

$$w(g \text{ diag}[b,1])\phi(v) = |b|_p w_{b^{-1}}(g)\phi(v)$$

for $g \in SL_2(F_p F)$ and $b \in F_p^\times$. The following is [AFG, 1.3 and 1.4] and [WRS, Proposition 2.27]:

**Lemma 2.** Assume that $p_F = pp^\sigma (p \neq p^\sigma)$ in $E$ with $D_{\sigma,p_F} \cong M_2(E_p)$ by the projection to the $p$-component. Identify $D_{\sigma,p_F}$ and $M_2(E_p)$. The above extension $w$ is a well defined representation, and for a given $\phi \in S(D_{\sigma,p_F})$ (for $D_{\sigma,p_F} := D_{\sigma,F_p F}$),

1. the stabilizer $U^\phi_{p_F}$ of $\phi$ in $GL_2(F_p F)$ under the extended action is an open subgroup of $GL_2(F_p F)$;
2. $U^\phi_{p_F}$ contains $GL_2(O_{p_F})$ if $\phi$ is a characteristic function of $D_{\sigma,p_F} \cap R_{p_F}$ for a maximal order $R_{p_F}$ of $D_{E_p F}$ and $p_F$ is prime to $2\partial$. 
§6. A corollary. Define $L(h)\phi(v) := \phi(h^tvh\sigma)$ and $L'(h)\phi(v) := \phi(h^{-1}vh\sigma)$ for $h \in B_{p_F}^\times$ and $\phi \in S(D_{\sigma,Fp_F})$.

Corollary 1. Let the notation and assumption be as in Lemma 2. Then we have $1|T_{p_F}^+ = 1|T_p$ and $1|T_{p_F}^+ = 1|T_p$ under the action $L'$ of $D_{E_p}^\times$ and $1|T_{p_F}^+ = 1|T_p^*$ and $1|T_{p_F}^+,* = 1|T_p$ under the action $L$ of $D_{E_p}^\times$, where we let $\text{GL}_2(E_p)$ act on $S(D_{\sigma,E_A})$ as in Lemma 2.

Proof. Changing $L'$ to $L$ brings $T_p$ to $T_p^*$, we prove the assertion for $T = T_p$. Recall $1|T_p = 1 + p1|p$. By the way of extending the Weil representation to $\text{GL}_2(F_{p_F})$ in Lemma 2, writing $(\bar{\varpi}^0 1 1) = v(u)\text{diag}[\varpi, 1]$ and $\text{diag}[1, \varpi] = \text{diag}[\varpi^{-1}, \varpi]\text{diag}[\varpi, 1]$, by the extension of the Weil representation $w$ in Lemma 2, we have

$$1|T_{p_F}^+(v) = T_1(v) + p1(\varpi^{-1}v)$$

for

$$T_1(v) := p^{-1} \sum_{u \mod p} e(u\varpi^{-1}N(v))1(v).$$
§7. Proof continues.

Note

\[
T_1(v) = \begin{cases} 
0 & \text{if } v \not\in M_2(O_{E_p}) \text{ or } N(v) \in O_{E_p}^\times, \\
1 & \text{if } v \in M_2(O_{E_p}) \text{ and } N(v) \in pO_{E_p}. 
\end{cases}
\]

Recall \( X_1 := \{ v \in M_2(O_{E_p}) | N(v) \in pO_{E_p} \} \). Thus, \( T_1(v) = 1_{X_1} \).

Since \( 1_{X_1} = 1|_{T_p} - p1|_{p} \) as in §2, and \( 1|p(v) = 1(\varpi^{-1}v) \),

\[
1|_{T_{pF}}^+(v) = (1|_{T_p})(v) - p1(\varpi^{-1}v) + p1(\varpi^{-1}v) = 1|_{T_p}(v)
\]

as desired. \( \square \)
§8. Non-split primes. Here is a version of Lemma 2 for non-split primes:

Lemma 3. Let $p_F$ be non-split in $E$ or ramified in $D$, and write $\Delta_{E/F}$ for the discriminant of $E/F$. Then for $\phi \in S(D_{\sigma,p_F})$, there exists an open subgroup $U^\phi_{p_F}$ of $\text{GL}_2(F_{p_F})$ such that

0. $\chi_E \circ \text{det}$ is trivial on $U^\phi_{p_F}$;
1. the action of $\text{SL}_2(F_{p_F})$ via $w$ extends to $\text{SL}_2(F_{p_F})U^\phi_{p_F}$ and $w(u)\phi = \phi$ for all $u \in U^\phi_{p_F}$;
2. If $p_F$ is prime to $2\partial \Delta_{E/F}$ and $\phi$ is a characteristic function of $D_{\sigma,p_F} \cap R_{p_F}$ for a maximal order $R_{p_F}$ of $D_{E_{p_F}}$, then $U^\phi_{p_F}$ contains $\text{GL}_2(O_{p_F})$.

Consider $U^\phi := \{u \in O_{p_F}^X | \phi(uv) = \phi(v) \text{ for all } v \in D_{\sigma,F_{p_F}}\}$, and let $S^\phi$ be an open subgroup of $\{s \in \text{SL}_2(O_{p_F}) | w(s)\phi = \phi\}$. By the smoothness of $w$, we may assume that $S^\phi$ is normalized by $\delta_u := \text{diag}[1,u]$ for $u \in O_{p_F}^X$ (i.e., a principal congruence subgroup).
§9. Proof continues. Then

\[ U = U^\phi := \{ s\delta_u | u \in U^\phi, s \in S^\phi \} \]

is an open subgroup of GL\(_2(\mathcal{O}_{p_F})\) and can be taken to contain GL\(_2(\mathcal{O}_{p_F})\) if \(p_F\) is prime to \(2\partial \Delta_{E/F}\) and \(\phi\) is a characteristic function of \(D_{\sigma,p_F} \cap R_{p_F}\) for a maximal order \(R_{p_F}\) of \(D_{E_{p_F}}\). Let \(w = w_{D_{\sigma,p_F}}\) (the local Weil representation on \(S(D_{\sigma,F_{p_F}})\)). We extend \(w\) to \(U\) by \(w(\delta_u)\phi = \phi\) for \(u \in U^\phi\). For this identity, we need \(\chi_E(u) = 1\) as \(r(diag[u,u^{-1}])\phi(v) = \chi_E(u)\phi(uv)\).

By the computation in the books [AFG, Lemma 1.4] and [WRS, Proposition 2.27], the conjugation by \(\delta_u\) induces an automorphism of \(\widetilde{SL}_2(F_{p_F})\) without changing the center if \(\chi_E(u) = 1\) and \(p_F\) is odd (if \(p_F|2\), we need to assume that \(u\) is square), and therefore, this extension is consistent.
§10. Archimedean primes.

**Lemma 4.** Write $\text{GL}_2^+(F_{\infty})$ for the identity connected component of $\text{GL}_2(F_{\infty})$. For a character $\psi_{\infty} : F_{\infty}^\times \to \mathbb{C}^\times$ and $\phi_{\infty} \in \mathcal{S}(D_{\sigma,F_{\infty}})$ with $\phi_{\infty}(\epsilon v) = \psi_{\infty}(\epsilon)\phi_{\infty}(v)$ for all $v \in D_{\sigma,F_{\infty}}$ and $\epsilon \in \mu_2(F_{\infty})$, we can extend $w$ defined on $\text{SL}_2(F_{\infty})$ to $\text{GL}_2^+(F_{\infty})$ so that the central character of $w$ at infinity is given by $\psi_{\infty}$.

**Proof.** By $\text{GL}_2(F_{\infty}) = \text{SL}_2(F_{\infty})F_{\infty}^\times$ with $\text{SL}_2(F_{\infty}) \cap F_{\infty}^\times = \mu_2(F_{\infty})$, we require for $\phi_{\infty}$ to satisfy $w(gz)\phi_{\infty} = \psi_{\infty}^{-1}(z)\phi_{\infty}$ for $z \in F_{\infty}^\times$. This gives the extension. □