Rankin convolution

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Lecture 6: We describe the adelic Fourier expansion of SL(2)-modular forms. Combined with the adelic Fourier expansion of metaplectic modular forms we covered in the last lecture, we study Rankin product of metaplectic and elliptic modular forms. We allow modular forms of mixed K-types for compact adelic subgroups K; so, at infinity, there may not be a well defined weight.

$\S$**0. Adelic cusp form and classical one.** Relation between the adelic version $\mathbf{f}$ and classical one $f \in S_k^{\pm}(\Gamma_0(C), \varphi)$:

$$(*) \qquad \mathbf{f}(\alpha u) = \varphi^*(u)^{-1} f(u_\infty(\sqrt{-1}))j(u_\infty, \sqrt{-1}^{\pm})^{-k}$$

for $\alpha \in \mathrm{SL}_2(\mathbb{Q})$ and $u \in \widehat{\Gamma}_0(C)\mathrm{SL}_2(\mathbb{R})$, where $\varphi^*(u)^{-1} = \varphi(u)$. Then, in addition to cuspidality and holomorphy/anti-holomorphy, we have the following automorphy

$$(**) \qquad \mathbf{f}(\alpha x u) = \varphi^*(u)^{-1}\mathbf{f}(x)j(u_\infty, \sqrt{-1}^{\pm})^{-k}$$

for all $\alpha \in \mathrm{SL}_2(\mathbb{Q})$ and $u \in \widehat{\Gamma}_0(C)\mathrm{SO}_2(\mathbb{R})$. $\mathrm{SO}_2(\mathbb{R})$ is the stabilizer $\pm\sqrt{-1}$. Write the expansion of f as $f(\tau) = \sum_{\alpha \in \mathbb{Q}} a_\alpha(f)\mathbf{e}(\pm n\tau^{\pm})$, where $a_\alpha(f) = 0$ if either $\alpha \le 0$ or $\alpha \notin \mathbb{Z}$. For $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B(\widehat{\mathbb{Z}})B(\mathbb{R})$ with $a \in \widehat{\mathbb{Z}}^\times \mathbb{R}_+^\times$, we get, for $\pm\tau^{\pm} = g_\infty(i) = a_\infty^2 i \pm a_\infty b_\infty$ and $\boldsymbol{\varphi}(a) = \varphi^*(a)|a|_{\mathbb{A}}^{-k}$,

$$\mathbf{f}(g) = \boldsymbol{\varphi}^{-1}(a)\sum_\alpha a_\alpha(f)\exp(-2\pi n a_\infty^2)\mathbf{e}(\pm n a_\infty b_\infty).$$

$\S$**1. Integral weight Fourier expansion, summary.** We describe briefly integral weight Fourier expansion on $\mathrm{SL}_2(\mathbb{A})$. Since the proof is similar to and easier than the half integral weight case, we only state the result. For $b = v(u)\,\mathrm{diag}[a, a^{-1}] \in B(\mathbb{A})$, write $\mathbf{f}(a,u) := \mathbf{f}(b)$. Since $\mathbf{f}(a, u+\alpha) = f(v(\alpha)b) = \mathbf{f}(a,u)$ for $\alpha \in \mathbb{Q}$, we have the adelic Fourier expansion of $\mathbf{f}(a,u)$ over $u \in \mathbb{A}$:

$$\mathbf{f}(a,u) = \sum_{\alpha\in\mathbb{Q}} a_{\mathbf{f}}(\alpha; a)\mathbf{e}(\alpha u).$$

Defining

$$\mathbf{a}_{\mathbf{f}}(x) := \varphi(a) a_{\mathbf{f}}(\alpha, a)\exp(2\pi\alpha_\infty a_\infty^2) = a_\alpha(f),$$

which is equal to $a_\alpha(f)$ if $0 < \alpha \in \mathbb{Z}$ and $a \in \widehat{\mathbb{Z}}^\times$, and $\mathbf{a}_{\mathbf{f}}(x) = 0$ if $x \notin \mathbb{Z}_+(\widehat{\mathbb{Z}}^\times\mathbb{R}^\times)^2$. The coefficients $\mathbf{a}_{\mathbf{f}}(x)$ is independent of chgoices and depends only on $x^{(\infty)}$. Then we have
$\mathbf{a}_{\mathbf{f}}(x) = a_\alpha(f) = \mathbf{a}_{\mathbf{f}}(xt^2)$ for $t \in \widehat{\mathbb{Z}}^\times\mathbb{R}^\times$ and $x = \alpha a^2$,

$$\mathbf{f}(a,u) = \varphi^{-1}(a)\sum_{\alpha\in\mathbb{Q}} \mathbf{a}_{\mathbf{f}}(\alpha a^2)\mathbf{e}(\alpha_\infty a_\infty^2\sqrt{-1})\mathbf{e}(\pm\alpha u).$$

§2. **First step in the pure weight case.** Let $f \in S_k^-(\Gamma_0(C), \varphi)$, $g \in M_{\ell/2}(\Gamma_0(M), \psi)$ ($C|M$ with $4|M$), and lift it to $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$. Take a continuous $\Phi : B(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$, and assume

(Φ1) $\Phi(x\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = (\varphi^*\psi^*)^{-1}(d_M)\Phi(x)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(M)$,

(Φ2) $\Phi(x(u_\infty, \zeta J(u_\infty, \tau))) = \Phi(x)\zeta^{-\ell} J(u_\infty, \sqrt{-1})^\ell j(u_\infty, -\sqrt{-1})^k$ for $(u_\infty, \zeta J(u_\infty, \tau)) \in C_\infty$,

(Φ3) $\Phi|_{B(\mathbb{A})}(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) = (\varphi^*\psi^*)(a)$ for $a \in \mathbb{A}^\times$ and $b \in \mathbb{A}$.

Then for $C_\infty = \pi_{\mathbb{A}}^{-1}(\mathrm{SO}_2(\mathbb{R})) \subset \mathrm{Mp}(\mathbb{A})$,

$$\mathbf{g}(g)\mathbf{f}(g)\Phi(g) = \mathbf{g}(gu)\mathbf{f}(gu)\Phi(gu)$$

for all $u \in \widehat{\Gamma}_0(M)C_\infty$. Thus $g \mapsto \mathbf{g}(g)\mathbf{f}(g)\Phi(g)$ is a function of

$$g \in \overline{B} := B(\mathbb{Q})\backslash B(\mathbb{A})\widehat{\Gamma}_0(M)C_\infty/\widehat{\Gamma}_0(M)C_\infty.$$

We want to compute first $\int_{\overline{B}} \mathbf{g}(g)\mathbf{f}(g)\Phi(g)d\mu(g)$ as an L-value. Since $B(\mathbb{A}) = T(\mathbb{A}) \ltimes U(\mathbb{A})$ for $T(A) = \{\mathrm{diag}[a, a^{-1}] | a \in A\}$ and $U(A) = \{v(u) | u \in A\}$, by $\mathbb{Q}^\times\widehat{\mathbb{Z}}^\times\backslash\mathbb{A}^\times \cong \mathbb{R}_+$ and $(\mathbb{Q} + \widehat{\mathbb{Z}})\backslash\mathbb{A} = \mathbb{R}/\mathbb{Z}$, we have $\overline{B} \cong [0, 1) \times \mathbb{R}_+^\times$.

§**3.** $B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty \cong \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})$. We first show that this is onto. By strong approximation, $\mathrm{SL}_2(\mathbb{Q})K = \mathrm{SL}_2(\mathbb{A}^{(\infty)})$ for any open subgroup K of $\widehat{\Gamma}_0(4)$ and by Iwasawa decomposition, $\mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})KC_\infty = \mathrm{Mp}(\mathbb{A})$. Writing $B_K := B(\mathbb{Q})\backslash B(\mathbb{A})KC_\infty$, this shows

$$\pi_K : B_K \twoheadrightarrow \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})KC_\infty$$

is onto. Passing to the projective limit with respect to $K \supset B(\widehat{\mathbb{Z}})$, we get surjectivity.

If $\pi_1(b) = \pi_1(b')$ for $b, b' \in B(\mathbb{A})$, then $\gamma b = b'u$ for $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ and $u \in B(\widehat{\mathbb{Z}})C_\infty$. By projecting down to $\mathrm{SL}_2(\mathbb{A})$, we find $\gamma b = b'\pi_{\mathbb{A}}(u)$, comparing the finite part, we conclude $\gamma \in B(\mathbb{Q})$; so, we find π_1 is an isomorphism.

See Lemma 5.3 for more details.

§**4. Convolution.** choose a fundamental domain $\mathcal{F}$ of $Y_0(M)$, so that $\mathcal{F} \subset B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}}) \cong [0,1) \times \mathbb{R}_+^\times \subset \mathfrak{H}$. So for $B\backslash \mathrm{SL}_2 = B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})$

$$\bigcup_{\gamma \in B\backslash \mathrm{SL}_2} \gamma\mathcal{F} \text{ is a fundamental domain of } B(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})/C_\infty\widehat{\Gamma}_0(M).$$

Taking the invariant measure $d\mu$ on $\mathrm{Mp}(\mathbb{A})$ inducing the Dirac measure on each point in $\mathrm{SL}_2(\mathbb{Q})$ with $\int_{\widehat{\Gamma}_0(M)C_\infty} d\mu = 1$, we have

$$\begin{aligned}
&\textstyle\int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})} \mathbf{g}(g)\mathbf{f}(g)E(\Phi)(g)d\mu(g)\\
&= \textstyle\int_{\mathcal{F}} \sum_{\gamma\in B\backslash \mathrm{SL}_2} \mathbf{g}(\gamma g)\mathbf{f}(\gamma g)\Phi(\gamma g)d\mu(g) = \int_{\cup_\gamma \gamma\mathcal{F}} \mathbf{g}(g)\mathbf{f}(g)\Phi(g)d\mu(g)\\
&= \textstyle\int_{B(\mathbb{Q})\backslash B(\mathbb{A})} \mathbf{g}(g)\mathbf{f}(g)\Phi(g)d\mu(g) + \int_{T(\mathbb{Q})\backslash JB(\mathbb{A})} \mathbf{g}(g)\mathbf{f}(g)\Phi(g)d\mu(g).
\end{aligned}$$

The last identity follows from

$$\begin{aligned}
B(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})/C_\infty\widehat{\Gamma}_0(M) &= B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})C_\infty\widehat{\Gamma}_0(M)/C_\infty\widehat{\Gamma}_0(M)\\
&= B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}}) \sqcup T(\mathbb{Q})\backslash JB(\mathbb{A})/B(\widehat{\mathbb{Z}}),
\end{aligned}$$

which in turn follows from

$$\mathrm{SL}_2(K) = B(K) \sqcup B(K)JB(K).$$

§**5. A vanishing condition.** We assume

(V) $\quad\int_{T(\mathbb{Q})\backslash J\cdot B(\mathbb{A})} \mathbf{g}(g)\mathbf{f}(g)\Phi(g)d\mu(g) = 0 \quad (\Leftarrow \Phi|_{J\cdot B(\mathbb{A})} = 0)$.

This follows from ϕ'_c in Choice B (see Lemma 5.28). Taking $d\mu_b = d\mu \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} = d^\times a^{(\infty)} \otimes |a_\infty|^{-1} da_\infty \otimes dx^{(\infty)} \otimes dx_\infty$ for the Lebesgue measure da_∞ and dx_∞ on $\mathbb{R}$ with $\int_{\widehat{\mathbb{Z}}} dx^{(\infty)} = 1$ and $d^\times a^{(\infty)}$ with $\int_{\widehat{\mathbb{Z}}^\times} d^\times a^{(\infty)} = 1$, we have

$$
\begin{aligned}
L(s) &:= \int_{B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty \widehat{\Gamma}_0(M)} \mathbf{g}(g)\mathbf{f}(g)\Phi_s(g)d\mu_g \\
&= \int_{B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})} \mathbf{g}(b)\mathbf{f}(b)\Phi_s(b)d\mu_b \\
= \sum_{\alpha,\beta\in\mathbb{Q}} &\int_{\mathbb{A}^\times/\mathbb{Q}^\times} \int_{\mathbb{A}/\mathbb{Q}} \mathbf{a_f}(\alpha a^2)\mathbf{a_g}(\beta a^2)\mathbf{e}((\alpha+\beta)u) \\
&\qquad \times \exp(-2\pi(-\alpha_\infty + \beta_\infty)a_\infty^2)du|a|_{\mathbb{A}}^{2s-1}da \\
= \sum_{\alpha\in\mathbb{Q}_+} &\int_{\mathbb{A}^\times/\mathbb{Q}^\times} \mathbf{a_f}(\alpha a^2)\mathbf{a_g}(\alpha a^2)\exp(-4\pi\alpha_\infty a_\infty^2)|a|_{\mathbb{A}}^{2s-1}da.
\end{aligned}
$$

§6. L-value. With variable change: $t = a_\infty^2$, we get

$$L(s) = 2\sum_{n=1}^{\infty}\int_0^\infty \mathbf{a_f}(n)\mathbf{a_g}(n)\exp(-4\pi nt)|t|^s t^{-1}dt$$
$$= 2(4\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\mathbf{a_f}(n)\mathbf{a_g}(n)n^{-s}$$
$$= 2(4\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}a_n(f)a_n(g)n^{-s} =: D(s;\mathbf{f},\mathbf{g}).$$

Choose g to be $\theta(\psi) = \theta_j(\psi) = \sum_{n\in\mathbb{Z}}\psi(n)n^j\mathbf{e}(n^2\tau)$ for $j \in \{0,1\}$ with $\psi(-1) = (-1)^j$ and write $D(s,f,\psi) := \sum_{n=1}^{\infty}a_n(f)a_n(\theta)n^{-s}$ and $\varphi = \psi^{-1}\chi_E = \left(\frac{\Delta}{}\right)$. Then θ has weight $\ell = j + \frac{1}{2}$ and $L(s) = D(s;\mathbf{f},\psi) = \prod_p L_p(s)$ with $L_p(s) := \sum_{n=0}^{\infty}\varphi\psi(p^n)a_{p^{2n}}(f)n^{-s}$. Let C_0 be given by the product of primes p ramified in E such that $\psi|_{\mathbb{Z}_p^\times} = \chi_E|_{\mathbb{Z}_p^\times}$ and the p-primary factor of C equals to the p-conductor of χ_E. Write $C_s(\psi)$ for the product of prime factors p of C with either $a_p = 0$ or $\psi(p) = 0$. Since $a_{p^{2n}}(f) = \frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}$ for eigenvalues α,β of $\rho_\lambda(\mathrm{Frob}_p)$, by computation (in §5.2.7), we have

$$\zeta^{(C)}(2s+2-2k)D(s;f,\psi) = L^{(C_s(\psi))}(s,\rho_\lambda^{sym\otimes 2}\otimes\psi)$$
$$= \left(\prod_{l|C_0}(1-\alpha_l^{-1}\overline{\alpha}_l l^{k-1-s})\right)L^{(C_s(\psi))}(s-k+1,Ad(\rho_\lambda)\otimes\chi_{D_\sigma}).$$

§**7. Mixed weight forms.** Let K be a compact subgroup of Mp($\mathbb{A}$). If $\mathbb{C}$-valued automorphic forms $\{\pi(u)\phi(g) = \phi(gu)\}_{u\in K}$ on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$ generate an irreducible K-representation space π which contains ρ as a direct constituent, we say ϕ has K-type ρ. If ρ is not irreducible, we say that ϕ has mixed K-type.

The modular form $\theta_j(\psi) = \sum_{n\in\mathbb{Z}} \psi(n)n^j\mathbf{e}(n^2\tau)$ with $j > 1$ has mixed weight.

If $K = C_\infty \cap \widetilde{\mathrm{SL}}_2(\mathbb{A})$, K is a two-fold covering of $\mathrm{SO}_2(\mathbb{R})$ isomorphic to S^1; so, for $c \in K$, we have $c^2 = r(\theta)$ with $r(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$. If $\chi(c^2) = e^{\ell\theta\sqrt{-1}}$ for an integer ℓ, we say that ϕ has weight $\ell/2$ if it has K-type given by χ as above. To indicate the weight, we sometimes write $\chi_{\ell/2}$ for this χ. The integer ℓ is odd if χ is non-trivial on $\mathrm{Ker}(\pi : K \to \mathrm{SO}_2(\mathbb{R})) \cong \{\pm 1\}$.

§8. **Mixed weight Rankin convolution.** We now enter the description of the Rankin product method in the mixed weight case. Suppose that we have a finite set of modular forms $\{\mathbf{g}_j\}$ on $\mathsf{SL}_2(\mathbb{Q})\backslash\mathsf{Mp}(\mathbb{A})$ which have $\widehat{\Gamma}_0(M)$-type ψ^* but may have a mixed C_∞-type. By saying modular forms, we mean that they are slowly increasing towards cusps in addition to the automorphic property. We assume the index set of j is integers $[0, k] \cap \mathbb{Z}$ for $0 < k \in \mathbb{Z}$. By the definition of mixed weight automorphic forms on $\mathsf{Mp}(\mathbb{A})$, the right translations of $\mathbf{g}_j$ by $\mathsf{Mp}(\mathbb{A})$ generate an admissible representation of $\mathsf{Mp}(\mathbb{A})$; so, the right translations of $\mathbf{g}_j$ by C_∞ span a finite dimensional space of functions on $\mathsf{SL}_2(\mathbb{Q})\backslash\mathsf{Mp}(\mathbb{A})$. Thus $\mathbf{g}_j$ is a finite sum of cusp forms of different C_∞-types. For example, $\mathbf{g}_j$ can be $\boldsymbol{\theta}_{\chi,j}$ for $0 \leq j$. We suppose to have $\{\Phi_j\}_j$ satisfying (V), (Φ1) and (Φ3) but not (Φ2).

§**9. Extra conditions.** We suppose the following extra assumptions:

(Φ0) the $\mathrm{SL}_2(\mathbb{A})$-representation generated by the right translations of $\mathbf{f}\sum_j \mathbf{g}_j E(\Phi_j)$ by $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ contains the trivial representation of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$,

(F) $\mathbf{g}_0|_{B(\mathbb{A})}$ has usual Fourier expansion for the index 0,

(G) for a multiple M of C, $\mathbf{f}\sum_j \mathbf{g}_j E(\Phi_j)$ factors through $Y_0(M) = \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})/\widehat{\Gamma}_0(M)C_\infty$.

Lemma 1. *Let $\pi_{\mathbf{f}}$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}^{(\infty)})$ generated by $\mathbf{f}$. Assume that $\pi_{\mathbf{f}}$ is irreducible cuspidal. The condition* (Φ0) *implies the following matching condition:*

(M) *the representation $\pi_{\mathbf{f}}$ has $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-type which is a contragredient of one of the $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-types of the automorphic representation generated by the set $\{\mathbf{g}_j E(\Phi_j)\}_j$ of modular forms.*

§**10. Convolution again.** As before, for $[B] = B(\mathbb{Q})\backslash B(\mathbb{A})$,

$$\int_{[B]} \mathbf{f}(g) \sum_j \mathbf{g}_j(g)\Phi_j(g)d\mu(g) = \int_{\cup_\gamma \gamma\mathcal{F}} \mathbf{f}(g) \sum_j \mathbf{g}_j(g)\Phi_j(g)d\mu(g)$$
$$= \int_{\mathcal{F}} \sum_{\gamma\in B(\mathbb{Q})\backslash \mathsf{SL}_2(\mathbb{Q})} \mathbf{f}(\gamma g) \sum_j \mathbf{g}_j(\gamma g)\Phi_j(\gamma g)d\mu(g)$$
$$= \int_{\mathcal{F}} \mathbf{f}(g) \sum_j \mathbf{g}_j(g) \sum_{\gamma\in B(\mathbb{Q})\backslash \mathsf{SL}_2(\mathbb{Q})} \Phi_j(\gamma g)d\mu(g)$$
$$= \int_{Y_0(M)} \mathbf{f} \sum_j \mathbf{g}_j E(\Phi_j)d\mu = \int_{Y_0(1)} \mathsf{Tr}(\mathbf{f} \sum_j \mathbf{g}_j E(\Phi_j))d\mu.$$

Now assume

(Key) $\Phi_0|_{B(\mathbb{A})}(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}) = \chi(a)$ with a matching character $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ for $a \in \mathbb{A}^\times$ and $b \in \mathbb{A}$ and $\Phi_j|_{B(\mathbb{A})} = 0$ if $j \neq 0$.

By (Key) and (V), the integral $\int_{[B]} \mathbf{f}(g) \sum_j \mathbf{g}_j(g)\Phi_j(g)d\mu(g)$ is reduced to $\int_{[B]} \mathbf{f}(b)\mathbf{g}_0(b)|a(b)|_{\mathbb{A}}^{2s}d\mu(b)$. Therefore, for any $0 \leq j \in \mathbb{Z}$, taking $\mathbf{g}_0$ to be the lift of $\theta_j(\psi)$, we get

$$\zeta^{(C)}(2s+2-2k)D(s; f, \psi) = L^{(C_s(\psi))}(s, \rho_\lambda^{sym\otimes 2} \otimes \psi)$$
$$= \left(\prod_{l|C_0}(1-\alpha_l^{-1}\overline{\alpha}_l l^{k-1-s})\right) L^{(C_s(\psi))}(s-k+1, Ad(\rho_\lambda) \otimes \chi_{D_\sigma}).$$

§**11. An example.** Let (V, Q) be a quadratic space with a decomposition $(V, Q) = (\mathbb{Q}, z^2) \oplus (W, Q')$ as a quadratic space. Take a Schwartz–Bruhat function ϕ on $V_{\mathbb{A}}$ and suppose that the Siegel–Weil theta series $\theta(\phi)$ has $\widehat{\Gamma}_0(M)$-type ψ^* and a unique C_∞-type k. Take

(i) $\phi = \phi^{(\infty)} \otimes \phi_\infty$ for $\phi^{(\infty)} \in \mathcal{S}(V_{\mathbb{A}^{(\infty)}})$ and $\phi_\infty \in \mathcal{S}(V_{\mathbb{R}})$,

(ii) $\phi_\infty = \sum_j \phi_{\mathbb{Q},j} \otimes \phi_{W,j}$ for $\phi_{\mathbb{Q},j}(z) = z^j \exp(-\pi z^2) \in \mathcal{S}(\mathbb{R})$ and $\phi_{W,j} \in \mathcal{S}(W_{\mathbb{R}})$,

(iii) $\phi^{(\infty)} = \phi_{\mathbb{Q}}^{(\infty)} \otimes \phi_W^{(\infty)}$ for $\phi_{\mathbb{Q}}^{(\infty)} \in \mathcal{S}(\mathbb{A}^{(\infty)})$ and $\phi_W^{(\infty)} \in \mathcal{S}(W_{\mathbb{A}^{(\infty)}})$.

Let $\phi_j^X := \phi_X^{(\infty)} \otimes \phi_{X,j}$ for $X = \mathbb{Q}, W$. We have a natural diagonal embedding of $\mathrm{O}_{\mathbb{Q}} \times \mathrm{O}_W$ into O_V. Note that $\mathrm{O}_{\mathbb{Q}} = \{\pm 1\}$; so, we may forget about it. Then we write $\theta(\phi)(g, h) = \sum_{j=0}^k \theta(\phi_j^{\mathbb{Q}})(g)\theta(\phi_j^W)(g, h)$ for $g \in \mathrm{Mp}(\mathbb{A})$ and $h \in \mathrm{O}_W(\mathbb{A})$. Consider

$$\int_{\mathrm{O}_W(\mathbb{Q})\backslash \mathrm{O}_W(\mathbb{A})} \theta(\phi_j^W)(g, h) d\mu(h).$$

§**12. We can take g_0 to be θ_j with any $j \geq 0$.** By the Siegel–Weil formula, for $\Phi_j(g) = (\mathbf{w}_W(g)\phi_j^W)(0)$, this integral is the Eisenstein series $E(\Phi_j)$, and

$$\int_{\mathrm{O}_W(\mathbb{Q})\backslash \mathrm{O}_W(\mathbb{A})} \theta(\phi)(g,h)d\mu(h) = \sum_{j=0}^{k} \theta(\phi_{k-j}^{\mathbb{Q}})(g)E(\Phi_j)(g)$$

is a modular form whose $\widehat{\Gamma}_0(M)$-type is given by ψ^* and has weight k as C_∞-type. However $\theta(\phi_j^{\mathbb{Q}})$ with $j > 1$ does not have a C_∞-type. We compute $\sum_j \theta(\phi_j^{\mathbb{Q}})(gu)E(\Phi_j)(gu)$ for $u \in C_\infty$:

$$\sum_j \theta(\phi_j^{\mathbb{Q}})(gu)E(\Phi_j)(gu) = \int_{\mathrm{O}_W(\mathbb{Q})\backslash \mathrm{O}_W(\mathbb{A})} \theta(\phi)(gu,h)d\mu(h)$$
$$= \int_{\mathrm{O}_W(\mathbb{Q})\backslash \mathrm{O}_W(\mathbb{A})} \theta(\phi)(g,h)\chi_\ell(u)d\mu(h) = \sum_j \theta(\phi_j^{\mathbb{Q}})(g)E(\Phi_j)(g)\chi_\ell(u).$$

Choosing a cusp form $\mathbf{f}$ with the inverse $\widehat{\Gamma}_0(M)C_\infty$-type and putting $\mathbf{g}_j = \theta(\phi_j^{\mathbb{Q}})$, the pair $\{\mathbf{g}_j, \Phi_j\}_j$ satisfies the conditions (Φ0) and (F). We will verify other conditions for a specific $\phi^{(\infty)}$ later.