L-value formula
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Lecture 4: We sketch the proof of the L-value formula for a division quaternion algebra $D/\mathbb{Q}$. The algebra $D$ can be definite or indefinite, though we describe mainly the details only for the indefinite case. Let $B = D \otimes_{\mathbb{Q}} E$ for a real semi-simple quadratic extension $E$. The non-trivial automorphism $\sigma \in \text{Gal}(E/\mathbb{Q})$ acts on $B$ through the factor $E$. Since the case $E = \mathbb{Q} \times \mathbb{Q}$ is easier, we mainly assume that $E$ is a field. A key point is the use of the see-saw principle for the decomposition $D_{\sigma} = Z \oplus D_0$, where $D_{\sigma} := \{v \in B| v^t = v^\sigma\}$ with the reduced norm $N : D_{\sigma} \to \mathbb{Q}$ and $Z = D_{\sigma} \cap E$ and $D_0 = \{v \in D_{\sigma}| \text{Tr}(v) = 0\}$. We need to use the Siegel–Weil formula for $D_0$. For simplicity, we assume $M = \partial$. The details are in Chapter 5, and the case $M = M_2(\mathbb{Q})$ is dealt with in Section 5.5 of the notes.
§0. An idea of Waldspurger. For an elliptic cusp form $f$, an idea of Waldspurger of computing the period of a theta lift of $f$ for a quadratic space $V = W \oplus W_{\perp}$ over an orthogonal Shimura subvariety $S_W \times S_W^{\perp} \subset S_V$ is two-folds:  
(S) Split $\theta(\phi')(\tau, h, h^{\perp}) = \theta(\phi)(\tau, h) \cdot \theta(\tau, \phi^{\perp})(h^{\perp})$ ($h^{?} \in O_{W^{?}}(A)$) for a decomposition $\phi' = \phi \otimes \phi^{\perp}$ ($\phi$ and $\phi^{\perp}$ Schwartz–Bruhat functions on $W_{A}$ and $W_{A}^{\perp}$);  
(R) For the theta lift $\theta^*(\phi)(f)(h) = \int_X f(\tau) \theta(\phi)(\tau, h) d\mu$ with an $\text{SL}(2)$-Shimura curve $X$, the period $P$ over the Shimura subvariety $S \times S^{\perp}$ ($S$ for $O(W)$ and $S^{\perp}$ for $O(W^{\perp})$) is given by:  
\begin{align*}
\int_{S \times S^{\perp}} \int_X f(\tau) \theta(\phi)(\tau; h) d\mu dh &= (d\mu = \eta^{-2} d\xi d\eta) \\
&= \int_X f(\tau) \left( \int_{S^{\perp}} \theta(\phi^{\perp})(\tau; h^{\perp}) dh^{\perp} \right) \cdot \left( \int_S \theta(\phi_0)(\tau; h_0) dh \right) d\mu.
\end{align*}
Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\phi)$ and $E(\phi^{\perp})$, reaching Rankin-Selberg integral  
$$P = \int_X f(\tau) E(\phi^{\perp}) E(\phi_0) d\mu = L\text{-value}.$$
§1. **Choice of $V$:** For a $\mathbb{Q}$-vector space $V$ and a $\mathbb{Q}$-algebra $A$, write $V_A := V \otimes_{\mathbb{Q}} A$. Let $E := \mathbb{Q}[\sqrt{\Delta}]$ be a quadratic extension of $\mathbb{Q}$ with discriminant $\Delta$. Pick a quaternion algebra $D$ over $\mathbb{Q}$ and put $B := D \otimes_{\mathbb{Q}} E$. We let $1 \neq \sigma \in \text{Gal}(E/\mathbb{Q})$ act on $D$ through the factor $E$. Recall

$$V = D_\sigma := \{v \in B|v^\sigma = v^t\} \text{ for } v^t = \text{Tr}_{B/E}(v) - v.$$ 

The quadratic form is given by $Q(v) = vv^\sigma = N(v) \in \mathbb{Q}$. We have two cases of isomorphism classes of $(D_\mathbb{R}, E_\mathbb{R})$. Note $E_\mathbb{R} = \mathbb{R} \times \mathbb{R}$; so, we have two cases Case I and Case H. The symbol “I” (resp. “H”) indicate $D$ is indefinite (resp. definite). The decomposition we take is

$$V = \mathbb{Z} \oplus D_0 \quad \mathbb{Z} = \mathbb{Q} \text{ with quadratic form } Q_\mathbb{Z}(z) = z^2,$$

$$D_0 := \{v_0 \in \sqrt{\Delta}D|\text{Tr}_{D/\mathbb{Q}}(v) = 0\} \text{ with } Q_0(v) = vv^\sigma = N(v)$$

Signature of $D_0$ is (1,2) in Case I and (3,0) in Case H, $O_{D_0}$ is almost $D^\times$ and the same for $O_{D_\sigma}$ and $B^\times$. 
§2. Bruhat functions and majorant. On $\mathbb{Z} = \mathbb{Q}$, for a Dirichlet character $\psi$ modulo $N$, we regard $\psi$ as a function supported on $\hat{\mathbb{Z}} \subset \mathbb{Z}_{\mathbb{A}(\infty)} = \mathbb{A}(\infty)$. This $\psi$ produces theta series $\sum_{n \in \mathbb{Z}} \psi(n)n^j e(n^2 \tau)$ on $\Gamma_0(4N^2)$ of character $\psi(\frac{-1}{\tau})$ and of weight $j + \frac{1}{2}$.

Take a maximal order $R$ of $D$ and take the characteristic function $\phi_0$ of $D_{0,\mathbb{A}} \cap \sqrt{\Delta} \hat{R}$. Here for any lattice $L$, $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. This $\phi_0$ produces theta series on $\Gamma_0(4\partial \Delta)$ of character $\psi(\frac{-\Delta}{\tau})$.

The theta series for $D_\sigma$ of $\psi \otimes \phi_0$ has level $M = [4N^2, 4\partial \Delta]$. We choose $M$ so that $C|M$ for the conductor $C$ of $F$.

A positive definite symmetric matrix $P \in M_n(\mathbb{R})$ (or the symmetric bilinear form on $V_\mathbb{R}$ associated to $P$) is a positive majorant of a symmetric matrix $S$ if $PS^{-1} = SP^{-1}$ ($\Leftrightarrow S^{-1}P = P^{-1}S$).
§3. **Schwartz function $\Psi$ on $D_{\sigma,R}$ in Case I.** The recipe of Hecke–Siegel is to put $\Psi(v) = H(v)e(\xi N(v) + P(v)\eta\sqrt{-1})$ for $e(x) = \exp(2\pi\sqrt{-1}x)$ and a harmonic polynomial $H$, where $P(v) = \frac{1}{2}p(v, v)$ with a positive majorant $p$ of $s(v, v') = \text{Tr}_{B/E}(v^tv')$. All positive majorants form the symmetric space $\mathcal{G}$ of $O_{D_\sigma}$.

We identify $(D_{\sigma,R}, N) = (M_2(\mathbb{R}), \text{det})$ by $M_2(\mathbb{R}) \ni v \mapsto (v, v^t) \in D_{\sigma,R} \subset D_\mathbb{R} \times D_\mathbb{R}$ and put for $(z, w) \in \mathfrak{h} \times \mathfrak{h}$ on which $B^\times \sim \text{GO}_{D_\sigma}$ acts by $\alpha(z, w) = (\alpha(z), \alpha^\sigma(w))$. For $(z, w) \in \mathfrak{h} \times \mathfrak{h}$, a standard harmonic polynomial of $v \in D_\sigma$ of degree $k$ is given by $[v; z, w]^k = s(v, p(z, w))^k$ for $p(z, w) = (\frac{z}{1})(w, 1)J$. For $0 < k \in \mathbb{Z}$, $\Psi(v; \tau, z, w) = \text{Im}(\tau)\frac{[v; z, w]^k}{(z-\bar{z})^k(w-\bar{w})^k}[e(N(v)\tau + i\frac{\text{Im}(\tau)}{2|\text{Im}(z)\text{Im}(w)|}[v; z, w]^2),$ for $(\alpha, \beta) \in GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ (see §3.2.3), $\alpha p(z, w)\beta^t = p(\alpha(z), \beta(w))j(\alpha, z)j(\beta, w)$. This formula is due to Shimura. This function is not a tensor product of functions on $Z_\mathbb{R}$ and $D_{0,R}$ which causes some difficulty later. For simplicity, we assume $k = 2$. See Section 3.2 for $\Psi$. 
§4. **Theta kernel.** Let $\phi$ be a Schwartz-Bruhat function on $D_{\sigma, \mathbb{A}}$. Let $\text{Mp}(\mathbb{A}) \to \text{SL}_2(\mathbb{A})$ be the metaplectic cover constructed by Weil, and $\phi \mapsto w(g)\phi$ the Weil representation. Noting $B^\times \to \text{GO}_{D_{\sigma}}$ by $v \mapsto h^\tau v h^\sigma$, Siegel–Weil theta series $\theta(g; h)$ is
\[
\sum_{\alpha \in D_{\sigma}} (w(g)\phi)(h^\tau \alpha h^\sigma) : \text{SL}_2(\mathbb{Q}) \backslash \text{Mp}(\mathbb{A}) \times B^\times \backslash B^\times_{\mathbb{A}} \to \mathbb{C}.
\]
Write $\hat{\Gamma} = \hat{\Gamma}_{\phi} = \{ u \in B^\times_{\mathbb{A}(\infty)} | \theta(g, u^\tau h u^\sigma) = \theta(g, h) \}$.

In Case I, choose $\phi = (\psi \otimes \phi_0)\Psi(v; \sqrt{-1}, \sqrt{-1}, \sqrt{-1})$ and for $g_\tau = \eta^{-1/2} \begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix}$ ($\tau = \xi + \eta \sqrt{-1} \in \mathfrak{H}$), we specialize $g$ to $g_\tau$ and $h$ to $(g_z, g_w)$ for $(\tau, z, w) \in \mathfrak{H} \times \mathfrak{H} \times \mathfrak{H}$. Then
\[
\theta(\tau; z, w) := \theta(g_\tau; g_z, g_w) = \sum_{\alpha \in D_{\sigma}} (\psi \otimes \phi_0)(\alpha)\Psi(\alpha; \tau, z, w).
\]
Set $\theta^*(\phi)(f) := \int_{X_0(M)} f(\tau) \theta(\phi)(\tau; z, w)\eta^{k-2} d\xi d\eta$ ($k = 2$). Then $\theta^*(\phi)(f)$ is a weight 2 quaternionic modular form on $B^\times$ holomorphic in $z$ and anti-holomorphic in $w$ for $f \in S^-_2(\Gamma_0(M), \psi^{-1}(\Delta))$. 
5. Theta differential form. To compute the period on
\[ Sh_D = D D^\times \setminus (D A_{\varphi}) \times H \subset Sh_B = B B^\times \setminus (B A_{\varphi}) \times B \], we convert
\[ \theta(\tau; z, w) \] into a sheaf valued differential 2–form. If \( n = k - 2 > 0 \),
the sheaf comes from the \( B^\times \)-module
\[ L_E(n; A) = \sum_{0 \leq i, j \leq n} AX^{n-j}Y^jX^{m-i}Y^i \]
with \( B^\times \)-action \( \gamma P(X, Y; X', Y') = P((X, Y)^t \gamma^t; (X', Y')^t \gamma^{\varphi^t}) \). As
we assumed \( k = 2 \) (i.e., \( n = 0 \)), we have \( L(n; A) = A \).

By putting \( \Theta = \theta(\phi)(\tau; z, w)dz \wedge dw \) for \( n = k - 2 \), we get \( \mathbb{C} \)-valued
\( \Gamma_{\phi} \)-invariant differential form. The period we like to compute is
\[ P = P_1(\theta^*(\phi)(f)) = \int_{Sh_D} \int_{X_0(M)} f(-\bar{\tau})\Theta(\tau; z, z)d\xi d\eta. \]
We integrate over \( Sh_D \) by a measure \( d\mu \) given by \( y^{-2}dxdy \) over
\( H \) and \( \int_{\Gamma} d\mu = 1. \)
§6. **Siegel–Weil Eisenstein series; §4.4.2.** Recall the explicit section $r : B \hookrightarrow \text{Mp}$ of the representation $w$ as follows:

$$r(\text{diag}[a, a^{-1}])\phi(v) = |a|^3/2 \phi(av), \quad r\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array}\right)\phi(v) = e(uN(v))\phi(v).$$

For the standard Borel subgroup $B \subset \text{SL}_2$, the function $g \mapsto (r(g)\phi)(0)$ is left $B(\mathbb{Q})$ invariant. Siegel–Weil Eisenstein series is

$$E(\phi)(g; s) = \sum_{\gamma \in B(\mathbb{Q}) \setminus \text{SL}_2(\mathbb{Q})} (w(\gamma g)\phi)(0)|a(\gamma g)|^{s}_A,$$

where $g = \text{diag}[a(g), a(g)^{-1}]\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array}\right)c$ for an element $c$ in the maximal compact subgroup by Iwasawa decomposition.

The Siegel–Weil formula by Kudla-Rallis and Sweet is

$$E(\phi)(g; 0) = \int_S \theta(\phi)(g, h)d\omega \quad \text{for the Tamagawa measure } d\omega.$$

The ratio $m = m(\widehat{\Gamma}) = d\mu/d\omega$ is the mass of Siegel–Shimura, which is an arithmetic rational number times $\zeta(2)/\pi$ in Case I and $\zeta(2)/\pi^2$ in Case H. Later we describe the Tamagawa measure, the metaplectic group and Weil representation in more details.
§7. Conclusion in Case I; §5.3. Decomposing \( v = a \oplus v_0 \) in \( D_\sigma = Z \oplus D_0 \), we have for \( k = 2 \)

\[
[a + v_0; z, \bar{z}]^k = ([a; z, \bar{z}] + [v_0; z, \bar{z}])^k = \sum_{j=0}^{k} \binom{k}{j} (a(z - \bar{z}))^{k-j} [v_0; z, \bar{z}]^j.
\]

Thus we have \( \phi = \sum_{j=0}^{k} \binom{k}{j} \phi_{k-j} Z \phi_{j} D_0 \) with infinity part \( \Psi_j \) of \( \phi_j \) given by

\[
\Psi_j^{D_0} := (z - \bar{z})^{-j} [v_0; z, \bar{z}]^j e(N(\tau)\tau + \frac{i \text{Im}(\tau) [w_0; z, \bar{z}]^2}{2 \text{Im}(z)^2}),
\]

\[
\Psi_j^{Z} := a^j e(a^2 \tau)
\]

with \((w(b)\phi_{j}^{D_0})(0) = 0\) and \( E(\phi_{j}^{D_0})|_{B(\mathbb{A})} = 0\) unless \( j = 0 \), and we reach Rankin convolution of \( \theta(\phi_k^Z) = \sum_{n \in \mathbb{Z}} \psi(n) n^k e(n^2 z) \) and \( f \) over \( B(\mathbb{Q}) \setminus B(\mathbb{A}) / B(\hat{\mathbb{Z}}) \cong [0, 1) \times \mathbb{R}_+^\times \), which produces (see [Sh75])

\[
\zeta(2) P = m 2^{-2k} * (2\pi)^{-k} \Gamma(k) L^{(s)}(1, \text{Ad}(\rho_f) \otimes (\frac{\Delta}{\cdot})
\]

with a simple constant \(*\). Here \( L^{(s)} \) means we remove Euler factors at \( p|C \) with either \( f|U(p) = 0 \) or \( \psi(p) = 0 \).
§8. Conclusion in Case H; §5.2. The choice of the Bruhat function \( \phi \) is the same as in Case I. As a \( \mathbb{C} \)-valued function, set

\[
\Psi(\tau; v; x) = e(N(v)\tau).
\]

Again in exactly the same way, for

\[
\theta^*(\phi)(f) := \int_{X_0(M)} \theta(\phi)(\tau; g)f(\tau)\eta^{k-2}d\xi d\eta \quad (k = 2)
\]

and \( P = \int_S \theta^*(\phi)(f)d\mu \), we conclude for a simple constant \( c' \)

\[
\zeta(2)P = 2m^*(2\pi)^{-k+1}\Gamma(k)L^s(1, Ad(\rho_f) \otimes \left( \frac{\Delta}{\cdot} \right)).
\]

Writing the point set \( S = \{x\}_{x \in Sh_R,} \quad m(\hat{\Gamma}) = \sum x \in Sh_R e_x^{-1} \div \zeta(2) \)

for \( e_x = |\hat{\Gamma} \cap O_{D_0}(\mathbb{Q})| \) and \( P = \sum x \in Sh_R t^* e_x^{-1} \theta^*(\phi)(f)(x) \).

Thus the period formula is an adjoint analogue of the mass formula of Siegel–Shimura. The determination of \( m(\hat{\Gamma}) \) was finished by Shimura in 1999 for an arbitrary quadratic space over a totally real field (see §5.2.8 for the explicit formula for the mass).
§9. Schwartz–Bruhat functions; §3.1.3. For a \( \mathbb{Q} \)-vector space \( V \), write \( V_p := V \otimes \mathbb{Z} \mathbb{Z}_p \) (which is a vector space over a local field \( \mathbb{Q}_p \)). A Bruhat function on \( V_p \) is a locally constant compactly supported function with values in \( \mathbb{C} \). Write \( S(V_p) \) for the space of Bruhat functions on \( V_p \). For a real vector space \( V_\infty \), we define \( S(V_\infty) \) to be the Schwartz space of functions on \( V_\infty \). Thus \( S(V_\infty) \) is made of \( C^\infty \)-class functions with all derivatives rapidly decreasing as Euclidean norm of \( v \in V_\infty \) grows. In other words, \( \phi \in S(V_\infty) \) if and only if \( \phi \) is of \( C^\infty \)-class and for any polynomial \( P(v) \) and any \( m \)-th derivative \( \Phi \) of \( \phi \), \( |P(v)\Phi(v)| \) goes to 0 as \( |x| \to \infty \). Writing \( V_\mathbb{A} \) for the adelization. We pick a lattice \( L \) of \( V \) and put \( \hat{L} = \prod_p L_p \subset V_\mathbb{A}(\infty) \) with \( L_p = L \otimes \mathbb{Z} \mathbb{Z}_p \). A Schwartz-Bruhat function on \( V_\mathbb{A} \) is a finite linear combination of the function of the form \( \phi(x) = \prod_v \phi_v(x_v) \) with \( \phi_v \in S(V_v) \) and \( \phi_p \) is the characteristic function of \( L_p \) for almost all \( p \).
§10. Weil representation. Let \( \nu(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \), \( \text{diag}[a, b] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \)

and \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Let \( e : \mathbb{A}/\mathbb{Q} \to S^1 \) be the additive character with \( e(x_\infty) := \exp(2\pi \sqrt{-1} x_\infty) \). We put \( e_v := e|_{\mathbb{Q}_v} \) and for a number field \( F \), we write \( e_F = e \circ \text{Tr}_{F/\mathbb{Q}} \) and \( e_{F_v} = e_v \circ \text{Tr}_{F_v/\mathbb{Q}_v} \). Let \( Q : V \to F \) be a non-degenerate quadratic form with symmetric bilinear form \( s(v, w) = Q(v + w) - Q(v) - Q(w) \) over a number field \( F \). We have the following operator \( r(g) \in \text{Aut}(S(V_?)) \) for \( m = \dim_F V, \ ? = v, \mathbb{A} \) with \( u \in F_v \) or \( F_\mathbb{A} \) and \( a \in F_v^\times \):

\[
\begin{align*}
\rho(v(u)) &= e_F(uQ(v))\phi(v), \\
\rho(\text{diag}[a, a^{-1}]\phi(v)) &= |a|_{F_\mathbb{A}}^{m/2} \phi(av) \quad \text{and} \\
\rho(J)\phi(v) &= \hat{\phi}(-v) := \int_{V_?} e_F(s(w, -v)) \phi(w) dw \quad \text{(Fourier transform)},
\end{align*}
\]

where \( dw \) is normalized so that \( \hat{\phi}(v) = \phi(-v) \). If \( b, b' \in B(F_\mathbb{A}) \) (upper triangular Borel subgroup), we extend \( \rho \) to \( \Omega = B(F_\mathbb{A})JB(F_\mathbb{A}) \) by \( \rho(bJb') := \rho(b)\rho(J)\rho(b') \). Then if \( g, h \in \text{SL}_2(F_?) \) either unipotent, diagonal or \( J, \rho(gh) = \kappa(g, h)\rho(g)\rho(h) \) for a 2-cocycle \( \kappa \) on \( \text{SL}_2 \) with values in \( S^1 \). Write \( \text{Mp}(F_?) \subset \text{Aut}(S(V_?)) \) for the group generated by these operators. We have an extension (*) \( 1 \to S^1 \to \text{Mp}(F_?) \xrightarrow{\pi_?} \text{SL}_2(F_?) \to 1 \) with \( \text{Mp} \ni w(g) \mapsto g \in \text{SL}_2 \). Therefore, the group \( \text{Mp} \) acts on \( S(V_?) \) by a representation \( w \).
§11. Weil’s theta series. The extension (*) is split in the following cases:
1. dim\(_F\)\(_V\) is even (the section is unique and if \(V = D_{\sigma,F}\), \(b \mapsto \chi_V(a)r(b)\) over \(B(F_\mathbb{A})\) if \(b = v(u)\text{diag}[a,a^{-1}]\));
2. \(b \mapsto r(b)\) and also \(b \mapsto \chi_V(a)r(b)\) as above over \(B(F_\mathbb{A})\);
3. Over \(\hat{\Gamma}_0(4)\) (canonical);
4. Over \(\text{SL}_2(F)\) (canonical and coincides with \(b \mapsto r(b)\) over \(B(F)\).

For the orthogonal group \(O_V\) for \(V\) and \(\phi \in S(V_\mathbb{A})\), we define a function on \(\text{Mp}(F_\mathbb{A}) \times O_V(F_\mathbb{A})\) by \(\theta(\phi)(g,h) = \sum_{\alpha \in V} w(g)L(h)\phi(\alpha)\), where \((L(h)\phi)(v) = \phi(vh)\) (as usual, \(O(F_\mathbb{A})\) acts on \(V_{F_\mathbb{A}}\) from the right). Weil showed that \(\theta(\phi)(g,h)\) is real analytic on \(\text{Mp}(F_\infty) \times O_V(F_\infty)\), left invariant under \(\text{SL}_2(F) \times O_V(F)\) and right invariant under an open subgroup of \(\text{Mp}(F_\mathbb{A}(\infty)) \times O_V(F_\mathbb{A}(\infty))\); in short, an automorphic form on \(\text{Mp} \times O_V\).

All the details are in Chapter 4.