Lecture 4: We sketch the proof of the L-value formula for a division quaternion algebra $D/Q$. The algebra $D$ can be definite or indefinite, though we describe mainly the details only for the indefinite case. Let $B = D \otimes Q E$ for a real semi-simple quadratic extension $E$. The non-trivial automorphism $\sigma \in \text{Gal}(E/Q)$ acts on $B$ through the factor $E$. Since the case $E = Q \times Q$ is easier, we mainly assume that $E$ is a field. A key point is the use of the see-saw principle for the decomposition $D_\sigma = Z \oplus D_0$, where $D_\sigma := \{v \in B | v^\rho = v^\sigma \}$ with the reduced norm $N : D_\sigma \to \mathbb{Q}$ and $Z = D_\sigma \cap E$ and $D_0 = \{v \in D_\sigma | \text{Tr}(v) = 0 \}$. We need to use the Siegel–Weil formula for $D_0$. For simplicity, we assume $M = \partial$. The details are in Chapter 5, and the case $M = M_2(\mathbb{Q})$ is dealt with in Section 5.5 of the notes.
§0. An idea of Waldspurger. For an elliptic cusp form $f$, an idea of Waldspurger of computing the period of a theta lift of $f$ for a quadratic space $V = W \oplus W^\perp$ over an orthogonal Shimura subvariety $S_W \times S_{W^\perp} \subset S_V$ is two-folds:

(S) Split $\theta(\phi')(\tau, h, h^\perp) = \theta(\phi)(\tau, h) \cdot \theta(\tau, \phi^\perp)(h^\perp)$ ($h^? \in O_{W^?}(\mathbb{A})$) for a decomposition $\phi' = \phi \otimes \phi^\perp$ ($\phi$ and $\phi^\perp$ Schwartz–Bruhat functions on $W_\mathbb{A}$ and $W^\perp_\mathbb{A}$);

(R) For the theta lift $\theta^*(\phi)(f)(h) = \int_X f(\tau)\theta(\phi)(\tau, h)d\mu$ with an $SL(2)$-Shimura curve $X$, the period $P$ over the Shimura subvariety $S \times S^\perp$ ($S$ for $O(W)$ and $S^\perp$ for $O(W^\perp)$) is given by:

$$\int_{S \times S^\perp} \int_X f(\tau)\theta(\phi)(\tau; h)d\mu dh \quad (d\mu = \eta^{-2}d\xi d\eta)$$

$$= \int_X f(\tau) \left( \int_{S^\perp} \theta(\phi^\perp)(\tau; h^\perp)dh^\perp \right) \cdot \left( \int_{S} \theta(\phi_0)(\tau; h_0)dh \right) d\mu.$$  

Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\phi)$ and $E(\phi^\perp)$, reaching Rankin-Selberg integral

$$P = \int_X f(\tau)E(\phi^\perp)E(\phi_0)d\mu = L\text{-value}.$$
§1. Choice of $V$: For a $\mathbb{Q}$-vector space $V$ and a $\mathbb{Q}$-algebra $A$, write $V_A := V \otimes \mathbb{Q} A$. Let $E := \mathbb{Q}[\sqrt{\Delta}]$ be a quadratic extension of $\mathbb{Q}$ with discriminant $\Delta$. Pick a quaternion algebra $D$ over $\mathbb{Q}$ and put $B := D \otimes \mathbb{Q} E$. We let $1 \neq \sigma \in \text{Gal}(E/\mathbb{Q})$ act on $D$ through the factor $E$. Recall

$$V = D_\sigma := \{v \in B|v^\sigma = v^t\} \quad \text{for} \quad v^t = \text{Tr}_{B/E}(v) - v.$$ 

The quadratic form is given by $Q(v) = vv^\sigma = N(v) \in \mathbb{Q}$. We have two cases of isomorphism classes of $(D_\mathbb{R}, E_\mathbb{R})$. Note $E_\mathbb{R} = \mathbb{R} \times \mathbb{R}$; so, we have two cases Case I and Case H. The symbol “I” (resp. “H”) indicate $D$ is indefinite (resp. definite). The decomposition we take is

$$V = Z \oplus D_0 \quad Z = \mathbb{Q} \quad \text{with quadratic form} \quad Q_Z(z) = z^2,$$

and

$$D_0 := \{v_0 \in \sqrt{\Delta}D|\text{Tr}_{D/\mathbb{Q}}(v) = 0\} \quad \text{with} \quad Q_0(v) = vv^\sigma = N(v)$$

Signature of $D_0$ is $(1,2)$ in Case I and $(3,0)$ in Case H, $\mathcal{O}_{D_0}$ is almost $D^\times$ and the same for $\mathcal{O}_{D_\sigma}$ and $B^\times$. 
§2. Bruhat functions and majorant. On $\mathbb{Z} = \mathbb{Q}$, for a Dirichlet character $\psi$ modulo $N$, we regard $\psi$ as a function supported on $\hat{\mathbb{Z}} \subset \mathbb{Z}_{\mathbb{A}(\infty)} = \mathbb{A}(\infty)$. This $\psi$ produces theta series $\sum_{n \in \mathbb{Z}} \psi(n)n^j e(n^2 \tau)$ on $\Gamma_0(4N^2)$ of character $\psi(1)$ and of weight $j + \frac{1}{2}$.

Take a maximal order $R$ of $D$ and take the characteristic function $\phi_0$ of $D_{0, \mathbb{A}} \cap \sqrt{\Delta} \hat{R}$. Here for any lattice $L$, $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. This $\phi_0$ produces theta series on $\Gamma_0(4\partial \Delta)$ of character $(-\Delta)$.

The theta series for $D_\sigma$ of $\psi \otimes \phi_0$ has level $M = [4N^2, 4\partial \Delta]$. We choose $M$ so that $C|M$ for the conductor $C$ of $F$.

A positive definite symmetric matrix $P \in M_n(\mathbb{R})$ (or the symmetric bilinear form on $V_\mathbb{R}$ associated to $P$) is a positive majorant of a symmetric matrix $S$ if $PS^{-1} = SP^{-1}$ ($\iff S^{-1}P = P^{-1}S$).
§3. **Schwartz function** \( \Psi \) on \( D_{\sigma, \mathbb{R}} \) in Case I. The recipe of Hecke–Siegel is to put \( \Psi(v) = H(v) e(\xi N(v) + P(v) \eta \sqrt{-1}) \) for \( e(x) = \exp(2\pi \sqrt{-1}x) \) and a harmonic polynomial \( H \), where \( P(v) = \frac{1}{2} p(v, v) \) with a positive majorant \( p \) of \( s(v, v') = \text{Tr}_{B/E}(v^t v') \). All positive majorants form the symmetric space \( \mathbb{G} \) of \( \mathbb{O}_{D_{\sigma}} \).

We identify \( (D_{\sigma, \mathbb{R}}, N) = (M_2(\mathbb{R}), \text{det}) \) by \( M_2(\mathbb{R}) \ni v \mapsto (v, v^t) \in D_{\sigma, \mathbb{R}} \subset D_{\mathbb{R}} \times D_{\mathbb{R}} \) and put for \( (z, w) \in \mathfrak{H} \times \mathfrak{H} \) on which \( B^* \sim \text{GO}_{D_\sigma} \) acts by \( \alpha(z, w) = (\alpha(z), \alpha^\sigma(w)) \). For \( (z, w) \in \mathfrak{H} \times \mathfrak{H} \), a standard harmonic polynomial of \( v \in D_{\sigma} \) of degree \( k \) is given by

\[
[v; z, w]_k = s(v, p(z, w))^k \quad \text{for} \quad p(z, w) = \begin{pmatrix} z & \bar{w} \\ \bar{w} & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J.
\]

For \( 0 < k \in \mathbb{Z} \),

\[
\Psi(v; \tau, z, w) = \text{Im}(\tau)[v; z, \bar{w}]^k e(N(v)\tau + i \frac{\text{Im}(\tau)}{2|\text{Im}(z)\text{Im}(w)|}|[v; z, \bar{w}]|^2),
\]

for \( (\alpha, \beta) \in GL_2(\mathbb{R}) \times GL_2(\mathbb{R}) \) (see §3.2.3),

\[
\alpha p(z, w) \beta^t = p(\alpha(z), \beta(w)) j(\alpha, z) j(\beta, w).
\]

This formula is due to Shimura. This function is not a tensor product of functions on \( Z_{\mathbb{R}} \) and \( D_{0, \mathbb{R}} \) which causes some difficulty later. For simplicity, we assume \( k = 2 \). See Section 3.2 for \( \Psi \).
\section{Theta kernel.} Let $\phi$ be a Schwartz-Bruhat function on $D_{\sigma,A}$. Let $\text{Mp}(\mathbb{A}) \to \text{SL}_2(\mathbb{A})$ be the metaplectic cover constructed by Weil, and $\phi \mapsto w(g)\phi$ the Weil representation. Noting $B^\times \to \text{GO}_{D_{\sigma}}$ by $v \mapsto h^tvh^\sigma$, Siegel–Weil theta series $\theta(g; h)$ is

$$\sum_{\alpha \in D_{\sigma}} (w(g)\phi)(h^t\alpha h^\sigma) : \text{SL}_2(\mathbb{Q})\backslash \text{Mp}(\mathbb{A}) \times B^\times \backslash B^\times_{\mathbb{A}} \to \mathbb{C}.$$ 

Write $\hat{\Gamma} = \hat{\Gamma}_\phi = \{ u \in B^\times_{\mathbb{A}(\infty)} | \theta(g, u^t hu^\sigma) = \theta(g, h) \}$. 

In Case I, choose $\phi = (\psi \otimes \phi_0)\Psi(v; \sqrt{-1}, \sqrt{-1}, \sqrt{-1})$ and for $g_\tau = \eta^{-1/2} \begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix} (\tau = \xi + \eta \sqrt{-1} \in \mathfrak{h})$, we specialize $g$ to $g_\tau$ and $h$ to $(g_z, g_w)$ for $(\tau, z, w) \in \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$. Then

$$\theta(\tau; z, w) := \theta(g_\tau; g_z, g_w) = \sum_{\alpha \in D_{\sigma}} (\psi \otimes \phi_0)(\alpha)\Psi(\alpha; \tau, z, w).$$

Set $\theta^*(\phi)(f) := \int_{X_0(M)} f(\tau)\theta(\phi)(\tau; z, w)\eta^{k-2}d\xi d\eta$ ($k = 2$). Then $\theta^*(\phi)(f)$ is a weight 2 quaternionic modular form on $B^\times$ holomorphic in $z$ and anti-holomorphic in $w$ for $f \in S^2_2(\Gamma_0(M), \psi^{-1}(\Delta)).$
§5. **Theta differential form.** To compute the period on $Sh_D = D_+^X \setminus (D^X_{\mathbb{A}(\infty)} \times \mathfrak{H}) \subset Sh_B = B^X \setminus (B^X_{\mathbb{A}(\infty)} \times \mathfrak{Z}_B)$, we convert $\theta(\tau; z, w)$ into a sheaf valued differential 2–form. If $n = k - 2 > 0$, the sheaf comes from the $B^\times$-module

$$L_E(n; A) = \sum_{0 \leq i, j \leq n} AX^{n-j}Y^j X^{m-i} Y^n$$

with $B^\times$-action $\gamma_P(X, Y; X', Y') = P((X, Y)^t \gamma^t; (X', Y')^t \gamma^o)$. As we assumed $k = 2$ (i.e., $n = 0$), we have $L(n; A) = A$.

By putting $\Theta = \theta(\phi)(\tau; z, w) dz \wedge d\bar{w}$ for $n = k - 2$, we get $\mathbb{C}$-valued $\Gamma_\phi$-invariant differential form. The period we like to compute is

$$P = P_1(\theta^*(\phi)(f)) = \int_{Sh_D} \int_{X_0(M)} f(-\bar{\tau}) \Theta(\tau; z, z) d\xi d\eta.$$ 

We integrate over $Sh_D$ by a measure $d\mu$ given by $y^{-2} dx dy$ over $\mathfrak{H}$ and $\int_{\Gamma} d\mu = 1$. 
§6. Siegel–Weil Eisenstein series; §4.4.2. Recall the explicit section \( r : B \hookrightarrow \text{Mp} \) of the representation \( w \) as follows:

\[
\begin{align*}
r(\text{diag}[a, a^{-1}]) \phi(v) &= |a|^{3/2} \phi(av), \\
r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \phi(v) &= e(uN(v)) \phi(v).
\end{align*}
\]

For the standard Borel subgroup \( B \subset \text{SL}_2 \), the function \( g \mapsto (r(g)\phi)(0) \) is left \( B(\mathbb{Q}) \) invariant. Siegel–Weil Eisenstein series is

\[
E(\phi)(g; s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{Q})} (w(\gamma g)\phi)(0) |a(\gamma g)|_A^s,
\]

where \( g = \text{diag}[a(g), a(g)^{-1}] \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} c \) for an element \( c \) in the maximal compact subgroup by Iwasawa decomposition.

The Siegel–Weil formula by Kudla-Rallis and Sweet is

\[
E(\phi)(g; 0) = \int_S \theta(\phi)(g, h) d\omega \quad \text{for the Tamagawa measure } d\omega.
\]

The ratio \( m = m(\hat{\Gamma}) = d\mu/d\omega \) is the mass of Siegel–Shimura, which is an arithmetic rational number times \( \zeta(2)/\pi \) in Case I and \( \zeta(2)/\pi^2 \) in Case H. Later we describe the Tamagawa measure, the metaplectic group and Weil representation in more details.
§7. Conclusion in Case I; §5.3. Decomposing \( v = a \oplus v_0 \) in 
\( D_\sigma = Z \oplus D_0 \), we have for \( k = 2 \)

\[
[a + v_0; z, \bar{z}]^k = ([a; z, \bar{z}] + [v_0; z, \bar{z}])^k = \sum_{j=0}^{k} \binom{k}{j} (a(z - \bar{z}))^{k-j} [v_0; z, \bar{z}]^j.
\]

Thus we have \( \phi = \sum_{j=0}^{k} \binom{k}{j} \phi_{k-j}^Z \phi_j^{D_0} \) with infinity part \( \Psi_j^? \) of \( \phi_j^? \) given by

\[
\Psi_j^{D_0} := (z - \bar{z})^{-j} [v_0; z, \bar{z}]^j e(N(x)\tau + \frac{i \text{Im}(\tau) \| [v_0; z, \bar{z}] \|^2}{2 \text{Im}(z)^2}), \quad \Psi_j^{Z} := a^j e(a^2 \tau)
\]

with \( (w(b) \phi_j^{D_0})(0) = 0 \) and \( E(\phi_j^{D_0})|_{B(\mathbb{A})} = 0 \) unless \( j = 0 \), and we reach Rankin convolution of \( \theta(\phi_k^Z) = \sum_{n \in \mathbb{Z}} \psi(n) n^k e(n^2 z) \) and \( f \) over \( B(\mathbb{Q}) \backslash B(\mathbb{A}) / B(\widehat{\mathbb{Z}}) \cong [0, 1) \times \mathbb{R}_+^\times \), which produces (see [Sh75])

\[
\zeta(2) P = m 2^{-2k} \ast (2\pi)^{-k} \Gamma(k) L^{(s)}(1, \text{Ad}(\rho_f) \otimes \left( \frac{\Delta}{-} \right))
\]

with a simple constant \( \ast \). Here \( L^{(s)} \) means we remove Euler factors at \( p|C \) with either \( f|U(p) = 0 \) or \( \psi(p) = 0 \).
§8. Conclusion in Case H; §5.2. The choice of the Bruhat function $\phi$ is the same as in Case I. As a $\mathbb{C}$-valued function, set

$$\Psi(\tau; v, x) = e(N(v)\tau).$$

Again in exactly the same way, for

$$\theta^*(\phi)(f) : = \int_{X_0(M)} \theta(\phi)(\tau; g) f(\tau) \eta^{k-2} d\xi d\eta \quad (k = 2)$$

and $P = \int_S \theta^*(\phi)(f) d\mu$, we conclude for a simple constant $c'$

$$\zeta(2) P = 2m^* (2\pi)^{-k+1} \Gamma(k) L(s)(1, \text{Ad}(\rho_f) \otimes \left( \frac{\Delta}{-} \right)).$$

Writing the point set $S = \{x\}_{x \in Sh_R}$, $m(\hat{\Gamma}) = \sum_{x \in Sh_R} e_x^{-1} \div \zeta(2)$ for $e_x = |\hat{\Gamma} \cap O_{D_0}(\mathbb{Q})|$ and $P \div \sum_{x \in Sh_R} e_x^{-1} \theta^*(\phi)(f)(x)$.

Thus the period formula is an adjoint analogue of the mass formula of Siegel–Shimura. The determination of $m(\hat{\Gamma})$ was finished by Shimura in 1999 for an arbitrary quadratic space over a totally real field (see §5.2.8 for the explicit formula for the mass).
§9. Schwartz–Bruhat functions; §3.1.3. For a $\mathbb{Q}$-vector space $V$, write $V_p := V \otimes \mathbb{Z} \mathbb{Z}_p$ (which is a vector space over a local field $\mathbb{Q}_p$). A Bruhat function on $V_p$ is a locally constant compactly supported function with values in $\mathbb{C}$. Write $S(V_p)$ for the space of Bruhat functions on $V_p$. For a real vector space $V_\infty$, we define $S(V_\infty)$ to be the Schwartz space of functions on $V_\infty$. Thus $S(V_\infty)$ is made of $C^\infty$–class functions with all derivatives rapidly decreasing as Euclidean norm of $v \in V_\infty$ grows. In other words, $\phi \in S(V_\infty)$ if and only if $\phi$ is of $C^\infty$–class and for any polynomial $P(v)$ and any $m$–th derivative $\Phi$ of $\phi$, $|P(v)\Phi(v)|$ goes to 0 as $|x| \to \infty$. Writing $V_\mathbb{A}$ for the adelization. We pick a lattice $L$ of $V$ and put $\hat{L} = \prod_p L_p \subset V_\mathbb{A}(\infty)$ with $L_p = L \otimes \mathbb{Z} \mathbb{Z}_p$. A Schwartz-Bruhat function on $V_\mathbb{A}$ is a finite linear combination of the function of the form $\phi(x) = \prod_v \phi_v(x_v)$ with $\phi_v \in S(V_v)$ and $\phi_p$ is the characteristic function of $L_p$ for almost all $p$. 
§10. Weil representation. Let $\nu(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, $\text{diag}[a, b] = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $e : \mathbb{A}/\mathbb{Q} \to S^1$ be the additive character with $e(x_\infty) := \exp(2\pi \sqrt{-1} x_\infty)$. We put $e_v := e|_{\mathbb{Q}_v}$ and for a number field $F$, we write $e_F = e \circ \text{Tr}_{F/\mathbb{Q}}$ and $e_{Fv} = e_v \circ \text{Tr}_{Fv/\mathbb{Q}_v}$. Let $Q : V \to F$ be a non-degenerate quadratic form with symmetric bilinear form $s(v, w) = Q(v + w) - Q(v) - Q(w)$ over a number field $F$. We have the following operator $r(g) \in \text{Aut}(S(V_?))$ for $m = \dim_F V$, $? = v, \mathbb{A}$ with $u \in F_v$ or $F_\mathbb{A}$ and $a \in F_v^\times$:

$$r(\nu(u)) = e_F(uQ(v))\phi(v), \quad r(\text{diag}[a, a^{-1}]\phi(v) = |a|^{m/2}_{F_\mathbb{A}} \phi(au)$$

and

$$r(J)\phi(v) = \hat{\phi}(-v) := \int_{V_?} e_F(s(w, -v))\phi(w)dw$$

(Fourier transform),

where $dw$ is normalized so that $\hat{\phi}(v) = \phi(-v)$. If $b, b' \in B(F_\mathbb{A})$ (upper triangular Borel subgroup), we extend $r$ to $\Omega = B(F_\mathbb{A})JB(F_\mathbb{A})$ by $r(bJb') := r(b)r(J)r(b')$. Then if $g, h \in \text{SL}_2(F_?)$ either unipotent, diagonal or $J$, $r(gh) = \kappa(g, h)r(g)r(h)$ for a 2-cocycle $\kappa$ on $\text{SL}_2$ with values in $S^1$. Write $\text{Mp}(F_?) \subset \text{Aut}(S(V_?))$ for the group generated by these operators. We have an extension $$(*) \quad 1 \to S^1 \to \text{Mp}(F_?) \xrightarrow{\pi_?} \text{SL}_2(F_?) \to 1$$ with $\text{Mp} \ni w(g) \mapsto g \in \text{SL}_2$.

Therefore, the group $\text{Mp}$ acts on $S(V_?)$ by a representation $w$. 
§11. Weil’s theta series. The extension (*) is split in the following cases:

1. \( \dim_F V \) is even (non-canonically but the section can be made to coincide with \( b \mapsto w(b) \) over \( B(F_A) \));
2. \( b \mapsto r(b) \) over \( B(F_A) \);
3. Over \( \tilde{\Gamma}_0(4) \) (canonical);
4. Over \( \text{SL}_2(F) \) (canonical and coincides with \( b \mapsto r(b) \) over \( B(F) \)).

For the orthogonal group \( O_V \) for \( V \) and \( \phi \in S(V_A) \), we define a function on \( \text{Mp}(F_A) \times O_V(F_A) \) by \( \theta(\phi)(g, h) = \sum_{\alpha \in V} w(g)L(h)\phi(\alpha) \), where \( (L(h)\phi)(v) = \phi(vh) \) (as usual, \( O(F_A) \) acts on \( V_{F_A} \) from the right). Weil showed that \( \theta(\phi)(g, h) \) is real analytic on \( \text{Mp}(F_{\infty}) \times O_V(F_{\infty}) \), left invariant under \( \text{SL}_2(F) \times O_V(F) \) and right invariant under an open subgroup of \( \text{Mp}(F_{A(\infty)}) \times O_V(F_{A(\infty)}) \); in short, an automorphic form on \( \text{Mp} \times O_V \).

All the details are in Chapter 4.