Tate conjecture for quaternionic Shimura varieties

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Lecture 13: We list our results towards the Tate conjecture and give a sketch of a proof. Pick an indefinite quaternion algebra B over a totally real field E.

§**0. Introduction.** We prove partially the Tate conjecture for $2r$-dimensional Shimura varieties Sh_B for a quaternion algebra $B_{/E}$ over a totally real field E under some restrictive conditions. Decompose

$$H^{2e}_{cusp}(Sh_B, \overline{\mathbb{Q}}_l) = \bigoplus_{\pi \in \mathcal{A}_B}(\Pi_\pi \otimes \pi^{(\infty)})$$

for $e \leq 2r$, Let X be a finite real extension of E and π is a cuspidal automorphic representation of $B^\times_{\mathbb{A}}$ contributing Tate classes. Pick a member ρ in the 2-dimensional compatible system ρ_π of π and define ${}^\nu\rho(g) := \rho(\nu^{-1}g\nu)$ for $\nu, g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Here is a list of our results a bit oversimplified (i.e., some assumptions are not stated here):

(1) Take $D_{/F}$ with $\dim Sh_D = 1$ and put $B = D^r = D \otimes_F E$ with $E = F^r$. Then $Sh_B = Sh^r_D$ and $\mathcal{A}_B = \overbrace{\mathcal{A}_D \otimes \cdots \otimes \mathcal{A}_D}^{r}$, and the Tate space $H^0(X, \Pi_\pi(e))$ in the factor $\Pi_\pi(e)$ for $H^{2e}_{cusp}(Sh^r_B, \overline{\mathbb{Q}}_l(e))$ with $0 \leq e \leq r$ is generated by the image of Shimura sub-varieties in Sh^r_B of codimension e (those are essentially partial Hecke correspondences) as long as π does **not** have an automorphic induction factor.

§**1. List for low dimesional cases.**

(2) If Sh_B is a Shimura surface and $B = D \otimes_F E$ for a subfield $F \subset E$ with $[E : F] = 2$, $H^0(X, H^2_{cusp}(Sh_B, \overline{\mathbb{Q}}_l(1)))$ is spanned by Shimura sub-curves. This is already done in Lecture 11.

(3) For $Sh_B \times Sh_B$ for Sh_B as in (2), $H^0(X, \Pi_{\pi\otimes\pi'}(e))$ with $\pi, \pi' \in \mathcal{A}_B$ is generated by Shimura sub-varieties in $Sh_B \times Sh_B$ of appropriate codimension as long as π and π' are **not** automorphic induction from a CM extension (non-CM).

(4) Assume that E is Galois over the reflex field $\mathcal{E}$ of Sh_B and $\mathrm{Gal}(E/\mathcal{E}) \cong \{\pm 1\}^2$ acting transitively on I_B. If Sh_B is a Shimura fourfold such that $B = D \otimes_{\mathcal{E}} E$ for a quaternion algebra $D_{/\mathcal{E}}$ producing a Shimura curve Sh_D, $H^0(X, \Pi_\pi(2)) \subset H^4(Sh_B, \overline{\mathbb{Q}}_l(2))$ is generated by Shimura sub-varieties in Sh_B of codimension 2 as long as π is non-CM.

§**2. General case.** (5) Suppose $2r = \dim Sh_B \geq 4$ and $E/\mathcal{E}$ for the reflex field $\mathcal{E}$ is a Galois extension with $\mathrm{Gal}(E/\mathcal{E})$ acting transitively on I_B. By transitivity of the Galois action on I_B, replacing B by a Galois conjugate over $\mathcal{E}$, we may assume that the identity embedding ι is in I_B. Then $H^0(X, \Pi_\pi(r)))$ is non-trivial for an appropriate extension $X_{/E}$ if and only if the partition $I_B = \bigsqcup_i J_i$ given by $\nu, \mu \in J_i \Leftrightarrow {}^\nu\rho_\pi \cong {}^\mu\rho_\pi$ over X satisfies the condition that $|J_i|$ is even. In this case, $|J_i|$ is independent of i. Write J_1 for the part containing ι. Assuming further $|J_1| = 2$ (a genericity condition), there exists a unique subfield $F \subset E$ such that $J_1 = \mathrm{Gal}(E/F)$, and if $B \cong D \otimes_F E$ for a quaternion F-subalgebra D of B, $H^0(X, \Pi_\pi(r))$ is generated by Shimura subvarieties $Sh_{D'}$ in Sh_B of codimension r, where D' runs over quaternion F-subalgebras of B. This contains the assertion (2) as a particular case.

§3. **Some definitions.**

Definition 1. *For each $\pi \in \mathcal{A}_B$, we consider the collection $\mathfrak{F}_\pi$ of the following subfields F of E satisfying*
(1) $[E:F]=2$; (2) Writing σ_F for the generator of $\mathrm{Gal}(E/F)$, $I_B\sigma_F = I_B$; (3) $JL_E(\pi\otimes\xi)$ for a suitable finite order Hecke character ξ of E is a base change from $\mathcal{A}_{M_2(F)}$ (so, $\pi\otimes\xi\in\mathcal{A}_B^{\sigma_F}$). The set $\mathfrak{F}_\pi$ can be empty. Put $\mathfrak{F}=\mathfrak{F}_B=\bigcup_{\pi\in\mathcal{A}_B}\mathfrak{F}_\pi$.

Definition 2. *For each $F\in\mathfrak{F}$, write $\mathfrak{Q}_F$ for the collection of quaternion subalgebras of B over F. Then we put $\mathfrak{Q}=\bigcup_{F\in\mathfrak{F}}\mathfrak{Q}_F$.*

Definition 3. *Choose $\Phi\subset I_B$ with $I_B=\Phi\sqcup\Phi\sigma_F$, and define $\epsilon_\Phi:\{\pm1\}^{I_B}\to\{\pm\}$ by $\epsilon_\Phi(\epsilon_\nu)=1\Leftrightarrow\nu\in\Phi$. For $\mathcal{F}\in S_2^{\epsilon_\Phi}(U)$, put $\omega_\Phi(\mathcal{F})=\mathcal{F}\bigwedge_{\nu\in\Phi}(dz_\nu\wedge d\overline{z}_{\nu\sigma_F})$ and $H^\Phi(Sh_B,\mathbb{C})$ to be the span of classes of $\omega_\Phi(\mathcal{F})$ and $H^{\sigma_F}(Sh_B,\mathbb{C})=\sum_\Phi H^\Phi(Sh_B,\mathbb{C})$ for U running over open compact subgroups of $B^\times_{\mathbb{A}^{(\infty)}}$.*

The action of $\pi^{(\infty)}$ (Hecke action) is already incorporated in the decomposition in §0. The only remaining automorphic action comes from π_∞ as we will see in the next section.

§**4. Archimedean Hecke operators.** Write C_∞ for the stabilizer of $\mathbf{i}_B = (\sqrt{-1}, \dots, \sqrt{-1}) \in \mathfrak{Z}_B$ in the norm 1 group B^1_∞, and $\mathfrak{Z}_B = B^\times_{\infty+}/E^\times_\infty C_\infty$. We then have a unique positive majorant P_0 of s such that $C_\infty = \mathrm{SO}_{P_0}(F_\infty)$. Let $\widetilde{C}_\infty = \mathrm{O}_{P_0}(\mathbb{R})$. An element $\epsilon \in \widetilde{C}_\infty$ acts on $\mathfrak{Z}_B$ by $z(E^\times_\infty C_\infty) \mapsto z\epsilon(E^\times_\infty C_\infty)$. This action factors through $\widetilde{C}_\infty/C_\infty \cong \{\pm 1\}^{I_B}$. We identify $\widetilde{C}_\infty/C_\infty$ with set of functions $\epsilon : I_B \to \{\pm\}$ as above, and for an automorphic form $\mathcal{F}$ on $B^\times_{\mathbb{A}}/B^\times$, we write $\mathcal{F}|T_\epsilon(h) = \mathcal{F}(h\epsilon)$. For a choice of Φ with $I_B = \Phi \sqcup \Phi\sigma_F$, we define a character $\mathrm{sgn}^\Phi_F : \{\pm 1\}^{I_D} \to \{\pm 1\}$ by $\mathrm{sgn}^\Phi_F(\epsilon) = \prod_{\nu\in\Phi} \epsilon_\nu$. Then sgn^Φ_F is independent of the lift Φ of I_D, and we write sgn_F for sgn^Φ_F for a choice of Φ and regard it as a character of $\{\pm 1\}^{I_D} \cong \mathrm{Inf}^E_F\{\pm 1\}^{I_D}$.

Lemma 1. *Let the notation as above. In $H^{2r}(Sh_B, \mathbb{C})$, T_ϵ for $\epsilon \in \mathrm{Inf}^E_F\{\pm 1\}^{I_D}$ acts on the fundamental class $[Sh_D]$ ($D \in \mathfrak{Q}_F$) by a character sgn_F; i.e., $T_\epsilon([Sh_D]) = \mathrm{sgn}_F(\epsilon)[Sh_D]$.*

§5. Tate cycles coming from F.

Proposition 1. *The image* $\Pi_\pi[\mathrm{sgn}_F]$ *of* $H^{\sigma_F}(Sh_B, \mathbb{C}))[\mathrm{sgn}_F]$ *in* Π_π *has dimension* 1*, and hence the projection to* $\Pi_\pi \otimes \pi^{(\infty)}$ *of the subspace of* $H^0(X, H^{2r}(Sh_B, \overline{\mathbb{Q}}_l(r)))$ *spanned by the fundamental classes of* Sh_D *has dimension at most* 1*, where* D *runs over* $\mathfrak{Q}_F$*.*

Assuming (Sp) in Lecture 10 for a fixed $F \in \mathfrak{F}$. Then if $\pi \in \mathcal{A}_B^{\sigma_F}$, we know that by the descent theorem combined with the L-value formula, we find $\Pi_\pi[\mathrm{sgn}_F] \neq 0$ containing the non-zero image of Sh_D for $D \in \mathfrak{Q}_F$. Thus

Corollary 1. *Assume* (Sp) *for* $F \in \mathfrak{F}_B$ *and* $\mathfrak{Q}_F \neq \emptyset$*. Then* $\Pi_\pi[\mathrm{sgn}_F] \neq 0$ *for* $\pi \in \mathcal{A}_B^{\sigma_F}$ *and is spanned by the image of* $[Sh_D]$ *for an appropriate* $D \in \mathfrak{Q}_F$*.*

§**6. Group theory.** We start giving a sketch of a proof of the general case (5).

We take the PE-partition $I_B = \bigsqcup_i J_i$ associated to ρ, and write $\overline{H}'_i$ for the stabilizer $\overline{H}'_i = \{h \in \overline{H}' | hJ_i = J_i\}$ with Gal$(E/\mathcal{E}_i) = \overline{H}'_i$. Pick a representative $\nu_i \in J_i$, and put $H_i := \nu_i H \nu_i^{-1}$, $H'_i :=$ Gal$(\overline{\mathbb{Q}}/\mathcal{E}_i)$ and $N_i := \bigcap_{\nu \in J_i} \nu H \nu^{-1}$. By the key lemma in the last lecture, we may assume that $|J_i| = 2e_i$ with $e_i \in \mathbb{Z}$ for all i. Just by a group theory, we can prove

Lemma 2. *Suppose* (B) *with* $|J_i| = 2e_i$ *and* $\overline{H}' =$ Gal$(E/\mathcal{E})$ *acts on* J *transitively. Then*
(1) $\overline{H}_i = \nu_i \overline{H}'_1 \nu_i^{-1}$ *acts on* J_i *transitively,*
(2) e_i *is independent of* i *and* $J_i = \nu_i J_1$ *for some* $\nu_i \in$ Gal$(E/\mathcal{E})$.

§**7. Theorem** Let X be a sufficiently large finite extension of $\mathcal{E}$ with the following property:
(nc) $X_{/\mathcal{E}}$ *has real* X^{cm} *if* π *is an automorphic induction.*
Assume that $|J_i| = 2e_i$ (as other wise no Tate cycles in Π_π) and that E is a Galois extension of $\mathcal{E}$. Here is a detailed restatement of the result (5):
Theorem 1. *Let the notation and the assumptions as above. Let* $\pi \in \mathcal{A}_B$ *of weight* 2 *and* $\rho \in \rho_\pi$ *be an* l*-adic member. If* $\mathrm{Gal}(E/\mathcal{E})$ *acts transitively on* $J := I_B$ *with* $\iota \in J_1$ *and* $|J_1| = 2$*, then modifying* π *by a suitable character twist,* $\mathfrak{F}_\pi$ *is a singleton* $\{F\}$ *with* $J_1 = \{\iota, \iota\sigma_F\}$*,* $\pi \in \mathcal{A}_B^{\sigma_F}$ *and assuming* $\mathfrak{Q}_F \neq \emptyset$*,* $H^0(X, \Pi_\pi(r)))$ *is generated by the* $\pi^{(\infty)}$*-projection of the fundamental classes of* Sh_α *with* α *running over* $\{\alpha \in D_{\sigma_F} | N(\alpha) \gg 0\}$*.*

By Lecture 12, §3, $\dim H^0(X, \Pi_\pi(r)) \leq 1$. Then the theorem follows from Corollary 1 in §5.

§8. r-Product of copies of Shimura curves.

Theorem 2. *Suppose that Sh_D is a Shimura curve over $\mathcal{E}$ and $Sh_B = Sh_D^r$ for $B = D \otimes_{\mathcal{E}} E$ for $E = \overbrace{\mathcal{E} \times \cdots \times \mathcal{E}}^{r}$. Then the Tate space $H^0(X, \Pi_\pi(e))$ in the factor $\Pi_\pi(e)$ for $H^{2e}_{cusp}(Sh_B, \overline{\mathbb{Q}}_l(e))$ with $0 \leq e \leq r$ is generated by the image of Shimura sub-varieties in Sh_B of codimension e (those are essentially partial Hecke correspondences) as long as π does* **not** *have an automorphic induction factor.*

The symmetric group $\mathfrak{S}_r = \mathfrak{S}_{I_B}$ acts on $Sh_B = Sh_D^{I_B}$ permuting factors Sh_D. Basically in the same way as in §4 of Lecture 12, for each $\pi = \otimes_{\nu \in I_B} \pi_\nu$ for $\pi_\nu \in \mathcal{A}_B$, the corresponding Galois representation is $\otimes_{\nu \in I_B} \rho_\nu$ for $\rho_\nu = \rho_{\pi_\nu}$. Decomposing $I_B = \bigsqcup_i J_i$ so that $\mu, \nu \in J_i \Leftrightarrow P\rho_\nu \cong P\rho_\mu$. Since $\otimes_{\nu \in J_i} \rho_\nu$ with the permutation action of $\mathfrak{S}_{J_i}$, assuming $|J_i| = 2e_i$ for all i, we find that 1-dimensional factor of $\otimes_{\nu \in I_B} \rho_\nu$ corresponds to $\bigotimes_i W_{e_i, e_i}$ and hence $H^0(X, \Pi_\pi(r))$ is irreducible under the action of $\prod_i \mathfrak{S}_{J_i}$. This essentially shows the result as it contains non-trivial image of a Shimura subvariety.

§9. Open questions.

(1) Remove the assumption that $E_{/\mathcal{E}}$ is Galois acting I_B transitively.

By scrutinizing my proof, perhaps one can determine the dimension of the subspace of Π_π spanned by Shimura subvarieties. In lucky cases, this shows the motivic Tate conjecture.

(2) Remove the assumption of non-automorphic induction.

One need to introduce perhaps new type of subvarieties. If $Sh_B = Sh_D^r$ with $\dim Sh_D = 1$, perhaps twisting operators provide them.

(3) Try to treat higher weight modular forms.

If B is totally indefinite (i.e., it is the case also for D if $B = D \otimes_F E$), then Sh_B carries a universal abelian variety with B-multiplication. Then try Kuga-Sato varieties associated to it.

If B is not totally indefinite, going up to a CM quadratic field $M_{/E}$, Sh_B gives components of a unitary Shimura variety, which carries the universal abelian variety of unitary type (which depends on M). We have L-value formula and Descent theorem in this case.

§10. More difficult open questions.

(4) Try prove the L-value formula and descent Fourier expansion theorem for Shimura variety associated to $\mathrm{O}(2n, 2)$.

This is perhaps doable, though computationally demanding.

(5) The case $\mathrm{O}(2n+1, 2)$ is more difficult, as the standard L-value at the central value could shows up as in the case $\mathrm{O}(2, 1)$ done by Waldspurger. However Siegel modular variety can be included as $\mathrm{O}(3, 2) \cong \mathrm{Sp}(4)$.

(6) Extend the question (4) to unitary Shimura varieties of $U(m, n)$.

First try $U(n, 1)$ which even have nontrivial H^1 but perhaps the easiest case.

(7) Try find good subvarieties which span the Tate space on Sh_B when B is a quaternion algebra which does not have any quaternion subalgebras.

Absolutely a new idea necessary.