Space of Tate cycles on Shimura varieties

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Lecture 12: We determine the dimension of $H^0(X, \Pi_\pi(r))$ for each $\pi \in \mathcal{A}_B$ contributing to Tate cycles, assuming that $E/\mathcal{E}$ is Galois and $I_B = \mathrm{Gal}(E/\mathcal{E})$. Some other cases are also treated.

§0. Serre's theory of l-adic Lie algebras; [LLG]. Let K be an l-adic field. A profinite K-analytic group chunk of dimension n is an analytic profinite space G together with a neutral element $e \in G$, and an analytic open neighborhood $U \hookrightarrow O_K^n$ of e in G and a pair of analytic maps $U \times U \to G; (u, v) \mapsto u \cdot v$ and $U \to U; u \mapsto u^{-1}$ such that

(i) for a neighborhood $V_1 \subset U$ of e, $x \in V_1 \Rightarrow x = x \cdot e = e \cdot x$;

(ii) for a neighborhood V_2 of e in U, $x \in V_2 \Rightarrow e = x \cdot x^{-1} = x^{-1} \cdot x$;

(iii) for a neighborhood V_3 of e in U, $V_3 \cdot V_3 \subset U$ and for all $x, y, z \in V_3$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

By analyticity, $x \cdot y = F(x, y)$ for a power series $F \in K[[X, Y]]$ for $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ and $x' = I(X)$ for $I(X) \in K[[X]]$. Write

$$F(X, Y) = X + Y + B(X, Y) + \text{ terms of homogeneous degree} \geq 3,$$

and define $[X, Y] := B(X, Y) - B(Y, X)$. Then the bracket $[\cdot, \cdot]$ gives rise to a Lie algebra structure on K^n. We write $L^0(G)$ for this Lie K-algebra.

§**1. Lie structure.** If G is a profinite $\mathbb{Z}_l$-analytic subgroup of $\mathrm{SL}_2(K)$, $\mathrm{GL}_2(K)$ or $\mathrm{GL}_2(K)$, we define $L(G)$ to be the K-span of $L^0(G)$; so, $L(G)$ is a Lie K-algebra even if G is just an l-adic analytic subgroup.

By [LLG, Part II, V.4, V.7],
Theorem 1 (Serre). *If $\mathfrak{g}$ is a finite dimensional Lie K-algebra, there exists a K-analytic group chunk G such that $L^0(G) = \mathfrak{g}$. If G_1 and G_2 are K-analytic group chunks with $L^0(G_1) \cong L^0(G_2)$, then G_1 and G_2 have isomorphic open subgroups.*

The Lie algebra $\mathfrak{sl}_2(K)$ has four types of non-trivial K-subalgebras: 1. $\mathfrak{sl}_2(K)$; 2. Normalizer of $K^\times \subset \mathrm{GL}_2(K)$ of a semi-simple quadratic extension of K (Cartan subalgebra); 3. Conjugate of the upper triangular subalgebra (Borel subalgebra); 4. Conjugate of the upper nilpotent subalgebra (nilpotent subalgebra).
Corollary 1. *If $L(G) = \mathfrak{sl}_2(K)$, G contains an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_l)$ up to conjugation in $\mathrm{SL}_2(K)$.*

§2. **Image of modular Galois representations.** Let ρ_π be the 2-dim compatible system of cuspidal π with coefficients in its Hecke field T.

Proposition 1. *If $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}/M_0)}$ is reducible over a finite extension M_0 of E for an $\mathfrak{l}$-adic member $\rho \in \rho_\pi$, $\rho \cong \mathrm{Ind}_M^E \eta_{\mathfrak{l}}$ for a member $\eta_{\mathfrak{l}}$ of a compatible system $\widehat{\eta}$ of characters associated to an arithmetic Hecke character η of a CM quadratic extension $M_{/E}$ inside M_0.*

This follows from Frobenius reciprocity.

Proposition 2. *If $JL_E(\pi)$ for $\pi \in \mathcal{A}_B$ is not an automorphic induction from a CM quadratic extension of E, then for any $\mathfrak{l}$-adic member $\rho = \rho_{\mathfrak{l}}$ of ρ_π, $\mathrm{Im}(\rho)$ contains an inner conjugate of an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_l)$ for the residual characteristic l of $\mathfrak{l}$ (i.e., $L(\mathrm{Im}(P\rho)) = \mathfrak{sl}_2(T_{\mathfrak{l}})$). Moreover the adelic image $\mathrm{Im}(\rho_\pi) := \mathrm{Im}(\prod_{\mathfrak{l}} \rho_{\mathfrak{l}}) \subset \mathrm{GL}_2(T_{\mathbb{A}}^{(\infty)})$ contains an adelic conjugate of an open subgroup of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$.*

§**3. Proof of Proposition 2.** Let $G := \mathrm{Im}(\rho)$. Then, by Proposition 1, its Lie algebra $L(G)$ cannot be a Cartan subalgebra as it is not an automorphic induction. It is neither a Borel subalgebra nor nilpotent as ρ is irreducible. Thus $L(G)$ is $\mathfrak{sl}_2(K)$ for $K = T_{\mathfrak{l}}/\mathbb{Q}_l$. In particular, $L^0(G)$ contains $\mathfrak{sl}_2(\mathbb{Z}_l)$ up to conjugate and therefore an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_l)$ up to conjugation by Corollary 1.

Write $\overline{\rho}_{\mathfrak{l}} := \rho_{\mathfrak{l}} \bmod \mathfrak{l}$. By Dimitrov [D05, Proposition 0.1], except for finitely many $\mathfrak{l}$s, $\mathrm{Im}(\overline{\rho}_{\mathfrak{l}}) \supset \mathrm{SL}_2(\mathbb{F}_l)$. This is sufficient to know that $\mathrm{SL}_2(\mathbb{Z}_l) \subset G$ up to conjugation. $\square$

Write $P : \mathrm{GL}(2) \twoheadrightarrow \mathrm{PGL}(2)$ for the projection. Let $\rho \in \rho_\pi$ be an $\mathfrak{l}$-adic member for a compatible system ρ_π for $\pi \in \mathcal{A}_B$. We define an equivalence relation on I_B so that $\nu \sim \mu \Leftrightarrow P^\mu\rho \cong P^\nu\rho$. Decompose $I_B = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_r$ for equivalence classes J_i.

§**4. Projective equivalence.** For a finite extension $X_{/\widetilde{E}}$, suppose the following condition if π is an automorphic induction:
(nc) $X_{/\widetilde{E}}$ *has real* X^{cm} *if* π *is an automorphic induction.*
Proposition 3. (i) *If there exists* J_i *such that* $|J_i|$ *is odd, then* $\Pi_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/X)}$ *does not have one dimensional constituent for a finite extension* $X_{/\mathcal{E}}$ *as above.*
(ii) *Suppose that every* $|J_i| = 2e_i$ *is even. If* π *is not an automorphic induction, the number of one dimensional factors appearing in* $\Pi^{ss}_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/X)}$ *counting with multiplicity is given by* $\prod_{i=1}^{e}\left(\binom{2e_i}{e_i} - \binom{2e_i}{e_i-1}\right)$.
(iii) *If* π *is an automorphic induction, the number of characters with value* -1 *at complex conjugation* c *is given by* $\prod_{i=1}^{e}\left(\binom{2e_i}{e_i}/2\right)$.

To prove the proposition, we need Goursat's lemma which is valid for groups and Lie algebras. We state it for groups.

§**5. Goursat's lemma.** Here is the original Goursat's lemma:

Lemma 1. *Let B_1 and B_2 be groups and suppose that A is a subgroup of $B_1 \times B_2$ for which the two projections $\pi_j : A \to B_j$ are surjective. Then the image of A in $B_1/\operatorname{Ker}(\pi_2) \times B_2/\operatorname{Ker}(\pi_1)$ is the graph of an isomorphism $B_1/\operatorname{Ker}(\pi_2) \cong B_2/\operatorname{Ker}(\pi_1)$.*

Proof. Let $\overline{A}$ be the image of A in $B_1/\operatorname{Ker}(\pi_2) \times B_2/\operatorname{Ker}(\pi_1)$. By symmetry it is enough to show that the projection $\overline{A} \to B_1/\operatorname{Ker}(\pi_2)$ is an isomorphism. By hypothesis, the projection is surjective. To show that it is injective is to show that the two maps $A \to \overline{A} \to B_1/\operatorname{Ker}(\pi_2)$ and $A \to \overline{A}$ have the same kernel. The kernels of the two maps are respectively $\{(u, u') \in A | u \in \operatorname{Ker}(\pi_2)\}$ and $\{(u, u') \in A | u \in \operatorname{Ker}(\pi_2), u' \in \operatorname{Ker}(\pi_1)\}$. But if $(u, u') \in A$ and $u \in \operatorname{Ker}(\pi_2)$, we know that $(u, 1) \in A$. Thus $(1, u') \in A$, so $u' \in \operatorname{Ker}(\pi_1)$. In other words, $(u, u') \in \operatorname{Ker}(\pi_2) \times \operatorname{Ker}(\pi_1)$, which shows that the two kernels are equal. $\square$

§6. Multiple Goursat's lemma.

Lemma 2 (K. Ribet). *Let $S_1, \dots, S_n$ be profinite groups with no non-trivial abelian quotients and N a closed normal subgroup of $S_1 \times \cdots \times S_n$ such that the projections $N \to S_i$ $(1 \le i \le n)$ are surjective. Then $N = S_1 \times \cdots \times S_n$.*

Proof. We argue inductively. By induction, the two projections $N \to S_n = B_2$ and $N \to S_1 \times \cdots \times S_{n-1} = B_1$ are surjective. Let K and K' be their kernels, respectively, so that $K' \subset S_n = B_2$, $K \subset S_1 \times \cdots \times S_{n-1}$. Then by Lemma 1, the image $\overline{N}$ of N in $B_1/K \times B_2/K'$ is the graph of an isomorphism $B_1/K \cong B_2/K'$. Since $\overline{N}$ is normal in the product, B_2/K' must be abelian. By hypothesis $K' = B_2$, and thus $N = B_1 \times B_2 = S_1 \times \cdots \times S_n$. $\square$

§7. A corollary.

Corollary 2. *Let $S_1, \ldots, S_n$ be profinite groups with finite abelian quotients and N a closed normal subgroup of $S_1 \times \cdots \times S_n$ such that the projections $N \to S_i$ $(1 \leq i \leq n)$ have finite cokernel. Then N is an open subgroup in $S_1 \times \cdots \times S_n$.*

Proof. Again we argue inductively. The projection $p_2 : N \to S_n$ has finite cokernel. By induction, the projection $p_1 : N \to S_1 \times \cdots \times S_{n-1}$ has finite cokernel. Let $B_j := \mathrm{Im}(p_j)$ for $j = 1, 2$. Let K and K' be their kernels, respectively, so that $K' \subset B_2$, $K \subset B_1$. Then by Lemma 1, the image $\overline{N}$ of N in $B_1/K \times B_2/K'$ is the graph of an isomorphism $B_1/K \cong B_2/K'$. Since $\overline{N}$ is normal in the product, B_2/K' must be abelian. By hypothesis B_2/K' and B_1/K are finite, and thus N is open in $B_1 \times B_2$, which is in turn open in $S_1 \times \cdots \times S_n$. Thus N is open in $S_1 \times \cdots \times S_n$. $\square$

§**8. Representations of** PGL(2) **and** $\mathfrak{S}_m$**.** Let $\mathcal{V}$ be two dimensional vector space over a characteristic 0 field K on which $\mathrm{GL}(\mathcal{V}) \cong \mathrm{GL}_2(K)$ act by the identity representation r. Consider the tensor representation $r^{\otimes J} = \overbrace{r \otimes \cdots \otimes r}^{J}$ acting on $\overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{J}$ for $J := \{1, 2, \ldots, m\}$. Plainly the action of $r^{\otimes J}$ and the permutation action of $s \in \mathfrak{S}_J := (v_1 \otimes \cdots \otimes v_m) \mapsto (v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(m)})$ commutes each other. As seen in [RTF, §4.1] from a result of Weyl and Schur, each absolutely irreducible representation $W_{(e_1,\ldots,e_k)}$ of $\mathfrak{S}_J$ is rational and has a canonical one-to-one correspondence to a Young diagram, which is a decreasing integer partition $m = e_1 + e_2 + \cdots + e_k$ $(e_1 \geq e_2 \geq \cdots \geq e_k > 0)$. In $\mathcal{V}^{\otimes J}$, since $\dim \mathcal{V} = 2$, $W_{(e_1,e_2)}$ only for 2-part partitions (e_1, e_2) including $(m, 0)$ shows up [RTF, Theorem 6.3 (1)]. Each $W_{(e_1,e_2)}$ factor is a sum of a unique irreducible representation ρ_{e_1,e_2} of GL(2). Then $\dim \rho_{e_1,e_2} = 1 \Leftrightarrow e_1 = e_2 = e$ (i.e., $m = 2e$), and

$$\dim W_{(e,e)} = \binom{2e}{e} - \binom{2e}{e-1}.$$

§**9. Key lemma.** Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $H = \mathrm{Gal}(\overline{\mathbb{Q}}/E)$, $H' = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{E})$ and $N = \mathrm{Gal}(\overline{\mathbb{Q}}/\widetilde{E})$. Here $\widetilde{E}$ is the field fixed by the stabilizer of every place in I_B. Write $\tau := \bigotimes_{\nu \in I_B} {}^{\nu}\rho$ and $\tau_i = \bigotimes_{\nu \in J_i} {}^{\nu}\rho_i$. For a class J_i with index i, picking $\nu_i H \in J_i$, we write ρ_i for ${}^{\nu_i}\rho$ as a representation of $H_i := \nu_i H \nu_i^{-1}$.

Lemma 3. *Let the notation be as above. Assume that $\rho_i|_{N_i}$ is absolutely irreducible and that $H'_i = \{g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) | gJ_i \subset J_i\}$ acts transitively on J_i.*

(1) *If $L(\mathrm{Im}(P\rho_i)) = \mathfrak{sl}_2(K)$ for $i = 1, 2, \ldots, n$, $\mathrm{Im}(\tau_i|_N)$ contains an open subgroup S_i of $\mathrm{SL}_2(\mathbb{Z}_p)$ for each i and hence $\mathrm{Im}(\tau|_N)$ contains $\prod_{i=1}^{n} S_i$ accordingly.*

(2) *If $|J_i|$ is odd for at least one index i and $L(\mathrm{Im}(\rho)) = \mathfrak{sl}_2(K)$, the representation $\tau|_{N'}$ does not have one dimensional constituent for all open subgroups N' of N.*

§**10. Proof.** Since $P\tau_i|_N \cong \overbrace{P\rho_i \otimes \cdots \otimes P\rho_i}^{|J_i|}$ for a character χ, we can apply the theory of Schur-Weyl in §8 as Im(ρ_i) contains an open subgroup of SL$_2(\mathbb{Z}_l)$.

Since $P\tau|_N = (P\tau_1 \otimes \cdots \otimes P\tau_e)|_N$, by Ribet's Corollary 2 applied to Im$(\tau) \subset S_1 \times \cdots \times S_e$ with $S_i :=$ Im(τ_i), Im$(\tau) \supset S$ for an open subgroup S of $S_1 \times \cdots \times S_e$.

We now prove the assertion (2). Since $\rho_i(N)$ contains an open subgroup S_i of SL$_2(\mathbb{Z}_l)$, we regard τ_i as a representation of S_i. Then as seen in §8, τ_i does not contain any one dimensional representation as a constituent. Since $\tau|_S = \tau_1|_{S_1} \otimes \cdots \otimes \tau_n|_{S_n}$ for $S = S_1 \times \cdots \times S_n$ and $\tau(N') \cap S$ is open in S, the representation $\tau|_{N'}$ itself cannot contain one dimensional constituent if $|J_i|$ is odd.