Galois action

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Lecture 11: For a quaternion algebra $D_{/F}$ over a general totally real field F and its base change $B := D \otimes_F E$ to a totally real quadratic extension E/F, we state the period integral formula over the Shimura subvariety $Sh_{\mathbf{1}} = \varprojlim_U D^\times \backslash D^\times_{F_{\mathbb{A}}}/U$ of $Sh_B = \varprojlim_S B^\times \backslash B^\times_{\mathbb{A}}/S$ as the adjoint L-value $L(1, Ad(\rho_{\mathbf{f}}) \otimes \chi_E)$ and the descent Fourier expansion formula involving Sh_α. Then we make explicit the Galois action Π^{ss}_π in terms of ρ_π.

§**0. Notation.** Choose $0 \neq \Delta \in O_F$ so that $E = F[\sqrt{\Delta}]$. For each finite place $\mathfrak{l}$ of F ramified in E, we choose $\Delta_{\mathfrak{l}}^- \in O_{F_{\mathfrak{l}}}$ such that $\sqrt{\Delta_{\mathfrak{l}}^-}$ is a uniformizer of $O_{E_{\mathfrak{l}}}$. If $\mathfrak{l}$ is odd inert unramified in $E_{\mathfrak{l}}$, we choose a unit $\Delta_{\mathfrak{l}}^- \in O_{F_{\mathfrak{l}}}^{\times}$ so that $O_{E_{\mathfrak{l}}} = O_{F_{\mathfrak{l}}}[\sqrt{\Delta_{\mathfrak{l}}^-}]$. If $\mathfrak{l}$ is even inert unramified in $E_{\mathfrak{l}}$, we choose a unit $\Delta_{\mathfrak{l}}^- \in O_{F_{\mathfrak{l}}}^{\times}$ so that $E_{\mathfrak{l}} = F_{\mathfrak{l}}[\sqrt{\Delta_{\mathfrak{l}}^-}]$. If $E_{\mathfrak{l}} \cong F_{\mathfrak{l}} \oplus F_{\mathfrak{l}}$, we take a unit $\Delta_{\mathfrak{l}}^- := 1$ so that $O_{E_{\mathfrak{l}}} = O_{F_{\mathfrak{l}}} \oplus O_{F_{\mathfrak{l}}} = O_{F_{\mathfrak{l}}}[\sqrt{\Delta_{\mathfrak{l}}^-}]$ (this means $\sqrt{\Delta_{\mathfrak{l}}^-} = (1, -1) \in O_{F_{\mathfrak{l}}} \oplus O_{F_{\mathfrak{l}}}$). Write $\boldsymbol{\Delta}$ for the relative discriminant of E/F and $\boldsymbol{\Delta}^-$ for its square-free part. Then $(\Delta)/\boldsymbol{\Delta}^-$ is a square integral ideal. Define $\Delta_{\mathfrak{l}}^+ = \sqrt{\Delta_{\mathfrak{l}}^+} = 1$ and $\sqrt{\boldsymbol{\Delta}^-}$ to be the O_E-ideal such that $\sqrt{\boldsymbol{\Delta}^-}^2 = \boldsymbol{\Delta}^- O_E$. We then define $\sqrt{\boldsymbol{\Delta}_-} := \{\xi \in \sqrt{\boldsymbol{\Delta}^-} | \xi^{\sigma} = -\xi\}$ which is an O_F-projective module of rank 1. We put $\sqrt{\boldsymbol{\Delta}^+} := O_E$ and $\sqrt{\boldsymbol{\Delta}_+} := O_F$. For an adele or idele a in $F_{\mathbb{A}}$, the projection to $\prod_{\mathfrak{l}|\mathfrak{n}} F_{\mathfrak{l}}$ is written as $a_{\mathfrak{n}}$ for a formal product $\mathfrak{n}$ of places (including Archimedean places) which can be identified with an integral ideal if it consists of finite places.

§**1. Notation continues.** Let $I_X = \mathrm{Isom}_{\mathsf{field}}(X, \mathbb{C})$ for a number field X. Choose $\Sigma \subset I_E$ so that $I_E = \Sigma \sqcup \Sigma\sigma$. Define $I^D = \{\nu \in I_F | D \otimes_{F,\nu} \mathbb{R} \cong \mathbb{H}\}$, $I_D = \{\nu \in I_F | D \otimes_{F,\nu} \mathbb{R} \cong M_2(\mathbb{R})\}$, $J_{\mathbb{R}} = \{\nu \in \Sigma | D \otimes_{F,\nu} \mathbb{R} \cong \mathbb{H}\}$, $I_{\mathbb{R}} = \{\nu \in \Sigma | D \otimes_{F,\nu} \mathbb{R} \cong M_2(\mathbb{R})\}$. We lift $I_B = I_{\mathbb{R}} \sqcup I_{\mathbb{R}}\sigma$ and $I^B = J_{\mathbb{R}} \sqcup J_{\mathbb{R}}\sigma$. We identify the corresponding symmetric domains as described below:

$$\mathfrak{Z}_F = \prod_{\nu \in I_F} \mathfrak{H}, \mathfrak{Z}_D = \prod_{\nu \in I_D} \mathfrak{H}, \mathfrak{Z}_B = \prod_{\nu \in I_{\mathbb{R}}} \mathfrak{H}_\nu \times \mathfrak{H}_{\nu\sigma}.$$

Here $\mathfrak{H}_\nu = \mathfrak{H}$ for $\nu \in I_B$. The standard Schwartz functions in $\mathcal{S}(D_{\sigma,F_\nu})$ at the place ν is $\Psi_\nu(v_\nu; \tau_\nu) = \mathbf{e}_\nu(N(v_\nu)\tau_\nu)$ ($\mathbf{e}_\nu(x) = \exp(2\pi i x)$ with $x \in F_\nu$) if $D_\nu \cong \mathbb{H}$, and $\Psi_\nu(v_\nu; \tau_\nu, z_\nu, z_{\nu\sigma})$ if $D_\nu \cong M_2(\mathbb{R})$ as defined in §1 in Lecture 9, where $\tau_\nu = \xi_\nu + \eta_\nu\sqrt{-1} \in \mathfrak{H}_\nu$. Then $\phi = \phi^{(\infty)}\Psi \in \mathcal{S}(D_{\sigma,F_{\mathbb{A}}})$ with

$$\Psi(v; \tau, z) := \prod_{\nu \in I_D} \Psi_\nu(v_\nu; \tau_\nu, z_\nu, z_{\nu\sigma}) \times \prod_{\nu \in I^D} \Psi_\nu(v_\nu; \tau_\nu) \in \mathcal{S}(D_{\sigma,\infty}).$$

§2. **Local factors** We follow the path described in the fundamental example in §11 of Lecture 6 for the decomposition $D_\sigma = Z \oplus D_0$ of $\{\mathbf{g}_j\}_j$ and $\{\Phi_j\}_j$, adjusting the setting to our base totally real field. For $\epsilon : I_B \to \{\pm\}$, write $S_2^\epsilon(\mathfrak{N}, \varphi)$ for the space of adelic quaternionic cusp form on $B_{\mathbb{A}}^\times$ of weight 2 of central character φ holomorphic in z_ν if $\epsilon(\nu) > 0$ and anti-holomorphic in z_ν if $\epsilon(\nu) < 0$. A natural choice of ϵ depending on the choice of $I_{\mathbb{R}}$ is $\epsilon(\nu) = + \Leftrightarrow \nu \in I_{\mathbb{R}}$, which is denoted by ϵ_0. We write $S_2^-(\mathfrak{N}, \varphi)$ for the space of Hilbert modular form anti-holomorphic everywhere. Pick a Hecke eigen new form $\mathbf{f} \in S_2^-(C, \varphi)$ and write

$$L(s,\mathbf{f}) = \prod_{\mathfrak{l}} \left[(1-\alpha_{\mathfrak{l}} N(\mathfrak{l})^{-s})(1-\beta_{\mathfrak{l}} N(\mathfrak{l})^{-s})\right]^{-1}.$$

This is normalized so that the symmetry is $s \leftrightarrow 2-s$. Let $\mathfrak{M}$ be the ideal of definition of ψ and C_P (resp. C_s) be given by the product of primes $\mathfrak{l}|\mathfrak{M}$ with $|\alpha_{\mathfrak{l}}| = N(\mathfrak{l})^{1/2}$ if $\beta_{\mathfrak{l}} = 0$ and $C(\chi_{E,\mathfrak{l}}\psi_{\mathfrak{l}}^{-1}) = C_{\mathfrak{l}}$ (resp. $\mathfrak{l}|\mathfrak{M}$ or $\alpha_{\mathfrak{l}}\beta_{\mathfrak{l}} = 0$). Define

$$E(s) := \prod_{\mathfrak{l}|C_P} (1 - \alpha_{\mathfrak{l}}^{-1}\overline{\alpha}_{\mathfrak{l}}\psi(\mathfrak{l})N(\mathfrak{l})^{-1-2s}).$$

§**3. Period.** Define the quaternionic period by

$$P_\alpha(\mathcal{F}) := \int_{Sh_\alpha} \omega(\mathcal{F}) = ([Sh_\alpha], [\omega(\mathcal{F})])$$

for $\mathcal{F} \in S_2^{\epsilon 0}(C, \psi)$ and $\omega(\mathcal{F}) := \mathcal{F}(z) \bigwedge_{\nu \in I_{\mathbb{R}}} dz_\nu \wedge d\overline{z}_{\nu\sigma}$. Write w_ν for $z_{\nu\sigma}$. When $\alpha = 1 \in Z$, we write $P_D(\mathcal{F}) := P_1(\mathcal{F})$. Recall the Eichler order $R(\mathfrak{N})$ of level $\mathfrak{N}$. Take two lattices in D_0 which are

$$L = R_0 = \{v \in \sqrt{\mathbf{\Delta}_-} R(\mathfrak{N}) | v + v^\iota = 0\} \subset D_0 \quad \text{and } \beta L \subset D_0$$

for $0 \ll \beta \in O_F$, As before, let $L = O_F \oplus R_0$ for the Eichler order R of level $\mathfrak{N} = \boldsymbol{\partial}\mathfrak{N}_0$ with $\mathfrak{N}_0 + \boldsymbol{\partial} = O_F$. Take $\phi = \phi^{(\infty)}\Psi$ so that $\phi^{(\infty)} = \psi \otimes \phi_0^{(\infty)}$ for a character ψ of $Cl_F(\mathfrak{N}_0)$ and

$$\phi_0^{(\infty)} = \frac{\phi_{\widehat{R}_0} - N(\beta)^3 \phi_{\beta\widehat{R}_0}}{1 - N(\beta)^3} \text{ and } \phi_0 = \phi_0^{(\infty)} \cdot \phi_\infty.$$

Then $\Gamma_\tau = \Gamma_0(M)$ for $M = 4\mathfrak{N}^2 \cap 4\beta^2 \mathbf{\Delta}^- \mathfrak{N}\boldsymbol{\partial}$. This choice guarantees the condition (V) (see Lemma 5.28), and $\theta(\phi)$ with $\phi_Z^{(\infty)} = \psi$ has Neben character $\psi\chi_E$.

§4. Adjoint L-value formula.

Theorem 1. *Let* $\mathbf{f}$ *be a Hilbert modular primitive Hecke eigenform of conductor* C *in* $S_2^-(M,\psi^{-1}\chi_E)$ *on* $\mathrm{GL}_2(F_{\mathbb{A}})$*, and put* $f := \mathbf{f}|_{\mathrm{SL}_2(F_{\mathbb{A}})}$*. Then we have for* $E(s)$ *as in* §2 *in this lecture*

$$P_D(\theta^*(\phi)(f)) = c\cdot\mathfrak{m}\cdot\mathbf{\Gamma}_F(2)E(0)\frac{L^{(C_s)}(1;Ad(\rho_{\mathbf{f}})\otimes\chi_E)}{\zeta_F^{([C,M])}(2)} \neq 0.$$

Here $\mathfrak{m}$ *is the mass factor* $\mathfrak{m}(\widehat{\Gamma}_D)$*,* c *is a product of local constant* $c = \prod_{\nu\in I_F} c_\nu$ *at infinite places given below, and the theta lift* $\theta^*(\phi)(f)$ *is the* $SL(2)$ *theta lift of* f*. The constants are given by*

$$c_\nu = \begin{cases}(-2\sqrt{-1})^3 & \text{if } \nu\in I_{\mathbb{R}},\\ 2^3 & \text{if } \nu\in J_{\mathbb{R}},\end{cases}$$

$$\mathbf{\Gamma}_F(2) := \prod_{\nu\in I_{\mathbb{R}}} 2^{-1}(4\pi)^{-2}\Gamma(2)\cdot\prod_{\nu\in J_{\mathbb{R}}} 2^{-1}(4\pi)^{-1}.$$

The proof is the same as Lecture 4.

§5. Descent Theorem.

Theorem 2. *Suppose* $\phi_\infty = \Psi(\tau; z, w)$ *as in* §1 *and that* $\mathcal{F} \in S_2^{B,\epsilon_0}(\widehat{\Gamma}_\phi)$ *is a cusp form on* $\mathrm{SO}_{D_\sigma}(F_{\mathbb{A}})$ *of weight 2 anti-holomorphic in* w *and holomorphic in* z *as above. Then*

$$\theta_{SL,*}(\phi)(\mathcal{F})(\tau) = \int_{\Gamma_\phi \backslash \mathfrak{Z}_B} \theta(\phi)(\tau; z, w)\mathcal{F}(z, w)(z - \overline{z})^{2I_{\mathbb{R}}}(w - \overline{w})^{2I_{\mathbb{R}}\sigma}\omega_{inv}$$

$$= (4\sqrt{-1})^{-|I_{\mathbb{R}}|} \sum_{\alpha \in D_\sigma/\Gamma_\phi; N(\alpha) \gg 0} \phi^{(\infty)}(\alpha) P_\alpha(\mathcal{F}) \mathbf{e}_F(N(\alpha)\tau),$$

where $\mathbf{e}_F(N(\alpha)\tau) = \mathbf{e}(\sum_{\nu \in I_F} N(\alpha)^\nu \tau_\nu)$ *and*

$$\omega_{inv} := \bigwedge_{\nu \in I_{\mathbb{R}}} ((z_\nu - \overline{z}_\nu)^{-2}(w_{\nu\sigma} - \overline{w}_{\nu\sigma})^{-2} dz_\nu \wedge d\overline{z}_\nu \wedge dw_{\nu\sigma} \wedge d\overline{w}_{\nu\sigma}).$$

As long as $\mathcal{F} \neq 0$ *is in the image of the theta lift, for a good choice of* ϕ, $\theta_{SL,*}(\phi)(\mathcal{F}) \neq 0$.

The proof is basically the same as in Lecture 9.

§**6. Reflex field.** For a quaternion algebra Q over a totally real field $K \subset \overline{\mathbb{Q}}$, a formal definition of the *reflex field* $\mathcal{E}_Q$ of the Shimura variety associated to $Q^\times$ is given as follows.

Definition 1. *Let* $H = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ *and identify* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/H$ *with* I_K. *Consider* $H' := \{\nu \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) | \nu I_Q = I_Q\}$. *Then*

$$\mathcal{E}_Q = H^0(H', \overline{\mathbb{Q}}).$$

Write $\mathcal{E}$ for $\mathcal{E}_B$.

- $\mathcal{E}_Q$ is generated over $\mathbb{Q}$ by $\sum_{\nu \in I_Q} \nu(\xi)$ for ξ running over K; so, $\mathcal{E}_{\tau(Q)} = \tau(\mathcal{E}_Q)$ for $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\tau(Q) = Q \otimes_{K,\tau} \tau(K)$.
- $\mathcal{E}_D = \mathcal{E}_B$ if $B = D \otimes_F E$.

Examples: Let D be a quaternion algebra over F.

1. If $I_D = \{\nu\}$, then $\mathcal{E}_D = \nu(F)$ for a quaternion algebra $D_{/F}$.
2. If $F_{/F_0}$ is Galois with $I_D = \mathrm{Gal}(F/F_0)$, then $\mathcal{E}_D = F_0 \subset F$.
3. If $F_{/\mathbb{Q}}$ is non-Galois of degree 3 and $I_D = \{\mathrm{id}, \tau\}$, then $\mathcal{E}_D$ is the Galois closure $F^{gal}_{/\mathbb{Q}}$; so, $\mathcal{E}_D \supset F$.

§**7. Quaternionic Shimura variety.** We define $Sh_Q(\mathbb{C}) = Q_+^\times \backslash (Q^\times_{\mathbb{A}^{(\infty)}} \times \mathfrak{Z}_Q)$. By Shimura, Sh_Q has a canonical model $Sh_{Q/\mathcal{E}_Q}$ defined over $\mathcal{E}_Q$. Writing ∂_Q (resp. $\mathbf{\Delta}_K$) for the discriminant of $Q_{/K}$ (resp. $K_{/\mathbb{Q}}$), for a level subgroup S of level $\mathfrak{N}$, $Sh_S := Sh_Q/S$ has good reduction outside $\mathfrak{N}\mathbf{\Delta}_K\partial_Q$ as proven by Deligne and Carayol. See [PAF, Chapter 7].

If $I_D = I_F$, Sh_D carries a universal abelian scheme with D-multiplication of dimension $[D : \mathbb{Q}]/2 = 2[F : \mathbb{Q}]$ (cf. [PAF, Section 7.1]). Otherwise, it is not a moduli of abelian schemes (see [PAF, Section 7.2]).

By definition, the Galois closure $F^{gal} \supset \mathcal{E}_D = \mathcal{E}_B$. As before, writing $\dim Sh_D = r > 0$ ($\Leftrightarrow \dim Sh_B = 2r$), we decompose

$$H^{2r}_{cusp}(Sh_B, \overline{\mathbb{Q}}_l) = \bigoplus_{\pi \in \mathcal{A}_B} \Pi_\pi \otimes \pi^{(\infty)}$$

for 2^{2r}-dimensional representation Π_π of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{E}_D)$ associated to π.

§**8. Automorphic Galois action.** Yoshida invented a functorial way of a tensor-multiplicatively inducing a compatible system of n-dimensional Galois representation ρ on GL(V) of Gal$(\overline{\mathbb{Q}}/E)$ to a n^{2r}-dimensional system $\bigotimes_{I_B} \mathrm{Ind}_H^{H'} \rho$ of Gal$(\overline{\mathbb{Q}}/\mathcal{E})$ (see §8.2.5). Here $H :=$ Gal$(\overline{\mathbb{Q}}/E)$ and $H' :=$ Gal$(\overline{\mathbb{Q}}/\mathcal{E}_B)$. Then we have

Theorem 3 (Langlands, Reimann, Yoshida)**.** *Let the notation be as above. Let* Π_π^{ss} *be the compatible system of the Galois representations associated to the cuspidal automorphic representation* π*. Then we have* $\Pi_\pi \cong \bigotimes_{I_B} \mathrm{Ind}_H^{H'} \rho_\pi$ *as compatible systems.*

An important point we need is

(∗) $$\Pi_\pi^{ss}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F^{gal})} \cong \bigotimes_{\nu \in I_B} {}^{\nu}\rho_\pi,$$

where ${}^{\nu}\rho(g) = \rho(\nu^{-1} g \nu)$.

§**9. History.** Langlands [L79] and Reimann [SZQ] and [Re00] try to identify the Hasse–Weil zeta function of $Sh_{B,S}$ with a product of Langlands L-functions of the form $L(s-2r,\pi,r_1)$ for a 2^{2r}-dimensional representation r_1 of the L-group associated to G^B. More precisely, Langlands proved this assuming a conjectural description of the set of points $Sh_{B,S}$ with values in $\overline{\mathbb{F}}_p$ together with the action of the Frobenius automorphism. Since this assumption of Langlands is verified by Zink and Reimann, Reimann in [SZQ] and [Re00] obtained the desired identity of the local factors at least for primes unramified in B and E. Brylinsky and Labesse [BL84] covered the Hilbert modular case earlier than Reimann. By Chebotarev density and strong multiplicity one, Yoshida in [Y94] via his tensor induction method created a compatible system of Galois representations out of $\rho_{\mathbf{f}}$ and identified the Hasse–Weil L-function with the L-function of $\bigotimes_{I_B} \mathrm{Ind}_H^{H'} \rho_{\mathbf{f}}$ without any assumption on the ramification in E and B.

§**10. Shimura surfaces.** Assume

(Sp) *all primes of $\mathcal{E}$ above 2 splits in E.*

(nc) *$X_{/\mathcal{E}}$ is finite and its maximal c-field X^{cm} is real.*

Here on a c-field, $ic = ci$ for any $i : X \hookrightarrow \mathbb{C}$.

Let D be a quaternion algebra over a totally real field F and $B = D \otimes_F E$ for a totally real quadratic extension $E_{/F}$. Assume $I_B = \{\nu, \mu = \nu\sigma\}$; thus, $\dim Sh_B = 2$ and $\mathcal{E} = \nu(F)$. We say that $X_{/\mathcal{E}}$ is *sufficiently large* for a level subgroup S if for any $\pi \in \mathcal{A}_B^\sigma$ with $H^0(S, \pi^{(\infty)}) \neq 0$, ${}^\nu\rho_\pi \cong {}^\mu\rho_\pi$ over $\mathrm{Gal}(\overline{\mathbb{Q}}/X)$ and $\det \rho_\pi$ is the cyclotomic character over $\mathrm{Gal}(\overline{\mathbb{Q}}/X)$. We now state

Theorem 4. *Let the notation and the assumption be as above. Let S be a level subgroup of $G^B(\mathbb{A}^{(\infty)})$. For a sufficiently large finite extension $X_{/\mathcal{E}}$ for S satisfying* (nc), *$H^0(X, H^2_{cusp}(Sh_B/S, \overline{\mathbb{Q}}_l(1)))$ is spanned by the fundamental class of Sh_α for $\alpha \in D_\sigma$ with totally positive $N(\alpha)$.*