Lecture 0: Strategy. We describe the Tate conjecture for varieties over number fields and its background. Then we sketch the strategy to prove the conjecture for a good amount of quaternionic Shimura varieties. A key point is a twisted adjoint L-value formula relative to each quaternion algebra $D/F$ for a totally real field $F$ and its scalar extension $B = D \otimes_F E$ for a totally real quadratic extension $E/F$. The theta base-change lift $\mathcal{F}$ of a Hilbert modular form $f$ to $B^\times$ has period integral over the Shimura subvariety $Sh_D \subset Sh_B$ given by $L(1, \text{Ad}(f) \otimes \left(\frac{E}{F}\right)) \neq 0$; so, $Sh_D$ gives rise to a non-trivial Tate cycle in $H^{2r}(Sh_B, \mathbb{Q}_l(r))$ for $r = \dim Sh_D = \dim Sh_B/2$. All quoted section numbers without specification are from the book manuscript.
§0. Algebraic Cycles. Let $S$ be a projective variety of dimension $r$ defined over a number field $K \subset \mathbb{C}$. Let $T \subset S$ be a closed subvariety of codimension $e$; so, $T(\mathbb{C})$ is of real dimension $2r - 2e$. Regard $S(\mathbb{C})$ as a complex manifold of dimension $r$; so, it is a real manifold of dimension $2r$. For a closed smooth differential form $\omega$ of degree $2r - 2e$, the integration $\omega \mapsto \int_T \omega$ induces a linear form $[T] : H^{2r-2e}_{\text{DR}}(S, \mathbb{C}) \to \mathbb{C}$ sending the de Rham cohomology class $[\omega]$ to its integral $\int_T \omega$. By Poincaré duality: $H^{2r-2e}_{\text{DR}}(S(\mathbb{C}), \mathbb{C}) \times H^{2e}_{\text{DR}}(S(\mathbb{C}), \mathbb{C}) \to \mathbb{C}$, we view $[T] = [T]_\infty \in H^{2e}_{\text{DR}}(S(\mathbb{C}), \mathbb{C})$. This class $[T]$ is called the algebraic cycle associated to $T$. Write $\text{Alg}^e(S, A)$ for the $A$-span inside $H^{2e}_{\text{DR}}(S(\mathbb{C}), \mathbb{C})$ of $[T]$ for $T$ running over all closed subvarieties of $S/\mathbb{C}$ of codimension $e$. Here $A$ is a subring of $\mathbb{C}$.
§1. **Hodge Cycles.** An obvious question is:

*Is there any good characterization of $\text{Alg}^e(S, \mathbb{Q})$?*

Define $H^{e,e}(S(\mathbb{C}), \mathbb{C})$ to be the degree $(e, e)$ subspace of the Hodge decomposition $H^{2e}(S(\mathbb{C}), \mathbb{C}) = \bigoplus_{i+j=2e} H^{i,j}(S(\mathbb{C}), \mathbb{C})$. The Hodge conjecture [D06] tells us

\[ \text{(H)} \quad \text{Alg}^e(S, \mathbb{Q}) \cong \text{Hdg}^e(S) := H^{e,e}(S(\mathbb{C}), \mathbb{C}) \cap H^{2e}(S(\mathbb{C}), \mathbb{Q}). \]

The Hodge conjecture (H) is known for divisors by Lefschetz [PAG, 1.2] and some small number of cases. Of course, one can further ask to find an explicit set of subvarieties which span $\text{Hdg}^e(S)$. 
§2. Étale Poincaré duality. Fix an algebraic closure $\overline{K} \subset \mathbb{C}$ and put $\overline{S} = S \otimes_K \overline{K}$. Let $\mu_N$ (resp. $\mathbb{Z}/N\mathbb{Z}$) be the sheaf of $N$-th roots of unity (resp. the constant sheaf of $\mathbb{Z}/N\mathbb{Z}$) on $S$; so, $\mu_N = \text{Spec}_S(\mathcal{O}_S[t]/(t^N - 1))$. Then the pairing $\mu_N \times \mathbb{Z}/N\mathbb{Z} \ni (\zeta, a) \mapsto \zeta^a \in \mu_N$ induces $\text{End}_{S\text{-gp}}(\mu_N) \cong \mathbb{Z}/N\mathbb{Z}$ canonically; i.e., the pairing is perfect. Let $\mathbb{Z}/N\mathbb{Z}(e) = \text{Hom}(\mu_N \otimes \cdots \otimes \mu_N, \mathbb{Z}/N\mathbb{Z})$ (the Pontryagin dual). By cup product, writing $H^{2e}(?) := H^{2e}_{\text{et}}(\overline{S}, ?)$, we have a perfect pairing

$$H^{2r}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}(r)) \times H^{0}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}) \rightarrow H^{2r}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}(r)).$$

By the comparison isomorphism between étale cohomology and Betti cohomology, $H^{2r}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}(r))$ is free of rank 1 over $\mathbb{Z}/N\mathbb{Z}$, and the pairing is perfect, inducing a canonical isomorphism $\text{Tr} : H^{2r}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}(r)) \cong \mathbb{Z}/N\mathbb{Z}$. By an abstract non-sence, this perfect duality extends to

$$H^{2e}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}(e)) \times H^{2r-2e}_{\text{et}}(\mathbb{Z}/N\mathbb{Z}) \rightarrow \mathbb{Z}/N\mathbb{Z} \quad (0 \leq e \leq r).$$
§3. **Tate conjecture.** The immersion \( i : T \hookrightarrow S \) induces \( i^* : H^{2r-2e}(\mathbb{Z}/N\mathbb{Z}(r-e)) \to H^{2r-2e}(\mathbb{T}, \mathbb{Z}/N\mathbb{Z}(r-e)) \cong \mathbb{Z}/N\mathbb{Z} \). By Poincaré duality, we have an inclusion \( i_* : \mathbb{Z}/N\mathbb{Z} \hookrightarrow H^{2e}(\mathbb{Z}/N\mathbb{Z}(e)) \). So we have the fundamental class \([T] = [T]_N = i_*(1)\). Taking \( N = l^n \) for a prime \( l \) and passing to the limit, \([T]_{et} = \lim_n [T]_{l^n} \in H^{2e}_{et}(\mathbb{Z}_l(e)) = \lim_n H^{2e}(\mathbb{Z}/l^n\mathbb{Z}(e))\). Extending scalars to \( \overline{\mathbb{Q}}_l \) and identifying \( \overline{\mathbb{Q}}_l \) with \( \mathbb{C} \), by the comparison isomorphism, \( H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)) \ni [T]_{et} \leftrightarrow [T]_{\infty} \in H^{2e}(\mathbb{C}) \). Thus we may regard \( \text{Alg}^e_K(S; \mathbb{Q}) \subset H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)) \). We define \( \text{Tate}^e_K(S) = \overline{\mathbb{Q}}_l \cdot \text{Alg}^e_K(S; \mathbb{Q}) \) (the \( \overline{\mathbb{Q}}_l \)-span of \( \text{Alg}^e_K(S; \mathbb{Q}) \) inside \( H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)) \)). The Tate conjecture states,

\[
(\text{T}) \quad \text{Tate}^e(S) \overset{?}{=} H^0(\overline{\mathbb{K}}/\mathbb{K}, H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)))
\]

Here \( H^0(\overline{\mathbb{K}}/\mathbb{K}, ?) = H^0(\text{Gal}(\overline{\mathbb{K}}/\mathbb{K}), ?) \) for a Galois module \(?\). Replacing \( \text{Tate}^e_K(S) \) by \( \mathbb{Q}_l \cdot \text{Alg}^e_K(S) \), we ask the same question for \( \mathbb{Q}_l \) in place of \( \overline{\mathbb{Q}}_l \).
§4. Known cases.
1. Divisors on abelian varieties, $K3$ surfaces and products of two curves (Faltings) [Ta94b, §5],
2. Hilbert modular surfaces by Harder–Langlands–Rapoport [HLR], Murty–Ramakrishnan [MR87] and Klingenberg [K87] (increasingly general cases),
3. non-CM motives on Hilbert modular fourfolds [R04] when the base totally real field is a Galois extension of $\mathbb{Q}$,
4. Picard surfaces by Blasius and Rogawski [BR92],
5. General Hilbert modular motives under some restrictive conditions by Getz and Hahn [GH14],
6. Good motives occurring on a product of a Hilbert-Siegel variety and a Hilbert modular variety by Sweeting [S22].

The results of [HLR], [GH14] and [S22] are explicit in the sense that Shimura subvarieties span $\text{Tate}^1_K(S')$, and in all the other cases, we do not know the generators explicitly.
\section{Explicit analog for Shimura varieties.} If $S$ is a Shimura variety associated to a reductive group $G$ defined over the reflex field $E$, the finite adele $G(\mathbb{A}^{(\infty)})$ acts on $S$ which induces an action of $G(\mathbb{A}^{(\infty)})$ on $H_{et}^{2e}(\overline{Q}_l)$ commuting with the Galois action. We can then decompose

$$H_{et}^{2e}(\overline{Q}_l) = \bigoplus \Pi_\pi \otimes \pi^{(\infty)}$$

for automorphic representations $\pi$ of $G(\mathbb{A})$ with its finite part $\pi^{(\infty)}$ and associated $l$-adic Galois representation $\Pi_\pi$. Then

$$H^0(X, H_{et}^{2e}(\overline{Q}_l(e))) = \bigoplus \Pi_\pi(e) \otimes H^0(X, \Pi_\pi(e)) \otimes \pi^{(\infty)}$$

for $\Pi_\pi(e) = \Pi_\pi \otimes_{Q_l} Q_l(e)$, where $H^0(X, ?) = H^0(\overline{X}/X, ?)$ for a finite extension $X/E$ with algebraic closure $\overline{X}$. As Langlands suggested, we may describe well $\Pi_\pi$ from the automorphic data; so, we might be able to compute the space $H^0(X, \Pi_\pi(e))$ of Tate cycles, or at least, we ask how much of it is spanned by Shimura subvarieties $[S_H]$ for reductive subgroups $H \subset G$. 
§6. Quaternionic Shimura varieties. In this series of lectures, we try to answer the questions in §5 for $G$ given by $G^B$ for a quaternion algebra $B$ over the totally real field $E$, as we have an explicit description of $\Pi_\pi$ as a tensor induction of 2-dimensional compatible system $\rho_\pi$. Our method is the use of the theta correspondence exploiting the following facts:

1. An appropriate theta lift $\theta^*(\phi)(f)$ of a Hilbert modular form $f$ gives the Doi–Naganuma lift of $f$ to a quaternionic automorphic form over a quadratic extension $E/F$ (see Section 4.7);
2. The integral over the Shimura subvariety $Sh_D$ of appropriate dimension in the quaternionic Shimura variety $Sh_B$ is equal to a nonzero twisted adjoint L-value of $f$ (see Chapter 5);
3. The theta descent $\theta_*(\phi)(\mathcal{F})$ of each quaternionic automorphic form $\mathcal{F}$ has Hilbert modular Fourier expansion whose coefficients are the integral of $\mathcal{F}$ over appropriate Shimura subvarieties $Sh_\alpha$. 
§7. **Twisted 4-dimensional quadratic spaces.** In the earlier lectures, we assume $F = \mathbb{Q}$. Let $D/\mathbb{Q}$ be a quaternion algebra with discriminant $\partial$. Choose a semi-simple quadratic extension $E = \mathbb{Q}[\sqrt{\Delta}]/\mathbb{Q}$ including $E = \mathbb{Q} \times \mathbb{Q}$ with $\sqrt{\Delta} = (1, -1)$, and let $B := D \otimes \mathbb{Q} E$. Write simply $\delta = \sqrt{\Delta}$ with square-free $\Delta \in \mathbb{Z}$. Let $\langle \sigma \rangle = \text{Gal}(E/\mathbb{Q})$ act on $B$ through the factor $E$. Then $D_\sigma := \{v \in B|v^\sigma = v^t\}$ for $v + v^t = \text{Tr}(v)$ is a 4-dimensional quadratic space with a quadratic $\mathbb{Q}$-form induced by the reduced norm $N : B \to E$. We have an orthogonal decomposition $D_\sigma = Z \oplus D_0$ where $Z = E \cap D_\sigma = (\mathbb{Q}, z^2)$ and

$$D_0 := \{v \in D_\sigma|\text{Tr}(v) = 0\} = \{\sqrt{\Delta}w|w \in D, \text{Tr}(w) = 0\}.$$

When $E = \mathbb{Q} \times \mathbb{Q}$, $\sigma(x, y) = (y, x)$ and $D_\sigma = \{(x, x^t)|x \in D\} \cong D$ by $(x, x^t) \leftrightarrow x$.

Let $R$ be a **Eichler order of level** $N$ of $B$; so, $N = \partial N_0$ with $N_0$ prime to $\partial$, maximal outside $\partial$ and $R/N_0 R \cong \{(*, *)\}$. Let $\hat{R} = \varprojlim_N R/N R$ (the profinite completion).
§8. Quaternion subalgebras of $B$. For each $\alpha \in D_\sigma \cap B^\times$, define the $\alpha$-twist $\sigma_\alpha$ of $\sigma$ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^{\sigma_\alpha}$. Then $\sigma_\alpha$ is another action of $\text{Gal}(E/\mathbb{Q})$ on $B$, and $D_\alpha = H^0(E/F, B)$ under this twisted action is a quaternion subalgebra of $B$.

- All quaternion $\mathbb{Q}$-subalgebras of $B$ are realized as $D_\alpha$ for some $\alpha \in D_\sigma$, and $D_z = D \iff z \in \mathbb{Z}$;
- $\alpha = \xi^{-1} \beta \xi^{-\iota} \sigma$ for $\xi \in B^\times \iff D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\alpha \cong D_\beta$ by an inner automorphism of $B$ if $N(\alpha) = N(\beta)$ and $D_{E_\infty} \cong M_2(E_\infty)$ (strong approximation);
- The even Clifford group $G_\alpha$ of $D_{\alpha,0} = \{v \in D_{\sigma,0} | \text{Tr}(v) = 0\}$ is $D_\alpha^\times$ and $B^\times$ is a covering of the similitude group $\text{GO}_{D_\sigma}$ of $D_\sigma$.

Let $\hat{\Gamma}_\phi = \{h \in D_{E_\infty}^\times | \phi(\infty)(h^{-1}vh^{-\iota}) = \phi(\infty)(v), \forall v \in D_{\sigma,\mathbb{A}(\infty)}\}$ for each Schwartz-Bruhat function $\phi$ on $D_{\sigma,\mathbb{A}(\infty)}$.

Let $Sh_B = B^\times \backslash D_{E_\mathbb{A}}^\times / E_\mathbb{A} \hat{\Gamma}_\phi C_\infty$ be the Shimura variety for $B^\times$ of level $\hat{\Gamma}_\phi$, and $Sh_\alpha$ be the image of $D_{\alpha,\mathbb{A}}^\times$ in $Sh_B$ for $\alpha \in D_\sigma$.

Let $d = \dim_{\mathbb{R}} Sh_\alpha \in \{0, 2\}$. Regard $Sh_\alpha \in H_d(Sh_B, \mathbb{Z})$ and write $(\cdot, \cdot) : H^d \times H_d \to \mathbb{C}$ for the Poincaré duality.
§9. Two formulas. Write \( f \mapsto \theta^*(\phi)(f) = \int_{Y_0(N)} \theta(\phi)(\tau, g) f(\tau) d\mu_\tau \) for the theta lift of \( f \in S_{new}^2(\Gamma_0(N)) \) to \( G \) and \( \theta^*(\phi)(F) \in S_2(\Gamma_0(N)) \) for the theta descent of a cuspidal harmonic differential form \( F \) on \( Sh_B \) of degree \( d \). The lift \( \theta^*(\phi)(f) \) is such a differential form of matching degree \( d' \) or \( d \) with \( d' + d = \dim_{\mathbb{R}} Sh_B \).

**Theorem A:** \( \theta^*(\Phi)(F) = \star \sum_{0 < n \in \mathbb{Q}} \sum_{\alpha, N(\alpha) = n} \Phi(\alpha)(F, Sh_\alpha) q^n \) for \( q = \exp(2\pi i \tau) \), where \( \star = (8\sqrt{-1})^{-1} \) if \( E_R := E \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R} \times \mathbb{R} \) and \( \star = 4 \) if \( E_R = \mathbb{C} \). If \( E_R = \mathbb{C} \), only \( \alpha \) with \( \dim Sh_\alpha = 1 \) appears.

Note \((F, Sh_\alpha) = \int_{Sh_\alpha} F\). The Tate conjectures only concern \( D \) and \( E \) with \( D_R = M_2(\mathbb{R}) \) and \( E_R = \mathbb{R} \times \mathbb{R} \).

**Theorem B:** \( i^{-3} E m_1 \frac{L(1, Ad(\rho_f) \otimes \chi_E)}{2^2 \pi^3} = (\theta^*(\phi')(f), Sh_1) \) for an explicit constant \( E \neq 0 \). Here choosing a Haar measure \( d\mu_1 \) on \( D_A^\times \) so that it is Dirac on \( D_A^\times \) and having volume 1 on \( \hat{\Gamma}_{\phi'} \cap D_A^\times C_\infty \), \( m = m_1 \zeta(2)/\pi \) is defined by \( d\mu_1 = (m/2)d\omega \) for the Tamagawa measure. Note \( 0 < m_1 \in \mathbb{Q}^\times \) by Siegel.
§10. Strategy.
1. We prove that $f \mapsto \theta^*(\phi)(f)$ is Hecke equivariant for $T(p)$ with $p$ splitting in $E/\mathbb{Q}$ if we choose a Schwartz–Bruhat function $\phi : D_{\sigma, \mathbb{A}} \to \mathbb{C}$ well.
2. By Chebotarev density, Item 1 is sufficient to see $\mathcal{F} = \theta^*(\phi)(f)$ is a Hecke eigenform for $T(p)$ for all prime $p$ of $E$ giving the Doi–Naganuma lift. Non-vanishing $\mathcal{F} \neq 0$ comes out from Theorem A and Rallis’ work (a bit over-simplified).
3. Analyze the Galois representation $\Pi_\pi$ for the automorphic representation $\pi$ generated by a general Hecke eigenform $\mathcal{F}$ on $Sh_B$ to see that $\dim \text{Tate}_K^1[\pi(\infty)]$ has dimension 1 only when $\mathcal{F} = \theta^*(\phi)(f)$ for some $f$.
4. By Theorem B, if $(Sh_\alpha, \mathcal{F}) = 0$ for all $\alpha \in D_\sigma \cap B^\times$, $\theta^*(\phi)(\mathcal{F}) = 0$, contradicting to $\theta^*(\phi)(f) \neq 0$. This proves $\sum_\alpha \overline{\mathbb{Q}}_l[Sh_\alpha] \supset \dim \text{Tate}_K^1[\pi(\infty)]$, proving the conjecture.
§11. Further generalizations. As we already mentioned, Sweeting [S22] applied theta correspondence to show the Tate conjecture to some extent for a product of a Hilbert-Siegel modular variety of genus 2 and a Hilbert modular variety.

The use of the theta correspondence for this type of problems seems quite effective. The case of orthogonal Shimura variety associated to quadratic forms of signature $(2n, 2)$ or $(2n, 0)$ at different infinite places of a totally real field $F$ can be treated to some extent; at least, an analog of the adjoint $L$-value formula is expected in these cases.