Adjoint L-value and the Tate conjecture

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Lecture 1: Strategy. We describe the Tate conjecture for varieties over number fields and its back ground. Then we sketch the strategy to prove the conjecture for a good amount of quaternionic Shimura varieties. A key point is a twisted adjoint L-value formula relative to each quaternion algebra $D_{/F}$ for a totally real field F and its scalar extension $B = D \otimes_F E$ for a totally real quadratic extension $E_{/F}$. The theta base-change lift ${\cal F}$ of a Hilbert modular form $f^{'}$ to $B^{ imes}$ has period integral over the Shimura subvariety $Sh_D \subset Sh_B$ given by $L(1, Ad(f) \otimes \left(\frac{E/F}{F}\right)) \neq 0$; so, Sh_D gives rise to a non-trivial Tate cycle in $H^{2r}(Sh_B, \mathbb{Q}_l(r))$ for $r = \dim Sh_D = \dim Sh_B/2$. All quoted section numbers without specification are from the book manuscript.

Algebraic Cycles. Let S be a projective variety of diξ**Ο**. mension r defined over a number field $K \subset \mathbb{C}$. Let $T \subset S$ be a closed subvariety of codimension e; so, $T(\mathbb{C})$ is of real dimension 2r - 2e. Regard $S(\mathbb{C})$ as a complex manifold of dimension r; so, it is a real manifold of dimension 2r. For a closed smooth differential form ω of degree 2r - 2e, the integration $\omega \mapsto \int_T \omega$ induces a linear form $[T] : H_{DR}^{2r-2e}(S,\mathbb{C}) \to \mathbb{C}$ sending the de Rham cohomology class $[\omega]$ to its integral $\int_T \omega$. By Poincaré duality: $H_{DR}^{2r-2e}(S(\mathbb{C}),\mathbb{C}) \times H_{DR}^{2e}(S(\mathbb{C}),\mathbb{C}) \to \mathbb{C}$, we view $[T] = [T]_{\infty} \in H^{2e}_{DR}(S(\mathbb{C}), \mathbb{C})$. This class [T] is called the algebraic cycle associated to T. Write $Alg^{e}(S, A)$ for the A-span inside $H^{2e}_{DR}(S(\mathbb{C}),\mathbb{C})$ of [T] for T running over all closed subvarieties of $S_{/\mathbb{C}}$ of codimension e. Here A is a subring of \mathbb{C}

\S **1. Hodge Cycles.** An obvious question is:

Is there any good characterization of $Alg^e(S, \mathbb{Q})$?

Define $H^{e,e}(S(\mathbb{C}),\mathbb{C})$ to be the degree (e,e) subspace of the Hodge decomposition $H^{2e}(S(\mathbb{C}),\mathbb{C}) = \bigoplus_{i+j=2e} H^{i,j}(S(\mathbb{C}),\mathbb{C})$. The Hodge conjecture [D06] tells us

(H)
$$\operatorname{Alg}^{e}(S, \mathbb{Q}) \stackrel{?}{=} \operatorname{Hdg}^{e}(S) := H^{e,e}(S(\mathbb{C}), \mathbb{C}) \cap H^{2e}(S(\mathbb{C}), \mathbb{Q}).$$

The Hodge conjecture (H) is known for divisors by Lefschetz [PAG, 1.2] and some small number of cases. Of course, one can further ask to find an explicit set of subvarieties which span $Hdg^e(S)$.

§2. Étale Poincaré duality. Fix an algebraic closure $\overline{K} \subset \mathbb{C}$ and put $\overline{S} = S \otimes_K \overline{K}$. Let μ_N (resp. $\mathbb{Z}/N\mathbb{Z}$) be the sheaf of N-th roots of unity (resp. the constant sheaf of $\mathbb{Z}/N\mathbb{Z}$) on S; so, $\mu_N =$ $\operatorname{Spec}_S(\mathcal{O}_S[t]/(t^N - 1))$. Then the pairing $\mu_N \times \mathbb{Z}/N\mathbb{Z} \ni (\zeta, a) \mapsto$ $\zeta^a \in \mu_N$ induces $\operatorname{End}_{S-\operatorname{gp}}(\mu_N) \cong \mathbb{Z}/N\mathbb{Z}$ canonically; i.e., the pairing is perfect. Let $\mathbb{Z}/N\mathbb{Z}(e) = \operatorname{Hom}(\mu_N \otimes \cdots \otimes \mu_N, \mathbb{Z}/N\mathbb{Z})$ (the

Ing is perfect. Let $\mathbb{Z}/N\mathbb{Z}(e) = \operatorname{Hom}(\mu_N \otimes \cdots \otimes \mu_N, \mathbb{Z}/N\mathbb{Z})$ (the Pontryagin dual). By cup product, writing $H^{2e}(?) := H^{2e}_{et}(\overline{S}, ?)$, we have a perfect pairing

 $H^{2r}_{et}(\mathbb{Z}/N\mathbb{Z}(r)) \times H^0_{et}(\mathbb{Z}/N\mathbb{Z}) \to H^{2r}_{et}(\mathbb{Z}/N\mathbb{Z}(r)).$

By the comparison isomorphism between étale cohomology and Betti cohomology, $H_{et}^{2r}(\mathbb{Z}/N\mathbb{Z}(r))$ is free of rank 1 over $\mathbb{Z}/N\mathbb{Z}$, and the paring is perfect, inducing a canonical isomorphism Tr : $H_{et}^{2r}(\mathbb{Z}/N\mathbb{Z}(r)) \cong \mathbb{Z}/N\mathbb{Z}$. By an abstract non-sence, this perfect duality extends to

 $H^{2e}_{et}(\mathbb{Z}/N\mathbb{Z}(e)) \times H^{2r-2e}_{et}(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}/N\mathbb{Z} \quad (0 \le e \le r).$

§3. Tate conjecture. The immersion $i : T \hookrightarrow S$ induces $i^* : H^{2r-2e}(\mathbb{Z}/N\mathbb{Z}(r-e)) \to H^{2r-2e}(\overline{T}, \mathbb{Z}/N\mathbb{Z}(r-e)) \cong \mathbb{Z}/N\mathbb{Z}$. By Poincaré duality, we have an inclusion $i_* : \mathbb{Z}/N\mathbb{Z} \hookrightarrow H^{2e}(\mathbb{Z}/N\mathbb{Z}(e))$. So we have the fundamental class $[T] = [T]_N = i_*(1)$. Taking $N = l^n$ for a prime l and passing to the limit, $[T]_{et} = \lim_{n} [T]_{l^n} \in H^{2e}_{et}(\mathbb{Z}_l(e)) = \lim_{n} H^{2e}(\mathbb{Z}/l^n\mathbb{Z}(e))$. Extending scalars to $\overline{\mathbb{Q}}_l$ and identifying $\overline{\mathbb{Q}}_l$ with \mathbb{C} , by the comparison isomorphism, $H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)) \ni [T]_{et} \leftrightarrow [T]_{\infty} \in H^{2e}(\mathbb{C})$. Thus we may regard $\operatorname{Alg}_K^e(S; \mathbb{Q}) \subset H^{2e}_{et}(\overline{\mathbb{Q}}_l(e))$. We define $\operatorname{Tate}_K^e(S) = \overline{\mathbb{Q}}_l \cdot \operatorname{Alg}_K^e(S; \mathbb{Q})$ (the $\overline{\mathbb{Q}}_l$ -span of $\operatorname{Alg}_K^e(S; \mathbb{Q})$ inside $H^{2e}_{et}(\overline{\mathbb{Q}}_l(e))$. The Tate conjecture states,

(T) Tate^e(S)
$$\stackrel{?}{=} H^0(\overline{K}/K, H^{2e}_{et}(\overline{\mathbb{Q}}_l(e))).$$

Here $H^0(\overline{K}/K,?) = H^0(\text{Gal}(\overline{K}/K),?)$ for a Galois module ?. Replacing $\text{Tate}_K^e(S)$ by $\mathbb{Q}_l \cdot \text{Alg}_K^e(S)$, we ask the same question for \mathbb{Q}_l in place of $\overline{\mathbb{Q}}_l$.

\S 4. Known cases.

1. Divisors on abelian varieties, K3 surfaces and products of two curves (Faltings) [Ta94b, $\S5$],

2. Hilbert modular surfaces by Harder–Langlands–Rapoport [HLR], Murty–Ramakrishnan [MR87] and Klingenberg [K87] (increasingly general cases),

3. non-CM motives on Hilbert modular fourfolds [R04] when the base totally real field is a Galois extension of \mathbb{Q} ,

4. Picard surfaces by Blasius and Rogawski [BR92],

5. General Hilbert modular motives under some restrictive conditions by Getz and Hahn [GH14],

6. Good motives occurring on a product of a Hilbert-Siegel variety and a Hilbert modular variety by Sweeting [S22].

The results of [HLR], [GH14] and [S22] are explicit in the sense that Shimura subvarieties span $Tate_{K}^{1}(S)$, and in all the other cases, we do not know the generators explicitly.

§5. Explicit analog for Shimura varieties. If S is a Shimura variety associated to a reductive group G defined over the reflex field \mathcal{E} , the finite adele $G(\mathbb{A}^{(\infty)})$ acts on S which induces an action of $G(\mathbb{A}^{(\infty)})$ on $H^{2e}_{et}(\overline{\mathbb{Q}}_l)$ commuting with the Galois action. We can then decompose

$$H^{2e}_{et}(\overline{\mathbb{Q}}_l) = \bigoplus_{\pi} \Pi_{\pi} \otimes \pi^{(\infty)}$$

for automorphic representations π of $G(\mathbb{A})$ with its finite part $\pi^{(\infty)}$ and associated *l*-adic Galois representation Π_{π} . Then

$$H^{0}(X, H^{2e}_{et}(\overline{\mathbb{Q}}_{l}(e)) = \bigoplus_{\pi} H^{0}(X, \Pi_{\pi}(e)) \otimes \pi^{(\infty)}$$

for $\Pi_{\pi}(e) = \Pi_{\pi} \otimes_{\mathbb{Q}_l} \mathbb{Q}_l(e)$, where $H^0(X,?) = H^0(\overline{X}/X,?)$ for a finite extension $X_{/\mathcal{E}}$ with algebraic closure \overline{X} . As Langlands suggested, we may describe well Π_{π} from the automorphic data; so, we might be able to compute the space $H^0(X, \Pi_{\pi}(e))$ of Tate cycles, or at least, we ask how much of it is spanned by Shimura subvarieties $[S_H]$ for reductive subgroups $H \subset G$.

§6. Quaternionic Shimura varieties. In this series of lectures, we try to answer the questions in §5 for G given by G^B for a quaternion algebra B over the totally real field E, as we have an explicit description of Π_{π} as a tensor induction of 2-dimensional compatible system ρ_{π} . Our method is the use of the theta correspondence exploiting the following facts:

1. An appropriate theta lift $\theta^*(\phi)(f)$ of a Hilbert modular form f gives the Doi–Naganuma lift of f to a quaternionic automorphic form over a quadratic extension $E_{/F}$ (see Section 4.7);

2. The integral over the Shimura subvariety Sh_D of appropriate dimension in the quaternionic Shimura variety Sh_B is equal to a nonzero twisted adjoint L-value of f (see Chapter 5);

3. The theta descent $\theta_*(\phi)(\mathcal{F})$ of each quaternionic automorphic form \mathcal{F} has Hilbert modular Fourier expansion whose coefficients are the integral of \mathcal{F} over appropriate Shimura subvarieties Sh_{α} . §7. Twisted 4-dimensional quadratic spaces. In the earlier lectures, we assume $F = \mathbb{Q}$. Let $D_{/\mathbb{Q}}$ be a quaternion algebra with discriminant ∂ . Choose a semi-simple quadratic extension $E = \mathbb{Q}[\sqrt{\Delta}]_{/\mathbb{Q}}$ including $E = \mathbb{Q} \times \mathbb{Q}$ with $\sqrt{\Delta} = (1, -1)$, and let $B := D \otimes_{\mathbb{Q}} E$. Write simply $\delta = \sqrt{\Delta}$ with square-free $\Delta \in \mathbb{Z}$. Let $\langle \sigma \rangle = \text{Gal}(E/\mathbb{Q})$ act on B through the factor E. Then $D_{\sigma} := \{v \in B | v^{\sigma} = v^{\iota}\}$ for $v + v^{\iota} = \text{Tr}(v)$ is a 4-dimensional quadratic space with a quadratic \mathbb{Q} -form induced by the reduced norm $N : B \to E$. We have an orthogonal decomposition $D_{\sigma} = Z \oplus D_0$ where $Z = E \cap D_{\sigma} = (\mathbb{Q}, z^2)$ and

 $D_0 := \{ v \in D_\sigma | \operatorname{Tr}(v) = 0 \} = \{ \sqrt{\Delta} w | w \in D, \operatorname{Tr}(w) = 0 \}.$ When $E = \mathbb{Q} \times \mathbb{Q}, \ \sigma(x, y) = (y, x)$ and $D_\sigma = \{ (x, x^\iota) | x \in D \} \cong D$ by $(x, x^\iota) \leftrightarrow x$.

Let R be a Eichler order of level N of B; so, $N = \partial N_0$ with N_0 prime to ∂ , maximal outside ∂ and $R/N_0R \cong \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \}$. Let $\widehat{R} = \varprojlim_N R/NR$ (the profinite completion).

§8. Quaternion subalgebras of *B*. For each $\alpha \in D_{\sigma} \cap B^{\times}$, define the α -twist σ_{α} of σ by $v \mapsto \alpha v^{\sigma} \alpha^{-1} =: v^{\sigma_{\alpha}}$. Then σ_{α} is another action of $\text{Gal}(E/\mathbb{Q})$ on *B*, and $D_{\alpha} = H^{0}(E/F, B)$ under this twisted action is a quaternion subalgebra of *B*.

• All quaternion \mathbb{Q} -subalgebras of B are realized as D_{α} for some

$$\alpha \in D_{\sigma}$$
, and $D_z = D \Leftrightarrow z \in Z$;

- $\alpha = \xi^{-1}\beta\xi^{-\iota\sigma}$ for $\xi \in B^{\times} \Leftrightarrow D_{\alpha} \cong D_{\beta}$ with $\xi D_{\alpha}\xi^{-1} = D_{\beta}$;
- $D_{\alpha} \cong D_{\beta}$ by an inner automorphism of B if $N(\alpha) = N(\beta)$ and $D_{E_{\infty}} \cong M_2(E_{\infty})$ (strong approximation);

• The even Clifford group G_{α} of $D_{\alpha,0} = \{v \in D_{\sigma_{\alpha}} | \operatorname{Tr}(v) = 0\}$ is D_{α}^{\times} and B^{\times} is a covering of the similitude group $\operatorname{GO}_{D_{\sigma}}$ of D_{σ} .

Let $\widehat{\Gamma}_{\phi} = \{h \in D_{E_{\mathbb{A}}^{(\infty)}}^{\times} | \phi^{(\infty)}(h^{-1}vh^{-\iota\sigma}) = \phi^{(\infty)}(v), \forall v \in D_{\sigma,\mathbb{A}^{(\infty)}}\}$ for each Schwartz-Bruhat function ϕ on $D_{\sigma,\mathbb{A}^{(\infty)}}$.

Let $Sh_B = B^{\times} \setminus D_{E_A}^{\times} / E_A^{\times} \widehat{\Gamma}_{\phi} C_{\infty}$ be the Shimura variety for B^{\times} of level $\widehat{\Gamma}_{\phi}$, and Sh_{α} be the image of $D_{\alpha,A}^{\times}$ in Sh_B for $\alpha \in D_{\sigma}$. Let $d = \dim_{\mathbb{R}} Sh_{\alpha} \in \{0,2\}$. Regard $Sh_{\alpha} \in H_d(Sh_B,\mathbb{Z})$ and write $(\cdot, \cdot) : H^d \times H_d \to \mathbb{C}$ for the Poincaré duality. §9. Two formulas. Write $f \mapsto \theta^*(\phi)(f) = \int_{Y_0(N)} \theta(\phi)(\tau, g) f(\tau) d\mu_{\tau}$ for the theta lift of $f \in S_2^{new}(\Gamma_0(N))$ to G and $\theta_*(\phi)(\mathcal{F}) \in S_2(\Gamma_0(N))$ for the theta descent of a cuspidal harmonic differential form \mathcal{F} on Sh_B of degree d. The lift $\theta^*(\phi)(f)$ is such a differential form of matching degree d' or d with $d' + d = \dim_{\mathbb{R}} Sh_B$.

Theorem A: $\theta_*(\Phi)(\mathcal{F}) = * \sum_{0 < n \in \mathbb{Q}}^{\infty} \sum_{\alpha, N(\alpha) = n} \Phi(\alpha)(\mathcal{F}, Sh_{\alpha})q^n$ for $q = \exp(2\pi i\tau)$, where $* = (8\sqrt{-1})^{-1}$ if $E_{\mathbb{R}} := E \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R} \times \mathbb{R}$ and * = 4 if $E_{\mathbb{R}} = \mathbb{C}$. If $E_{\mathbb{R}} = \mathbb{C}$, only α with dim $Sh_{\alpha} = 1$ appears.

Note $(\mathcal{F}, Sh_{\alpha}) = \int_{Sh_{\alpha}} \mathcal{F}$. The Tate conjectures only concern Dand E with $D_{\mathbb{R}} = M_2(\mathbb{R})$ and $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$.

Theorem B: $i^{-3}E\mathfrak{m}_1 \frac{L(1,Ad(\rho_f)\otimes\chi_E)}{2^2\pi^3} = (\theta^*(\phi')(f),Sh_1)$ for an explicit constant $E \neq 0$. Here choosing a Haar measure $d\mu_1$ on D_A^{\times} so that it is Dirac on D^{\times} and having volume 1 on $\widehat{\Gamma}_{\phi'} \cap D_A^{\times}C_{\infty}$, $\mathfrak{m} = \mathfrak{m}_1\zeta(2)/\pi$ is defined by $d\mu_1 = (\mathfrak{m}/2)d\omega$ for the Tamagawa measure. Note $0 < \mathfrak{m}_1 \in \mathbb{Q}^{\times}$ by Siegel.

$\S10.$ Strategy.

1. We prove that $f \mapsto \theta^*(\phi)(f)$ is Hecke equivariant for T(p) with p splitting in E/\mathbb{Q} if we choose a Schwartz–Bruhat function $\phi: D_{\sigma,\mathbb{A}} \to \mathbb{C}$ well.

2. By Chebotarev density, Item 1 is sufficient to see $\mathcal{F} = \theta^*(\phi)(f)$ is a Hecke eigenform for $T(\mathfrak{p})$ for all prime \mathfrak{p} of E giving the Doi–Naganuma lift. Non-vanishing $\mathcal{F} \neq 0$ comes out from Theorem A and Rallis' work (a bit over-simplified).

3. Analyze the Galois representation Π_{π} for the automorphic representation π generated by a general Hecke eigenform \mathcal{F} on Sh_B to see that dim Tate $_K^1[\pi^{(\infty)}]$ has dimension 1 only when $\mathcal{F} = \theta^*(\phi)(f)$ for some f.

4. By Theorem B, if $(Sh_{\alpha}, \mathcal{F}) = 0$ for all $\alpha \in D_{\sigma} \cap B^{\times}$, $\theta_{*}(\phi)(\mathcal{F}) = 0$, contradicting to $\theta^{*}(\phi)(f) \neq 0$. This proves $\sum_{\alpha} \overline{\mathbb{Q}}_{l}[Sh_{\alpha}] \supset dim \operatorname{Tate}_{K}^{1}[\pi^{(\infty)}]$, proving the conjecture.

§11. Further generalizations. As we already mentioned, Sweeting [S22] applied theta correspondence to show the Tate conjecture to some extent for a product of a Hilbert-Siegel modular variety of genus 2 and a Hilbert modular variety.

The use of the theta correspondence for this type of problems seems quite effective. The case of orthogonal Shimura variety associated to quadratic forms of signature (2n, 2) or (2n, 0) at different infinite places of a totally real field F can be treated to some extent; at least, an analog of the adjoint L-value formula is expected in these cases.