* Local p-indecomposability of modular p-adic Galois representations.

Haruzo Hida
Department of Mathematics, UCLA,
Los Angeles, CA 90095-1555, U.S.A.

Second talk at TIFR, November 24, 2022.

*Abstract: A question of R. Greenberg asks if a modular 2-dimensional p-adic Galois representation of a cusp form of weight larger than or equal to 2 is indecomposable over the p-inertia group unless it is induced from an imaginary quadratic field. I start with a survey of the known results and try to reach a brief description of new cases of indecomposability.

§0. **Set-up, assumptions and notations.** Fix a prime $p \geq 3$.

- Fix an absolutely irreducible odd representation $\overline{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ ($\mathbb{F}_{/\mathbb{F}_p}$ finite). $F(\rho) := \overline{\mathbb{Q}}^{\mathrm{Ker}(\rho)}$ for a representation ρ.

(S) $\boxed{\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \overline{\epsilon} \oplus \overline{\delta}}$ (split locally at p); $\boxed{\overline{\delta} \neq \overline{\epsilon}}$; $\overline{\delta}|_{I_p} = 1$.

(T) Ramification index of primes in $F(\overline{\rho})_{/\mathbb{Q}}$ is prime to p;

- $\mathfrak{F}$: the maximal p-profinite extension of $F(\overline{\rho})$ unramified outside p, $\mathcal{G} := \mathrm{Gal}(\mathfrak{F}/\mathbb{Q})$ and $\mathcal{H} := \mathrm{Gal}(\mathfrak{F}/F(\overline{\rho}))$. Fix a decomposition subgroup $D = D_l \subset \mathcal{G}$ of l with inertia I_l.
- The p-ordinary universal deformation $(R, \boldsymbol{\rho}: \mathcal{G} \to \mathrm{GL}_2(R))$ over the category CL of local p-profinite W-algebras with residue field $\mathbb{F}$. Let $W = W(\mathbb{F})$ (Witt vectors). So the functor $\mathcal{D}(A)$ given by

$$\{\rho: \mathcal{G} \to \mathrm{GL}_2(A) | \rho \bmod \mathfrak{m}_A = \overline{\rho},\ \rho|_{D_p} = \begin{pmatrix} \epsilon & u_\rho \\ 0 & \delta \end{pmatrix}, \delta|_{I_p} = 1\}/\Gamma(\mathfrak{m}_A)$$

is isomorphic to $A \mapsto \mathrm{Hom}_{CL}(R, A)$. We assume $\boxed{R = \mathbb{T}}$ for a Hecke algebra $\mathbb{T}$; so, $\boxed{R \text{ is free of finite rank over } \Lambda}$.

- $\nu_p: \mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \twoheadrightarrow \mathbb{Z}_p^\times$: the p-adic cyclotomic character.

§**1. Greenberg's conjecture:** If f of weight $k \geq 2$ without CM, then its p-adic Galois representation $\rho_f|_{I_p} \in \mathcal{D}(W(\overline{\mathbb{F}}_p))$ is indecomposable.

Let $\mathfrak{m} = \{x \in W(\overline{\mathbb{F}}_p) : |x|_p < 1\}$. For an ordinary elliptic curve $E_{/W(\overline{\mathbb{F}}_p)}$ with $\overline{E}_{/\overline{\mathbb{F}}_p} = E \otimes_{W(\overline{\mathbb{F}}_p)} \overline{\mathbb{F}}_p$, by Serre and Tate,

$$\mathcal{D}_E := \frac{\{\mathcal{E}_{/W(\overline{\mathbb{F}}_p)} : \text{ elliptic curve}|\mathcal{E} \mod \mathfrak{m} \cong \overline{E}\}}{\text{isomorphisms}} \xrightarrow[i]{\sim} (1+\mathfrak{m}).$$

Any $\mathcal{E} \in \mathcal{D}_E$ has Galois representation $\rho_{\mathcal{E}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathrm{GL}_2(\mathbb{Z}_p)$ ($K :=$ Frac$(W(\overline{\mathbb{F}}_p))$ such that $\rho_{\mathcal{E}} = \begin{pmatrix} \epsilon_{\mathcal{E}} & u_{\mathcal{E}} \\ 0 & \delta_{\mathcal{E}} \end{pmatrix}$ with 1-cocycle $u_{\mathcal{E}}$ with values in $\mathbb{Z}_p(1)_{/W(\overline{\mathbb{F}}_p)}$. If $i(\mathcal{E}) = u$, $u_{\mathcal{E}} \mod p^n$ is a Kummer cocycle over the p-inertia subgroup associated to $\sqrt[p^n]{u}$ for all n; so, $i(\mathcal{E}) = 0 \Leftrightarrow \mathcal{E}$ has CM by an order maxmal at p of an imaginary quadratic field.

$$(1+\mathfrak{m}) : - - - - - - - - - - - - \overset{\text{CM}}{1} - - - - \overset{E}{\bullet} - - - - - - - - - .$$

§**2. Abelian variety of type** GL(2). Assume that an abelian variety A over a number field K is of GL(2)-type (i.e., $O_A := \mathrm{End}(A_{/\overline{\mathbb{Q}}})$ is the integer ring of another number field with degree $\dim A$). Let $\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2(O_{\mathfrak{p}})$ be the $\mathfrak{p}$-adic Galois representation of A for a prime $\mathfrak{p}|p$ of O_A.

Theorem 1 (H., JAMS 26 (2013), Zhao AIF 64 (2014))**.** *If $\rho_{\mathfrak{p}} = \left(\begin{smallmatrix} \epsilon_A & u_A \\ 0 & \delta_A \end{smallmatrix}\right)$ over a decomposition group at p with unramified δ_A, then $\rho_{\mathfrak{p}}$ is indecomposable over p-inertia group. So if f has weight 2 without CM, then $\rho_f|_{I_p}$ is indecomposable.*

Such abelian varieties are parameterized $\mathcal{D}_A = (1+\mathfrak{m}) \otimes_{\mathbb{Z}} O_A = \prod_{\mathfrak{p}} (1+\mathfrak{m}) \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$ which has axis corresponding $\mathfrak{p}$'s. So one needs to show that A is not on any of the axis (so, the proof is far involved via the use of rationality of Hilbert modular variety).

§3. Local one generator theorem. Choosing $\phi \in \mathcal{G}$ with $\phi|_{\mathbb{Q}_p^{ab}} = [p, \mathbb{Q}_p]$, we have $\boxed{\rho_{\mathbb{T}}(\phi) = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}}$. Let $\Lambda[a^2] \subset \mathbb{T}$ be the closed $\mathbb{Z}_p[[T]]$-subalgebra generated by a^2 inside $\mathbb{T}$. By Iwasawa's local one generator theorem (1973), for the wild inertia subgroup $I_p^w \subset I_p$, we know

$$\rho_{\mathbb{T}}(I_p^w) = \left\{ \begin{pmatrix} t^{\mathbb{Z}_p} & \Lambda[a^2]\theta \\ 0 & 1 \end{pmatrix} \right\} \text{ for } \theta \in \mathbb{T}, \quad \boxed{\text{What is this } \theta?}$$

Then I_p-indecomposability for $\rho_\lambda = \lambda \circ \rho_{\mathbb{T}}$ is equivalent to: for any $\lambda \in \mathrm{Hom}_{CL}(\mathbb{T}, A)$ of weight $k \geq 2$,

$$\boxed{\mathrm{Ker}(\lambda) \nmid \theta \Leftrightarrow \lambda(\theta) \neq 0.}$$

The density of weight 2 forms combined with Zhao's local indecomposability theorem, θ is a non-zero divisor of the non-CM component $\mathbb{T}^{ncm}$ of $\mathbb{T}$, and hence there is only finitely many $\lambda \in \mathrm{Spec}(\mathbb{T}^{ncm})(\overline{\mathbb{Q}}_p)$ with $\lambda(\theta) = 0$. This slightly generalizes a result of Ghate–Vatsal in 2004:

Theorem 2. *Without assuming* (S) *and* (T), *for almost all* $\lambda \in \mathrm{Spec}(\mathbb{T}^{ncm})(\overline{\mathbb{Q}}_p)$, *if ρ_λ is non CM, then $\rho_\lambda|_{I_p}$ is indecomposable.*

§**4. Global one generator theorem.** The number r of generators of $\mathbb{T}$ over Λ is given by $r = \dim_{\mathbb{F}} \mathrm{Sel}_{\mathbb{Q}}(Ad(\overline{\rho}))$, where $Ad(\overline{\rho})$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ with Galois action $x \mapsto \overline{\rho}(g)x\overline{\rho}(g)^{-1}$. The Selmer group $\mathrm{Sel}_M(Ad(\rho))$ for $M \subset \mathfrak{F}$ with $\mathcal{G}_M := \mathrm{Gal}(\mathfrak{F}/M)$ is

$$\mathrm{Sel}_M(Ad(\overline{\rho})) := \mathrm{Ker}(H^1(\mathcal{G}_M, Ad(\overline{\rho})) \to \prod_{\mathfrak{p}|p} \frac{H^1(D_{\mathfrak{p}}, Ad(\overline{\rho}))}{F_-^+ H^1(D_{\mathfrak{p}}, Ad(\overline{\rho}))}),$$

where $F_-^+ H^1(D_{\mathfrak{p}}, Ad(\overline{\rho}))$ is made of cohomology classes upper triangular over $D_{\mathfrak{p}}$ and upper nilpotent over $I_{\mathfrak{p}}$. Let $F := F(Ad(\overline{\rho}))$ with integer ring O. Let $\widehat{O_p^\times} = \varprojlim_n O_p^\times/(O_p^\times)^{p^n}$.

Theorem 3. *Suppose* $\overline{\epsilon}\overline{\delta}^{-1} \neq \mathbb{F}(1)$ *and* $Cl_F \otimes_{\mathbb{Z}[G]} Ad(\overline{\rho}) = 0$ *for* $G := \mathrm{Gal}(F/\mathbb{Q})$*, where* Cl_F *is the class group of* F*. Then* $r \leq 1$*, and if* $r = 1$*,* $\mathbb{T} = \Lambda[X]/(D(X)) = \Lambda[\Theta]$ *for the characteristic polynomial* $D(X)$ *of* $\mathbb{T} \ni x \mapsto \Theta x$ *for* $\Theta \in \mathfrak{m}_{\mathbb{T}}$.

What is this Θ? Are there infinitely many locally split p for a given f? Is the vanishing of $Cl_F[Ad(\overline{\rho})] := Cl_F \otimes_{\mathbb{Z}[G]} Ad(\overline{\rho}) = 0$ true for infinitely many locally split p?

§5. Proof/example. By CFT, $\widehat{O}_p^\times \to \mathcal{G}_F^{ab} \to Cl_{F,p} \to 0$ and

$0 = \mathrm{Hom}_{\mathbb{Z}[G]}(Cl_F, Ad(\overline{\rho})) \hookrightarrow \mathrm{Sel}_F(Ad(\overline{\rho}))^G \xrightarrow{\pi} \mathrm{Hom}_{\mathbb{Z}_p[G]}(\widehat{O_p^\times}, Ad(\overline{\rho}))$

for $\widehat{O}_p^\times = \varprojlim_n (O_p^\times)/(O_p^\times)^{p^n}$ are exact. By tameness of p in $F/\mathbb{Q}$ and $\mu_p(\overline{\mathbb{Q}}_p) \not\subset O_\mathfrak{p}^\times$, $\widehat{O_p^\times} \cong \mathbb{Z}_p[G] = \mathrm{Ind}_1^G \mathbb{Z}_p$. By Shapiro's lemma

$$\mathrm{Hom}_{\mathbb{Z}_p[G]}(\widehat{O_p^\times}, Ad(\overline{\rho})) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, Ad(\overline{\rho})) \cong Ad(\overline{\rho})$$

where $\pi(\mathrm{Sel}(Ad(\overline{\rho}))) \hookrightarrow \mathfrak{n} = \{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\} \subset Ad(\overline{\rho})$. □

An example: Let $K = \mathbb{Q}[\sqrt{D}]$ be a quadratic field with split $(p) = \mathfrak{p}\mathfrak{p}^\sigma$. Take $\overline{\rho} = \mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}$ ($\Rightarrow p \nmid [F : \mathbb{Q}]$).

Theorem 4. *Assume K real with a fundamental unit ε. Then $r \geq 1$ and $\theta | (\langle\varepsilon\rangle - 1)$ $(t = 1 + T)$ for $\langle\varepsilon\rangle = t^{\log_p(\varepsilon)/\log_p(1+p)} \in \Lambda$; so, ρ_f in $\mathcal{D}(W)$ is indecomposable over I_p if $k \geq 2$. If $r = 1$, $(\theta) = (\Theta)$.*

§6. Real quadratic field.

Let $K = \mathbb{Q}[\sqrt{D}]$ be real with $\alpha := \left(\frac{K/\mathbb{Q}}{}\right)$ in which p splits into $\mathfrak{p}\mathfrak{p}^{\sigma}$.

Assume $\overline{\rho} = \mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}$ (so, p is tame in $F(\overline{\rho})$). Since $\overline{\rho} \otimes \alpha \cong \overline{\rho}$, $\rho_{\mathbb{T}} \otimes \alpha$ is still in $\mathcal{D}(\mathbb{T})$; so, we have an involution ι : Aut$(\mathbb{T}_{/\Lambda})$ such that $\iota \circ \rho_{\mathbb{R}} \cong \rho_{\mathbb{T}} \otimes \alpha$. The $R \cong \mathbb{T}$ implies $\mathrm{Sel}(Ad(\rho_\lambda))^\vee \cong \Omega_{\mathbb{T}/\Lambda} \otimes_{\mathbb{T},\lambda} A$. If $\rho_\lambda \leftrightarrow f \in S_k$ with weight $k \geq 2$, $\rho_\lambda \otimes \alpha \ncong \rho_\lambda$; so, $\iota \neq$ id; hence $\mathbb{T} \neq \Lambda$ implying $0 < \dim_{\mathbb{F}} \Omega_{\mathbb{T}/\Lambda} \otimes_{\mathbb{T}} \mathbb{F} = \dim_{\mathbb{F}} \mathrm{Sel}(Ad(\overline{\rho}))$.

Put $\overline{\varphi}^-(g) = \overline{\varphi}(g)\overline{\varphi}(\sigma g \sigma^{-1})^{-1}$ for $\sigma \in \mathcal{G}$ non-trivial over K. Since $Ad(\overline{\rho}) = \overline{\alpha} \oplus \mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}^-$, $\mathrm{Sel}(Ad(\overline{\rho})) = \mathrm{Sel}(\overline{\alpha}) \oplus \mathrm{Sel}(\mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}^-)$ with $\iota = 1$ on $\mathrm{Sel}(\overline{\alpha})$ and -1 on $\mathrm{Sel}(\mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}^-)$, and if $p \nmid h_F$,

$$\mathrm{Sel}(\alpha) \cong \mathrm{Hom}(Cl_K, \mathbb{F}) = 0 \quad \text{with} \quad \mathrm{Sel}(\mathrm{Ind}_K^{\mathbb{Q}} \overline{\varphi}^-) = \mathbb{F} d\Theta \neq 0.$$

We may assume that for the image Θ of X in $\mathbb{T}$, $\iota(\Theta) = -\Theta$.

§7. $D(0) = \langle\varepsilon\rangle - 1$.

Let $I = \mathbb{T}(\iota - 1)\mathbb{T} = (\Theta)$ (the different of $\mathbb{T}$ over $\mathbb{T}_{\pm} = \mathbb{T}^{\iota=\pm 1}$). If $\rho \in \mathcal{D}(\mathbb{T}/I)$, then $\rho \otimes \alpha \cong \iota \circ \rho = \rho$, which implies $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \phi$. By $\rho \leftrightarrow \phi$, writing $\rho_{\mathbb{T}}$ mod $I = \mathrm{Ind}_K^{\mathbb{Q}} \Phi$, $(\mathbb{T}/I, \Phi)$ is the universal ring deforming $\overline{\varphi}$ unramified outside $\mathfrak{p}$. Under $p \nmid h_F$, Φ induces $C := Cl(\mathfrak{p}^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \widehat{O_{\mathfrak{p}}^{\times}}/\varepsilon^{(p-1)\mathbb{Z}_p} \cong \Gamma/\Gamma^{\log_p(\varepsilon)/\log_p(1+p)}$. So,

$$\mathbb{T}/(\Theta) = \mathbb{T}/I \cong W[C] \cong W[[T]]/(\langle\varepsilon\rangle - 1).$$

We may assume that $D(X)$ is a monic distinguished polynomial by Weierstrass preparation theorem for $\Lambda[[X]]$. Then

$$\Lambda/(\langle\varepsilon\rangle - 1) \cong \mathbb{T}/(\Theta) = \Lambda[[X]]/(X, D(X)) = \Lambda/(D(0)).$$

Thus we may assume that $D(0) = \langle\varepsilon\rangle - 1$.

Decompose $\mathbb{T} = \mathbb{T}_+ \oplus \mathbb{T}_-$ for the $\pm$-eigenspace of ι.

§**8.** $D(X)$ **is an Eisenstein polynomial.** For $t = 1 + T$, we have $(\langle\varepsilon\rangle - 1) = (t^{p^m} - 1)$ for the minimal m such that $\varepsilon^{(p-1)} \equiv 1 \mod \mathfrak{p}^m$. Thus $(\langle\varepsilon\rangle - 1)$ is square-free non-trivial, $m = 0 \Leftrightarrow \mathbb{T} = W[[\Theta]]$, and $\mathbb{T}/\sqrt{(\langle\varepsilon\rangle - 1)} \cong \Lambda/(\langle\varepsilon\rangle - 1)$ for the radical $\sqrt{(\langle\varepsilon\rangle - 1)}$ of $(\langle\varepsilon\rangle - 1)$ in $\mathbb{T}$. Thus $\mathbb{T}$ fully ramifies in $\mathbb{T}_{/\Lambda}$. After localizing at $P|(\langle\varepsilon\rangle - 1)$, $\mathbb{T}_P = \mathbb{T} \otimes_\Lambda \Lambda_P$ has rank over Λ_P equal to $e = \text{rank}_\Lambda \mathbb{T} = \deg D(X)$; so, $D(X)$ is the characteristic polynomial of $x \mapsto \Theta x$ on $\mathbb{T}/\Lambda$ and also $\mathbb{T}_P/\Lambda_P$. We find $\mathbb{T}_P = \Lambda_P[X]/(D(X))$ and Weierstrass preparation applied to Λ_P, we find $D(X)$ is an Eisenstein polynomial with respect to P; so, $\mathbb{T}[\frac{1}{p}]$ is a Dedekind domain.

For $j = \text{diag}[-1, 1]$, normalize $\rho_\mathbb{T} \cdot \alpha = j\iota \circ \rho_\mathbb{T} j^{-1}$. So, $\rho_\mathbb{T}|_{\text{Ker}(\alpha)}$ has values in $\begin{pmatrix} \mathbb{T}_+ & \mathbb{T}_- \\ \mathbb{T}_- & \mathbb{T}_+ \end{pmatrix} = \begin{pmatrix} \mathbb{T}_+ & \Theta\mathbb{T}_+ \\ \Theta\mathbb{T}_+ & \mathbb{T}_+ \end{pmatrix}$; so, $\theta = \Theta u$ for $u \in \mathbb{T}_+$. Thus $(\Theta\mathbb{T}_+/\theta\mathbb{T}_+) \otimes_{\mathbb{T}_+} \mathbb{F} = \mathbb{T}_+/((u) + \mathfrak{m}_{\mathbb{T}_+}) \hookrightarrow \text{Hom}_{\mathbb{Z}_p[G]}(Cl_F, Ad(\overline{\rho}))$. If $Cl_F[Ad(\overline{\rho})] = 0$, u is a unit; so, $\boxed{Cl_F[Ad(\overline{\rho})] = 0 \Rightarrow \Theta = \theta}$ and without $Cl_F[Ad(\overline{\rho})] = 0$, we can actually show $(\theta)|(\langle\varepsilon\rangle - 1)$. $\square$

§9. **Wall-Sun-Sun primes** (Zhi-Hong Sun and Zhi-Wei Sun). $\mathbb{T}$ is a regular local ring $\Leftrightarrow$ $\langle\varepsilon\rangle - 1$ is a prime $\Leftrightarrow$ $\mathfrak{p} \parallel \varepsilon^{p-1} - 1$. So $\mathbb{T}$ is non-regular $\Leftrightarrow$ $\mathfrak{p}^2 | \varepsilon^{p-1} - 1$.

Take $K = \mathbb{Q}[\sqrt{5}]$. Note that $\mathfrak{p}^2 | \varepsilon^{p-1} - 1 \Leftrightarrow \mathfrak{p}^2 | \varepsilon^{2(p-1)} - 1$, and

$$\varepsilon^{2(p-1)} - 1 = \varepsilon^{p-1}(\varepsilon^{p-1} - \varepsilon^{\sigma(p-1)}) = \varepsilon^{p-1} f_{p-1}\sqrt{5}$$

for Fibonacci number f_{p-1} of index $p-1$; so,

$\mathbb{T}$ is non-regular $\Leftrightarrow p^2 | f_{p-1} \Leftrightarrow p$ is a Wall-Sun-Sun prime.

No Wall-Sun-Sun prime is known up to 2.6×10^{17}.

However for $K = \mathbb{Q}[\sqrt{10}]$, $p = 191,643$ satisfies $\mathfrak{p}^2 | \varepsilon^{p-1} - 1$. An analytic number theorist (J. Klaška) conjectures that there are infinitely many Wall-Sun-Sun primes (but density 0). If true, for the Artin representation $\mathrm{Ind}_K^{\mathbb{Q}} \varphi$, $\mathbb{T}$ is non-regular for infinitely many but very thinly populated primes.

§10 What happens if K is imaginary?

As in the real case, we assume that $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ in imaginary K. Let $L_{\mathfrak{p}}$ be the anticyclotomic Katz p-adic L-function with branch character $\overline{\varphi}^{-}(g) = \overline{\varphi}(g)\overline{\varphi}(cgc)^{-1}$ for complex conjugation c, giving the characteristic ideal of the Iwasawa module unramified outside $\mathfrak{p}$ over the anti-cyclotomic tower. The power series lives in $\Lambda_{\overline{W}} := \overline{W}[[T]] = \Lambda\widehat{\otimes}_{\mathbb{Z}_p}\overline{W}$ for the Witt vector ring $\overline{W}$ of $\overline{\mathbb{F}}_p$.

In this case, $\mathrm{Spec}(\mathbb{T}) = \mathrm{Spec}(\mathbb{T}^{cm}) \cup \mathrm{Spec}(\mathbb{T}^{ncm})$ with $\mathrm{Spec}(\mathbb{T}^{cm})$ is a union of CM components. We still have the involution ι with $\mathbb{T}(\iota - 1)\mathbb{T} = (\Theta) \subset \mathbb{T}^{ncm}$ if $Cl_F[Ad(\overline{\rho})] = 0$. We need to extend scalar to W and to replace $\langle\varepsilon\rangle - 1$ by the anticyclotomic Katz p-adic L-function $L_{\mathfrak{p}}$ Then $\mathbb{T}^{ncm} = \Lambda[\Theta]$ as before for Θ with $\Theta^{\iota} = -\Theta$.

§**11. Imaginary K.** For simplicity, assume $\overline{\varphi}^-$ has order ≥ 3.

Another Greenberg's conjecture: $W[[T]]/(L_{\mathfrak{p}}, L_{\overline{\mathfrak{p}}})$ is finite for the Katz p-adic $L_?$ for the Coates Wiles tower unramified outside $? = \mathfrak{p}, \overline{\mathfrak{p}}$.

Theorem 5. *Assume $\mathbb{T}^{ncm} \neq 0$.*
(1) *If $Cl_F[Ad(\overline{\rho})] = 0$, $\mathbb{T}^{ncm} = \Lambda[X]/(D(X))$ is an integral domain, $(\theta) = (\Theta)$, $(\theta)|(L_{\mathfrak{p}})$ and $\rho_f|_{I_p}$ is indecomposable if ρ_f does not have CM.*
(2) *Suppose the above conjecture. Then $\rho_f|_{I_p}$ is indecomposable if ρ_f does not have CM.*

When K is real, we always have $\mathbb{T} \neq \Lambda$. When K is imaginary and $p \nmid h_K$, $\mathbb{T} \neq \Lambda \Leftrightarrow \mathbb{T}^{ncm} \neq 0$. For a fixed finite order character φ with ordinary $\mathrm{Ind}_K^{\mathbb{Q}} \varphi$, are there infinitely many p such that $\mathbb{T} \neq \Lambda$?