* Adjoint L-value formula and its relation to Tate conjecture

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*First talk at TIFR, Mumbai (India), November 22, 2022.

Abstract: For a Hecke eigenform f, we state an adjoint L-value formula relative to each quaternion algebra D over $\mathbb{Q}$ with discriminant ∂ and reduced norm N. A key to prove the formula is the theta correspondence for the quadratic $\mathbb{Q}$-space (D, N). Under the $R = \mathbb{T}$-theorem, p-part of the Bloch-Kato conjecture is known; so, the formula is an adjoint Selmer class number formula. We also describe how to relate the formula to a consequence of the Tate conjecture for quaternionic Shimura varieties.

§**0. Setting of the simplest case.** Assume that D is definite. Let $G_D(\mathbb{Q}) = \{(h_l, h_r) \in D^\times \times D^\times | N(h_l) = N(h_r)\}$ which acts on D by $v \mapsto h_l^{-1} v h_r$; so, $\mathrm{SO}_D(\mathbb{Q}) = G_D(\mathbb{Q})/Z_D(\mathbb{Q})$ for the center $Z_D \subset G_D$. For a Bruhat function ϕ on $D_{\mathbb{A}}^{(\infty)}$ and $\mathbf{e}(N(v_\infty)\tau)$ ($\tau \in \mathfrak{H}$: the upper half complex plane), the theta series

$$\theta(\phi;\tau;h_l,h_r) = \sum_{\alpha \in D} \phi(h_l^{-1}\alpha h_r)\mathbf{e}(N(\alpha_\infty)\tau) \quad \text{on } \mathfrak{H} \times G_D(\mathbb{A})$$

can be extended to an automorphic form on $Y_\Gamma \times Sh \times Sh$ for $Sh := D^\times \backslash D_{\mathbb{A}}^\times / D_\infty^\times$ and $Y_\Gamma := \Gamma \backslash \mathfrak{H}$ for a congruence subgroup Γ. For a weight 2 cusp form $f \in S_2(\Gamma)$ and automorphic forms $\mathcal{F}, \mathcal{G} : Sh \to \mathbb{C}$, we define

$$\theta^*(\phi)(f)(h_l,h_r) = \int_{Y_\Gamma} f(\tau)\theta(\phi)(\tau;h_l,h_r)y^{-2}dxdy, \ (h_l,h_r \in D_{\mathbb{A}}^\times)$$

$$\theta_*(\phi)(\mathcal{F}\otimes\mathcal{G})(\tau) = \int_{Sh\times Sh} \theta(\phi)(\tau;h_l,h_r)(\mathcal{F}(h_l)\cdot\mathcal{G}(h_r))d\mu_l d\mu_r.$$

We choose the Haar measure $d\mu_?$ on $D_{\mathbb{A}}^\times$ suitably. We call $\theta^*(\phi)(f) : Sh \times Sh \to \mathbb{C}$ a theta lift and $\theta_*(\phi)(\mathcal{F}\otimes\mathcal{G}) \in M_2(\Gamma)$ a theta descent.

§**1. Two good choices of ϕ.** Let R be an Eichler order of level N for N with $(N, \partial) = 1$.

First choice: At N, we identify $R/NR = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \right\}$. Let ϕ_R be the characteristic function of

$$\left\{ x \in \widehat{R} | x \bmod NR = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},\ d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

Then the first choice is

$$\phi(v) = \phi_R(v^{(\infty)})\mathbf{e}(N(v_\infty)\tau)$$

as a Schwartz-Bruhat function on $D_\mathbb{A}$. In this case, $\Gamma = \Gamma_0(\partial N)$.

Second choice: Let ϕ_L be the characteristic function of $\widehat{L}$. Choose $0 < c \in \mathbb{Z}$ and $L := \widehat{\mathbb{Z}} \oplus \widehat{R}_0$ for $R_0 = \{v \in R | \mathrm{Tr}(v) = 0\}$, and put $\phi'_{R_0} = (1 - c^3)^{-1}(\phi_{R_0} - \phi_{cR_0})$. Then we define, writing $v = z \oplus w$ with $z \in Z_\mathbb{A}$ and $w \in D_{0,\mathbb{A}}$

$$\phi'(v) = \phi_\mathbb{Z}(z^{(\infty)})\phi'_{R_0}(w^{(\infty)})\mathbf{e}(N(v_\infty)\tau).$$

We have $\Gamma = \Gamma_0(4c^2\partial N)$.

§2. **Two theorems.** Let $Sh_R = Sh/\widehat{R}^\times$ and $\delta(Sh_R)$ is the diagonal image of Sh_R in $Sh_R \times Sh_R$.

Theorem A: *Assume that* $\int_{Sh_R} \mathcal{F} d\mu = \int_{Sh_R} \mathcal{G} d\mu = 0$ *(a cuspidal condition). If* $L = R_0(N)$, *then* $\theta_*(\phi)(\mathcal{F} \otimes \mathcal{G}) = \sum_{n=1}^\infty (\mathcal{F}, \mathcal{G}|T(n))q^n$ *for* $q = \exp(2\pi i\tau)$ *with* $(\mathcal{F}, \mathcal{G}) = \int_{\delta(Sh_R)} \mathcal{F}(h)\mathcal{G}(h) d\mu_h$. *So the maps* $\theta_*(\phi)$ *and* $\theta^*(\phi)$ *are Hecke equivariant.*

Theorem B: *If* f *is a Hecke eigen new form of* $S_2(\Gamma_0(\partial N))$, *then for the canonical period* $\Omega_\pm$ *of* f,

$$\prod_{p|\partial}(1-p^{-2})^{-1}\mathfrak{m}_1 \frac{L(1, Ad(\rho_f))}{2\pi^3 \Omega_+ \Omega_-} = \int_{\delta(Sh_R)} \frac{\theta^*(\phi')(f)(h)}{\Omega_+ \Omega_-} d\mu,$$

where $\mathfrak{m}_1$ *is the mass factor of* $L \cap D_0$*:* $\mathfrak{m}_1 \frac{\zeta(2)}{\pi^2} = \int_{Sh_R} d\mu \in \mathbb{Q}$ (Siegel's mass formula) *and if* $\partial = p$ *with* $N = 1$, $\mathfrak{m}_1 = (p-1)/2$.

Theorem B is independent of c. Under $R = \mathbb{T}$ theorem at a prime p, p-primary Bloch-Kato conjecture known for $Ad(\rho_f)$; so, Theorem B is an adjoint Selmer class number formula.

§3. Analogy to Class number formula and Mass formula.

The Shimura set Sh_R is isomorphic canonically to the set Cl_D of right ideal classes of R. For each representative $\mathfrak{a} \in Cl_D$, write $R_{\mathfrak{a}} := \{\alpha \in D | \alpha\mathfrak{a} \subset \mathfrak{a}\}$ for the left order of $\mathfrak{a}$. Let $w(R_{\mathfrak{a}}) := |R_{\mathfrak{a}}^{\times}|$. Similarly, for an imaginary quadratic field $K := \mathbb{Q}[\sqrt{-d}]$ with discriminant $-d < 0$. Write Cl_K for its class group, and put $w = |O_K^{\times}|$ for the integer ring O_K of K. We have the following three analogous formulas:

$$\prod_{p|\partial}(1-p^{-2})^{-1}\mathfrak{m}_1\frac{L(1,Ad(\rho_f))}{2\pi^3} = \sum_{\mathfrak{a}\in Cl_D}\frac{\theta^*(\phi')(f)(h)}{w(R_{\mathfrak{a}})} \text{ (Theorem B)},$$

$$\mathfrak{m}_1\frac{\zeta(2)}{\pi^2} = \sum_{\mathfrak{a}\in Cl_D}\frac{1}{w(R_{\mathfrak{a}})} \text{ (Siegel)},$$

$$\frac{\sqrt{d}L(1,\left(\frac{-d}{}\right))}{2\pi} = \sum_{\mathfrak{a}\in Cl_K}\frac{1}{w} \text{ (Dirichlet)}$$

§**4. Higher weight k and indefinite case.** For a higher weight k or an indefinite case, we need to replace the Schwartz function $\mathbf{e}(N(v)\tau)$ by a standard Schwartz function of Siegel–Shimura by multiplying a vector valued spherical function for (D, N) and then in the indefinite case, we modify $\theta(\varphi)$ (for $\varphi = \phi, \phi'$) and $\mathcal{F}$ and $\mathcal{G}$ into vector valued differential forms by the Eichler–Shimura map. Then $\mathcal{F}$ and $\mathcal{G}$ are closed harmonic 1-forms with values in a locally constant sheaf $\mathcal{L}_{n/A}$ whose fiber is the symmetric n-th tensor representation over an appropriate ring A for $n = k - 2$. We replace $(\mathcal{F}, \mathcal{G})$ by the cup product $(\mathcal{F}, \mathcal{G})_n := \int_{\delta(Sh_R)} \mathcal{F} \cup \mathcal{G}$ of $H^*(Sh_R, \mathcal{L}_{n/\mathbb{C}}) \times H^*(Sh_R, \mathcal{L}_{n,\mathbb{C}}) \to H^{2*}(Sh_R, \mathbb{C}) = \mathbb{C}$ in Theorem A ($* = 0, 1$ definite or indefinite).

In Theorem B, we pull back the class $\theta(\phi')^*(f)$ on $Sh_R \times Sh_R$ to $\delta(Sh_R)$ and integrate over $\delta(Sh_R)$. Then Theorem B is valid in general.

§**5. Canonical periods.** If D is indefinite, Sh_R is a Shimura curve defined over $\mathbb{Z}[\frac{1}{\partial}]$. Let A be a DVR at a prime $\mathfrak{p}$ such that $\mathbb{Z}[\frac{1}{\partial},\lambda] = \mathbb{Z}[\frac{1}{\partial},\lambda(T(n))|n \in \mathbb{Z}] \subset A \subset \mathbb{Q}[\lambda] :=$ Frac$(\mathbb{Z}[\frac{1}{\partial},\lambda])$ of f (i.e., $f|T(n) = \lambda(T(n))f$). Define $\mathcal{F}_\pm$ by

$$H_\lambda := H^1(Sh_R, \mathcal{L}_{n/A})[\lambda,\pm] = A[\mathcal{F}_\pm],$$

where $\pm$ means the $\pm$-eigenspace of complex conjugation. Put $H = H_\pm := H^1(Sh_R, \mathcal{L}_{n/A})[\pm]$. Define $\mathcal{F}$ by $H^0(Sh_{R/A}, \omega^k) = A\mathcal{F}$ for the weight k Hodge bundle ω^k. Then we project it to a unique element $\omega^\pm(\mathcal{F})$ of the $\pm$-eigenspace $H^1(Sh_R, \mathcal{L}_{n/\mathbb{C}})[\lambda,\pm]$ of complex conjugation and define $\Omega_\pm^D \in \mathbb{C}^\times$ by $\omega^\pm(\mathcal{F}) = \Omega_\pm^D[\mathcal{F}_\pm]$. The period $\Omega_\pm$ in Theorem B is $\Omega_\pm^{M_2(\mathbb{Q})}$. We just put $\Omega_\pm^D = 1$ if D is definite.

Tate conjecture predicts $\Omega_\pm^D/\Omega_\pm \in \mathbb{Q}[\lambda]^\times$ if D is indefinite.

The conjecture is known for $k = 2$ by Faltings and by Prasanna for $k \geq 2$ but our formulation is more explicit.

§**6. Relation to Tate conjecture.** Assume that D is indefinite. Let E be one of $H^1(Sh_R,\mathcal{L}_{n/\mathbb{Q}[\lambda]})[\pm]$. Decompose $E\otimes_A\mathbb{Q}=E_\lambda\oplus E_\lambda^\perp$ into λ-eigenspace E_λ and its Hecke stable complement, and write $\widetilde{H}_\lambda$ for the projection of H to E_λ. Define $c_D := (\mathcal{F}_+,\mathcal{F}_-)_n$ which is called cohomological D-congruence number, and $\widetilde{H}_\lambda/H_\lambda \cong A/c_DA$. Under the $R=\mathbb{T}$-theorem,

$$(*)\quad (\mathcal{F}_+,\mathcal{F}_-)_n = c_D \overset{R=\mathbb{T}}{=} c_{M_2(\mathbb{Q})} = \frac{L(1,Ad(\rho_f))}{2\pi^3\Omega_+\Omega_-}\quad \text{(up to A-units).}$$

By Theorem A, for $u_\pm^D\in\mathbb{C}^\times$, $\theta_D^*(\phi)(f)=u_+^D\mathcal{F}_+\otimes u_-^D\mathcal{F}_-$. Thus

$$L(1,Ad(\rho_f)) \overset{\text{Theorem B}}{\doteqdot} \int_{\delta(Sh_R)}\theta_D^*(\phi')(f)$$
$$= u_+^Du_-^D(\mathcal{F}_+,\mathcal{F}_-)_n \overset{(*)}{\doteqdot} u_+^Du_-^D\frac{L(1,Ad(\rho_f))}{\Omega_+\Omega_-}.$$

Thus $u_+^Du_-^D/\Omega_+\Omega_-\in A^\times$. Thus if $u_+^Du_-^D=\Omega_+^D\Omega_-^D$ (i.e. $u_\pm^D\mathcal{F}_\pm=\omega^\pm(\mathcal{F})\Leftrightarrow\theta_D^*(\phi')=\omega^+(\mathcal{F})\otimes\omega^-(\mathcal{F})$), the A-integral Tate conjecture in this case holds (which Prasanna studied in a different way).

§7. Proof of Theorem A. Let $h_k(\partial N; A)$ be the subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_0(\partial N)))$ generated over A by Hecke operators $T(n)$ and $S_k(\Gamma_0(\partial N); A) = S_k(\Gamma_0(\partial N)) \cap A[[q]]$. Recall

Duality theorem *The space $S := S_k(\Gamma_0(\partial N); A)$ is A-dual of $H := h_k(\partial N; A)$ such that for a linear form $\phi : h_k(\partial N; A) \to A$, $\boxed{\sum_{n=1}^{\infty} \phi(T(n))q^n} \in S_k(\Gamma_0(\partial N); A)$. Writing $f = \sum_{n=1}^{\infty} a(n, f)q^n \in S$, the pairing $\langle \cdot, \cdot \rangle : H \times S \to A$ is given by $\langle h, f \rangle = a(1, f|h)$.*

By Jacquet-Langlands correspondence, $H^*(Sh_R, \mathcal{L}_{n/A})$ is a module over $h_k(\partial N; A)$. Then applying the above theorem to the linear form $h_k(\partial N; A) \ni h \mapsto (\mathcal{F}, \mathcal{G}|h)_n$, we get Theorem A.

For the proof of Theorem B, we resort to an idea of Waldspurger.

§**8. An idea of Waldspurger.** Computing the period of $\theta^*(\phi')(f)$ for a quadratic space $V = W \oplus W^\perp$ over an orthogonal Shimura subvariety $S_W \times S_{W^\perp} \subset S_V$ has two steps:
(S) Split $\theta(\phi')(\tau, h, h^\perp) = \theta(\varphi)(\tau, h) \cdot \theta(\tau, \varphi^\perp)(h^\perp)$ ($h^? \in \mathrm{O}_{W^?}(\mathbb{A})$) for a decomposition $\phi' = \varphi \otimes \varphi^\perp$ (φ and $\varphi^\perp$ Schwartz–Bruhat functions on $W_\mathbb{A}$ and $W_\mathbb{A}^\perp$);
(R) For the theta lift $(\theta^*(\phi')(f))(h) = \int_Y f(\tau)\theta(\phi')(\tau, h)d\mu$ with a modular curve Y, the period P over the Shimura subvariety $S \times S^\perp$ (S for $\mathrm{O}(W)$ and $S^\perp$ for $\mathrm{O}(W^\perp)$) is given by:

$$\int_{S\times S^\perp}\int_Y f(\tau)\theta(\phi')(\tau;h)d\mu(\tau)dh \quad (d\mu(\tau) = y^{-2}dxdy)$$
$$= \int_Y f(\tau)\left(\int_{S^\perp}\theta(\varphi^\perp)(\tau;h^\perp)dh^\perp\right)\cdot\left(\int_S \theta(\varphi)(\tau;h_0)dh\right)d\mu.$$

Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\varphi)$ and $E(\varphi^\perp)$, reaching Rankin-Selberg integral

$$P = \int_Y f(\tau)E(\varphi^\perp)E(\varphi)d\mu = L\text{-value}.$$

§**9.** D **definite and** $n = 0$**.** For simplicity, we assume that D is definite and $n = 0$. Then $D = Z \oplus D_0$ for the center Z and $D_0 := \{v \in D | \mathrm{Tr}(v) = 0\}$. So $W = Z$ and $W^\perp = D_0$.

Identifying $Z = \mathbb{Q}$, $N|_Z(v) = v^2$ and for $\varphi := \phi_{\mathbb{Z}}$, $E(\varphi) = \sum_{n=-\infty}^{\infty} q^{n^2}$ (Jacobi's theta series).

Write $\mathbf{w}$ for the Weil representation of the metaplectic group $\mathrm{Mp}(\mathbb{A})$ fits into the exact sequence $S^1 \hookrightarrow \mathrm{Mp}(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$ on the space of Schwartz-Bruhat functions on $D_{0,\mathbb{A}}$. The rational subgroup $\mathrm{SL}_2(\mathbb{Q})$ can be lifted into the metaplectic cover $\mathrm{Mp}(\mathbb{A})$ canonically. Then for the upper triangular Borel subgroup B, Siegel–Weil Eisenstein series is defined by

$$E(\varphi^\perp)(g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} (\mathbf{w}(\gamma g)\varphi^\perp)(0) \ (g \in \mathrm{Mp}(\mathbb{A})).$$

As Weil verified, $g \mapsto (\mathbf{w}(g)\varphi^\perp)(0)$ is $B(\mathbb{Q})$ left invariant. In the definite case, $E(\varphi^\perp)$ converges to a weight $\frac{3}{2}$ Eisenstein series.

§**10. Siegel–Weil formula.** Note that the even Clifford group G_{D_0} of D_0 is equal to $D^\times$ which acts on D_0 by $v \mapsto g^{-1}vg$. Thus $\mathrm{SO}_{D_0} = G_{D_0}/Z(G_{D_0})$; so, $S^\perp = Sh$ for our choice of ϕ'_{R_0}. Let $d\omega$ be the Tamagawa measure on $\mathrm{SO}_{D_0}(\mathbb{A})$.

Siegel–Weil formula: $\int_{Sh} \theta(\varphi^\perp)(\tau; h)d\omega(h) = E(\varphi^\perp)$.

In the indefinite case, $E(\varphi^\perp)$ does not converge; so, we need an auxiliary complex variable s to analytically continue the Eisenstein series to $s \in \mathbb{C}$ and evaluate at $s = 0$ to have $E(\varphi^\perp)$.

Let $d\mu$ be the Haar measure on $\mathrm{SO}_{D_0}(\mathbb{A})$ with volume 1 on $\widehat{R}^\times$ (which we used already). Basically by definition, $d\mu = (\mathfrak{m}/2)d\omega$ for the mass $\mathfrak{m} := \int_{Sh_R} d\mu = \mathfrak{m}_1 \frac{\zeta(2)}{\pi^2}$. This is the reason to have $\mathfrak{m}_1 \in \mathbb{Q}^\times$ in front of the formula in Theorem B. The value $\mathfrak{m}_1$ is computed explicitly by Shimura in 1999 for general N after the early work of Siegel for the maximal order.

§**11. Conclusion.** For general $\varphi^\perp$, $E(\varphi^\perp)$ is the sum of the Eisenstein series $E_\infty(\varphi^\perp)$ induced from B (the infinity cusp) and $E_0(\varphi^\perp)$ induced from ${}^tB$ (the zero cusp). For the Rankin convolution, $\int_Y f\theta(\varphi)E_0(\varphi^\perp)d\mu_\tau$ causes a trouble. Our choice of ϕ'_{R_0} introducing $0 < c \in \mathbb{Z}$ is made to have the vanishing $E_0(\phi'_{R_0}) = 0$ and the identity $E_\infty(\phi'_{R_0}) = E_\infty(\phi_{R_0})$. The Rankin convolution $\int_Y f\theta(\varphi)E_\infty(\phi_{R_0})d\mu_\tau$ is computed in 1976 by Shimura and produces the adjoint L-value in Theorem B. All computation can be generalized to the Hilbert modular case over a totally real field F and a quaternion algebra $D_{/F}$.

A refinement of Tate conjecture is conjectured by Shimura-Blasius-Yoshida and studied by Harris over $\overline{\mathbb{Q}}$. I hope to be able to factorize $\Omega_+^D\Omega_-^D$ into $\prod_\sigma \Omega_+^{D_\sigma}\Omega_-^{D_\sigma}$ integrally over A for infinite places σ with $D\otimes_{\sigma,F}\mathbb{R} \cong M_2(\mathbb{R})$, where D_σ is a quaternion algebra with $D \otimes_{F,\sigma} \mathbb{R} \cong M_2(\mathbb{R})$ only at σ (Period factorization).