* Adjoint L-value as period integrals

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Abstract: For a quaternion algebra $D/\mathbb{Q}$ and a quadratic field $E = \mathbb{Q}[\sqrt{\Delta}]/\mathbb{Q}$, we compute as $L(1, \text{Ad}(\rho_F) \otimes (\Delta))$ the period of Doi-Naganuma lift of an elliptic Hecke eigen new form $F$ (of conductor $C$) to the quaternionic Shimura variety associated to $D \otimes_{\mathbb{Q}} E$ over Shimura subvarieties associated to $D$. Here $\rho_F$ is the compatible system of Galois representations associated to $F$. 
§0. An idea of Waldspurger. For an elliptic cusp form $F$, an idea of Waldspurger of computing the period of a theta lift of $F$ for a quadratic space $V = W \oplus W^\perp$ over an orthogonal Shimura subvariety $S_W \times S_{W^\perp} \subset S_V$ is two-folds:

(S) Split $\theta(\Phi)(\tau, h, h^\perp) = \theta(\phi)(\tau, h) \cdot \theta(\tau, \phi^\perp)(h^\perp)$ ($\tau = \xi + \eta \sqrt{-1} \in \mathfrak{H}$ with $\eta > 0$ and $h^? \in O_{W^?}(\mathbb{A})$) for a decomposition $\Phi = \phi \otimes \phi^\perp$ ($\phi$ and $\phi^\perp$ Schwartz–Bruhat functions on $W_\mathbb{A}$ and $W^\perp_\mathbb{A}$);

(R) For the theta lift $\Theta(F)(h) = \int_X F(\tau) \theta(\phi)(\tau, h) d\mu$ with an $SL(2)$-Shimura curve $X$, the period $P$ over the Shimura subvariety $S \times S^\perp$ ($S$ for $O(W)$ and $S^\perp$ for $O(W^\perp)$) is given by:

$$\int_{S \times S^\perp} \int_X F(\tau) \theta(\phi)(\tau; h) d\mu dh \quad (d\mu = \eta^{-2} d\xi d\eta)$$

$$= \int_X F(\tau) \left( \int_{S^\perp} \theta(\phi^\perp)(\tau; h^\perp) dh^\perp \right) \cdot \left( \int_S \theta(\phi_0)(\tau; h_0) dh \right) d\mu.$$

Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\phi)$ and $E(\phi^\perp)$, reaching Rankin-Selberg integral

$$P = \int_X F(\tau) E(\phi^\perp) E(\phi_0) d\mu = L\text{-value}.$$
1. Choice of $V$: For a $\mathbb{Q}$-vector space $V$ and a $\mathbb{Q}$-algebra $A$, write $V_A := V \otimes_{\mathbb{Q}} A$. Let $E := \mathbb{Q}[\sqrt{\Delta}]$ be a semi-simple quadratic extension of $\mathbb{Q}$ with discriminant $\Delta$. It can be $\mathbb{Q} \times \mathbb{Q}$ with $\Delta = 1$. Pick a quaternion algebra $D$ over $\mathbb{Q}$ and put $D_E := D \otimes_{\mathbb{Q}} E$. We let $1 \neq \sigma \in \text{Gal}(E/\mathbb{Q})$ act on $D$ through the factor $E$. Then

$$V = D_{\sigma} := \{v \in D_E | v^\sigma = v^t\} \text{ for } v^t = \text{Tr}_{D_E/E}(v) - v.$$  

The quadratic form is given by $Q(v) = vv^\sigma = N(v) \in \mathbb{Q}$. We have four cases of isomorphism classes of $(D_{\mathbb{R}}, E_{\mathbb{R}})$. For simplicity, we assume $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$; so, we have two cases Case I and Case H. The symbol “I” (resp. “H”) indicate $D$ is indefinite (resp. definite). The decomposition we take is

$$V = Z \oplus D_0 \quad Z = \mathbb{Q} \text{ with quadratic form } Q_Z(a) = a^2,$$

and

$$D_0 := \{v_0 \in \sqrt{\Delta}D | \text{Tr}_{D/\mathbb{Q}}(v) = 0\} \text{ with } Q_0(v) = vv^\sigma = N(v).$$

Signature of $D_0$ is $(1,2)$ in Case I and $(3,0)$ in Case H, $\mathcal{O}_{D_0}$ is almost $D_{\times}$ and the same for $\mathcal{O}_{D_{\sigma}}$ and $D_E^{\times}$. 
§2. Bruhat functions $\phi_Z$ and $\phi_0$. On $Z = \mathbb{Q}$, for a Dirichlet character $\psi$ modulo $N$, we regard $\psi$ as a function supported on $\hat{\mathbb{Z}} \subset Z_\mathbb{A}(\infty) = \mathbb{A}(\infty)$. For $e(x) = \exp(2\pi \sqrt{-1}x)$, this $\psi$ produces theta series $\sum_{n \in \mathbb{Z}} \psi(n)n^j e(n^2 \tau)$ on $\Gamma_0(4N^2)$.

On $D_0$, first we fix a maximal order $R$ of $D$. Writing $\partial^2$ for the discriminant of the quadratic space $(D, N)$ with respect to a $\mathbb{Z}$-basis of $R$. Take the Eichler order $R(N_0)$ of level $N_0$ (prime to $\partial$) and take the characteristic function $\phi_0$ of $D_0, \mathbb{A} \cap \sqrt{\Delta} \hat{R}(N_0)$. Here for any lattice $L$, $\hat{L} = L \otimes_\mathbb{Z} \hat{\mathbb{Z}}$. This $\phi_0$ produces theta series on $\Gamma_0(4\partial \Delta N_0)$ of character $(\frac{-\Delta}{\cdot})$.

The theta series for $D_{\sigma}$ of $\psi \otimes \phi_0$ has level $M = [4N^2, 4\partial \Delta N_0]$. We choose $M$ so that $C|M$ for the conductor $C$ of $F$ and we write always $v = a \oplus v_0$ for $a \in \mathbb{Z}$ and $v_0 \in D_0$. 
§3. **Schwartz function on** $D_{\sigma,\mathbb{R}}$. The recipe is $\phi = (\phi_Z \otimes \phi_0) \Psi$ for $\Psi(v_\infty) = H(v_\infty)e(P[v_\infty]\sqrt{-1})$ for $\tau = \xi + \sqrt{-1}\eta \in \mathcal{H}$ and a harmonic polynomial $H$, where $P[v] = 2^{-1}P(v,v)$ for a positive majorant $P(v,v') = P_Z(a,a') + P_0(v_0,v'_0)$ of $s(v,v') = \text{Tr}_{D_E/E}(v^tv')$ and $H(v) = s(v,v_P)^v$ for a well chosen $v_P \in D_{\sigma,\mathbb{C}}$. Let $H_Z(a) = s(a,v_P)$ and $H_0(v_0) = s(v_0,v_P)$. By $H(a \oplus v_0) = (H_Z(a) + H_0(v_0))^v = \sum_{j=0}^\nu \binom{\nu}{j} H_Z(a)^j H_0(v_0)^{\nu-j}$,

$$\phi = \sum_{j=0}^\nu \binom{\nu}{j} \phi_Z^j \otimes \phi_0^{k-j} \text{ with } \phi_0^{k-j}(0) = 0 \text{ unless } j = k$$
on $D_{\sigma,\mathbb{A}}$, where $\phi_Z^j(a) = \phi_Z(a^{(\infty)}) H_Z(a_\infty)^j e(a_\infty^{2} \tau)$ and $\phi_0^j(v_0) = \phi_0(v_0^{(\infty)}) H_0(v_0)^j e(P[v_0]\sqrt{-1})$. All positive majorants form the symmetric space $\mathcal{G}$ of $O_{D_\sigma}$ and $\mathcal{G} = \{P \circ g | g \in O_{D_\sigma}(\mathbb{R})\}$. In Case H, $\mathcal{G} = \{s\} = \{P\}$. In Case I, $\mathcal{G} = \mathcal{H} \times \mathcal{H} = \{(z,w) \in \mathcal{H}\}$, as $O_{D_\sigma} = SO_{D_\sigma} \sqcup SO_{D_\sigma,\sigma}$ and by $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$

$$SO_{D_\sigma}(\mathbb{R}) = \{h = (h,h^\sigma) \in GL_2(E_{\mathbb{R}}) | \det(h)/\det(h^\sigma) = 1\}/\mathbb{R}^\times$$

and $h = (h,h^\sigma)$ acts on $D_{\sigma,\mathbb{R}} \cong M_2(\mathbb{R})$ by $x \mapsto h^{-1}xh^\sigma$. 

§4. **Theta kernel.** Let $\text{Mp}(\mathbb{A}) \rightarrow \text{SL}_2(\mathbb{A})$ be the metaplectic cover constructed by Weil, and $\phi \mapsto r(g)\phi$ the Weil representation. Siegel–Weil theta series $\theta(g; h)$ is

$$\sum_{\alpha \in D_\sigma} (r(g)\phi)(h^{-1}\alpha h^\sigma) : \text{SL}_2(\mathbb{Q}) \setminus \text{Mp}(\mathbb{A}) \times \mathcal{O}_{D_\sigma}(\mathbb{Q}) \setminus \mathcal{O}_{D_\sigma}(\mathbb{A}) \rightarrow \mathbb{C}.$$ 

In Case I, choose $\phi = (\psi \otimes \phi_0)\Psi(v)$ and for $g_\tau = \eta^{-1/2} \begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix}$ ($\tau = \xi + \eta\sqrt{-1} \in \mathfrak{h}$), we specialize $g$ to $g_\tau$ and $h$ to $P \circ (g_z, g_w)$ for $(\tau, z, w) \in \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$. Then

$$\theta_k(\tau; z, w) := \theta(g_\tau; g_z, g_w) = \eta \sum_{\alpha \in D_\sigma} (\psi \otimes \phi_0)(\alpha)r(g_\tau)\Psi(g_z^{-1}\alpha g_w).$$

Set $\theta(F) := \int_{X_0(M)} F(-\tau)\Theta(\phi)(\tau; z, w)\eta^{k-2}d\xi d\eta$, and assume

$$\boxed{\theta(F) \neq 0}.$$ 

Then $\theta(F)$ is a weight $(k, k)$ quaternionic modular form for $D_\mathbb{E}_E \sim \mathcal{O}_{D_\sigma}$ holomorphic in $z$ and anti-holomorphic in $w$ for $F \in S_k(\Gamma_0(M), \psi^{-1}(\Delta)).$
§5. **Theta differential form.** To compute the period on $S = \mathcal{O}_{D_0}(\mathbb{Q}) \backslash \mathcal{O}_{D_0}(\mathbb{A}) \subset S_E = D_E^\times \backslash D_{\sigma,\mathbb{A}}^\times$, we convert $\theta(\tau; z, w)$ into a sheaf valued differential 2–form. The sheaf corresponds to the $D_{E}^\times$-representation $L_{E}(n; A) = \sum_{0 \leq i, j \leq n} AX^{n-j}Y^j X^{m-i}Y^i$ on which $\gamma \in D_{E}^\times$ acts by $\gamma P(X, Y; X', Y') = P((X, Y)^t \gamma^t; (X', Y')^t \gamma \sigma^t)$. Then $L_{E}(n; A)$ has a canonical Clebsch-Gordan projection to $(n!)^{-2} \nabla^n : L_{E}(n; A)|_{\mathcal{O}_{D_0}} \to A$ for $\nabla := \frac{\partial^2}{\partial X \partial Y} - \frac{\partial^2}{\partial X' \partial Y'}$.

By putting $\Theta = \theta_k(\tau; z, w)(X - zY)^n(X' - wY')^n dz \wedge dw$ for $n = k - 2$, we get $L_{E}(n; \mathbb{C})$-valued invariant differential form. The period we like to compute is

$$\int_S \Theta(F) = \int_S \int_{X_0(M)} F(-\tau)n!^{-2} \nabla^n \Theta(\tau; z, z) \eta^{k-2} d\xi d\eta.$$  

Note $n!^{-2} \nabla^n (X - zY)^n(X' - \bar{z}Y')^n = (\bar{z} - z)^n$, and we integrate therefore over $S$ by a measure $d\mu$ given by $(z - \bar{z})^{-2} dz \wedge d\bar{z}$ over $\mathcal{H}$ and $\int_{\hat{R}(N_0) \cdot SO_2(\mathbb{R})} d\mu = 1$. After lifting $\theta(\phi)$ to $\mathcal{O}_{D_0}(\mathbb{A})$, this factor disappears as $\theta(\phi)$ is weight $k + k = 2k$ in $z$.  


§6. Siegel–Weil Eisenstein series. By Weil,
\[ r(\text{diag}[a, a^{-1}])\phi_j^0 (v_0) = |a|_{\mathbb{A}}^{3/2} \phi_j^0 (av_0), \quad r\left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \phi_j^0 (v_0) = e(uN(v_0))\phi_j^0 (v_0) \]

For the Borel subgroup \( B = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \} \subset \text{SL}_2 \), the function \( g \mapsto (r(g)\phi)(0) \) is left \( B(\mathbb{Q}) \) invariant. Siegel–Weil Eisenstein series is
\[
E(\phi_j^0)(g; s) = \sum_{\gamma \in B(\mathbb{Q})/\text{SL}_2(\mathbb{Q})} (r(\gamma g)\phi_j^0)(0)|a(\gamma g)|_A^s,
\]
where \( g = \text{diag}[a(g), a(g)^{-1}] \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} c \) for \( c \in \text{SL}_2(\mathbb{Z})\text{SO}_2(\mathbb{R}) \). Note
\[
E(\phi_j^{k-j})|_{B(\mathbb{A})} = 0 \text{ unless } k = j.
\]

For \( S = \text{O}_{D_0}(\mathbb{Q})\backslash \text{O}_{D_0}(\mathbb{A}) \), the Siegel–Weil formula by Kudla-Rallis/Sweet is
\[
E(\phi_0^{k-j})(g; 0) = \int_{S} \theta(\phi_0)(g, h)d\omega \quad \text{for the Tamagawa measure } d\omega.
\]

The ratio \( m = m(\hat{R}(N_0)) = d\mu/d\omega \) is the mass of Siegel–Shimura, which is a rational number times \( \zeta(2)/\pi \) in Case I and \( \zeta(2)/\pi^2 \) in Case H.
§7. Conclusion in Case I. We have

\[
\int_S \Theta(F)(h) d\mu(h) = \int_{\text{SL}_2(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{A}) / \hat{\Gamma}_0(M) \text{SO}_2(\mathbb{R})} \tilde{F}(g) \sum_j \binom{k}{j} \theta(\phi_Z^j) E(\phi_0^{k-j}) d\mu(h) d\mu(g)
\]

\[
= \int_{\text{B}(\mathbb{Q}) \backslash \text{B}(\mathbb{A}) / (\text{B}(\hat{\mathbb{Z}}) \cap \text{SO}_2(\mathbb{R}))} \theta(\phi_Z^k) (r(g_\tau) \phi_0^0)(0) \eta^{-1} d\xi d\eta
\]

\[
= \int_0^\infty \int_0^1 \sum_{n \in \mathbb{Z}} \psi(n) n^k e(n^2 \tau) F(-\tau) d\xi \eta^{k-1} d\eta
\]

\[
= c_D (m\pi / \zeta(2))(2\pi)^{-k} \Gamma(k) L(1, \text{Ad}(F) \otimes (\Delta))
\]

for a simple constant \(0 \neq c_D \in \mathbb{Q}\) depending on \(D\).
§8. Conclusion in Case H. The choice of the Bruhat function $\phi$ is the same as in Case I. We regard $\Psi$ as a polynomial having values in $L_E(n; \mathbb{C})$ as $H(v) \in L_E(n; \mathbb{C})$. Again in exactly the same way, for $\Theta(F) := (n!)^{-2} \nabla^n \int_{X_0(M)} \theta(\phi)(\tau; g) F(\tau) \eta^{k-2} d\xi d\eta$ and we conclude

$$\int_S \Theta(F) d\mu = c_D(m\pi^2/\zeta(2))(2\pi)^{-k} \Gamma(k) L(1, Ad(\rho_F) \otimes \left(\frac{\Delta}{\cdot}\right)).$$

Writing the point set $S = \{x\}_{x \in Cl_D(\hat{R}(N_0))}$,

$$m(\hat{R}(N_0)) = \sum_{x \in Cl_D} e_x^{-1} \div \zeta(2)$$

for $e_x = |\hat{R}(N_0) \cap O_{D_0}(\mathbb{Q})|$ and $P \div \sum_{x \in Cl_D(\hat{R}(N_0))} e_x^{-1} \Theta(F)(x)$.

Thus the period formula is an adjoint analogue of the mass formula of Siegel–Shimura. The determination of $m(\hat{R}(N_0))$ was finished by Shimura in 1999 for an arbitrary quadratic space over a totally real field (Siegel had an ambiguous factors even for $D_0$ over $\mathbb{Q}$).