Number Theory
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On Selmer Groups of Adjoint Modular Galois Representations

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Warning : There is a misstatement in the main result; see, correction attached to this file (added on September 15, 1999)

0. — Introduction

Let p be an odd prime. Starting from a modular Galois representation φ into $\mathrm{GL}_2(\mathbb{I})$ for an irreducible component $\mathrm{Spec}(\mathbb{I})$ of the spectrum of the universal ordinary Hecke algebra of prime-to-p level N, we study the Selmer group $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$ of Greenberg [G] for the adjoint representation of $\mathrm{Ad}(\varphi)$ on the trace zero subspace $V(\mathrm{Ad}(\varphi))$ of $M_2(\mathbb{I})$ and the universal character ν unramified outside p deforming the trivial character of $\mathcal{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The pontryagin dual of $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ is basically known to be a torsion $\mathbb{I}$-module of finite type by a result of Flach [F] and Wiles [W] under a suitable assumption on φ. The key point of the proof is to show for an arithmetic height 1 prime P, the subgroup $\mathrm{Sel}(\mathrm{Ad}(\varphi))[P]$ killed by P is finite. Our Selmer group $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})$ is naturally a module over $\mathbb{I}[[\Gamma]]$ for $\Gamma = \mathrm{Im}(\nu)$ ($\cong \mathbb{Z}_p$). However, it is well known that for the augmentation ideal P of $\mathbb{I}[[\Gamma]]$, $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})[P]$ has non-trivial $\mathbb{I}$-divisible subgroup, and hence the co-torsionness of $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})$ over $\mathbb{I}[[\Gamma]]$ does not follow from the co-torsionness of $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ over $\mathbb{I}$. In this paper, under a suitable assumption, we prove a control theorem giving the following exact sequence :

$$0 \to \mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}} \to \mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})^{\Gamma} \to \mathbb{I}^* \to 0\ ,$$

where $\mathbb{I}^*$ is the Pontryagin dual module of $\mathbb{I}$ on which Γ acts trivially. Actually this assertion is valid for more general 2-dimensional representations φ not necessarily modular (Theorems 2.2 and 3.2) and also for $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/F}$ for a general number field F. Although the above exact sequence does not

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directly yield the co-torsionness of $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})$, when φ is modular, we can deduce it from the co-torsionness of $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ using the fact that the p-th Hecke operator $T(p)$ is transcendental over $\mathbb{Z}_p$ in the universal ordinary Hecke algebra (see Theorem 3.3). For these, we consider the universal ordinary deformation ring R_F of φ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ as in [W]. Then we have natural projection $\pi_F : R_F \to \mathbb{I}$, and we can identify $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/F}$ with the module (called a Mazur module) of 1-differentials of R_F as in [MT] which gives a tool of proving the above control theorem. To help the reader to understand the formal but subtle argument dealing with various deformation rings, we added to the main text a lengthy Appendix which describes a general theory of controlling deformation rings.

1. — Control of differential modules

In this section, we describe how a group action on a ring induces a group action on its differential modules.

1.1. Functoriality of differential modules. We start with a noetherian integral domain A with quotient field K. Let H be an A-algebra, and $\lambda : H \to B$ be an A-algebra homomorphism. The differential module is then defined by

$$(\mathrm{C}_1)\qquad \begin{aligned} C_1(\lambda;B) = \mathrm{Tor}_1^H(\mathrm{Im}(\lambda),B) &\cong \mathrm{Ker}(\lambda)\otimes_{H,\lambda} B \\ &\cong (\mathrm{Ker}(\lambda)/\mathrm{Ker}(\lambda)^2)\otimes_{H,\lambda} B. \end{aligned}$$

See [H2] Section 6 and [H3] Section 1 for a general theory of these modules including above isomorphisms. Suppose that we have two surjective A-algebra homomorphisms : $H \xrightarrow{\theta} T \xrightarrow{\mu} B$ with $\lambda = \mu\circ\theta$. Anyway, these modules are torsion modules over A if B is of finite type as an A-module. Then we recall Theorem 6.6 in [H2] :

Proposition 1.1. — *Suppose the surjectivity of θ and μ. Then we have the following canonical exact sequences of H-modules :*

$$\mathrm{Tor}_1^T(B,\mathrm{Ker}(\mu)) \to C_1(\theta;T)\otimes_T B \to C_1(\lambda;B) \to C_1(\mu;B) \to 0\ ;$$

Proof : we have an exact sequence of H-modules :

$$0 \to \mathrm{Ker}(\theta) \to \mathrm{Ker}(\lambda) \xrightarrow{\theta} \mathrm{Ker}(\mu) \to 0\ .$$

Tensoring B over H with the above sequence, we obtain the desired result.

We now suppose that a finite group G acts on H through A-algebra automorphisms. Thus the finite group G acts on $\mathrm{Spec}(H)$. We consider the following condition :

(Nt) $\mathrm{Spec}(T)$ is the fixed point subscheme of G in $\mathrm{Spec}(H)$.

Let $\mathfrak{a}$ be the augmentation ideal of $\mathbb{Z}[G]$. Then the condition (Nt) is equivalent to

(Nt′) $\mathrm{Ker}(\theta)$ is generated over H by $g(x) - x$ for $x \in H$ and $g \in G$, that is, $\mathrm{Ker}(\theta) = H\mathfrak{a}H$.

Let $\sigma \in G$. Then, under (Nt), the action of $\sigma - 1$ induces an A-linear map : $\mathrm{Ker}(\lambda) \to \mathrm{Ker}(\theta)$. If $x, y \in \mathrm{Ker}(\lambda)$, then

$$\begin{aligned}\sigma(xy) - xy &= (\sigma(x) - x)(\sigma(y) - y) + x(\sigma(y) - y) + y(\sigma(x) - x) \\ &\equiv (\sigma(x) - x)(\sigma(y) - y) \equiv 0 \bmod \mathrm{Ker}(\lambda)\,\mathrm{Ker}(\theta) \ .\end{aligned}$$

Note that $C_1(\theta; T) \otimes_{T,\mu} B \cong (\mathrm{Ker}(\theta)/\mathrm{Ker}(\lambda)\,\mathrm{Ker}(\theta)) \otimes_{H,\lambda} B$. Thus the A-linear map induces a B-linear map $[\sigma - 1] : C_1(\lambda; B) \to C_1(\theta; T) \otimes_{T,\mu} B$. Under (Nt), $\sigma(x) - x$ for $x \in H$ and $\sigma \in G$ generates $\mathrm{Ker}(\theta)$ over H. Now assume that

(Sec) λ has a section $\iota : B \to H$ of $A[G]-$modules.

Then for each $y \in H$, we can write $y = x \oplus \iota\lambda(y)$ for $x = y - \iota\lambda(y) \in \mathrm{Ker}(\lambda)$, and hence $\sigma(y) - y = \sigma(x) - x \in (\sigma - 1)\,\mathrm{Ker}(\lambda)$. Thus $[\sigma - 1](x)$ for $\sigma \in G$ and $x \in \mathrm{Ker}(\lambda)$ generates $C_1(\theta; T) \otimes_{T,\mu} B$ over B, and

$$\oplus_{\sigma\in G}[\sigma - 1] : \oplus_{\sigma\in G} C_1(\lambda; B) \to C_1(\theta; T) \otimes_{T,\mu} B \text{ is surjective.}$$

This shows that the image of $C_1(\theta; T) \otimes_{T,\mu} B$ in $C_1(\lambda; B)$ is equal to $\mathfrak{a}C_1(\lambda; A)$, and we have

Corollary 1.1. — *Suppose* (Nt) *and* (Sec). *Then we have*

$$C_1(\mu; B) \cong C_1(\lambda; B)/\mathfrak{a}C_1(\lambda; B) = H_0(G, C_1(\lambda; B)) \ ,$$

where $\mathfrak{a}$ *is the augmentation ideal of* $\mathbb{Z}[G]$.

We now put ourselves in a bit more general setting where μ is not necessarily surjective. We write B_0 for $\mathrm{Im}(\mu)$ and consider the following three algebra homomorphisms :

$$H \otimes_A B \xrightarrow{\theta\otimes\mathrm{id}} T \otimes_A B \xrightarrow{\mu\otimes\mathrm{id}} B_0 \otimes_A B \xrightarrow{m} B \ ,$$

where $m(a \otimes b) = ab$. Since $\mathrm{Ker}(\mu \otimes \mathrm{id})^j$ is a surjective image of $\mathrm{Ker}(\mu)^j \otimes_A B$, the natural map : $C_1(\lambda; B) \otimes_A B \to C_1(\lambda \otimes \mathrm{id}; B \otimes_A B)$ is surjective. When B is flat over A, the map is an isomorphism of $B \otimes_A B$-modules. Similarly, the natural maps

$$\text{(Ext1)} \qquad \begin{gathered} C_1(\lambda; B_0) \otimes_A B \to C_1(\lambda \otimes \mathrm{id}; B_0 \otimes_A B)\ , \\ C_1(\mu; B_0) \otimes_A B \to C_1(\mu \otimes \mathrm{id}; B_0 \otimes_A B) \text{ and} \\ C_1(\theta; T) \otimes_A B \to C_1(\theta; T \otimes_A B) \end{gathered}$$

are all surjective and are isomorphisms if B is flat over A. By Proposition 1.1, we get an exact sequence, writing B' for $B_0 \otimes_A B$,

$$\text{(Ext2)} \qquad \begin{gathered} \mathrm{Tor}_1^{B'}(\mathrm{Ker}(m), B) \to C_1(\mu \otimes \mathrm{id}; B') \otimes_{B'} B \to \\ C_1(m \circ (\mu \otimes \mathrm{id}); B) \to C_1(m; B) \to 0\ . \end{gathered}$$

We get from the short exact sequence : $0 \to \mathrm{Ker}(\mu) \to T \to B_0 \to 0$, an exact sequence : $\mathrm{Tor}_1^A(B_0, B_0) \to \mathrm{Ker}(\mu) \otimes_A B_0 \to T \otimes_A B_0 \to B_0 \otimes_A B_0 \to 0$, and as a part of it, we know the exactness of the following sequence :

$$\mathrm{Tor}_1^A(B_0, B_0) \to \mathrm{Ker}(\mu) \otimes_A B_0 \to \mathrm{Ker}(\mu \otimes \mathrm{id}) \to 0\ .$$

Applying $\otimes_{T'} B$ to the last sequence, writing T' for $T \otimes_A B$, we have another exact sequence :

$$\text{(Ext2}'\text{)} \qquad \begin{gathered} \mathrm{Tor}_1^A(B_0, B_0) \otimes_{B'} B \to (\mathrm{Ker}(\mu) \otimes_A B_0) \otimes_{T'} B \to \\ C_1(\mu \otimes \mathrm{id}; B') \otimes_{B'} B \to 0 \end{gathered}$$

and

$$\begin{aligned} (\mathrm{Ker}(\mu) \otimes_A B_0) \otimes_{T'} B &= (\mathrm{Ker}(\mu)/\mathrm{Ker}(\mu)^2) \otimes_A B_0 \otimes_{B'} B \\ &= ((\mathrm{Ker}(\mu)/\mathrm{Ker}(\mu)^2) \otimes_{B_0} B_0) \otimes_A B \otimes_{B'} B \\ &= (\mathrm{Ker}(\mu)/\mathrm{Ker}(\mu)^2) \otimes_{B_0} B = C_1(\mu; B_0) \otimes_{B_0} B\ . \end{aligned}$$

This combined with (Ext2) shows the exactness of

$$\text{(Ext3)} \qquad C_1(\mu; B_0) \otimes_{B_0} B \to C_1(m \circ (\mu \otimes \mathrm{id}); B) \to C_1(m; B) \to 0\ .$$

If B_0 is A-flat, then $\mathrm{Tor}_1^A(B_0, B_0) = 0$. When $B = B_0$, the above sequence is nothing but the well known exact sequence for the closed immersion μ

of $X = \mathrm{Spec}(B_0)$ into $Y = \mathrm{Spec}(T)$ over $S = \mathrm{Spec}(A) : \mathrm{Ker}(\mu)/\mathrm{Ker}(\mu)^2 \to \mu^*\Omega_{Y/S} \to \Omega_{X/S} \to 0$.

2. — Control Theorems of universal ordinary deformation rings

We fix a prime $p \geq 3$. For a number field X in $\overline{\mathbb{Q}}$, we write $\mathcal{G}_X = \mathrm{Gal}(\overline{\mathbb{Q}}/X)$ for the absolute Galois group over X. Let $\mathfrak{O}$ be a valuation ring finite flat over $\mathbb{Z}_p$ with residue field $\mathbb{F}$. We consider a p-ordinary deformation problem $\mathcal{D} = \mathcal{D}_X$ defined on the category $\mathrm{CNL}_{\mathfrak{O}}$ of complete noetherian local $\mathfrak{O}$-algebras with residue field $\mathbb{F}$. Morphisms of $\mathrm{CNL}_{\mathfrak{O}}$ are assumed to be local $\mathfrak{O}$-algebra homomorphisms. See [T] and Appendix for a general theory of such deformation problems. Let (R_X, ρ_X) be the universal couple of the deformation problem $\mathcal{D}_X$ of representations of $\mathcal{G}_X$. We study how the Galois action controls R_X.

2.1. Deformation problems. Let $\overline{\rho}$ be a continuous representation of $\mathcal{G}_E$ into $\mathrm{GL}_2(\mathbb{F})$ for a number field E. We consider the following condition for an algebraic extension F/E :

(AI_F) $\qquad$ $\overline{\rho}$ restricted to $\mathcal{G}_F$ is absolutely irreducible .

We assume (AI_F). For each prime ideal $\mathfrak{l}$, we write $F_{\mathfrak{l}}$ for the $\mathfrak{l}$-adic completion of F and $\mathcal{G}_{F_{\mathfrak{l}}}$ for the absolute Galois group over $F_{\mathfrak{l}}$. Let C be an integral ideal of E prime to p and write $Cl_E(Cp)$ for the strict ray class group of E modulo Cp. We also pick a character $\chi : Cl_E(Cp) \to \mathfrak{O}^\times$ such that the order of χ is prime to p. We write $C(\chi)$ for the conductor of χ and assume that $C|C(\chi)$. By class field theory, we may regard χ as a character of $\mathcal{G}_E$. We write $\chi_{\mathfrak{q}}$ for the restriction of χ to the decomposition subgroup $\mathcal{G}_{E_{\mathfrak{q}}}$ at each prime $\mathfrak{q}$. Let $\mathcal{M}$ be a finite set of primes outside p. We assume that $\mathcal{M}$ contains all prime factors of $C(\chi)$ outside p. We write $\mathcal{M}(\chi)$ for the set of primes in $\mathcal{M}$ dividing $C(\chi)$ and put $\mathcal{M}' = \mathcal{M} - \mathcal{M}(\chi)$. We write Σ for the union of $\mathcal{M}$ and the set of all prime factors of p. Then for $\mathfrak{q} \in \Sigma$, write $I_{\mathfrak{q}}$ (resp. $\mathcal{N}_{\mathfrak{q}}$) for the inertia subgroup of $\mathcal{G}_{E_{\mathfrak{q}}}$ (resp. the p-adic cyclotomic character $\mathcal{N}$ restricted to $\mathcal{G}_{E_{\mathfrak{q}}}$). Here we normalize cyclotomic characters so that they take the *geometric* Frobenius at each unramified prime ideal $\mathfrak{l}$ to the norm of the ideal $\mathfrak{l}$. Under this convention, we consider a deformation problem of $\overline{\rho}$ on $\mathrm{CNL}_{\mathfrak{O}}$. A deformation $\rho : \mathcal{G}_E \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ is called of type $\mathcal{D} = \mathcal{D}_E$ if ρ satisfies the following five conditions (UNR), (χ_p), $(\mathrm{Reg}_{\mathfrak{p}})$ for each prime $\mathfrak{p}|p$, $(\chi_{\mathfrak{q}})$ for each prime $\mathfrak{q} \in \mathcal{M}(\chi)$ and $(\mathcal{N}_{\mathfrak{q}})$ for each prime $\mathfrak{q} \in \mathcal{M}'$:

(UNR) $\qquad$ π is unramified outside Σ ;

$(\chi_{\mathfrak{p}})$ We have an exact sequence of $\mathcal{G}_{E_{\mathfrak{p}}}$-modules :
$0 \to V(\rho_{1;\mathfrak{p}}) \to V(\rho) \to V(\rho_{2,\mathfrak{p}}) \to 0$ with $V(\rho_{2,\mathfrak{p})}$ A-free of rank 1 , $\rho_{1,\mathfrak{p}}$ unramified and $\rho_{2,\mathfrak{p}} \mod m = \chi_{\mathfrak{p}} \bmod m$ on the inertia subgroup $I_{\mathfrak{p}}$.

Writing $\overline{\rho}_i$ for $\rho_i \mod m$, we assume

(Reg$_{\mathfrak{p}}$) $$\bar{\rho}_{1,\mathfrak{p}} \neq \bar{\rho}_{2,\mathfrak{p}} \ .$$

We assume the following conditions for $\mathfrak{q} \in \mathcal{M}$:

$(\chi_{\mathfrak{q}})$ As $I_{\mathfrak{q}}$ – modules, $V(\rho) \cong V(\mathrm{id}) \oplus V(\chi_{\mathfrak{q}})$ with $V(\chi_{\mathfrak{q}})$ A-free of rank 1 for $\mathfrak{q}|C(\chi)$,

$(\mathcal{N}_{\mathfrak{q}})$ For $\mathfrak{q} \in \mathcal{M}'$, we have an exact sequence, of $\mathcal{G}_{E_{\mathfrak{q}}}$-modules, non-split over $I_{\mathfrak{q}}$.

$$0 \to V(\rho_{1,\mathfrak{q}}) \to V(\rho) \to V(\rho_{2,\mathfrak{q}}) \to 0$$

where $V(\rho_{2,\mathfrak{q}})$ is A-free of rank 1, $\rho_{i,\mathfrak{q}}$ $(i = 1,2)$ is unramified and $\rho_{1,\mathfrak{q}}\rho_{2,\mathfrak{q}}^{-1} = \mathcal{N}_{\mathfrak{q}}$.

Since the order of χ is prime to p, $\chi_{\mathfrak{q}}$ for $\mathfrak{q} \in \mathcal{M}(\chi)$ is non-trivial. To make our deformation problem $\mathcal{D}_E$ non-empty, we assume that $\overline{\rho}$ satisfies the above five conditions. The contragredient of this deformation problem is studied in [W] and is denoted by $D = (\mathrm{Ord}, \Sigma, \mathfrak{O}, \mathcal{M})$ (for $E = \mathbb{Q}$) there. As shown in [W], the problem D is representable, and hence $\mathcal{D}_E$ is also representable. See [T] and Appendix for the proof in more general case. To apply the argument in [T] and Appendix to our situation here, we note the following facts : for the maximal extension F_Σ of F unramified outside Σ, the Galois group $G = \mathrm{Gal}(F_\Sigma/E)$ satisfies the condition (pF) in Appendix; any deformation of type $\mathcal{D}_E$ factors through G; the group $D \in S$ (resp. its subgroup I) is given by a choice of decomposition subgroups (resp. its inertia subgroup) at each $\mathfrak{p}^{\cdot}$, and the condition $(\mathrm{Reg}_{\mathfrak{p}})$ is the same as (RG_D) in Section A.2.2 for the decomposition subgroup D of $\mathfrak{p}$ in G. We write (R_E, ρ_E) for the universal couple for $\mathcal{D}_E$. Thus for each deformation $\rho : \mathcal{G}_E \to \mathrm{GL}_2(A)$ of type $\mathcal{D}_E$, there exists a unique local $\mathfrak{O}$-algebra homomorphism $\varphi : R_E \to A$ such that ρ is strictly equivalent to $\varphi\rho_E$. Here we say ρ is strictly equivalent to ρ' if $\rho(\tau) = x\rho'(\tau)x^{-1}$ for $x \in \widehat{\mathrm{GL}}_2(A) = 1 + m_A M_2(A)$ independent of τ. We write $\rho \approx \rho'$ if ρ is strictly equivalent to ρ'.

Let F be a finite extension of E. Write $\overline{\rho}_F$ (resp. χ_F) for the restriction of $\overline{\rho}$ (resp. χ) to $\mathcal{G}_F$. We consider the deformation problem $\mathcal{D}_F$ of $\overline{\rho}_F$ on $\mathrm{CNL}_{\mathfrak{O}}$

given as follows. Let $\mathcal{M}_F(\chi)$ (resp. $\mathcal{M}'_F$) be the set of prime ideals dividing $C(\chi_F)$ and prime to p (resp. the set of primes dividing primes in $\mathcal{M}'$). We write Σ_F for the union of $\mathcal{M}_F(\chi)$, $\mathcal{M}'_F$ and the set of all prime factors of p in F. A deformation ρ of $\overline{\rho}_F$ is a continuous representation $\rho : \mathcal{G}_F \to \mathrm{GL}_2(A)$ with $\rho \bmod m_A = \overline{\rho}_F$ for an object A in $\mathrm{CNL}_{\mathfrak{O}}$. A deformation ρ of $\overline{\rho}_F$ is of type $\mathcal{D}_F$ if ρ satisfies the following five conditions (UNR_F), $(\chi_{\mathcal{P},F})$, $(\mathrm{Reg}_{\mathcal{P},F})$ for each prime $\mathcal{P}$ of F dividing p, $(\chi_{\mathcal{Q},F})$ for $\mathcal{Q} \in \mathcal{M}_F(\chi)$ and $(\mathcal{N}_Q)$ for $\mathcal{Q} \in \mathcal{M}'_F$:

(UNR$_F$) ρ is unramified outside Σ_F ;

$(\chi_{\mathcal{P},F})$ We have an exact sequence of $\mathcal{G}_{F_\mathcal{P}}$-modules for each prime ideal $\mathcal{P}|p$:

$0 \to V(\rho_{1,\mathcal{P}}) \to V(\rho) \to V(\rho_{2,\mathcal{P}}) \to 0$ with $V(\rho_{2,\mathcal{P}})$ A-free of rank 1 ,

$\rho_{1,\mathcal{P}}$ unramified and $\rho_{2,\mathcal{P}} \bmod m = \chi_\mathcal{P} \bmod m$ on $I_\mathcal{P}$;

(Reg$_{\mathcal{P},F}$) $\overline{\rho}_{1,\mathcal{P}} \neq \overline{\rho}_{2,\mathcal{P}}$ for each prime ideal $\mathcal{P}|p$.

where $\overline{\rho}_{i,\mathcal{P}} = \rho_{i,\mathcal{P}} \bmod m$. Writing $\chi_\mathcal{Q}$ for the restriction of χ to $\mathcal{G}_{F_\mathcal{Q}}$, we assume :

$(\chi_{\mathcal{Q},F})$ As $I_\mathcal{Q}$−modules, $V(\rho) \cong V(\mathrm{id}) \oplus V(\chi_\mathcal{Q})$ with A-free of rank 1 for $\mathcal{Q} \in \mathcal{M}_F$,

$(\mathcal{N}_\mathcal{Q})$ For $\mathcal{Q} \in \mathcal{M}'_F$ we have an exactsequence, of $\mathcal{G}_{F_\mathcal{Q}}$-modules, non-split over $I_\mathcal{Q}$,

$$0 \to V(\rho_{1,\mathcal{Q}}) \to V(\rho) \to V(\rho_{2,\mathcal{Q}}) \to 0$$

where $V(\rho_{2,\mathcal{Q}})$ is A-free of rank 1, $\rho_{i,\mathcal{Q}}$ ($i = 1, 2$) is unramified and $\rho_{1,\mathcal{Q}}\rho_{2,\mathcal{Q}}^{-1} = \mathcal{N}_\mathcal{Q}$.

Then $\mathcal{D}_F$ is representable under (AI_F). We write $H = \mathrm{Gal}(F_\Sigma/F)$. Then H is a normal subgroup of G with $G/H = \Delta = \mathrm{Gal}(F/E)$. We take $S_G = \{D_\mathfrak{p}\}_{\mathfrak{p}|p}$ and $S_H = \{D_\mathcal{P}\}_{\mathcal{P}|p}$ for the decomposition subgroups $D_?$ for each prime "?". Then to the quadruple (G, H, S_G, S_H) the theory described in Appendix (Sections A.2.1-3) applies, and it is easy to deduce the representability from the argument in Section A.2.3. Hereafter, assuming (AI_F), we write (R_F, ρ_F) for the universal couple representing the problem $\mathcal{D}_F$.

2.2. Controlling universal deformation rings and Mazur modules. We now suppose that F/E is cyclic of degree d. Since ρ_E restricted to $H = \mathrm{Gal}(F_\Sigma/F)$ is a solution of the deformation problem $\mathcal{D}_F$, we have a non-trivial algebra homomorphism $\alpha : R_F \to R_E$ such that $\alpha\rho_F \approx \rho_E$. If ρ is a deformation of type $\mathcal{D}_E$, then we have a unique $\varphi : R_E \to A$ such that $\varphi\rho_E \approx \rho$. Then $\rho|_H \approx \varphi\rho_E|_H \approx \varphi\alpha\rho_F$. Let us write $\mathcal{F}_F$ (resp. $\mathcal{F}$) for the deformation functor of the problem $\mathcal{D}_F$ (resp. $\mathcal{D}_E$). Then the Galois group $\Delta = \mathrm{Gal}(F/E)$ naturally acts on $\mathcal{F}_F$ as follows : taking $\sigma \in G$ and define $\rho^\sigma(g) = \rho(\sigma g \sigma^{-1})$ for $\rho \in \mathcal{F}_F(A)$ and $g \in H$. We take $c(\sigma) \in \mathrm{GL}_2(\mathfrak{O})$ such that $c(\sigma) \equiv \overline{\rho}(\sigma) \bmod m_{\mathfrak{o}}$ and define $\rho^{[\sigma]} = c(\sigma)^{-1}\rho^\sigma c(\sigma) \in \mathcal{F}_F(A)$. The strict equivalence class of $\rho^{[\sigma]}$ is well defined depending only on the class of σ in Δ, and in this way Δ acts on $\mathcal{F}_F$ and R_F through $\mathfrak{O}$-algebra automorphisms. We define a new functor $\mathcal{F}_F^\Delta$ by $\mathcal{F}_F^\Delta(A) = \{\rho \in \mathcal{F}_F(A) | \rho^{[\sigma]} \approx \rho\}$. Since $\varphi\alpha$ is the unique homomorphism bringing ρ_F down to $\rho|_H$, the deformation subfunctor :

$$\mathcal{F}_{E,F}(A) = \{\rho|_H \in \mathcal{F}_F(A) \mid \rho \in \mathcal{F}(B) \text{ for a flat } A\text{-algebra } B \text{ in } \mathrm{CNL}_{\mathfrak{O}}\}$$

is representable by $(\mathrm{Im}(\alpha), \alpha\rho_F)$ under (AI_F), as long as $\alpha\rho_F$ can be extended to an element of $\mathcal{F}(B)$ for an algebra B flat over $\mathrm{Im}(\alpha)$ in $\mathrm{CNL}_{\mathfrak{O}}$. The argument proving this is the same as the proof of Theorem A.2.3. To check the extendibility of $\alpha\rho_F$, we introduce the following assumptions. Let F_0 be the maximal subfield of F such that $d' = [F_0 : E]$ is prime to p. Let $\mathbb{S}$ be the set of primes of E ramifying in F. For each prime $\mathfrak{q}$ of E, we write $I_{\mathfrak{q}}$ (resp. $I(\mathfrak{q})$) for the inertia group of $\mathfrak{q}$ in G (resp. Δ). We also write $D(\mathfrak{q})$ for the decomposition subgroup of Δ at $\mathfrak{q}$. We assume for $\mathfrak{q}$ outside p

$(\mathrm{TR}_{\mathfrak{q}})$ $\qquad |I(\mathfrak{q})|$ is prime to p .

For $\mathfrak{p}|p$, we assume either $(\mathrm{TR}_{\mathfrak{p}})$ or

$(\mathrm{Ex}_{\mathfrak{p}})$ $\qquad$ Every character of $I(\mathfrak{p}) \cap \mathrm{Gal}(F/F_0)$ with values in $A^\times$ can be extended to a character of Δ having values in $B^\times$ for a flat extension B of A such that it is unramified outside $\mathfrak{p}$.

These conditions correspond the conditions (TR_D) and (Ex_D) in Section A.2.3. If $I(\mathfrak{p}) \cap I(\mathfrak{q}) \cap \mathrm{Gal}(F/F_0) = \{1\}$ for any two primes $\mathfrak{p}$ dividing p and an arbitrary $\mathfrak{q}$, then $(\mathrm{Ex}_{\mathfrak{p}})$ is satisfied. In particular, if F is a subfield of $\mathbb{Q}_\infty$, $(\mathrm{Ex}_{\mathfrak{p}})$ is satisfied, where $\mathbb{Q}_\infty$ is the unique $\mathbb{Z}_{\mathfrak{p}}$-extension of $\mathbb{Q}$.

Take $\pi \in \mathcal{F}_F^\Delta(A)$. By Corollary A.1.2 combined with the argument in A.2.1, there exists a faithfully flat A-algebra B in $\mathrm{CNL}_{\mathfrak{O}}$ such that π extends to a representation $\pi_E : G \to \mathrm{GL}_2(B)$ with $\pi_E \equiv \overline{\rho} \bmod m_{\mathfrak{O}}$. By $(\mathrm{TR}_{\mathfrak{q}})$, the unramifiedness at $q \nmid p$ in $\mathbb{S} - \mathcal{M}$ of $\overline{\rho}$ implies the unramifiedness of π_E, where $\mathbb{S}$ is the set of primes ramifying in F/E. We now look at the restriction of π_E to the inertia group $I = I_{\mathfrak{q}}$ for primes $\mathfrak{q}$ in $\mathcal{M}$ or dividing p. By $(\mathrm{TR}_{\mathfrak{q}})$ and the fact that the decomposition group at $\mathfrak{q}$ acts through conjugation on the maximal tame quotient of the inertia subgroup by the cyclotomic character $\mathcal{N}$, the conditions $(\chi_{\mathcal{Q}})$ (resp. $(\mathcal{N}_Q)$) for $\overline{\rho}$ implies $(\chi_{\mathfrak{q}})$ (resp. $(\mathcal{N}_{\mathfrak{q}})$) for π_E at $\mathfrak{q} = \mathcal{Q} \cap E$. We look at the restriction of π_E to $I_{\mathfrak{p}}$. Since the characteristic polynomial of $\overline{\rho}(\sigma)$ for an element σ of $I_{\mathfrak{p}}$ has two distinct roots in $\mathbb{F}$ by $(\mathrm{Reg}_{\mathfrak{p}})$ and $(\chi_{\mathfrak{p}})$, that of $\pi_E(\sigma)$ again has two distinct roots a and b in B by Hensel's lemma. Then writing V for $\mathrm{Ker}(\pi_E(\sigma) - a \cdot id)$, $V(\pi_E)/V$ is B-free of rank 1, and on V, $D_{\mathfrak{p}}$ acts by a character $\eta_{\mathfrak{p}}$. Replacing a by b if necessary, we may assume that $\overline{\eta}_{\mathfrak{p}}$ is trivial on $I(\mathfrak{p})$. If $\mathfrak{p}$ satisfies $(\mathrm{TR}_{\mathfrak{p}})$, the argument is the same as $\mathfrak{q}$ above. Now suppose $(\mathrm{Ex}_{\mathfrak{p}})$. Since $\overline{\eta}_{\mathfrak{p}}$ is trivial on $I(\mathfrak{p})$, $\eta_{\mathfrak{p}}$ factors through the p-primary quotient of $I(\mathfrak{p})$. Thus by $(\mathrm{Ex}_{\mathfrak{p}})$, we can lift $\eta_{\mathfrak{p}}$ to a p-power order character ξ of Δ unramified outside $\mathfrak{p}$. Replacing π_E by $\pi_E \otimes \xi^{-1}$, we may assume that π_E satisfies $(\chi_{\mathfrak{p}})$ because $\xi \equiv 1 \bmod m$. Thus π_E is a deformation of type $\mathcal{D}_E$, and we get

Proposition 2.1. — *Assume* (AI_F), $(\mathrm{TR}_{\mathfrak{q}})$ *for* $\mathfrak{q}$ *outside* p *and one of* $(\mathrm{Ex}_{\mathfrak{p}})$ *or* $(\mathrm{TR}_{\mathfrak{p}})$ *for* $\mathfrak{p}|p$. *Then* $\pi \in \mathcal{F}_F^{\Delta}(A)$ *can be extended to an element* π_E *of* $\mathcal{F}(B)$ *for a faithfully flat* A*-algebra* B *in* $\mathrm{CNL}_{\mathfrak{O}}$. *Moreover the functor* $\mathcal{F}_{E,F}$ *is represented by* $(\mathrm{Im}(\alpha), \alpha\rho_F)$.

For each integral ideal C of a number field X, we write $\mathrm{Cl}_X(Cp^e)$ for the strict ray class group modulo Cp^e. We allow $e = \infty$, and $\mathrm{Cl}_X(Cp^\infty) = \varprojlim_e \mathrm{Cl}_X(Cp^e)$. Then by class field theory, there exists an abelian extension X_∞/X unramified outside Cp such that $\mathrm{Gal}(X_\infty/X) \cong \mathrm{Cl}_X(Cp^\infty)$. We consider the character $\det(\rho_F) : \mathcal{G}_F \to R_F^\times$. By $(\chi_{\mathcal{Q}})$, the restriction of this character to the inertia subgroup $I_{\mathcal{Q}}$ factors through a finite quotient. Thus there exists an integral ideal C prime to p of F such that $\det(\rho_F)$ factors through $Z_F = \mathrm{Cl}_F(Cp^\infty)$, and hence there exists an algebra homomorphism : $\mathfrak{O}[[Z_F]] \to R_F$ taking $z \in Z_F$ to $\det(\rho_E(z))$. We take C maximal among the ideals satisfying the above condition. We write Λ_F for the image of $\mathfrak{O}[[Z_F]]$ in R_F which is an object in $\mathrm{CNL}_{\mathfrak{O}}$.

We now modify the deformation problem $\mathcal{D}_E$ on $\mathrm{CNL}_{\mathfrak{O}}$ and create a new one $\mathcal{D}_\Lambda$ defined over the category CNL_Λ of complete noetherian local Λ_E-algebras with residue field $\mathbb{F}$ by adding the following condition to the conditions of $\mathcal{D}_E$:

(det) $\det(\rho)$ for each deformation $\rho : \mathcal{G}_E \to \mathrm{GL}_2(A)$ of type $\mathcal{D}_E$ coincides with $\det(\rho_E)$ composed with the inclusion $i : \Lambda_E \to A$.

For any deformation $\rho : \mathcal{G}_E \to \mathrm{GL}_2(A)$ of type $\mathcal{D}_\Lambda$, it is automatically a deformation of type $\mathcal{D}_E$. Thus we have a unique $\mathfrak{O}$-algebra homomorphism $\phi : R_E \to A$ such that $\phi\rho_E \approx \rho$ and $\phi(\det(\rho_E)) = \det(\rho)$, which implies that

ϕ is actually a Λ_E-algebra homomorphism. Therefore (R_E, ρ_E) represents the new problem $\mathcal{D}_\Lambda$. We consider another deformation functor $\mathcal{F}_{\Lambda,E,F}$ defined on CNL_Λ by

$$\mathcal{F}_{\Lambda,E,F}(A) = \{\rho|_H \in \mathcal{F}_{\Lambda,F}(A) | \rho \in \mathcal{F}_\Lambda(B) \text{ for a flat } A\text{-algebra } B \text{ in } \mathrm{CNL}_\Lambda\} .$$

By Lemma A.2.1 in Appendix, actually $\mathcal{F}_{\Lambda,E,F}(A) = \{\rho|_H \in \mathcal{F}_{\Lambda,F}(A)|\rho \in \mathcal{F}_\Lambda(A)\}$. By the argument proving Theorem A.2.1, we can conclude that this functor is represented by $(\mathrm{Im}(\alpha)\Lambda_E, \rho_E|_H)$, where we write $\mathrm{Im}(\alpha)\Lambda_E$ for the image of $\mathrm{Im}(\alpha)\widehat{\otimes}_{\mathfrak{O}}\Lambda$ in R_E. Here is the argument : let ρ and ρ' be two deformations of $\overline{\rho}$ over E of type $\mathcal{D}_\Lambda$ with values in $\mathrm{GL}_2(A)$. Suppose that $\rho \approx \rho'$ on H. Under (AI_F), $\rho \cong \rho' \otimes \xi$ for some character ξ of $\mathrm{Gal}(F/E)$ (see Corollary A.2.1). Since in our deformation problem, the determinant is fixed, we have $\xi^2 = 1$. Since $\overline{\rho} \cong \overline{\rho} \otimes \xi$ and ξ is quadratic, if $\xi \neq 1$, $\xi \bmod m$ is non-trivial, because p is odd. By [DHI] Proposition 4.1, $\overline{\rho}$ is an induced representation of a character of $\mathrm{Ker}(\xi)$, which violates (AI_F). Thus ξ is trivial. The algebra $\mathrm{Im}(\alpha)$ may not be a Λ-algebra for $\Lambda = \Lambda_E$. We thus find

$$\begin{aligned}\mathrm{Hom}_{\Lambda-\mathrm{alg}}(\mathrm{Im}(\alpha)\Lambda_E, A) &\cong \{\pi|_H \mid \pi \in \mathcal{F}(A),\ \det(\pi) = \det(\rho_E)\}/\approx \\ &\cong \mathcal{F}_\Lambda(A) \cong \mathrm{Hom}_{\Lambda-\mathrm{alg}}(R_E, A) \text{ for } \Lambda\text{-algebras } A.\end{aligned}$$

Thus under (AI_F), we have $\mathrm{Im}(\alpha)\Lambda_E = R_E$.

THEOREM 2.1. — *Assume* (AI_F) *and one of* $(\mathrm{Ex}_\mathfrak{p})$ *or* $(\mathrm{TR}_\mathfrak{p})$ *for* $\mathfrak{p}|p$, $(\mathrm{TR}_\mathfrak{q})$ *for* $\mathfrak{q}$ *outside* p. *Then we have* $\mathrm{Im}(\alpha)\Lambda_E = R_E$. *If a prime factor* $\mathfrak{p}$ *of* p *ramifies totally in the maximal* p*-extension of* E *in* F, *then* $\mathrm{Im}(\alpha) = R_E$.

Proof : We only need to prove the last assertion. Here we give a short argument restricting ourselves to our special case. See Appendix for the treatment in more general cases. We argue similarly as above by replacing the deformation problem $\mathcal{D}_\Lambda$ by $\mathcal{D}_E$. Thus we pick two deformations ρ and ρ' of $\overline{\rho}$ over E of type $\mathcal{D}_E$ with values in $\mathrm{GL}_2(A)$. Suppose that $\rho \approx \rho'$ on H. Under (AI_F), $\rho \cong \rho' \otimes \xi$ for some character ξ of $\mathrm{Gal}(F/E)$. If $\xi \neq 1$, by our assumptions and $(\chi_\mathfrak{p})$, $\xi\rho_{1,\mathfrak{p}} = \rho'_{2,\mathfrak{p}}$ and $\xi\rho_{2,\mathfrak{p}} = \rho'_{1,\mathfrak{p}}$. Since $\xi \bmod m$ is trivial (because ξ is of p-power order), this contradicts to $(\mathrm{Reg}_\mathfrak{p})$. Thus $\xi = 1$ by the total ramification of $\mathfrak{p}$, and we find

$$\mathrm{Hom}_{\mathfrak{O}-\mathrm{alg}}(\mathrm{Im}(\alpha), A) \cong \mathcal{F}_{E,F}(A) \supset \mathrm{Hom}_{\mathfrak{O}-\mathrm{alg}}(R_E, A) \text{ for } \mathfrak{O}\text{-algebras } A ,$$

which shows the result, because $\mathrm{Im}(\alpha) \subset R_E$.

Let $\mathfrak{a}$ be the ideal of R_F generated by $[\sigma](x) - x$ for all $x \in R_F$ and a generator $\sigma \in \Delta$. Then $\pi = \rho_F \bmod \mathfrak{a} : H \to \mathrm{GL}_2(A)$ for $A = R_F/\mathfrak{a}$ satisfies

$\pi^{[\sigma]} \approx \pi$ and hence, by proposition 2.1, π extends to a representation π_E in $\mathcal{F}(B)$ for a faithfully flat A-algebra B in $\mathrm{CNL}_{\mathfrak{O}}$. On the other hand, we have the Galois representation ρ_E attached to R_E with $\alpha \circ \rho_F \approx \rho_E$ on H. By the universality of (R_E, ρ_E), we have an $\mathfrak{O}$-algebra homomorphism $\theta : R_E \to R$ such that $\theta \circ \rho_E \approx \pi_E$. We conclude from $\theta\alpha\rho_F \approx \theta\rho_E \approx \pi_E$ that $\theta\alpha$ coincides with the inclusion map of A into B. Thus α is injective on $A = R/\mathfrak{a}$. This shows (Nt) in 1.1 for $(R_F, \mathrm{Im}(\alpha))$ in place of (H, T) there. See Theorem A.2.3 for more general result of this type.

By definition, the ideal C defining Λ_F is invariant under Δ, because $\overline{\rho}$ is Δ-invariant. Thus Δ naturally acts on Λ_F. Identifying $Z_F = \mathrm{Gal}(F_\infty/F)$, we have the restriction map $\mathrm{res} : Z_F \to Z_E\ (= \mathrm{Gal}(E_\infty/E))$. This induces a Δ-equivariant algebra homomorphism $\mathrm{res} : \Lambda_F \to \Lambda_E$. We now take a closed $\mathfrak{O}$-subalgebra Λ'_F in R_F which is stable under the action of Δ. We can take Λ'_F to be Λ_F, but some other choice is also possible. The map $\alpha : \Lambda'_F \to \alpha(\Lambda'_F)$ coincides with res on the image of Λ_F and is equivariant under the Galois action. Let $\mathbb{I}$ be a normal integral Λ'_F-algebra which is a member of $\mathrm{CNL}_{\mathfrak{O}}$. Let $\pi : R_E \to \mathbb{I}$ be a Λ'_F-algebra homomorphism. We put

$$\lambda'_F = m(\pi\alpha \otimes_{\Lambda'_F} \mathrm{id}) : R_F \widehat{\otimes}_{\Lambda'_F} \mathbb{I} \to \mathbb{I} \text{ and } \mu'_F = m(\pi \otimes_{\Lambda'_F} \mathrm{id}) : R_E \widehat{\otimes}_{\Lambda'_F} \mathbb{I} \to \mathbb{I}$$

for the multiplication $m : \mathbb{I}\widehat{\otimes}_{\Lambda'_F}\mathbb{I} \to \mathbb{I}$, where $\mathbb{I}_0 = \mathrm{Im}(\pi)$. Here "$\widehat{\otimes}$" indicates the completion of the algebraic tensor product under the adic topology of the maximal ideal of the algebraic tensor product. Since the condition (Nt) is insensitive to tensor product, $\pi\alpha \otimes \mathrm{id} : R_F\widehat{\otimes}_{\lambda'_F}\mathbb{I} \to R_E\widehat{\otimes}_{\lambda'_F}\mathbb{I}$ again satisfies (Nt) if α is surjective. Note that $R_F\widehat{\otimes}_{\lambda'_F}\mathbb{I}$ is an $\mathbb{I}$- algebra and hence λ'_F has a trivial section of $\mathbb{I}$-modules. Thus we get from Corollary 1.1.

THEOREM 2.2. — *Assume* (AI_F) *and one of* $(\mathrm{Ex})_{\mathfrak{p}}$ *or* $(\mathrm{TR}_{\mathfrak{p}})$ *for* $\mathfrak{p}|p$, $(\mathrm{TR}_{\mathfrak{q}})$ *for* $\mathfrak{q}$ *outside* p. *Let* $\mathbb{I}$ *be a normal integral* Λ'_F*-algebra in* $\mathrm{CNL}_{\mathfrak{O}}$ *and* $\pi : R_E \to \mathbb{I}$ *be a* Λ'_F*-algebra homomorphism which is a morphism in* CNL. *Then we have, for* $\Delta = \mathrm{Gal}(F/E)$,

(i) $\mathrm{Spec}(\mathrm{Im}(\alpha))$ *is the maximal subscheme of* $\mathrm{Spec}(R_F)$ *fixed under the action of* Δ*;*

(ii) *If* α *is surjective, then* $C_1(\mu'_F; \mathbb{I}) \cong C_1(\lambda'_F; \mathbb{I})/(\sigma - 1)C_1(\lambda'_F; \mathbb{I})$, *where* σ *is a generator of* Δ.

Suppose that α is surjective and $\mathbb{I}$ is a Λ'_F-module of finite type. We now take a subalgebra Λ'_E of R_E containing $\alpha(\Lambda'_F)$. Note here that $C_1(\mu'_F; \mathbb{I})$ can be different from $C_1(\mu'_E; \mathbb{I})$ for $\mu'_E = m(\pi \otimes_{\Lambda'_E} \mathrm{id}) : R_E \otimes_{\Lambda'_E} \mathbb{I} \to \mathbb{I}$. We now compute the difference using the following diagram :

$$\begin{array}{ccc}\Lambda'_E \otimes_{\alpha(\Lambda'_F)} \mathbb{I} & \xrightarrow{\theta} & \Lambda'_E \otimes_{\Lambda'_E} \mathbb{I} \to \mathbb{I} \\ \downarrow & & \downarrow \\ R_E \otimes_{\alpha(\Lambda'_F)} \mathbb{I} & \xrightarrow{\mathrm{id}\otimes\theta} & R_E \otimes_{\Lambda'_E} \mathbb{I} \to \mathbb{I}\ .\end{array}$$

We have an exact sequence, writing $T = R_E \otimes_{\Lambda'_E} \mathbb{I}$,

$$\mathrm{Tor}_1^T(\mathbb{I}, \mathrm{Ker}(\mu'_E)) \to C_1(\mathrm{id}\otimes\theta; T) \otimes_T \mathbb{I} \to C_1(\mu'_F; \mathbb{I}) \to C_1(\mu'_E; \mathbb{I}) \to 0\ .$$

By the diagram, since the multiplication map has a section of Λ'_E-modules, we have

$$C_1(\mathrm{id}\otimes\theta; T) \cong C_1(\theta; \Lambda'_E \otimes_{\Lambda'_E} \mathbb{I}) \otimes_{\Lambda'_E} R_E \text{ and}$$

$$C_1(\theta; \Lambda'_E \otimes_{\Lambda'_E} \mathbb{I}) \cong C_1(\mathrm{id}\otimes\theta; T) \otimes_T \mathbb{I}\ .$$

Since θ is a scalar extension to $\mathbb{I}$ of the multiplication map : $\Lambda'_E \otimes_{\alpha(\Lambda'_F)} \Lambda'_E \to \Lambda'_E$. Thus we see that $C_1(\theta; \Lambda'_E \otimes_{\Lambda'_E} \mathbb{I}) \cong \Omega_{\Lambda'_E/\alpha(\Lambda'_F)} \otimes_{\Lambda'_E} \mathbb{I}$, and we have an exact sequence :

(Ext4) $\mathrm{Tor}_1^T(\mathbb{I}, \mathrm{Ker}(\mu'_E)) \to \Omega_{\Lambda'_E/\alpha(\Lambda'_F)} \otimes_{\Lambda'_E} \mathbb{I} \to$

$$C_1(\lambda'_F; \mathbb{I})/(\sigma - 1)C_1(\lambda'_F; \mathbb{I}) \to C_1(\mu'_E; \mathbb{I}) \to 0\ .$$

Then as seen in [H3] Lemma 1.11, if $\Lambda'_F \cong \mathfrak{O}[[W_F]]$ and $\Lambda'_E \cong \mathfrak{O}[[W_F]]$ for a p-profinite subgroup W_X of Z_X, we have $\Omega_{\Lambda'_E/\Lambda'_F} \cong \Lambda'_E \otimes_Z W_E/\operatorname{res}(W_F)$, and hence $C_1(\mathrm{id}\otimes\theta; T) \otimes_T \mathbb{I} \cong \mathbb{I} \otimes_Z W_E/\operatorname{res}(W_F)$. We write λ_X and μ_X for λ'_X and μ'_X when $\Lambda'_X = \Lambda_X$. Assume that Λ'_F contains Λ_F. By a similar argument using the following diagram with exact rows :

$$\begin{array}{ccc}\Lambda'_F \widehat{\otimes}_{\Lambda_F} \mathbb{I} & \xrightarrow{\theta} & \Lambda'_F \otimes_{\Lambda'_F} \mathbb{I} \to \mathbb{I} \\ \downarrow & & \downarrow \\ R_F \widehat{\otimes}_{\Lambda_F} \mathbb{I} & \xrightarrow{\mathrm{id}\otimes\theta} & R_F \otimes_{\Lambda'_F} \mathbb{I} \to \mathbb{I}\ ,\end{array}$$

we get the following exact sequence :

(Ext′4) $$\Omega_{\Lambda'_F/\Lambda_F} \widehat{\otimes}_{\Lambda_F} \mathbb{I} \to C_1(\lambda_F; \mathbb{I}) \to C_1(\lambda'_F; \mathbb{I}) \to 0\ .$$

Here $\mathbb{I}$ may not be a Λ_F-module of finite type, but we assume it is Λ'_F-module of finite type. The exactness of the above sequence follows from the compactness of these modules and [EGA] IV, 0.20.7.18.

3. — Selmer groups

Keeping notation introduced in the previous section, under a suitable assumption, we deduce the control theorem for the Selmer group $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$ from the control theorem of the deformation rings (Theorem 2.2). Then assuming the transcendence of $\varphi_{1,p}(\phi_p)$ over $\mathbb{Z}_p$ for the geometric Frobenius ϕ_p at p and the co-torsionness of $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ over $\mathbb{I}$, we prove the co-torsionness of $\mathrm{Sel}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$ over $\mathbb{I}[[\Gamma]]$.

3.1. Selmer groups. Here we suppose that $E = \mathbb{Q}$; so, instead of writing primes in E as $\mathfrak{q}$, we use the Roman character q for that. Let $\mathbb{I}$ be an integral normal local domain complete under $m_{\mathbb{I}}$-adic topology for the maximal ideal $m_{\mathbb{I}}$. We assume that $\mathbb{I}$ is an algebra over $\mathfrak{O}$ and $\mathbb{I}/m_{\mathbb{I}} = \mathbb{F}$. Let $\varphi : \mathcal{G}_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{I})$ be a continuous Galois representation. Thus $E = \mathbb{Q}$ in the notation of the previous section. Let $V(\varphi) = \mathbb{I}^n$ be the representation space of φ and $W(\varphi)$ be a subspace of $V(\varphi)$ stable under $\mathcal{G}_{\mathbb{Q}_p}$. We define two Galois modules $V(\varphi)^* = V\otimes_{\mathbb{I}}\mathbb{I}^*$ and $W(\varphi)^* = W(\varphi)\otimes_{\mathbb{I}}\mathbb{I}^*$, where $\mathbb{I}^*$ is the Pontryagin dual module of $\mathbb{I}$. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the unique $\mathbb{Z}_p$-extension. We identify $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ with $\Gamma = 1 + p\mathbb{Z}_p$ by the cyclotomic character $\mathcal{N}$. Here $\mathcal{N}$ satisfies $\mathcal{N}(\phi_q) = q$ for the geometric Frobenius ϕ_q at q. We write $\nu = \nu_\infty$ for the inclusion of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ into $\mathfrak{O}[[\Gamma]]$. Let Γ_m be the subgroup of Γ of index p^m, and write $\mathbb{Q}_m$ for the fixed field of Γ_m. We write the projection $\pi_m : \mathfrak{O}[[\Gamma]] \to \mathfrak{O}[[\Gamma/\Gamma_m]]$ and put $\nu_m = \pi_m\circ\nu$. We put for $m = 1, 2, \dots, \infty$,

$$V(\varphi\otimes\nu_m^{-1}) = V(\varphi)\widehat{\otimes}_{\mathfrak{O}}V(\nu_m^{-1}),\ W(\varphi\otimes\nu_m^{-1}) = W(\varphi)\widehat{\otimes}_{\mathfrak{O}}V(\nu_m^{-1})\ ,$$

where $V(\nu_m^{-1}) = \mathfrak{O}[\Gamma/\Gamma_m]$ for finite m.

LEMMA 3.1. — *We have the following isomorphisms of* $\mathbb{I}[[\Gamma]]$*-modules for* $m = 1, 2, \dots, \infty$,

$$H^1(\mathcal{G}_{\mathbb{Q}_m}, V(\varphi)^*) \cong H^1(\mathcal{G}_{\mathbb{Q}}, V(\varphi\otimes\nu_m^{-1})^*)\ ,$$

$$H^1(I_p\cap\mathcal{G}_{\mathbb{Q}_m}, V(\varphi)^*/W(\varphi)^*) \cong H^1(I_p, V(\varphi\otimes\nu_m^{-1})^*/W(\varphi\otimes\nu_m^{-1})^*)\ ,$$

$$\Pi_{\gamma\in\Gamma/\Gamma_m}H^1(\gamma I_q\gamma^{-1}, V(\varphi)^*/W(\varphi)^*) \cong H^1(I_q, V(\varphi\otimes\nu_m^{-1})^*/W(\varphi\otimes\nu_m^{-1})^*)\quad \text{for } q\neq p\ ,$$

where I_q *is the inertia subgroup at* q *of* $\mathcal{G}_{\mathbb{Q}}$, $\mathbb{I}[[\Gamma]]$ *acts through coefficients on the right-hand side, and on the left-hand side* $\mathbb{I}$ *acts through coefficients but* Γ *acts through the group* $\mathcal{G}_{\mathbb{Q}_\infty}$ *by conjugation.*

Proof : Note that $V(\varphi \otimes \nu_m^{-1})^*$ is the induced representation of (injective type) of $V(\varphi)^*$ restricted to $\mathcal{G}_{\mathbb{Q}_m}$ to $\mathcal{G}_{\mathbb{Q}}$, that is, $V(\varphi)\widehat{\otimes}_{\mathfrak{O}}V(\nu_m^{-1})$ is isomorphic to the space of continuous functions $\mathrm{Cont}(\Gamma/\Gamma_m, V(\varphi)^*)$ on Γ with values in $V(\varphi)^*$, on which $\mathcal{G}_{\mathbb{Q}}$ acts by $g\phi(\gamma) = \varphi(g)\phi(g^{-1}\gamma)$ for $\phi \in \mathrm{Cont}(\Gamma/\Gamma_m, V(\varphi)^*)$. Thus by Shapiro's lemma, we have the first isomorphism in the lemma. The second isomorphism can be proven in the same manner because $I_p/I_p \cap \mathcal{G}_{\mathbb{Q}_m} \cong \Gamma/\Gamma_m$. Since $I_q \subset \mathcal{G}_{\mathbb{Q}_\infty}$, the third isomorphism is obvious and induced by the first.

We now fix a representation $\overline{\rho}$ satisfying the conditions defining $\mathcal{D}$. Let $(R_{\mathbb{Q}}, \rho_{\mathbb{Q}})$ be the universal couple representing the problem $\mathcal{D}$ for $\overline{\rho}$. We assume that $\mathrm{Spec}(\mathbb{I})$ gives a closed subscheme of the normalization of the reduced part of $\mathrm{Spec}(R_{\mathbb{Q}})$. Let $\pi : R_{\mathbb{Q}} \to \mathbb{I}$ be the projection and $\varphi = \pi\rho_{\mathbb{Q}}$ be the representation of $\mathcal{G}_{\mathbb{Q}}$ into $\mathrm{GL}_2(\mathbb{I})$. Since p is odd, we can decompose $V(\varphi \otimes_{\mathbb{I}} \varphi^{\vee}) = \mathrm{End}_{\mathbb{I}}(V(\varphi))$ into the sum of trace zero space $\mathfrak{sl}_2(V(\varphi))$ and the center $Z(\varphi)$. We write the representation on $\mathfrak{sl}_2(V(\varphi))$ as $\mathrm{Ad}(\varphi)$, that is, $V(\mathrm{Ad}(\varphi)) = \mathfrak{sl}_2(V(\varphi))$. We have a filtration (χ_p) and $(\mathcal{N}_q)$ of the Galois representation of $\mathcal{G}_{\mathbb{Q}_q}$ for $q \in \mathcal{M}' \cup \{p\} : V(\rho_{\mathbb{Q},1,q}) \subset V(\rho_{\mathbb{Q}})$. This induces the filtration on $\mathfrak{sl}_2(V(\varphi)) : 0 \subset V_q^+(\mathrm{Ad}(\varphi)) \subset V_q^-(\mathrm{Ad}(\varphi)) \subset V(\mathrm{Ad}(\varphi))$ given as follows (see [H6]) :

$$V_q^+(\mathrm{Ad}(\varphi)) = \{\phi \in \mathfrak{sl}_2(V(\varphi)) | \phi(V(\rho_{\mathbb{Q},1,q})) = 0\} \text{ and}$$
$$V_q^-(\mathrm{Ad}(\varphi)) = \{\phi \in \mathfrak{sl}_2(V(\varphi)) | \phi(V(\rho_{\mathbb{Q},1,q})) \subset V(\rho_{\mathbb{Q},1,q})\} .$$

Let F be an algebraic extension of $\mathbb{Q}$. Suppose (AI_F) and that $\overline{\rho}$ satisfies the condition defining $\mathcal{D}_F$. Then the above filtration stable under $\mathcal{G}_{F_{\mathcal{Q}}}$ induces a filtration for each prime $\mathcal{Q}$ in $\mathcal{M}'_F \cup \{\mathcal{P}|p\} : 0 \subset V_{\mathcal{Q}}^+(\mathrm{Ad}(\varphi)) \subset V_{\mathcal{Q}}^-(\mathrm{Ad}(\varphi)) \subset V(\mathrm{Ad}(\varphi))$. In other words, if $\sigma\mathcal{G}_{\mathbb{Q}_q}\sigma^{-1} \supseteq \mathcal{G}_{F_{\mathcal{Q}}}$, then $V_{\mathcal{Q}}^{\pm}(\mathrm{Ad}(\varphi)) = \sigma V_q^{\pm}(\mathrm{Ad}(\varphi))$. For each $\mathbb{I}$-direct summand W of $V(\mathrm{Ad}(\varphi))$ or $V(\mathrm{Ad}(\varphi) \otimes \nu_m^{-1})$ for $0 \le m < \infty$, we define $W^* = W \otimes_{\mathbb{I}} \mathbb{I}^*$ for the Pontryagin dual $\mathbb{I}^*$ of $\mathbb{I}$. When $m = \infty$, for each $\mathbb{I}[[\Gamma]]$-submodule W of $V(\mathrm{Ad}(\varphi) \otimes \nu^{-1}) = V(\mathrm{Ad}(\varphi))\widehat{\otimes}_{\mathfrak{O}}V(\nu^{-1})$, we define $W^* = W \otimes_{\mathbb{I}[[\Gamma]]} \mathbb{I}[[\Gamma]]^*$ for the Pontryagin dual $\mathbb{I}[[\Gamma]]^*$ of $\mathbb{I}[[\Gamma]]$. We also put

$$V_{\mathcal{Q}}^{\pm}(\mathrm{Ad}(\varphi)) \otimes \nu_m^{-1}) = V_{\mathcal{Q}}^{\pm}(\mathrm{Ad}(\varphi))\widehat{\otimes}_{\mathfrak{O}}V(\nu_m^{-1}) \text{ for } m = 0, 1, \ldots, \infty .$$

We let the Galois group act on W^* through W and write Φ for one of $\mathrm{Ad}(\varphi)$ and $\mathrm{Ad}(\varphi) \otimes \nu_m^{-1}$ for $m = 0, 1, \ldots, \infty$. For each prime ideal $\mathcal{Q}$ of F and a subset $\mathcal{L}$ of $\mathcal{M}'_F$, we consider the following subgroup $L_{\mathcal{Q}}$ of $H^1(\mathcal{G}_{F_{\mathcal{Q}}}, V(\Phi)^*)$:

$$L_{\mathcal{Q}} = \ker(H^1(\mathcal{G}_{F_{\mathcal{Q}}}, V(\Phi)^*) \to H^1(I_{\mathcal{Q}}, V(\Phi)^*/V_{\mathcal{Q}}^+(\Phi)^*)) \text{ for } \mathcal{Q} \in \mathcal{L} \cup \{\mathcal{P}|p\} ,$$
$$L_{\mathcal{Q}} = \ker(H^1(\mathcal{G}_{F_{\mathcal{Q}}}, V(\Phi)^*) \to H^1(I_{\mathcal{Q}}, V(\Phi)^*)) \text{ for } \mathcal{Q} \text{ outside } \mathcal{L} \cup \{\mathcal{P}|p\} .$$

Then associated Selmer group of Φ over F is defined by

(Sel) $$\mathrm{Sel}_{\mathcal{L}}(\Phi)_{/F} = \cap_{\mathcal{Q}} \ker(H^1(\mathcal{G}_F, V(\Phi)^*) \to H^1(\mathcal{G}_{F_{\mathcal{Q}}}, V(\Phi)^*)/L_{\mathcal{Q}})) .$$

The Selmer group defined in [G] Section 4 is equal to $\mathrm{Sel}_{\emptyset}(\mathrm{Ad}(\varphi))_{/F}$, which we write simply as $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/F}$. By a general theory due to Greenberg [G] p. 217, the Pontryagin dual $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$ of $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/F}$ is of finite type over $\mathbb{I}$. In this case, starting from a modular Galois representation $\overline{\rho}$, it is basically known by Flach [F] and Wiles [W] that $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ is an $\mathbb{I}$-torsion module of finite type if $\mathrm{Spec}(\mathbb{I})$ gives an irreducible component of $R_{\mathbb{Q}}$. It has been conjectured by Greenberg that $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$ (resp. $\mathrm{Sel}^*(\mathrm{Ad}(\varphi)) \otimes \nu^{-1})_{/F}$) is an $\mathbb{I}$-torsion (resp. $\mathbb{I}[[\Gamma]]$-torsion) module of finite type if $\mathbb{I}$ is sufficiently large.

If $F/\mathbb{Q}$ is a Galois extension, by definition, $\mathrm{Gal}(F/\mathbb{Q})$ naturally acts on $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$, and the restriction map of cohomology takes $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/E}$ into $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/F}$. By Lemma 3.1, for the subfield $\mathbb{Q}_m$ of $\mathbb{Q}_\infty$, we have the following commutative diagram for $n > m$:

$$\begin{array}{ccc}
\mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}_m} & \cong & \mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi) \otimes V_m^{-1})_{/\mathbb{Q}} \\
\downarrow \mathrm{res} & & \downarrow i_{m,n} \\
\mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}_n} & \cong & \mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi) \otimes V_n^{-1})_{/\mathbb{Q}}
\end{array}$$

where $i_{m,n}$ is induced by the natural inclusion $V(\mathrm{Ad}(\varphi) \otimes \nu_m^{-1})^* \subset V(\mathrm{Ad}(\varphi) \otimes \nu_n^{-1})^*$ induced by the dual map of the projection $\mathfrak{O}[\Gamma/\Gamma_n] \to \mathfrak{O}[\Gamma/\Gamma_m]$. Since the formation of Galois cohomology commutes with injective limit of coefficients, we get the following version of a result of Greenberg [G] Proposition 3.2 :

(Sel1) $$\varinjlim_m \mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}_m} = \varinjlim_m \mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi) \otimes \nu_m^{-1})_{/\mathbb{Q}}$$
$$= \mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})_{/\mathbb{Q}} = \mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}_\infty} .$$

3.2. Mazur modules and Selmer groups. We return to the situation in 2.1. We assume (AI_F) and the conditions defining $\mathcal{D}_F$ for $\overline{\rho}$. Take a Λ-algebra homomorphism $\pi : R_F \to \mathbb{I}$ and suppose that $\mathbb{I}$ is a Λ_F-module of finite type. Let $\varphi = \pi\rho_F$. We consider the scalar extension $R = R_F \otimes_{\Lambda_F} \mathbb{I}$, which is naturally an $\mathbb{I}$-algebra. Then we consider the module $\Omega_{R/\mathbb{I}}$ of m_R-adically continuous 1-differentials over $\mathbb{I}$. Then

$$\mathrm{Hom}_R(\Omega_{R/\mathbb{I}}, M) = \mathrm{Hom}_{R_F}(\Omega_{R_F/\Lambda_F}, M) = \mathrm{Der}_{\Lambda_F}(R_F, M)$$

for each topological $\mathbb{I}$-module M of finite type or an injective limit of such modules. Here every homomorphism and derivation as above is supposed to be continuous under the $m_{\mathbb{I}}$-adic topology. We consider the ring $R_F[M] = R_F \oplus M$ with $M^2 = 0$. Then for $\delta \in \mathrm{Der}_{\Lambda_F}(R_F, M)$, we have an Λ_F-algebra homomorphism $\iota(\delta) : R_F \to R_F[M]$ given by $r \mapsto r \oplus \delta(r)$. Any Λ_F-algebra homomorphism, inducing the identity modulo M, is of the form $\iota(\delta) : R_F \to R_F[M]$ for a derivation δ.

We consider the deformation $\rho : \mathcal{G}_F \to \mathrm{GL}_2(R_F[M])$ of ρ_F. Then we can write down $\rho = \rho_F \oplus u'$ for $u' : \mathcal{G}_F \to \mathrm{End}_{\mathbb{I}}(M \oplus M)$. Define $u(\sigma) = u'(\sigma)\rho_F(\sigma)^{-1}$. Then

$$\begin{aligned} u(\sigma\tau) = u'(\sigma\tau)\rho_F(\sigma\tau)^{-1} &= (\rho_F(\sigma)u'(\tau) + u'(\sigma)\rho_F(\tau))\rho_F(\sigma\tau)^{-1} \\ &= \mathrm{Ad}(\rho_F)(\sigma)u(\tau) + u(\sigma)\ . \end{aligned}$$

Note that $\det(1 \oplus u) = \mathrm{Tr}(u)$ for $u \in \mathrm{End}_{\mathbb{I}}(M \oplus M)$. Thus by $(\det)$ in 2.2, u is a 1-cocycle, under the adjoint action $\mathrm{Ad}(\rho_F)$ on $\mathrm{Ad}(M) = V(\mathrm{Ad}(\varphi)) \otimes_{\mathbb{I}} M$ in $\mathrm{End}_{\mathbb{I}}(M \oplus M)$, having values in $\mathrm{Ad}(M)$. One can check that the map : $\rho \mapsto$ the cohomology class of u from the set of deformations of ρ_F of type $\mathcal{D}_F$ is an injection. For primes $\mathcal{Q}$ in $\mathcal{M}_F(\chi)$, by $(\chi_{\mathcal{Q}})$, the order of $\rho(I_{\mathcal{Q}})$ is prime to p, and thus (a) $u(I_{\mathcal{Q}}) = 0$. For $\mathcal{Q} \in \mathcal{M}'_F$ or $\mathcal{Q}|p$, we have (b) $u(\mathcal{G}_{F_{\mathcal{Q}}}) \subset V^-_{\mathcal{Q}}(\mathrm{Ad}(\varphi)) \otimes_{\mathbb{I}} M$ and (c) $u(I_{\mathcal{Q}}) \subset V^+_{\mathcal{Q}}(\mathrm{Ad}(\varphi)) \otimes_{\mathbb{I}} M$. Since $\mathcal{G}_{F_{\mathcal{Q}}}$ normalizes $I_{\mathcal{Q}}$, the non-splitting of the exact sequence in $(\mathcal{N}_{\mathcal{Q}})$ or $(\mathrm{Reg}_{\mathcal{P}})$ shows that (c) implies (b). It is obvious that if we are given a 1-cocycle u satisfying (a) and (c), $\rho = \rho_F \oplus u\rho_F$ is a deformation of type $\mathcal{D}_\Lambda$. Thus we get a version of the results in [MT] Proposition 25 and [HT] Proposition 2.3.10 :

THEOREM 3.1. — *Suppose that $\mathbb{I}$ is a Λ_F-module of finite type. We have*

$$\mathrm{Hom}_R(\Omega_{R/\mathbb{I}}, \mathbb{I}^*) = C_1(\lambda; \mathbb{I})^* \cong \mathrm{Sel}_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/F}\ ,$$

where λ is given by $m \circ (\pi \otimes \mathrm{id})$ for the multiplication $m : \mathbb{I}_0 \otimes_{\Lambda_F} \mathbb{I} \to \mathbb{I}$ with $\mathbb{I}_0 = \mathrm{Im}(\pi)$ and "$*$" indicates the Pontryagin dual module.

3.3. Control theorem of Selmer groups of $\mathrm{Ad}(\varphi)$. We now assume that $E = \mathbb{Q}$ and $F = \mathbb{Q}_j \subset \mathbb{Q}_\infty$. Then it is easy to check that if $\overline{\rho}$ satisfies the condition of $\mathcal{D}_E$, then $\overline{\rho}$ restricted to $\mathcal{G}_F$ satisfies the conditions of $\mathcal{D}_F$. We assume $(\mathrm{AI}_{\mathbb{Q}_\infty})$. Then we write α_j (resp. Λ_j, m_j) for the morphism $\alpha : R_j = R_F \to R_{\mathbb{Q}}$ (resp. $\mathfrak{O}[[\Gamma_j]]$ in Λ_F, the multiplication $m : \mathbb{I}_0 \otimes_{\Lambda_j} \mathbb{I} \to \mathbb{I}$). We write ρ_j for the universal representation realized on R_j. Now let $\mathrm{Spec}(\mathbb{I})$ be an irreducible component of the normalization of $\mathrm{Spec}(R_{\mathbb{Q}})$ and write

$\pi : R_{\mathbb{Q}} \to \mathbb{I}$ for the projection map. We assume that $\mathbb{I}$ is a *torsion-free* Λ_0-module of finite type. We apply the above theorem to $\pi_j = \pi\alpha_j : R_j \to \mathbb{I}$. We write $\lambda_j = m_j \circ (\pi_j \otimes \mathrm{id})$. Under $(\mathrm{AI}_{\mathbb{Q}_\infty})$, by Theorem 2.1, $\mathrm{Im}(\pi_j) = \mathrm{Im}(\pi) = \mathbb{I}_0$ independent of j. Then similarly to α_j, we can construct an algebra homomorphism $\alpha_{j,k} : R_k \to R_j$ so that $\alpha_{j,k} \circ \rho_k \approx \rho_j$. Then $\alpha_{j,k}$ induces a projection map

$$C_1(\pi_k; \mathbb{I}_0) \to C_1(\pi_j; \mathbb{I}_0), C_1(m_k; \mathbb{I}) \to C_1(m_j; \mathbb{I}) \text{ and } C_1(\lambda_k; \mathbb{I}) \to C_1(\lambda_j; \mathbb{I})\ .$$

Since R_F is topologically of finite type over Λ_j, all these modules are made of compact modules. Note that projective limit is an exact functor on the category of compact modules. Then we take the projective limit of these modules, and write them as $C_1(\lambda_\infty; \mathbb{I})$, $C_1(\pi_\infty; \mathbb{I})$ and $C_1(m_\infty; \mathbb{I})$. Then we have an exact sequence from (Ext4) in 2.2, for the generator γ_j of $\Gamma_j = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}_j)$, $k \geq j$ and $T_j = R_j \otimes_{\Lambda_j} \mathbb{I}$

(Ext5) $\mathrm{Tor}_1^{T_j}(\mathbb{I}, \mathrm{Ker}(\mu_{\mathbb{Q}_j})) \to \mathbb{I} \otimes_{\mathbb{Z}} \Gamma_j/\Gamma_k \to C_1(\lambda_k; \mathbb{I})/(\gamma_j - 1)C_1(\lambda_k; \mathbb{I}) \to C_1(\lambda_j; \mathbb{I}) \to 0\ .$

Since these are compact modules, after taking projective limit with respect to k, we still have the following exact sequence :

(Ext6) $\mathrm{Tor}_1^{T_j}(\mathbb{I}, \mathrm{Ker}(\mu_{\mathbb{Q}_j})) \to \mathbb{I}\otimes_{\mathbb{Z}_p}\Gamma_j (\cong \mathbb{I}) \to C_1(\lambda_\infty; \mathbb{I})/(\gamma_j - 1)C_1(\lambda_\infty; \mathbb{I}) \to C_1(\lambda_j; \mathbb{I}) \to 0\ .$

Suppose $R_{\mathbb{Q}}$ is reduced. Then if $R_{\mathbb{Q}}$ is a $\Lambda_{\mathbb{Q}}$- module of finite type, $C_1(\lambda_0; \mathbb{I})$ is a torsion $\mathbb{I}$-module of finite type. We now show that $C_1(\lambda_\infty; \mathbb{I})/(\gamma - 1)C_1(\lambda_\infty; \mathbb{I})$ contains actually a copy of $\mathbb{I}$. For that, we look into the following exact sequence obtained from (Ext3) in 1.1 :

$$C_1(\pi_j; \mathbb{I}_0) \otimes_{\mathbb{I}_0} \mathbb{I} \xrightarrow{\iota_j} C_1(\lambda_j; \mathbb{I}) \to C_1(m_j; \mathbb{I}) \to 0 \text{ for } j = 0, 1, \ldots, \infty\ .$$

We study $\mathrm{Ker}(\iota_j)$. By definition, Λ_j is isomorphic to $\mathfrak{O}[[\Gamma_j]]$. We write Λ for $\mathfrak{O}[[\Gamma_j]]$. By (Ext2-3) in 1.1, this module is a surjective image of $\mathrm{Tor}_1^{\mathbb{I}^\circ}(\mathrm{Ker}(m_j), \mathbb{I})$ if $\mathbb{I}_0$ is flat over Λ, where we write $\mathbb{I}^\circ$ for $\mathbb{I}_0 \otimes_{\Lambda_j} \mathbb{I}$. Suppose for the moment that $\mathbb{I}_0$ is Λ-flat. We write Λ° for $\Lambda \otimes_{\Lambda_j} \Lambda$. We first compute $\mathrm{Tor}_1^{\Lambda^\circ}(\mathrm{Ker}(m_j), \Lambda)$ when $\mathbb{I} = \Lambda$. Identifying Λ with $\mathfrak{O}[[T]]$, we see easily that $\Lambda^\circ \cong \mathfrak{O}[[T]][S]/((1 + S)^q - (1 + T)^q)$ for $q = p^j$. Then $\mathrm{Ker}(m_j)$ is a principal ideal generated by $S - T$, and decomposing $(1 + S)^q - (1 + T)^q = (S-T)f(S, T)$, we have an isomorphism $\mathrm{Ker}(m_j) \cong \Lambda^\circ/(f(S, T))$. Thus we have an exact sequence :

$$0 \to (S - T)\Lambda^\circ \to \Lambda^\circ \xrightarrow{f(S,T)} \Lambda^\circ \to \mathrm{Ker}(m_j) \to 0\ .$$

Tensoring Λ, we get $\mathrm{Tor}_1^{\Lambda^\circ}(\mathrm{Ker}(m_j),\Lambda)=0$ and $\varprojlim_j \mathrm{Tor}_1^{\Lambda^\circ}(\mathrm{Ker}(m_j),\Lambda)$ $=0$. Since $\mathbb{I}$ is a Λ-module of finite type, $\mathrm{Ker}(\iota_j)$ is a torsion $\mathbb{I}$-module in general. In particular, if $C_1(\lambda_0,\mathbb{I})$ is of $\mathbb{I}$-torsion, $C_1(\pi_0,b_0)\otimes_{\mathbb{I}_0}\mathbb{I}$ is of $\mathbb{I}$-torsion.

Writing $\mathbb{I}_j$ for $\mathbb{I}_0\widehat{\otimes}_{\Lambda_j}\mathbb{I}$, we have projective systems of surjective homomorphisms $\mathbb{I}_j\to\mathbb{I}_{j-1}$ and $\mathrm{Ker}(m_j)\to\mathrm{Ker}(m_{j-1})$. Writing $\mathbb{I}_\infty$ for $\varprojlim_j \mathbb{I}_j$, we know that $\mathrm{Ker}(m_\infty)=\varprojlim_j \mathrm{Ker}(m_j)$ and that $\mathrm{Tor}_1^{\mathbb{I}_\infty}(\mathrm{Ker}(m_\infty),\mathbb{I})=$ $\varprojlim_j \mathrm{Tor}_1^{\mathbb{I}_j}(\mathrm{Ker}(m_j),\mathbb{I})$. Note that $\mathbb{I}_\infty=\mathbb{I}_0\widehat{\otimes}_{\mathfrak{O}}\mathbb{I}$. It is obvious that $\mathrm{Ker}(m_\infty)$ is the ideal of the diagonal Δ in $\mathrm{Spec}(\mathbb{I}_0)\times_{\mathrm{Spec}(\mathfrak{O})}\mathrm{Spec}(\mathbb{I})$. Obviously Δ is irreducible and spanned by $S-T$. Thus $\mathrm{Ker}(m_\infty)$ is $\mathbb{I}_\infty$-free. This implies that $\mathrm{Tor}_1^{\mathbb{I}_\infty}(\mathrm{Ker}(m_\infty),\mathbb{I})=0$. Thus we have an exact sequence of $\mathbb{I}[[\Gamma]]$-modules :

$$0\to C_1(\pi_\infty;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I}\xrightarrow{\iota_\infty}C_1(\lambda_\infty;\mathbb{I})\to C_1(m_\infty;\mathbb{I})\to 0\ .$$

When $\mathbb{I}_0$ is not flat over Λ but Λ-torsion free, then $\mathbb{I}_0$ can be embedded into a Λ-free subalgebra $\mathbb{I}'$ of $\mathbb{I}$ such that $\mathbb{I}'/\mathbb{I}_0$ is pseudo-null. Thus we can repeat the above argument in the category of $\mathbb{I}$-modules with pseudo-morphisms. We then get the above exact sequence with pseudo-null $\mathrm{Ker}(\iota_\infty)$.

We now study $C_1(m_\infty;\mathbb{I}_0)$. Then, by definition, we have $C_1(m_j;\mathbb{I}_0)=$ $\Omega_{\mathbb{I}_0/\Lambda_j}$, where $\mathrm{Hom}_{\mathbb{I}_0}(\Omega_{\mathbb{I}_0/\Lambda_j},M)$ is naturally isomorphic to the module $\mathrm{Der}_{\Lambda_j}(\mathbb{I}_0,M)$ of continuous derivations over Λ_j for all compact $\mathbb{I}_0$-modules M. As we have seen (see also [H3] Lemma 1.11), we have $\mathrm{Ker}(m_\infty)=$ $(S-T)\mathbb{I}_0\widehat{\otimes}_{\mathfrak{O}}\mathbb{I}_0$. This shows that $C_1(m_\infty;\mathbb{I}_0)\cong\mathbb{I}_0$ and that $C_1(m_\infty;\mathbb{I}_0)$ is a torsion $\mathbb{I}[[\Gamma]]$-module of finite type.

Suppose either that $R_\mathbb{Q}$ is reduced and is a Λ-module of finite type or $R_\mathbb{Q}$ is reduced and $\mathrm{Spec}(\mathbb{I}_0)$ is an irreducible component of $\mathrm{Spec}(R_\mathbb{Q})$. Then $C_1(\lambda_0;\mathbb{I})$ is of torsion and cannot contain a submodule isomorphic to $\mathbb{I}$. Thus the inclusion of $\mathbb{I}$ into $C_1(\lambda_\infty;\mathbb{I})$ composed with the projection from $C_1(\lambda_j;\mathbb{I})$ onto $\mathbb{I}$ is a non-zero $\mathbb{I}$-linear map, and it is injective. Writing M for $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$, we have a commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
 & & \mathbb{I}\otimes_{\mathbb{Z}_p}\Gamma_j & \xrightarrow{\varepsilon_j} & M/(\gamma^{p^j}-1)M & \to & C_1(\lambda_j;\mathbb{I}) & \to & 0\\
 & & \cap & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{I}\otimes_{\mathbb{Z}_p}\Gamma & \xrightarrow{\varepsilon_0} & M/(\gamma-1)M & \to & C_1(\lambda_0;\mathbb{I}) & \to & 0
\end{array},$$

where the last two vertical arrows are surjective and the first vertical map is injective induced by the $inclusion : \Gamma_j \subset \Gamma$. Thus ε_j is injective.

As is remarked in Section A.1 (see (AI) after Corollary A.1.3), the condition $(\mathrm{AI}_{\mathbb{Q}_\infty})$ is equivalent to $(\mathrm{AI}_{\mathbb{Q}})$. We record what we have proven.

THEOREM 3.2. — *Suppose* $(\mathrm{AI}_{\mathbb{Q}})$, *the conditions of* $\mathcal{D}$ *for* $\overline{\rho}$ *and that* $\mathbb{I}$ *is a torsion-free* Λ_0*-module of finite type giving the normalization of an irreducible component of* $\mathrm{Spec}(R_{\mathbb{Q}})$. *Let* $\mathrm{Sel}^*_{\mathcal{L}}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})_{/\mathbb{Q}}$ *be the Pontryagin dual module of the Selmer group* $\mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})_{/\mathbb{Q}}$. *We have the following two exact sequences of* $\mathbb{I}$*-modules :*

$$\mathbb{I} \otimes_{\mathbb{Z}_p} \Gamma_j \xrightarrow{\varepsilon_j} \mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})_{/\mathbb{Q}} / (\gamma^{p^j} - 1)\, \mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})_{/\mathbb{Q}} \to C_1(\lambda_j; \mathbb{I}) \to 0$$

$$C_1(\pi_\infty; \mathbb{I}_0) \otimes_{\mathbb{I}_0} \mathbb{I} \xrightarrow{\iota_\infty} \mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})_{/\mathbb{Q}} \to \mathbb{I} \to 0$$

with pseudo-null kernel $\mathrm{Ker}(\iota_\infty)$, *which vanishes when* $\mathbb{I}_0$ *is flat over* Λ. *Moreover suppose that* $R_{\mathbb{Q}}$ *is reduced and either that* $R_{\mathbb{Q}}$ *is a* Λ*-module of finite type or that* $\mathrm{Spec}(\mathbb{I}_0)$ *is an irreducible component of* $\mathrm{Spec}(R_{\mathbb{Q}})$. *Then* ε_j *is injective.*

3.4. Cotorsionness of the Selmer group over $\mathbb{I}[[\Gamma]]$. We write Λ'_j for the subalgebra of R_0 topologically generated over $\mathfrak{O}$ by $\alpha(\rho_{F,1,\mathcal{P}}(\phi_{\mathcal{P}}))$ for $F = \mathbb{Q}_j$, where $\phi_{\mathcal{P}}$ is the geometric Frobenius element in $D_{\mathcal{P}}/I_{\mathcal{P}}$. Taking a unit u in $\mathfrak{O}$ such that $\rho_{\mathbb{Q},1,p}(\phi_p) \equiv u \bmod m_{\mathbb{Q}}$, we assume that, with the notation of $(\chi_{\mathcal{P},F})$,

(Ind) *the subalgebra of* $\mathbb{I}_0$ *topologically generated over* $\mathfrak{O}$ *by* $\varphi_{1,p}(\phi_p)$ *is isomorphic to the one variable power series ring* $\mathfrak{O}[[X]]$ *via* $\varphi_{1,p}(\phi_p) - u \mapsto X$.

Since π takes $\rho_{\mathbb{Q},1,p}(\phi_p)$ to $\varphi_{1,p}(\phi_p)$, $\rho_{\mathbb{Q},1,p}(\phi_p) - u$ is analytically independent over $\mathfrak{O}$. Since p totally ramified in $\mathbb{Q}_\infty$, α takes $\rho_{F,1,\mathcal{P}}(\phi_{\mathcal{P}})$ to $\rho_{\mathbb{Q},1,p}(\phi_p)$. Thus (Ind) implies

(Ind_j) *the subalgebra of* R_j *topologically generated over* $\mathfrak{O}$ *by* $\rho_{F,1,\mathcal{P}}(\phi_{\mathcal{P}})$ *is isomorphic to* $\mathfrak{O}[[X]]$ *via* $\rho_{F,1,\mathcal{P}}(\phi_{\mathcal{P}}) - u \mapsto X$ *and is stable under* $\mathrm{Gal}(\mathbb{Q}_j/\mathbb{Q})$.

Thus $\Lambda' = \Lambda'_j$ is independent of j and R_j is naturally an algebra over Λ'. We also suppose that $\mathbb{I}_0$ is a Λ_0-torsion free module of finite type. Since $\mathbb{I}_0$ is an integral domain, $\mathbb{I}_0$ is a Λ'-torsion free module. We write λ'_j for the composite of $\pi_j \otimes \mathrm{id} : R_j \widehat{\otimes}_{\Lambda'} \mathbb{I} \to \mathbb{I}_0 \widehat{\otimes}_{\Lambda'} \mathbb{I}$ and the multiplication : $\mathbb{I}_0 \widehat{\otimes}_{\Lambda'} \mathbb{I} \to \mathbb{I}$, where "$\widehat{\otimes}$" indicates the completion under the adic topology of the maximal ideal of the algebraic tensor product. We prove the following theorem in this section :

THEOREM 3.3. — *Suppose* $(\mathrm{AI}_{\mathbb{Q}})$, (Ind), *that* $\mathbb{I}$ *is a torsion-free* Λ*-module of finite type giving the normalization of an irreducible component of* $\mathrm{Spec}(R_{\mathbb{Q}})$ *and that* $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ *is a torsion* $\mathbb{I}$*-module. Then we have*

(i) $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ *is a torsion* $\mathbb{I}[[\Gamma]]$*-module of finite type;*

(ii) *There is a pseudo-isomorphism of* $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ *into* $M\times\mathbb{I}$ *for a torsion* $\mathbb{I}[[\Gamma]]$*-module* $M=C_1(\lambda'_\infty;\mathbb{I})$ *such that* $M/(\gamma-1)M$ *is a torsion* $\mathbb{I}$*-module and* M *is pseudo-isomorphic to* $C_1(\pi_\infty;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I}$*;*

(iii) *If* $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ *is a pseudo-null* $\mathbb{I}$*-module and* $\Lambda'=\mathbb{I}$*, then* $\mathrm{Sel}^*_{\mathcal{M}}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ *is pseudo isomorphic to* $\mathbb{I}$*, on which* Γ *acts trivially;*

(iv) *If* $\mathbb{I}_0$ *is formally smooth over* $\mathfrak{O}$*, then we have the following exact sequence of* $\mathbb{I}[[\Gamma]]$*-modules :*

$$0\to C_1(\pi_\infty;\mathbb{I})\to C_1(\lambda'_\infty;\mathbb{I})\to\widehat{\Omega}_{\mathbb{I}/\Lambda'}\to 0\ ,$$

where $\widehat{\Omega}_{\mathbb{I}/\Lambda'}$ *is the module of continuous* 1*-differentials or equivalently is the* $m_{\mathbb{I}}$*-adic completion of* $\Omega_{\mathbb{I}/\Lambda'}$ *(which is a torsion* $\mathbb{I}$*-module of finite type by* (Ind)*).*

By the theorem, $\mathrm{Sel}^*_{\mathcal{L}}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ is a torsion $\mathbb{I}[[\Gamma]]$-module of finite type for any subset $\mathcal{L}$ of $\mathcal{M}'$. The theorem combined with Theorem 3.2 reduces the study of $\mathrm{Sel}^*_{\mathcal{L}}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ to the study of $C_1(\pi_\infty;\mathbb{I})$ if $\mathbb{I}_0$ is formally smooth over $\mathfrak{O}$.

Here is a concrete case where the theorem applies. For a positive integer N prime to p, let $h^{\mathrm{ord}}(N;\mathfrak{O})$ be the universal ordinary Hecke algebra for $\mathrm{GL}(2)_{/\mathbb{Q}}$. Then $h^{\mathrm{ord}}(N;\mathfrak{O})$ is an algebra over $\mathfrak{O}[[\Gamma\times(\mathbb{Z}/Np\mathbb{Z})^\times]]$. The algebra structure is given so that $h^{\mathrm{ord}}(N;\mathfrak{O})/P_\kappa h^{\mathrm{ord}}(N;\mathfrak{O})$ is isomorphic to the ordinary Hecke algebra of weight $\kappa+1$ of level Np, where P_κ is the prime ideal of Λ generated by $\gamma-\mathcal{N}(\gamma)^\kappa$. Take a primitive character ψ modulo Np and suppose that ψ has order prime to p. We take the algebra direct summand $h(\psi)$ of $h^{\mathrm{ord}}(N;\mathfrak{O})$ on which $(\mathbb{Z}/Np\mathbb{Z})^\times$ acts by ψ. Take a maximal ideal m of $h(\psi)$ with residue field $\mathbb{F}$ and write H for the m-adic completion of the Hecke algebra $h(\psi)$. Then we have a unique isomorphism class of Galois representations $\rho:G\to\mathrm{GL}_2(H)$ as in [H1] (see also [DHI] Section 1) if $\overline{\rho}=\rho \bmod m$ is absolutely irreducible. In this case, $\overline{\rho}$ and ρ satisfy the requirement of the deformation problem $\mathcal{D}_{\mathbb{Q}}$. Since H is reduced and Λ-free of finite rank (see [H1] and [H] Chapter 7), the reducedness and the finiteness of $R_{\mathbb{Q}}$ over $\Lambda_{\mathbb{Q}}$ follows from Wiles' result [W] Theorem 3.3 asserting that $(R_{\mathbb{Q}},\rho_{\mathbb{Q}})\cong(H,\rho)$ under the assumption (AI_F) for $F=\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. Thus under this condition, for each irreducible component $\mathrm{Spec}(\mathbb{I})$ of the normalization of $\mathrm{Spec}(H)$ and the projection $\pi : H \to \mathbb{I}$, $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ for $\varphi = \pi\rho$ is a torsion $\mathbb{I}$-module of finite type. This also follows from a result of Flach [F] in some special cases. The condition (Ind) can be verified in this case as follows. Note that $\rho_{\mathbb{Q},1,p}(\phi_p) = T(p)$ in H. Thus $\varphi_{1,p}(\phi_p)$ specializes to an algebraic integer a_κ in $\mathfrak{O}$ modulo P_κ with $|a_\kappa| = p^{\kappa/2}$ for any archimedean absolute value $|\ |$ on $\mathbb{Q}(a_\kappa)$ for infinitely many κ. Thus $\varphi_{1,p}(\phi_p)$ is transcendental over $\mathfrak{O}$ and hence (Ind) is satisfied. Thus in this case $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi) \otimes \nu^{-1})$ is a torsion $\mathbb{I}[[\Gamma]]$-module of finite type.

We give two proofs of the theorem. The first one is just a repetition of the argument in the previous section replacing Λ_j by Λ', which is easy but we need to assume an additional assumption that $\mathbb{I}_0$ is a Λ'-module of finite type. The other one works in general, but we need to use the theory of imperfection modules in [EGA] IV.0.20.6. Anyway we look into the following exact sequence obtained from (Ext3) in 1.1 for $j = 0, 1, \dots, \infty$:

$$C_1(\pi_j; \mathbb{I}_0) \otimes_{\mathbb{I}_0} \mathbb{I} \xrightarrow{\iota'_j} C_1(\lambda'_j; \mathbb{I}) \to C_1(m'_j; \mathbb{I}) \to 0$$

where $m' : \mathbb{I}_0 \widehat{\otimes}_{\Lambda'} \mathbb{I} \to \mathbb{I}$ is the multiplication map and $\lambda'_j : R_j \widehat{\otimes}_{\Lambda'} \mathbb{I} \to \mathbb{I}$ is the composition $m' \circ (\pi\alpha_j \otimes \mathrm{id})$. Note here that the first term $C_1(\pi_j; \mathbb{I}_0) \otimes_{\mathbb{I}_0} \mathbb{I}$ is the same as the case studied in 3.3.

We study $\mathrm{Ker}(\iota'_j)$. As we remarked, here we assume that $\mathbb{I}_0$ is a Λ'-module of finite type and will deal with the general case later. Then $\mathrm{Ker}(\iota'_j)$ is a surjective image of $\mathrm{Tor}_1^{\mathbb{I}'}(\mathrm{Ker}(m'), \mathbb{I})$ if $\mathbb{I}_0$ is flat over Λ and is a surjective image up to pseudo-null error in general (see (Ext2) and (Ext2$'$)), where we write $\mathbb{I}'$ for $\mathbb{I}_0 \otimes_{\Lambda'} \mathbb{I}$. Note that this fact holds independently of j. We have the long exact sequence for $M' = \mathrm{Tor}_1^{\mathbb{I}'_0}(\mathrm{Ker}(m'), \mathbb{I}_0)$ and $\mathbb{I}'_0 = \mathbb{I}_0 \otimes_{\Lambda'} \mathbb{I}_0$:

$$0 \to M' \to I \otimes_{\mathbb{I}'} I \to I \to \Omega_{\mathbb{I}_0/\Lambda'} \to 0$$

obtained out of the following short exact sequence :

$$0 \to I \to \mathbb{I}'_0 \to \mathbb{I}_0 \to 0 .$$

Thus M' is an $\mathbb{I}_0$-module of finite type. Since $\Omega_{\mathbb{I}_0/\Lambda'}$ is a torsion $\mathbb{I}_0$-module, localizing at a prime P outside $\mathrm{Supp}(\Omega_{\mathbb{I}_0/\Lambda'})$, we have $I_P/I_P^2 = 0$, and by Nakayama's lemma, $I_P = 0$, which implies $M'_P = 0$. Thus $\mathrm{Supp}(M') \subset \mathrm{Supp}(\Omega_{\mathbb{I}_0/\Lambda'})$, which shows that M' is a torsion $\mathbb{I}_0$-module of finite type. Thus $C_1(\pi_\infty; \mathbb{I}_0) \otimes_{\mathbb{I}_0} \mathbb{I}$ is pseudo-isomorphic to $C_1(\lambda'_\infty; \mathbb{I})$ as $\mathbb{I}[[\Gamma]]$-modules. By Theorem 2.2, we have

$$C_1(\lambda'_\infty; \mathbb{I})/(\gamma - 1)C_1(\lambda'_\infty; \mathbb{I}) \cong C_1(\lambda'_0; \mathbb{I}) .$$

Thus if $C_1(\lambda_0;\mathbb{I}) \cong \mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ is of $\mathbb{I}$-torsion, $C_1(\pi_0;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I}$ and hence $C_1(\lambda'_0;\mathbb{I})$ are $\mathbb{I}$-torsion modules of finite type. The Krull dimension of $C_1(\lambda'_0;\mathbb{I})$ over $\mathbb{I}$ satisfies $\dim_{\mathbb{I}}(C_1(\lambda'_0;\mathbb{I})) < \dim(\mathbb{I})$. By [EGA] IV.0.16.2.3.1, we have

$$\begin{aligned}\dim(\mathbb{I}[[\Gamma]]) = \dim(\mathbb{I}) + 1 &> \dim_{\mathbb{I}[[\Gamma]]}(C_1(\lambda'_\infty;\mathbb{I})/(\gamma-1)C_1(\lambda'_\infty;\mathbb{I})) + 1 \\ &\geq \dim_{\mathbb{I}[[\Gamma]]}(C_1(\lambda'_\infty;\mathbb{I})).\end{aligned}$$

Thus $C_1(\lambda'_\infty;\mathbb{I})$ is a torsion $\mathbb{I}[[\Gamma]]$-module, and hence $C_1(\pi_\infty;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I}$ is a torsion $\mathbb{I}[[\Gamma]]$-module of finite type. Then Theorem 3.2 tells us that $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$ is a torsion $\mathbb{I}[[\Gamma]]$-module of finite type.

A principal ingredient of the second proof is the theory of imperfection modules in [EGA] IV § 0.20.6; so, we recall the theory here and generalize it in the case of compact adic rings. Here $\mathfrak{O}$ is a valuation ring finite and flat over $\mathbb{Z}_p$ with residue field $\mathbb{F}$. Let Λ be a base ring which is an object of $\mathrm{CNL} = \mathrm{CNL}_{\mathfrak{O}}$. If $X \to Y$ is a morphism in CNL, we write $\widehat{\Omega}_{Y/X}$ for the m_X-adic completion of the module of one differentials of Y over X. Then $\widehat{\Omega}_{Y/X}$ is a Y-module of finite type and hence is compact. We consider local algebra homomorphisms in $\mathrm{CNL} : \Lambda \to A \xrightarrow{u} B \xrightarrow{v} C$. Then by [EGA] 0.20.7.18 we have an exact sequence

$$\widehat{\Omega}_{B/A}\widehat{\otimes}_B C \xrightarrow{v_*} \widehat{\Omega}_{C/A} \xrightarrow{u_*} \widehat{\Omega}_{C/B} \to 0\ ,$$

where "$\widehat{\otimes}$" indicates the m_C-adic completion. We define the imperfection module $\Upsilon_{C/B/A}$ following [EGA] 0.20.6.1.1 by

$$\Upsilon_{C/B/A} = \mathrm{Ker}(\widehat{\Omega}_{B/A}\widehat{\otimes}_B C \xrightarrow{v_*} \widehat{\Omega}_{C/A})\ .$$

Then we have the following commutative diagram with exact rows (see [EGA] 0.20.6.16) :

$$\begin{array}{ccccccccc} 0 & \to & \widehat{\Omega}_{A/\Lambda}\widehat{\otimes}_A C & \xrightarrow{j_1} & (\widehat{\Omega}_{A/\Lambda}\widehat{\otimes}_A C)\oplus(\widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C) & \xrightarrow{p_1} & \widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C & \to & 0 \\ & & \downarrow{\scriptstyle u_*\otimes\mathrm{id}} & & \downarrow{\scriptstyle (vu)_*\oplus\mathrm{id}} & & \downarrow{\scriptstyle v_*} & & \\ 0 & \to & \widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C & \xrightarrow{j_0} & (\widehat{\Omega}_{C/\Lambda}\oplus(\widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C) & \xrightarrow{p_0} & \widehat{\Omega}_{C/\Lambda} & \to & 0, \end{array}$$

where, writing $f = u_*\otimes\mathrm{id}$ and $g = v_*$, $j_1(x) = x\oplus f(x)$, $p_1(y\oplus z) = z - f(y)$, $j_0(x) = g(x)\oplus x$ and $p_0(y\oplus z) = g(z) - y$. We put

$$\Upsilon^C_{B/A/\Lambda} = \mathrm{Ker}(\widehat{\Omega}_{A/\Lambda}\widehat{\otimes}_A C \to \widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C)\ .$$

Then by the snake lemma, we have an exact sequence (cf. [EGA] 0.20.6.17) :
(E1)
$$0 \to \Upsilon^C_{B/A/\Lambda} \to \Upsilon_{C/A/\Lambda} \to \Upsilon_{C/B/\Lambda} \to \widehat{\Omega}_{B/A}\widehat{\otimes}_B C \to \widehat{\Omega}_{C/A} \to \widehat{\Omega}_{C/B} \to 0 .$$

Now suppose that v is surjective, and hence $\widehat{\Omega}_{C/B} = 0$. By [EGA] 0.20.7.20, we have another exact sequence :

(E1′) $$\mathrm{Ker}(v)/\mathrm{Ker}(v)^2 \to \widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C \to \widehat{\Omega}_{C/\Lambda} \to 0 .$$

If $B/\mathrm{Ker}(v)^2 \to C$ has a section of Λ-algebras (for example, if C is formally smooth over Λ), then

$$0 \to \mathrm{Ker}(v)/\mathrm{Ker}(v)^2 \to \Omega_{B/\Lambda} \otimes_B C \to \Omega_{C/\Lambda} \to 0$$

is exact (cf. [EGA] 0.20.6.10). Since taking m_B-adic completion is a left exact functor, we have the exactness of

$$0 \to \mathrm{Ker}(v)/\mathrm{Ker}(v)^2 \to \widehat{\Omega}_{B/\Lambda}\widehat{\otimes}_B C \to \widehat{\Omega}_{C/\Lambda} \to 0 .$$

This shows that if $B/\mathrm{Ker}(v)^2 \to C$ has a section of Λ-algebras, then

$$\Upsilon_{C/B/\Lambda} \cong \mathrm{Ker}(v)/\mathrm{Ker}(v)^2$$

and the following sequence is exact :

(E2) $$0\to\Upsilon^C_{B/A/\Lambda}\to\Upsilon_{C/A/\Lambda} \to \mathrm{Ker}(v)/\mathrm{Ker}(v)^2\to\widehat{\Omega}_{B/A}\widehat{\otimes}_B C\to\widehat{\Omega}_{C/A}\to 0 .$$

Let $\mathbb{K}$ be a finite extension of the quotient field $\mathbb{L}$ of $A = \mathfrak{O}[[t]]$ for a variable t. Let $\mathbb{I}$ be an A-subalgebra of $\mathbb{K}$ integral over A with quotient field $\mathbb{K}$. Thus $\dim(\mathbb{I}) = 2$. Since A is a Japanese ring, $\mathbb{I}$ is an A-module of finite type. Thus we have a surjection $v : A[[X_1, \dots, X_r]] \to \mathbb{I}$. This yields an exact sequence

$$\mathrm{Ker}(v)/\mathrm{Ker}(v)^2 \to \oplus_i \mathbb{I} dX_i \to \Omega_{\mathbb{I}/A} \to 0 .$$

Since $\Omega_{\mathbb{K}/\mathbb{L}} = 0$, $\Omega_{\mathbb{I}/A}$ is a torsion $\mathbb{I}$-module of finite type. We also have the following exact sequence :

$$0 \to \Upsilon_{\mathbb{I}/A/\mathfrak{O}} \to \widehat{\Omega}_{A/\mathfrak{O}} \otimes_A \mathbb{I} \to \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}} \to \Omega_{\mathbb{I}/A} \to 0 .$$

Since any continuous derivation of A can be extended to $\mathbb{K}$, the image of $\widehat{\Omega}_{A/\mathfrak{O}} \otimes_A \mathbb{I} = \mathbb{I}dT \cong \mathbb{I}$ in $\widehat{\Omega}_{\mathbb{I}/\mathfrak{O}}$ is $\mathbb{I}$-free of rank 1. Since $\Omega_{\mathbb{I}/A}$ is a torsion

$\mathbb{I}$-module, $\widehat{\Omega}_{A/\mathfrak{O}} \otimes_A \mathbb{I}$ has to inject into $\widehat{\Omega}_{\mathbb{I}/\mathfrak{O}}$. Thus $\Upsilon_{\mathbb{I}/A/\mathfrak{O}} = 0$. Let $s \in m_{\mathbb{I}}$, and suppose s is analytically independent over $\mathfrak{O}$. Let $\Lambda = \mathfrak{O}[[s]] \subset \mathbb{I}$. We consider $\widehat{\Omega}_{\mathbb{I}/\Lambda}$. Then we have, taking a surjective algebra homomorphism $v' : \Lambda[[X_1, \dots, X_r]] \to \mathbb{I}$,

$$\mathrm{Ker}(v')/\mathrm{Ker}(v')^2 \to \oplus_i \mathbb{I} dX_i \to \Omega_{\mathbb{I}/A} \to 0 .$$

If t and s are analytically independent over $\mathfrak{O}$, then $\mathbb{I}$ becomes integral over the power series ring $\mathfrak{O}[[T,S]] \cong \mathfrak{O}[[t,s]]$, which is impossible because $\dim(\mathbb{I}) = 2$. Thus t and s are analytically dependent, and the evaluation map $v'' : \mathfrak{O}[[T,S]] \to \mathbb{I}$ at (t,s) has non-trivial kernel P. The prime P cannot have height 2 or more because $\dim(\mathrm{Im}(v'')) = 2$. Thus P is of height 1, and it is therefore generated by a single element $f(T,S)$ because $\mathfrak{O}[[T,S]]$ is a unique factorization domain. Then we have

$$\frac{\partial f}{\partial T}(t,s)dt + \frac{\partial f}{\partial S}(t,s)ds = 0 \text{ in } \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}}, \text{ and}$$
$$\widehat{\Omega}_{\mathfrak{O}[[t,s]]/\mathfrak{O}} \cong (\mathfrak{O}[[t,s]]dt \oplus \mathfrak{O}[[t,s]]ds)/(\frac{\partial f}{\partial T}(t,s)dt + \frac{\partial f}{\partial S}(t,s)ds) .$$

Suppose $\frac{\partial f}{\partial S}(t,s) = 0$. Then $\frac{\partial f}{\partial S}(T,S)$ is divisible by $f(T,S)$, that is, $\frac{\partial f}{\partial S} = fg$ for $g \in \mathfrak{O}[[T,S]]$ and hence $\frac{\partial^2 f}{\partial S^2}(t,s) = (g\frac{\partial f}{\partial S} + f\frac{\partial g}{\partial S})(t,s) = 0$. Repeating this argument, we find $\frac{\partial^n f}{\partial S^n}(t,s) = 0$ for all n, and hence $f(T,S) \in \mathfrak{O}[[T]]$ because $\mathfrak{O}$ is of characteristic 0. This is in contradiction to the analytic independence of t. Thus $\frac{\partial f}{\partial S}(t,s) \neq 0$. Similarly we know that $\frac{\partial f}{\partial T}(t,s) \neq 0$. Since ds and dt has a linear relation, ds is $\mathbb{I}$-linearly independent. We thus conclude for an analytically independent s, $\widehat{\Omega}_{\mathbb{I}/\mathfrak{O}[[s]]}$ is a torsion $\mathbb{I}$-module. We now look at the following exact sequence :

$$0 \to \Upsilon_{\mathbb{I}/\mathfrak{O}[[s]]/\mathfrak{O}} \to \widehat{\Omega}_{\mathfrak{O}[[s]]/\mathfrak{O}} \widehat{\otimes}_{\mathfrak{O}[[s]]} \mathbb{I} \to \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}} \to \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}[[s]]} \to 0 .$$

Since s is analytically independent, $\widehat{\Omega}_{\mathfrak{O}[[s]]/\mathfrak{O}} \widehat{\otimes}_{\mathfrak{O}[[s]]} \mathbb{I} \cong \mathbb{I}$ via $ds \mapsto 1$. Since $\widehat{\Omega}_{\mathbb{I}/\mathfrak{O}[[s]]}$ is a torsion $\mathbb{I}$-module and $\widehat{\Omega}_{\mathbb{I}/\mathfrak{O}} \otimes_{\mathbb{I}} \mathbb{K}$ is of dimension 1, $\widehat{\Omega}_{\mathfrak{O}[[s]]/\mathfrak{O}} \widehat{\otimes}_{\mathfrak{O}[[s]]} \mathbb{I}$ has to inject into $\widehat{\Omega}_{\mathbb{I}/\mathfrak{O}}$. This shows that $\Upsilon_{\mathbb{I}/\mathfrak{O}[[s]]/\mathfrak{O}} = 0$.

We now consider the situation where we have a surjective $\mathfrak{O}[[s]]$-algebra homomorphism $\pi : R \to \mathbb{I}$ for an object R of $\mathrm{CNL}_{\mathfrak{O}}$. By (E1), we have the following exact sequence :

$$0 \to \Upsilon^{\mathbb{I}}_{R/\mathfrak{O}[[s]]/\mathfrak{O}} \to \Upsilon_{\mathbb{I}/\mathfrak{O}[[s]]/\mathfrak{O}} \to \Upsilon_{\mathbb{I}/R/\mathfrak{O}} \to \widehat{\Omega}_{R/\mathfrak{O}[[s]]} \widehat{\otimes}_R \mathbb{I} \to \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}[[s]]} \to 0 ,$$

and $\Upsilon_{\mathbb{I}/R/\mathfrak{O}} \cong \mathrm{Ker}(\pi)/\mathrm{Ker}(\pi)^2 = C_1(\pi;\mathbb{I})$ if $\mathbb{I}$ is formally smooth over $\mathfrak{O}$. By the above result, this yields a short exact sequence :

$$0 \to \Upsilon_{\mathbb{I}/R/\mathfrak{O}} \to \widehat{\Omega}_{R/\mathfrak{O}[[s]]}\widehat{\otimes}_R\mathbb{I} \to \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}[[s]]} \to 0 \ .$$

Now we study how large the difference of $\Upsilon_{\mathbb{I}/R/\mathfrak{O}}$ and $C_1(\pi,\mathbb{I}) = \mathrm{Ker}(\pi)/\mathrm{Ker}(\pi)^2$, when $\mathbb{I}$ is not formally smooth over $\mathfrak{O}$. The key point here is that $\Upsilon_{\mathbb{I}/R/\mathfrak{O}}$ is independent of the choice of s. We pick $t' \in R$ so that $\pi(t') = t$ as above. We regard R as an $\mathfrak{O}[[t]]$-algebra through the algebra homomorphism of $\mathfrak{O}[[t]]$ into R taking t to t'. Then we have again an exact sequence :

$$0 \to \Upsilon_{\mathbb{I}/R/\mathfrak{O}} \to \widehat{\Omega}_{R/\mathfrak{O}[[t]]}\widehat{\otimes}_R\mathbb{I} \to \widehat{\Omega}_{\mathbb{I}/\mathfrak{O}[[t]]} \to 0 \ .$$

We have a surjective $\mathbb{I}$-linear map $r : C_1(\pi,\mathbb{I}) \to \Upsilon_{\mathbb{I}/R/\mathfrak{O}}$ from (E1) and (E1′). Let $\mathbb{I}' = \mathbb{I}\otimes_{\mathfrak{O}[[t]]}\mathbb{I}$ and $m : \mathbb{I}' \to \mathbb{I}$ be the multiplication. In this case, if $\mathbb{I}$ is flat over $\mathfrak{O}[[t]]$, by (Ext2 and Ext2′), $\mathrm{Ker}(r)$ is a surjective image of $\mathrm{Tor}_1^{\mathbb{I}'}(\mathrm{Ker}(m),\mathbb{I})$, which is a torsion $\mathbb{I}$-module of finite type, because $\mathbb{I}$ is an $\mathfrak{O}[[t]]$-module of finite type. Even if $\mathbb{I}$ is not flat over $\mathfrak{O}[[t]]$, one can embed $\mathbb{I}$ into an $\mathfrak{O}[[t]]$-flat module with pseudo-null cokernel. Thus $\mathrm{Ker}(r)$ is a surjective image of $\mathrm{Tor}_1^{\mathbb{I}'}(\mathrm{Ker}(m),\mathbb{I})$ up to pseudo-null error. The error is annihilated by a power $m_{\mathbb{I}}^M$ for a positive M independently of R and the choice of t' with $\pi(t') = t$ (but depending on $\mathbb{I}$ and t). Thus without any assumption, $\mathrm{Ker}(r)$ is a torsion $\mathbb{I}$-module of finite type killed by a non-trivial ideal $\mathfrak{a}$ of $\mathbb{I}$ independent of R.

We now give the second proof. Here we do not assume the integrality of $\mathbb{I}$ over Λ'. Since the result over $\mathbb{I}$ is just a scalar extension of the result over $\mathbb{I}_0$, we only need to prove the assertions (i)-(iv) replacing $\mathbb{I}$ by $\mathbb{I}_0$. Thus hereafter, we assume that $\mathrm{Im}(\pi_0) = \mathbb{I}$ and discard the assumption that $\mathbb{I}$ is integrally closed. Thus hereafter, we write $\mathbb{I}$ instead of $\mathbb{I}_0$ for $\mathrm{Im}(\pi_0)$. We pick $t \in \mathbb{I}$ so that $\mathbb{I}$ is integral over $\mathfrak{O}[[t]]$. We take $t_j \in R_j$ so that $\alpha_{j,k}(t_k) = t_j$ and $\pi_0(t_0) = t$. Then we apply the above theory to $R = R_j$, $s = \rho_{F,1,\mathcal{P}}(\phi_{\mathcal{P}}) \in R_j$ for $F = \mathbb{Q}_j$ and $t' = t_j$. Then we have an exact sequence of compact modules :

$$C_1(\pi_j,\mathbb{I}) \xrightarrow{r_j} \Upsilon_{\mathbb{I}/R_j/\mathfrak{O}} \to 0 \ ,$$

where $\mathrm{Ker}(r_j)$ is the image of a torsion $\mathbb{I}$-module $X = \mathrm{Tor}_1^{\mathbb{I}'}(\mathrm{Ker}(m),\mathbb{I})$ of finite type (independent of j) up to a bounded $\mathbb{I}$-pseudo-null error. Taking the projective limit with respect to j, we find that $r_\infty : C_1(\pi_\infty;\mathbb{I}) \to \Upsilon_{\mathbb{I}/R_\infty/\mathfrak{O}} = \varprojlim_j \Upsilon_{\mathbb{I}/R_j/\mathfrak{O}}$ is surjective because of the compactness of these modules and that $\mathrm{Ker}(r_\infty)$ is an $\mathbb{I}$-torsion module of finite type. Thus r_∞

is an $\mathbb{I}[[\Gamma]]$-pseudo-isomorphism. By our assumption : $\mathbb{I} = \mathrm{Im}(\pi_0)$, we have $\widehat{\Omega}_{R_j/\Lambda'}\widehat{\otimes}_R\mathbb{I} \cong C_1(\lambda'_j;\mathbb{I})$. By taking the projective limit of the exact sequences : $0 \to \Upsilon_{\mathbb{I}/R_j/\mathfrak{O}} \to \widehat{\Omega}_{R_j/\Lambda'}\widehat{\otimes}_R\mathbb{I}\ (\cong C_1(\lambda'_j;\mathbb{I})) \to \widehat{\Omega}_{\mathbb{I}/\Lambda'} \to 0$ for $\Lambda' = \mathfrak{O}[[s]]$, we get another exact sequence :

$$0 \to \Upsilon_{\mathbb{I}/R_\infty/\mathfrak{O}} \to C_1(\lambda'_\infty;\mathbb{I}) \to \widehat{\Omega}_{\mathbb{I}/\Lambda'} \to 0 \text{ for } R_\infty = \varprojlim_j R_j\ .$$

Since $\widehat{\Omega}_{\mathbb{I}/\Lambda'}$ is a torsion I-module of finite type, $\Upsilon_{\mathbb{I}/R_\infty/\mathfrak{O}}$ is pseudo-isomorphic to $C_1(\pi_\infty;\mathbb{I})$ as $\mathbb{I}[[\Gamma]]$-modules. Thus $C_1(\pi_\infty;\mathbb{I})$ is pseudo-isomorphic to $C_1(\lambda'_\infty;\mathbb{I})$ as $\mathbb{I}[[\Gamma]]$-modules. By Theorem 2.2, we have

$$C_1(\lambda'_\infty;\mathbb{I})/(\gamma-1)C_1(\lambda'_\infty;\mathbb{I}) \cong C_1(\lambda'_0;\mathbb{I})\ .$$

As in the first proof, the assumption that $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ is a torsion $\mathbb{I}$-module tells us that $C_1(\pi_0;\mathbb{I})$ is of $\mathbb{I}$-torsion. Again by the exact sequence :

$$\mathrm{Ker}(r_0) \to C_1(\pi_0;\mathbb{I}) \to C_1(\lambda'_0;\mathbb{I}) \to \widehat{\Omega}_{I/\Lambda'} \to 0\ ,$$

the $\mathbb{I}$-torsionness of $\mathrm{Ker}(r_0)$, $C_1(\pi_0;\mathbb{I})$ and $\widehat{\Omega}_{\mathbb{I}/\Lambda'}$ tells us the same for $C_1(\lambda'_0;\mathbb{I})$. Then we conclude the assertions (i)-(iii) as in the first proof. If $\mathbb{I}$ is formally smooth over $\mathfrak{O}$, $\mathrm{Ker}(r_j) = 0$ for all j. The assertion (iv) follows from this immediately.

Appendix : control of universal deformation rings of representations

In this appendix, we give a general theory of controlling the deformation rings of representations of a normal subgroup under the action of the quotient finite group.

A.1. — Extending representations

Let G be a profinite group with a normal closed subgroup H of finite index. We put $\Delta = G/H$. In this section, we describe when we can extend a representation π of a profinite group H to G (keeping the dimension of π). The theory is a version of Schur's theory of projective representations [CR] Section 11E.

A.1.1. — Representations with invariant trace

Let $\mathcal{O}$ be a complete noetherian local ring over $\mathbb{Z}_p$ with residue field $\mathbb{F}$. We consider the category $CNL = CNL_{\mathcal{O}}$ of complete noetherian local $\mathcal{O}$–algebras with residue field $\mathbb{F}$. Any algebra A in this section will be assumed to be an object of CNL. For each continuous representation $\rho : H \to GL_n(A)$ and $\sigma \in G$, we define $\rho^\sigma(g) = \rho(\sigma g \sigma^{-1})$.

We take a representation $\pi : H \to GL_n(A)$ for an artinian local $\mathcal{O}$–algebra A with residue field $\mathbb{F}$. We assume one of the following conditions :

- (AI_H) *$\overline{\rho} = \pi \bmod m_A$ is absolutely irreducible for the maximal ideal m_A of A ;*
- (Z_H) *The centralizer of $\overline{\rho}(H)$ as an algebraic subgroup of $GL(n)_{/\mathbb{F}}$ is the center of $GL(n)$.*

Of course the first condition implies the second. There are some other cases where the last condition is satisfied; for example, (Z_H) holds if the following condition is satisfied :

- (Red_H) *$\overline{\rho}$ is upper triangular with distinct n characters ρ_i at diagonal entries, and its image contains a unipotent subgroup U' such that $U'/(U', U') = U/(U, U)$ for the unipotent radical U.*

LEMMA A.1.1. — *Suppose (Z_H). Then the centralizer of π in $GL_n(A)$ is $A^\times$.*

We assume the following condition :

(C) $\qquad \pi = c(\sigma)^{-1}\pi^\sigma c(\sigma)$ with some $c(\sigma) \in GL_n(A)$ for each $\sigma \in G$.

If we find another $c'(\sigma) \in GL_n(A)$ satisfying $\pi = c'(\sigma)^{-1}\pi^\sigma c'(\sigma)$, we have

$$\pi = c'(\sigma)^{-1}c(\sigma)\pi c(\sigma)^{-1}c'(\sigma)\,,$$

and hence by Lemma 1.1, $c(\sigma)^{-1}c'(\sigma)$ is a scalar. In particular, for σ, $\tau \in G$,

$$c(\sigma\tau)^{-1}\pi^{\sigma\tau}c(\sigma\tau) = \pi = c(\tau)^{-1}\pi^{\tau}c(\tau) = c(\tau)^{-1}c(\sigma)^{-1}\pi^{\sigma\tau}c(\sigma)c(\tau)\,,$$

and hence, $b(\sigma,\tau) = c(\sigma)c(\tau)c(\sigma\tau)^{-1} \in A^{\times}$. Thus $c(\sigma)c(\tau) = b(\sigma,\tau)c(\sigma\tau)$. This shows by the associativity of the matrix multiplication that

$$(c(\sigma)c(\tau))c(\rho) = b(\sigma,\tau)c(\sigma\tau)c(\rho) = b(\sigma,\tau)b(\sigma\tau,\rho)c(\sigma\tau\rho) \text{ and}$$
$$c(\sigma)(c(\tau)c(\rho)) = c(\sigma)b(\tau,\rho)c(\tau\rho) = b(\tau,\rho)b(\sigma,\tau\rho)c(\sigma\tau\rho)\,,$$

and hence $b(\sigma,\tau)$ is a 2–cocycle of G. If $h \in H$, then

$$\pi(g) = c(h\tau)^{-1}\pi(h\tau g\tau^{-1}h^{-1})c(h\tau) =$$
$$c(h\tau)^{-1}\pi(h)c(\tau)\pi(g)c(\tau)^{-1}\pi(h)^{-1}c(h\tau)\,.$$

Thus $c(h\tau)^{-1}\pi(h)c(\tau) \in A^{\times}$. Thus if we let $h \in G$ act on the space $C(G; M_n(A))$ of continuous functions $f : G \to M_n(A)$ by $f|h(g) = \pi(h)^{-1}f(hg)$, then c is an eigenfunction belonging to a character $\xi : H \to A^{\times}$. Now we take $\eta : G \to A^{\times}$ such that $\eta(h\tau) = \xi^{-1}(h)\eta(\tau)$ for all $h \in H$. For example, writing $G = \bigsqcup_{\tau\in R} H\tau$ (disjoint), we may define $\eta(h\tau) = \xi^{-1}(h)$. We replace c by ηc. Then c satisfies that

$$(\pi) \qquad c(h\tau) = \pi(h)c(\tau) \text{ for all } h \in H\,.$$

Since $c(1)$ commutes with $\mathrm{Im}(\pi)$, $c(1)$ is scalar. Thus we may also assume

$$(\mathrm{Id}) \qquad c(1) = 1\,.$$

Note that for h, $h' \in H$,

$$b(h\sigma, h'\tau) = c(h\sigma)c(h'\tau)c(h\sigma h'\tau)^{-1} =$$
$$\pi(h)c(\sigma)\pi(h')c(\tau)c(\sigma\tau)^{-1}\pi(h\sigma h'\sigma^{-1})^{-1}$$
$$= \pi(h)\pi^{\sigma}(h')b(\sigma,\tau)\pi(h\sigma h'\sigma^{-1})^{-1} = b(\sigma,\tau)\,.$$

Thus b is a 2–cocycle factoring through Δ.

If $b(\sigma,\tau) = \zeta(\sigma)\zeta(\tau)\zeta(\sigma\tau)^{-1}$ is further a coboundary of $\zeta : \Delta \to A^{\times}$, we modify c by $\zeta^{-1}c$. Since ζ factors through Δ, this modification does not destroy (π). Then $c(\sigma\tau) = c(\sigma)c(\tau)$ and $c(h\tau)c(\tau)$ for $h \in H$. Thus c extends π to G. Let d be another extension of π. Then $\chi(\sigma) = c(\sigma)d(\sigma)^{-1} \in A^{\times}$ is a character of G. Thus $c = d \otimes \chi$.

We consider another condition

$$(\mathrm{Inv}) \qquad \mathrm{Tr}(\pi) = \mathrm{Tr}(\pi^{\sigma}) \text{ for all } \sigma \in G\,.$$

Under (AI_F), it has been proven by Carayol and Serre [C] that (Inv) is actually equivalent to (C). Thus we have

THEOREM A.1.1. — *Let $\pi : H \to GL_n(A)$ be a continuous representation for a p–adic artinian local ring A. Suppose either (AI_H) and (Inv) or (Z_H) and (C). Then we can choose c satisfying (π). Then $b(\sigma,\tau) = c(\sigma)c(\tau)c(\sigma\tau)^{-1}$ is a 2–cocycle of Δ with values in $A^\times$, and if its cohomology class in $H^2(\Delta, A^\times)$ vanishes, then there exists a continuous representation π_E of G into $GL_n(A)$ extending π. Moreover all other extensions of π are of the form $\pi_E \otimes \chi$ for a character χ of Δ with values in $A^\times$. In particular, if $H^2(\Delta, A^\times) = 0$, then any representation π satisfying either (AI_H) and (Inv) or (Z_H) and (C) can be extended to G.*

COROLLARY A.1.1. — *If Δ is a p–group, then any representation π with values in $GL_n(\mathbb{F})$ for a finite field of characteristic p satisfying either (AI_H) and (Inv) or (Z_H) and (C) can be extended to G.*

This follows from the fact that $|\mathbb{F}^\times|$ is prime to p, and hence $H^2(\Delta, \mathbb{F}^\times) = 0$. When Δ is cyclic, then $H^2(\Delta, A^\times) \cong A^\times/(A^\times)^d$ for $d = |\Delta|$. If for a generator σ of G, $\xi = c(\sigma^d)\pi(\sigma^d)^{-1} \in (A^\times)^d$, then b is a coboundary of $\zeta(\sigma^j) = \xi^{j/d}$. By extending scalar to $B = A[X]/(X^d - \xi)$, in $H^2(G, B^\times)$, the class of b vanishes. Thus we have

COROLLARY A.1.2. — *Suppose either (AI_F) and (Inv) or (Z_F) and (C). If Δ is a cyclic group of order d, then π can be extended to a representation of G into $GL_n(B)$ for a local A–algebra B which is A–free of rank at most d.*

Let $\overline{\rho} = \pi \bmod m_A$. We suppose that $\overline{\rho}$ can be extended to G. Then we may assume that the cohomology class of $b(\sigma,\tau) \bmod m_A$ vanishes in $H^2(G, \mathbb{F}^\times)$. Thus we can find $\zeta : G \to A^\times$ such that

$$a(\sigma,\tau) = b(\sigma,\tau)\zeta(\sigma)\zeta(\tau)\zeta(\sigma\tau)^{-1} \bmod m_A \equiv 1\,.$$

Then a has values in $\widehat{\mathbb{G}}_m(A) = 1 + m_A$. In particular, if the Sylow p–subgroup S of G is cyclic, we have $H^2(S, \widehat{\mathbb{G}}_m(A)) \cong \widehat{\mathbb{G}}_m(A)/\widehat{\mathbb{G}}_m(A)^{|S|}$. Write ξ for the element in $\widehat{\mathbb{G}}_m(A)$ corresponding to a. Then for $B = A[X]/(X^{|S|} - \xi)$, the cohomology class of a vanishes in $H^2(S, \widehat{\mathbb{G}}_m(B))$. This implies that in $H^2(S, B^\times)$, the cohomology class of b vanishes.

COROLLARY A.1.3. — *Suppose either (AI_H) and (Inv) or (Z_H) and (C). Suppose Δ has a cyclic Sylow p–subgroup of order g. If $\overline{\rho}$ can be extended to G, then π can be extended to a representation of G into $GL_n(B)$ for a local A–algebra B which is A–free of rank at most g.*

We now prove the following fact :

(AI) *When Δ is cyclic of odd order and $n = 2$, the condition (AI_H) is equivalent to (AI_G).*

We start a bit more generally. Let ρ be an absolutely irreducible representation of G into $GL_n(K)$ for a field K. For the moment, n is arbitrary. We assume that Δ is cyclic of order prime to n. We prove that ρ cannot contain a character of H as a representation of H, which shows the equivalence when $n = 2$. Suppose by absurdity that ρ restricted to H contains a character χ. If χ is invariant under the conjugate action of Δ, χ can be extended to a character of G, and it is easy to see in this case, ρ has to contain an extension of χ, and hence reducible. Thus χ is not invariant under Δ. If χ is invariant under a subgroup $H' \supset H$ of G, again by the same argument as above, ρ gets reducible on H' containing a character χ' of H' extending χ. Thus we may assume that conjugates of χ' under $\Delta' = G/H'$ are all distinct. By Mackey's theorem, the induced representation $\mathrm{Ind}(\chi')$ for H' to G is irreducible. By Frobenius reciprocity or Shapiro's lemma, the induced representation $\mathrm{Ind}(\chi')$ has a unique quotient isomorphic to ρ. Thus $|\Delta'| = n$, which contradicts to the assumption that the order of Δ is prime to n. Of course, one can generalize the above argument for more general Δ not necessarily cyclic.

A.2. — Deformation functors of group representations

We suppose that G satisfies the following condition (cf. [T]) :

(pF) *All open subgroup of G has finite p–Frattini quotient*

We fix a representation $\overline{\rho} : G \to GL_n(\mathbb{F})$ satisfying (Z_H). In this section, we study various deformation problems of $\overline{\rho}$ and relation among the universal rings.

A.2.1. — Full deformations

We consider a deformation functor $\mathcal{F}_H : CNL \to SETS$ given by

$$\mathcal{F}_H(A) = \{\rho : H \to GL_n(A) \mid \rho \equiv \overline{\rho} \bmod m_A\}/\approx$$

where "$\approx$" is the strict equivalence in $GL_n(A)$, this is, the conjugation by elements in $\widehat{GL}_n(A) = 1 + m_A M_n(A)$. The functor $\mathcal{F}_H$ is representable ([T] Theorem 3.3) under (Z_H). We write (R_H, ρ_H) for the universal couple. Since ρ_G restricted to H is an element in $\mathcal{F}_H(R_H)$, we have an $\mathcal{O}$–algebra homomorphism $\alpha : R_H \to R_G$ such that $\alpha\rho_H = \rho_G|_H$.

We like to determine $\mathrm{Ker}(\alpha)$ and $\mathrm{Im}(\alpha)$ in terms of Δ. By choosing a lift $c_0(\sigma) \in GL_n(\mathcal{O})$ for $\sigma \in G$ such that $c_0(\sigma) \equiv \overline{\rho}(\sigma) \bmod m_{\mathcal{O}}$, we can define for any $\rho \in \mathcal{F}_G(A)$, $\rho^\sigma(g) = \rho(\sigma g \sigma^{-1})$ and $\rho^{[\sigma]}(g) = c_0(\sigma)^{-1}\rho^\sigma(g)c_0(\sigma)$ in $\mathcal{F}_H(A)$. In this way, Δ acts via $\sigma \longmapsto [\sigma]$ on $\mathcal{F}_H$ and R_H. Then as seen in Section 1, we can attach a 2–cocycle b on Δ with values in $\widehat{\mathbb{G}}_m(A)$ to any representation $\rho \in \mathcal{F}_H(A)$ with $\rho^{[\sigma]} \approx \rho$ in the following way.

First choose a lift $c(\sigma)$ of $\overline{\rho}(\sigma)$ in $GL_n(A)$ for each $\sigma \in G$ such that $\rho = c(\sigma)^{-1}\rho^{\sigma}c(\sigma)$ and $c(h\tau) = \rho(h)c(\tau)$ for $h \in H$ and $\tau \in G$. Then we know that $c(\sigma)c(\tau) = b(\sigma,\tau)c(\sigma\tau)$ for a 2–cocycle b of Δ with values in $\widehat{\mathbb{G}}_m(A)$. If we change c by c' such that $c'(\sigma) = c(\sigma)\zeta(\sigma)$ for $\zeta(\sigma) \in \widehat{\mathbb{G}}_m(A)$, we see from $c(\sigma)c(\tau) = b(\sigma,\tau)c(\sigma\tau)$ that $c'(\sigma)c'(\tau) = b(\sigma,\tau)\zeta(\sigma)\zeta(\tau)c'(\sigma\tau)\zeta(\sigma\tau)^{-1}$. Thus the cocycle b' attached to c' is cohomologous to b, and the cohomology class $[b] = [\rho] \in H^2(\Delta, \widehat{\mathbb{G}}_m(A))$ is uniquely determined by ρ. If $[\rho] = 0$, then $b(\sigma,\tau) = \zeta(\sigma)^{-1}\zeta(\tau)^{-1}\zeta(\sigma\tau)$ for a 1–cochain ζ. We then modify c by $c\zeta$ and by constant so that $c(1) = 1$. Then c extends the representation ρ to a representation π of G (Theorem A.1.1).

LEMMA A.2.1. — *Suppose* (Z_H) *and that* n *is prime to* p *and* $\rho^{[\sigma]} \approx \rho$ *for* $\rho \in \mathcal{F}_H(A)$. *If* $\det(\rho)$ *can be extended to a character of* G *having values in an* A*–algebra* B *containing* A, *then* ρ *can be extended uniquely to a representation* $\pi : G \to GL_n(B)$ *whose determinant coincides with the extension to* G *of* $\det(\rho)$.

Proof : by applying "det" to c and b, we know that $[\det(\rho)] = [\det(b)] = [\rho]^n$. If n is prime to p, the vanishing of $[\rho]^n$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ is equivalent to the vanishing of $[\rho]$. Thus if $\det(\rho)$ extends to G (that is $[\rho]^n = 0$), then ρ extends to a representation π of G which has determinant equal to the extension of $\det(\rho)$ prearranged. We now show the uniqueness of π. We get, out of π, other extensions $\pi \otimes \chi \in \mathcal{F}_G(B)$ for $\chi \in H^1(\Delta, \widehat{\mathbb{G}}_m(B)) = \mathrm{Hom}(\Delta, \widehat{\mathbb{G}}_m(B))$. Conversely, if π and π' are two extensions of ρ in $\mathcal{F}_G(B)$, then for $h \in H$, $\pi'(\sigma)\rho(h)\pi'(\sigma)^{-1} = \pi(\sigma)\rho(h)\pi(\sigma)^{-1}$ and hence $\pi(\sigma)^{-1}\pi'(\sigma)$ commutes with ρ. Then by Lemma A.1.1, $\chi(\sigma) = \pi(\sigma)^{-1}\pi'(\sigma)$ is a scalar in $\widehat{\mathbb{G}}_m(B)$.

$$\begin{aligned}\chi(\sigma\tau) &= \pi(\sigma\tau)^{-1}\pi'(\sigma\tau) = \pi(\tau)^{-1}\pi(\sigma)^{-1}\pi'(\sigma)\pi'(\tau)\\ &= \pi(\tau)^{-1}\chi(\sigma)\pi'(\tau) = \chi(\sigma)\chi(\tau)\,.\end{aligned}$$

Thus χ is an element in $H^1(\Delta, \widehat{\mathbb{G}}_m(B))$ and $\pi' = \pi \otimes \chi$, which shows that $\det(\pi')$ is equal to $\det(\pi)\chi^n$. If $\det(\pi') = \det(\pi)$, then $\chi^n = 1$. Since χ is of p–power order, if n is prime to p, $\chi = 1$.

Here is a consequence of the proof of the lemma :

COROLLARY A.2.1. — *Let* $\pi_0 \in \mathcal{F}_G(B)$ *be an extension of* $\rho \in \mathcal{F}_H(A)$ *for an* A*–algebra* B *containing* A. *Then we have*

$$\{\pi_0 \otimes \chi \mid \chi \in \mathrm{Hom}(\Delta, \widehat{\mathbb{G}}_m(B))\} = \{\pi \in \mathcal{F}_G(B) \mid \pi_{|H} = \rho\}\,.$$

It is easy to see that if $H^2(\Delta, \mathbb{F}) = 0$, then $H^2(\Delta, \widehat{\mathbb{G}}_m(A)) = 0$ for all A in CNL. Therefore we see, if $H^2(\Delta, \mathbb{F}) = 0$,

(*) $\mathcal{F}_H^{\Delta}(A) = H^0(\Delta, \mathcal{F}_H(A)) \cong \mathcal{F}_G(A)/\widehat{\Delta}(A)$ for $\widehat{\Delta}(A) = \mathrm{Hom}(\Delta, \widehat{\mathbb{G}}_m(A))\,.$

Here we let $\chi \in \widehat{\Delta}(A)$ act on $\mathcal{F}_G(A)$ via $\pi \longmapsto \pi \otimes \chi$. Suppose that $\mathcal{F}_H^{\Delta}$ is represented by a universal couple $(R_{H,\Delta}, \rho_{H,\Delta})$ and $[\rho_{H,\Delta}] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(R_{H,\Delta}))$. Then for each $\rho \in \mathcal{F}_H^{\Delta}(A)$, we have $\varphi : R_{H,\Delta} \to A$ such that $\varphi\rho_{H,\Delta} \approx \rho$. Then $\varphi_*[\rho_{H,\Delta}] = [\rho]$ and therefore, $[\rho] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(A))$. This shows again (*).

Let us show that the functor $\mathcal{F}_H^{\Delta}$ is representable by applying the Schlessinger criterion (see [Sch] and [T] Proposition 2.5). For a Cartesian diagram in $\mathrm{CNL}_{\mathcal{O}}$:

$$\begin{array}{ccc} A_3 = A_1 \times_A A_2 & \xrightarrow{\pi_1} & A_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ A_2 & \xrightarrow{\alpha_2} & A\,, \end{array}$$

we need to check the bijectivity of the natural map

$$\gamma_H^{\Delta} : \mathcal{F}_H^{\Delta}(A_1 \times_A A_2) \longrightarrow \mathcal{F}_H^{\Delta}(A_1) \times_{\mathcal{F}_H^{\Delta}(A)} \mathcal{F}_H^{\Delta}(A_2)\,.$$

We already know from the representability of $\mathcal{F}_H$ that

$$\gamma_H : \mathcal{F}_H(A_1 \times_A A_2) \cong \mathcal{F}_H(A_1) \times_{\mathcal{F}_H(A)} \mathcal{F}_H(A_2)\,.$$

Since $\mathcal{F}_H^{\Delta}$ is a subfunctor of $\mathcal{F}_H$, $\mathcal{F}_H^{\Delta}(A_1) \times_{\mathcal{F}_H^{\Delta}(A)} \mathcal{F}_H^{\Delta}(A_2)$ is a subset of $\mathcal{F}_H(A_1) \times_{\mathcal{F}_H(A)} \mathcal{F}_H(A_2)$, and hence γ_H^{Δ} is injective. Take an element (ρ_1, ρ_2) of $\mathcal{F}_H^{\Delta}(A_1) \times_{\mathcal{F}_H^{\Delta}(A)} \mathcal{F}_H^{\Delta}(A_2)$. Then $\alpha_1\rho_1 \approx \alpha_2\rho_2$, that is, there exists $x \in \widehat{GL}_n(A)$ such that $x\alpha_1\rho_1 x^{-1} = \alpha_2\rho_2$. We may assume that α_1 is surjective (cf. [T] Proposition 2.5). Then we can lift x to $x' \in \widehat{GL}_n(A_1)$. Then replacing ρ_1 by $x'\rho_1 x'^{-1}$, we may assume that $\alpha_1\rho_1 = \alpha_1\rho_2$. Thus $\rho = \rho_1 \times_A \rho_2$ has values in $GL_n(A_1 \times_A A_2)$. It is easy to see that ρ is invariant under Δ. Thus $\gamma_H^{\Delta}(\rho) = (\rho_1, \rho_2)$, and therefore γ_H^{Δ} is surjective. Then it is obvious that $\mathcal{F}_H^{\Delta}$ is represented by $R_{H,\Delta} = R_H/\Sigma_{\sigma\in\Delta} R_H([\sigma] - 1)R_H$.

PROPOSITION A.2.1. — *Suppose* (Z_H). *Then* $\mathcal{F}_H^{\Delta}$ *is represented by* $(R_{H,\Delta}, \rho_{H,\Delta})$ *for* $R_{H,\Delta} = R_H/\mathfrak{a}$ *with* $\mathfrak{a} = \Sigma_{\sigma\in\Delta} R_H([\sigma] - 1)R_H$ *and* $\rho_{H,\Delta} = \rho_H \bmod \mathfrak{a}$. *If either* $[\rho_{H,\Delta}] = 0$ *in* $H^2(\Delta, \widehat{\mathbb{G}}_m(R_{H,\Delta}))$ *or* $H^2(\Delta, \mathbb{F}) = 0$, *then we have* $\mathcal{F}_G/\widehat{\Delta} \cong \mathcal{F}_H^{\Delta}$ *via* $\pi \longmapsto \pi|_H$.

We now consider the following subfunctor $\mathcal{F}_{G,H}$ of $\mathcal{F}_H$ given by

$$\mathcal{F}_{G,H}(A) = \{\rho|_H \in \mathcal{F}_H(A) \mid \rho \in \mathcal{F}_G(B) \text{ for a flat } A\text{–algebra } B \text{ in } CNL_{\mathcal{O}}\}\,.$$

Here the algebra B may not be unique and depends on A. Let us check that $\mathcal{F}_{G,H}$ is really a functor. If $\varphi : A \to A'$ is a morphism in CNL and $\rho|_H \in \mathcal{F}_{G,H}(A)$ with $\rho \in \mathcal{F}_G(B)$, B being flat over A, then $A'\widehat{\otimes}_A B$ is a flat A'–algebra in CNL. Then $(\varphi \otimes id)\rho \in \mathcal{F}_G(A'\widehat{\otimes}_A B)$ such that $\varphi(\rho|_H) = ((\varphi \otimes id)\rho)|_H$. Thus $\mathcal{F}_H(\varphi)$ takes $\mathcal{F}_{G,H}(A)$ into $\mathcal{F}_{G,H}(A')$, which shows that $\mathcal{F}_{G,H}$ is a well defined functor. For each $\rho \in \mathcal{F}_{G,H}(A)$, we have an extension $\rho \in \mathcal{F}_G(B)$. By the universality of (R_G, ρ_G), we have $\varphi : R_G \to B$ such that $\varphi\rho_G = \rho$. Then $\rho|_H = (\varphi\rho_G)|_H = \varphi(\rho_G|_H) = \varphi\alpha\rho_H$. This shows that $\varphi\alpha$ is uniquely determined by $\rho|_H \in \mathcal{F}_{G,H}(A)$. Therefore φ restricted to $\mathrm{Im}(\alpha)$ has values in A and is uniquely determined by $\rho|_H \in \mathcal{F}_{G,H}(A)$. Conversely, supposing that $[\alpha\rho_H] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha)$ in CNL, for a given $\varphi : \mathrm{Im}(\alpha) \to A$ which is a morphism in CNL, we shall show that $\rho = \varphi\alpha\rho_H$ is an element of $\mathcal{F}_{G,H}(A)$. Anyway $\alpha\rho_H$ can be extended to G as an element in $\mathcal{F}_G(B)$, and hence $\alpha\rho_H \in \mathcal{F}_{G,H}(\mathrm{Im}(\alpha))$. We note that ρ can be extended to G because $[\varphi\alpha\rho_H] = \varphi_*[\alpha\rho_H]$ which vanishes in $H^2(\Delta, \widehat{\mathbb{G}}_m(B'))$ for $B' = B\widehat{\otimes}_{\mathrm{Im}(\alpha),\varphi}A$. Thus $\rho \in \mathcal{F}_{G,H}(A)$, and $\mathcal{F}_{G,H}$ is represented by $(\mathrm{Im}(\alpha), \alpha\rho_H)$ as long as $[\alpha\rho_H] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha)$ in CNL.

We have the following inclusions of functors : $\mathcal{F}_H \supset \mathcal{F}_H^{\Delta} \supset \mathcal{F}_{G,H} \supset \mathcal{F}_G/\widehat{\Delta}$, the last inclusion being given by $\rho \longmapsto \rho|_H$. The functor $\mathcal{F}_H^{\Delta}$ is represented by $R_H/\mathfrak{a}$ for $\mathfrak{a} = \Sigma_{\sigma\in\Delta} R_H([\sigma]-1)R_H$. Because of the above inclusion, if $[\alpha\rho_H] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha)$ in CNL, the ring $\mathrm{Im}(\alpha)$ is a surjective image of $R_H/\mathfrak{a} = R_{H,\Delta}$. If $[\rho_{H,\Delta}] = 0$ (for $\rho_{H,\Delta} = \rho \bmod \mathfrak{a}$) in $H^2(\Delta, \widehat{\mathbb{G}}_m(B'))$ for a flat extension B' of $R_{H,\Delta}$ in CLN, then $\rho_{H,\Delta} \in \mathcal{F}_{G,H}(R_{H,\Delta})$ and thus $\mathcal{F}_H^{\Delta} = \mathcal{F}_{G,H}$.

Proposition A.2.2. — *Assume (Z_H) and that $[\alpha\rho_H] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha)$ in CNL. Then $\mathcal{F}_{G,H}$ is represented by $(\mathrm{Im}(\alpha), \alpha\rho_H)$. If further $[\rho_{H,\Delta}] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B'))$ for a flat extension B' of $R_{H,\Delta}$, then we have $\mathcal{F}_{G,H} = \mathcal{F}_H^{\Delta}$.*

The character $\det(\rho_H)$ induces an $\mathcal{O}$–algebra homomorphism : $\mathcal{O}[[H^{ab}]] \to R_H$ for the maximal continuous abelian quotient H^{ab} of H. We write its image as Λ_H and write simply Λ for Λ_G. Thus we have a character $\det(\rho_H) : H \to \Lambda_H^{\times}$. We consider the category CNL_{Λ_H} of complete noetherian local Λ_H–algebras with residue field $\mathbb{F}$. We consider the functor $\mathcal{F}_{\Lambda_H,H} : CNL_{\Lambda_H} \to SETS$ given by

$$\mathcal{F}_{\Lambda_H,H}(A) = \{\rho : H \to GL_n(A) \mid \rho \equiv \overline{\rho} \bmod m_A \text{ and } \det(\rho) = \det(\rho_H)\}/\approx .$$

Pick $\rho : H \to GL_n(A) \in \mathcal{F}_{\Lambda_H,H}(A)$. Then regarding A as an $\mathcal{O}$–algebra naturally, we know that $\rho \in \mathcal{F}_H(A)$. Thus there is a unique morphism

$\varphi : R_H \to A$ such that $\varphi\rho_H \approx \rho$. Then $\varphi(\det(\rho_H)) = \det(\rho)$, and φ is a morphism in CNL_{Λ_H}. Therefore (R_H, ρ_H) represents $\mathcal{F}_{\Lambda_H}$. Similarly to $\mathcal{F}_{G,H}$, we consider another functor on CNL_Λ :

$$\mathcal{F}_{\Lambda,G,H}(A) = \{\rho|_H \mid \rho \in \mathcal{F}_{\Lambda,G}(B) \text{ for a flat } A\text{–algebra } B \text{ in } CNL_\Lambda\}/\approx\ .$$

Take $\rho \in \mathcal{F}_{\Lambda,G,H}(A)$ such that $\rho = \rho'|_H$ for $\rho' \in \mathcal{F}_{\Lambda,G}(B)$. Then there exists a unique $\varphi : R_G \to B$ with $\det(\rho') = \varphi(\det(\rho_G))$. Since the Λ–algebra structure of B is given by $\det(\rho')$, φ induces a Λ–algebra homomorphism of $\mathrm{Im}(\alpha)\Lambda$ into B for the algebra $\mathrm{Im}(\alpha)\Lambda$ generated by $\mathrm{Im}(\alpha)$ and Λ. From $\rho = (\varphi\rho_G)|_H = \varphi(\rho_G|_H) = \varphi\alpha\rho_H$, we see that the Λ–algebra homomorphism φ restricted $\mathrm{Im}(\alpha)\Lambda$ is uniquely determined by ρ. Supposing that $[\alpha\rho_H]$ vanishes in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha)$, we knows that $[\alpha\rho_H]$ vanishes in $H^2(\Delta, \widehat{\mathbb{G}}_m(\mathrm{Im}(\alpha)\Lambda \otimes_{\mathrm{Im}(\alpha)} B))$. For any morphism $\varphi : \mathrm{Im}(\alpha)\Lambda \to A$ in CNL_Λ, $[\varphi\alpha\rho_H] = \varphi_*[\alpha\rho_H]$ vanishes in $H^2(\Delta, \widehat{\mathbb{G}}_m(B'))$ for $B' = A \otimes_{\mathrm{Im}(\alpha)} B$ which is flat over A. Thus we have an extension π of ρ to G having values in B'. Suppose further that n is prime to p. In this case, as already remarked, we can always extend ρ without extending A and without assuming the vanishing of $[\alpha\rho_H]$, because $\det(\rho)$ can be extended to G by $\varphi \circ \det(\rho_G)$. Thus we know :

$$\mathcal{F}_{\Lambda,G,H}(A) = \{\rho|_H \mid \rho \in \mathcal{F}_{\Lambda,G}(A)\}/\approx\ .$$

Since $\det(\rho)$ can be extended to G without changing A, there is a unique extension of π with values in $GL_n(A)$ such that $\det(\pi) = \iota \circ (\det(\rho_G))$, which implies that $\pi \in \mathcal{F}_{\Lambda,G}(A)$ and hence $\pi|_H \in \mathcal{F}_{\Lambda,G,H}(A)$. Thus $\mathcal{F}_{\Lambda,G,H}$ is represented by $(\mathrm{Im}(\alpha)\Lambda, \alpha\rho_H)$ if n is prime to p. We consider the natural transformation : $\mathcal{F}_{\Lambda,G} \to \mathcal{F}_{\Lambda,G,H}$ sending π to $\pi|_H$. As we have already remarked, the extension of $\rho \in \mathcal{F}_{\Lambda,G,H}(A)$ to $\pi \in \mathcal{F}_\Lambda(A)$ is unique if n is prime to p. Thus in this case, the natural transformation is an isomorphism of functors. Therefore $(R_G, \rho_G) \cong (\mathrm{Im}(\alpha)\Lambda, \alpha\rho_H)$. Thus we get

THEOREM A.2.1. — *Suppose (Z_H) and that either n is prime to p or $[\alpha\rho_H]$ vanishes in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha)$. Then $\mathcal{F}_{\Lambda,G,H}$ is representable by $(\mathrm{Im}(\alpha)\Lambda_G, \alpha\rho_H)$. Moreover if n is prime to p, we have the equality $R_G = \mathrm{Im}(\alpha)\Lambda_G$.*

Since α restricted to Λ_H coincides with the algebra homomorphism induced by the inclusion $H \subset G$, $\alpha(\Lambda_H) \subset \Lambda$. We put $R' = \mathrm{Im}(\alpha) \otimes_{\Lambda_H} \Lambda$. By definition, the character $1 \otimes \det(\rho_G)$ of G coincides on H with $(\alpha \circ \det(\rho_H)) \otimes 1$ in R'. Thus $\alpha\rho_H$ can be extended uniquely to $\rho'_G : G \to GL_n(R')$ such that $\det(\rho'_G) = 1 \otimes \det(\rho_G)$ if n is prime to p. Thus we have a natural map $\iota : R_G \to R'$ such that $\iota\rho_G = \rho'_G$. Since R_G is an algebra over Λ and $\mathrm{Im}(\alpha)$, it is an algebra over R'. Thus we have the structural morphism $\iota' : R' \to R_G$. By Theorem A.2.1, ι' is surjective. By definition, $\iota\alpha\rho_H = \iota\rho_H|_H = \iota\rho'_G|_H = \alpha\rho_H \otimes 1$ and $\iota\det(\rho_G) = \det(\rho'_G) = 1 \otimes \det(\rho_G)$. Thus $\iota'\iota\alpha\rho_H = \iota'(\alpha\rho_H \otimes 1) = \alpha\rho_H$ and $\iota'\iota\det(\rho_G) = \iota'(1 \otimes \det(\rho_G)) = \det(\rho_G)$. Thus $\iota'\iota$ is identity on Λ and $\mathrm{Im}(\alpha)$, and hence $\iota'\iota = id$. Similarly, $\iota\iota'\rho'_g = \iota\rho_G = \rho'_G$. This shows that

$$\iota\iota'(\alpha\rho_H \otimes 1) = \iota(\alpha\rho_H) = (\alpha\rho_H \otimes 1) \text{ and}$$
$$\iota\iota'(1 \otimes \det(\rho_G)) = \iota(\det(\rho_G)) = 1 \otimes \det(\rho_G)\,.$$

Thus $\iota\iota'$ is again identity on $\mathrm{Im}(\alpha) \otimes 1$ and $1 \otimes \Lambda$, and $\iota\iota' = id$. Let X_p (resp. $X^{(p)}$) indicate the maximal p–profinite (resp. prime–to–p profinite) quotient of each profinite group X. Write ω for the restriction of $\det(\rho_G)$ to $(G^{ab})^{(p)}$. Define $\kappa : G^{ab} \to \mathcal{O}[[G^{ab}_p]]^\times$ by $\kappa(g) = \omega(g)[g_p]$ for the projection g_p of g into G^{ab}_p, where $[x]$ denotes the group element of $x \in G^{ab}_p$ in the group algebra. Assuming that $\mathbb{F}$ is big enough to contain all g–th roots of unity for the order g of $\mathrm{Im}(\omega)$, we can perform the same argument replacing $(\Lambda_H, \Lambda_G, \det(\rho_G))$ by $(\mathcal{O}[[H^{ab}_p]], \mathcal{O}[[G^{ab}_p]], 1 \otimes \kappa)$. Thus we get

COROLLARY A.2.2. — *Suppose* (Z_H) *and that* n *is prime to* p. *Then we have*

$$(R_G, \rho_G) \cong (\mathrm{Im}(\alpha) \otimes_{\Lambda_H} \Lambda_G, \alpha\rho_H \otimes \det(\rho_G)) \cong$$
$$(\mathrm{Im}(\alpha) \otimes_{\mathcal{O}[[H^{ab}_p]]} \mathcal{O}[[G^{ab}_p]], \alpha\rho_H \otimes \kappa)\,.$$

In particular, R_G *is flat over* $\mathrm{Im}(\alpha)$.

By Hochschild–Serre spectral sequence, we have an exact sequence

$$\begin{array}{ccccccccc} H_2(\Delta, \mathbb{Z}_p) & \longrightarrow & H_0(\Delta, H_1(H, \mathbb{Z}_p)) & \longrightarrow & H_1(G, \mathbb{Z}_p) & \longrightarrow & H_1(\Delta, \mathbb{Z}_p) & \longrightarrow & 0 \\ & & \| \wr & & \| \wr & & \| \wr & & \\ & & H_0(\Delta, H^{ab})_p & \longrightarrow & G^{ab}_p & \longrightarrow & \Delta^{ab}_p & \longrightarrow & 1\,, \end{array}$$

where the subscript “p” indicates the maximal p–profinite quotient. Suppose that $\mathbb{F}$ is big enough to contain all d_0–th roots of unity for the prime–to–p part d_0 of the order d of Δ. Then the inclusion $H \subset G$ induces the following commutative diagram :

$$\begin{array}{ccc} \alpha' : \mathcal{O}[[H^{ab}_p]] & \longrightarrow & \mathcal{O}[[G^{ab}_p]] \\ \downarrow & & \downarrow \\ \alpha : \Lambda_H & \longrightarrow & \Lambda_G\,. \end{array}$$

As seen in Corollary 2.2, this diagram is Cartesian. Thus Λ_G is flat over Λ_H. If $H_2(\Delta, \mathbb{Z}_p) = 0$ ($\Leftrightarrow H^2(\Delta, \mathbb{Q}_p/\mathbb{Z}_p) = 0$), $\mathrm{Spec}(\mathrm{Im}(\alpha')) \cong \mathrm{Spec}(\mathcal{O}[[H_p^{ab}]])^{\Delta}$, and hence $\mathrm{Spec}(\alpha(\Lambda_H)) = \mathrm{Spec}(\Lambda_H)^{\Delta}$. From the exact sequence, if $\mathcal{O}$ contains a primitive g–th roots of unity for the order g of Δ^{ab}, we get

$$\mathrm{Im}(\alpha') = H^0(\widehat{\Delta}(\mathcal{O}), \mathcal{O}[[G_p^{ab}]]),$$

where $\chi \in \widehat{\Delta}(\mathcal{O})$ takes $\Sigma_{g \in G_p^{ab}} a(g)[g]$ to $\Sigma_{g \in G_p^{ab}} a(g)\chi(g)[g]$.

A.2.2. — Nearly ordinary deformations

Now we impose the following additional condition to our deformation problem : let $S = S_G$ be a finite set of closed subgroups of G. For each $D \in S$, let $S(D)$ be a complete representative set for H–conjugacy classes of $\{gDg^{-1} \cap H \mid g \in G\}$. In the main text (Section 2), the data S is given by a choice of decomposition subgroups of $G = \mathrm{Gal}(F_\Sigma/E)$ at primes dividing p. For simplicity, we assume that $D \cap H \in S(D)$ always. Then the disjoint union $S_H = \bigsqcup_{D \in S} S(D)$ is a finite set, because $|S(D)| = |H \backslash G/D|$. Let P_D be a proper parabolic subgroup of $GL(n)_{/\mathcal{O}}$ defined over $\mathcal{O}$ indexed by $D \in S$. For each $D' \in S(D)$ such that $D' = H \cap gDg^{-1}$, we define $P_{D'} = c(g)P_D c(g)^{-1}$ for a lift $c(g) \in GL_n(\mathcal{O})$ of $\overline{\rho}(g)$. We assume

(NO) $$\overline{\rho}(D) \subset P_D(\mathbb{F}) \text{ for each } D \in S_G\,.$$

Then we consider the following condition :

(NO_H) $$\begin{aligned} &\text{there exists } g_D \in \widehat{GL}_n(A) \text{ for each } D \in S_H \\ &\qquad \text{such that } g_D \rho(D) g_D^{-1} \subset P_D(A)\,, \end{aligned}$$

where $\widehat{GL}_n(A) = 1 + m_A M_n(A)$. We define a subfunctor $\mathcal{F}_?^{n.o}$ of the functor $\mathcal{F}_?$, with various restriction "?" introduced in the previous section, by

$$\mathcal{F}_?^{n.o}(A) = \{\rho \in \mathcal{F}_?(A) \mid \rho \text{ satisfies } (NO_X)\}\,,$$

where X denotes either G or H depending on the group concerned. Then by (NO), (NO_X) and our choice of P_D, $\mathcal{F}_?^{n.o}(\mathbb{F}) = \{\overline{\rho}|_X\} \neq \emptyset$. Let us write $\mathfrak{gl}$ (resp. $\mathcal{P}_D$) for the Lie algebra of $GL_n(\mathbb{F})$ (resp. $P_D(\mathbb{F})$). Note that D acts on $\mathfrak{gl}$ and $\mathcal{P}_D$ by conjugation. We can identify $\mathfrak{gl}$ with $V \otimes V^* = \mathrm{Hom}_{\mathbb{F}}(V, V)$ for the representation space V of $\overline{\rho}$, where V^* is the contragredient of V. Then $\mathcal{P}_D$ can be identified with

$$\{g \in \mathrm{Hom}_{\mathbb{F}}(V, V) \mid gF_D \subset F_D\}\,,$$

for a filtration $F_D : \{0\} = F_0 \subset F_1 \subset \cdots \subset F_r = V$. Here $gF_D \subset F_D$ implies that $gF_i \subset F_i$ for all i. The filtration F_D naturally induces a double filtration F_{AD} on $\mathfrak{gl}$ and $\mathcal{P}_D$. Since this filtration is compatible with that of $\mathcal{P}_D$, it induces a filtration of $\mathfrak{gl}/\mathcal{P}_D$, which is stable under the adjoint action $Ad(\overline{\rho})$ of $\overline{\rho}$. As shown in [T] Proposition 6.2, under the following regularity condition for every $D \in S_X$,

$$(Reg_D) \qquad H^0(D, \mathfrak{gl}/\mathcal{P}_D) = 0\,,$$

$\mathcal{F}_X^{n.o}$ is representable for $X = H$ or G. We can think of a stronger condition :

$$(RG_D) \qquad H^0(D, gr(\mathfrak{gl}/\mathcal{P}_D)) = 0$$

This condition is stronger than the condition (Reg_D), because on $gr(\mathfrak{gl}/\mathcal{P}_D)$, D acts through the Levi–quotient of P_D. Writing the representation of D on F_i/F_{i-1} as $\overline{\rho}_{D,i}$, (RG_D) is equivalent to

$$(RG'_D) \qquad \mathrm{Hom}_D(\overline{\rho}_{D,i}, \overline{\rho}_{D,j}) = 0 \quad \text{if } i > j\,.$$

In the same manner as in the previous section, we can check that Δ acts on $\mathcal{F}_H^{n.o}$. Take $D \in S$ and put $D' = D \cap H \in S(D)$. Since $\overline{\rho}$ is invariant under Δ and $\overline{\rho} \in \mathcal{F}_G^{n.o}(\mathbb{F})$,

$$(\text{Inv}) \qquad \overline{\rho}_{D',i}^{[\sigma]} = \overline{\rho}_{D',i} \text{ in } gr(V) \text{ for all } i \text{ and } \sigma \in D\,.$$

For $\rho \in \mathcal{F}_X^{n.o}(A)$, we have $g_D \in \widehat{GL}_n(A)$ such that $\rho(D) \subset g_D^{-1} P_D(A) g_D$. This implies that $V(\rho)$ has a filtration $F_D(\rho) : \{0\} = F_0(\rho) \subset F_1(\rho) \subset \cdots \subset F_r(\rho) = V(\rho)$ stable under D such that $F_i(\rho)$ is a direct A–summand of $V(\rho)$ for all i and $F_D(\rho) \otimes_A \mathbb{F} = F_D$. We write $\rho_{D,i}$ for the representation of D on $F_i(\rho)/F_{i-1}(\rho)$. Now suppose $\rho \in \mathcal{F}_H^{\Delta, n.o}(A)$ and $[\rho] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat A–algebra B. Then we find an extension $\pi : G \to GL_n(B)$ of ρ. Let $\sigma \in D$ and $D' = H \cap D$. Thus $\pi(\sigma)\rho(d')\pi(\sigma)^{-1} = \rho(\sigma d' \sigma^{-1}) \in g_{D'}^{-1} P_D(A) g_{D'}$ for all $d' \in D'$ and hence $\pi(\sigma)\rho(d')\pi(\sigma)^{-1}\rho(d')^{-1} \in g_{D'}^{-1} P_D(A) g_{D'}$. From this and $(Reg_{D'})$, it follows that $\pi(\sigma) \in g_{D'}^{-1} P_D(B) g_{D'}$ for $\sigma \in D$ (see [T] Proof of Proposition 6.2). Thus, taking $g_D = g_{D'}$, we confirm that $\pi \in \mathcal{F}_G^{n.o}(A)$. Since $\mathcal{F}_G^{n.o}$ is stable under the action of $\widehat{\Delta}$, all the arguments given for $\mathcal{F}_X$ in the previous paragraph are valid for $\mathcal{F}_X^{n.o}$. Writing $(R_X^{n.o}, \rho_X^{n.o})$ for the universal couple representing $\mathcal{F}_X^{n.o}$, we conclude

THEOREM A.2.2. — *Suppose* (Z_H), (Reg_D) *for all* $D \in S_H$ *and that* n *is prime to* p. *Then we have the equality* $R^{n.o}_G = \mathrm{Im}(\alpha^{n.o})\Lambda^{n.o}_G$, *where* $\alpha^{n.o} : R^{n.o}_H \to R^{n.o}_G$ *is an* $\mathcal{O}$*–algebra homomorphism given by* $\alpha^{n.o}\rho^{n.o}_H \approx \rho^{n.o}_G|_H$ *and* $\Lambda^{n.o}_G$ *is the image of* $\mathcal{O}[[G^{ab}_p]]$ *in* $R^{n.o}_G$. *Moreover we have*

$$\begin{aligned}(R^{n.o}_G, \rho^{n.o}_G) &\cong (\mathrm{Im}(\alpha^{n.o}) \otimes_{\Lambda_H} \Lambda^{n.o}_G, \alpha^{n.o}\rho^{n.o}_H \otimes \det(\rho^{n.o}_G)) \\ &\cong (\mathrm{Im}(\alpha^{n.o}) \otimes_{\mathcal{O}[[H^{ab}_p]]} \mathcal{O}[[G^{ab}_p]], \alpha^{n.o}\rho^{n.o}_H \otimes \kappa)\,.\end{aligned}$$

A.2.3. — Ordinary deformations

Fix a normal closed subgroup $I = I_D$ of each $D \in S$. For $D' = gDg^{-1} \cap H \in S(D)$, we put $I_{D'} = gI_Dg^{-1} \cap H$. We call $\rho \in \mathcal{F}^{n.o}_X(A)$ ordinary if ρ satisfies the following conditions :

(Ord_X) $\quad \rho_{D,1}$ is of rank 1 over A and $I \subset \mathrm{Ker}(\rho_{D,1})$ for every $D \in S_X$.

We then consider the following subfunctor $\mathcal{F}^{ord}_X$ of $\mathcal{F}^{n.o}_X$:

$$\mathcal{F}^{ord}_X(A) = \{\rho \in \mathcal{F}^{n.o}_X(A) \mid \rho \text{ is ordinary}\}\,.$$

It is easy to see that the functor $\mathcal{F}^{ord}_X$ is representable by $(R^{ord}_X, \rho^{ord}_X)$ under (Reg_D) for every $D \in S_X$.

Let $\rho \in \mathcal{F}^{ord}_H(A)$. Suppose $[\rho] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat A–algebra B. Then we have at least one extension π of ρ in $\mathcal{F}^{n.o}_G(B)$. We consider $\pi_{D,1} : D \to A^\times$ for $D \in S$. We suppose one of the following two conditions for each $D \in S$:

(TR_D) $|I_D/I_D \cap H|$ is prime to p;
(Ex_D) Every p–power order character of $I_D/I_D \cap H$ can be extended to a character of Δ having values in a flat extension B' of B so that it is trivial on $I_{D'}$ for all $D' \in S$ different from D.

Under (TR_D), as a homomorphism of groups, $\pi_{D,1}$ restricted to I_D factors through $\overline{\rho}_{D,1}$ which is trivial on I. Thus $\pi_{D,1}$ is trivial on I_D. We note that $\pi_{D,1}$ is of p–power order on $I_D/H \cap I_D$ because $\overline{\rho}_{D,1}$ is trivial on I_D and $\rho_{D,1}$ is trivial on $I_D \cap H$. Thus we may extend $\pi_{D,1}$ to a character η of Δ congruent 1 modulo $m_{B'}$. Then we twists π by η^{-1}, getting an extension $\pi' = \pi \otimes \eta^{-1}$ such that $\pi'_{D,1}$ is trivial on I_D. Repeating this process for the D's satisfying (Ex_D), we find an extension $\pi \in \mathcal{F}^{ord}_G(B)$ for a flat extension B of A. We now consider

$$\mathcal{F}^{ord}_{G,H}(A) = \{\rho|_H \in \mathcal{F}^{ord}_H(A) \mid \rho \in \mathcal{F}^{ord}_G(B) \text{ for a flat extension } B \text{ of } A\}\,.$$

In the same manner as in Section 2, if either n is prime to p or $[\alpha^{ord}\rho^{ord}_H] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $\mathrm{Im}(\alpha^{ord})$ in $CNL_{\mathcal{O}}$, we know that

$\mathcal{F}^{ord}_{G,H}$ is represented by $(\mathrm{Im}(\alpha^{ord}), \alpha^{ord}\rho^{ord}_H)$, where $\alpha^{ord} : R^{ord}_H \to R^{ord}_G$ is an $\mathcal{O}$–algebra homomorphism given by $\alpha^{ord}\rho^{ord}_H \approx \rho^{ord}_G|_H$.

Let $\rho \in \mathcal{F}^{ord}_{G,H}(A)$ and π be its extension in $\mathcal{F}^{ord}_G(B)$ for a flat A–algebra B in $CNL_{\mathcal{O}}$. The character $\det(\pi)$ is uniquely determined by ρ on the subgroup of G^{ab}_p generated by all $I_{D,p}$, because another choice is $\pi \otimes \chi$ for a character χ of Δ and $(\pi \otimes \chi)_{D,1} = \chi$ on $I_{D,p}$. If G^{ab}_p is generated by the $I_{D,p}$'s and H_p, $\det(\pi)$ is uniquely determined by ρ. Thus assuming that n is prime to p, π itself is uniquely determined by ρ. Therefore the natural transformation : $\mathcal{F}^{ord}_G \to \mathcal{F}^{ord}_{G,H}$ given by $\rho \longmapsto \rho|_H$ identifies $\mathcal{F}^{ord}_G$ with a subfunctor of $\mathcal{F}^{ord}_{G,H}$, inducing a surjective $\mathcal{O}$–algebra homomorphism $\beta : \mathrm{Im}(\alpha^{ord}) \to R^{ord}_G$ such that $\rho^{ord}_G|_H = \beta\alpha\rho^{ord}_H$. Since $\rho^{ord}_G|_H = \alpha\rho^{ord}_H$, β is the identity on $\mathrm{Im}(\alpha^{ord})$, and we conclude that $\mathrm{Im}(\alpha^{ord}) = R^{ord}_G$. This implies

THEOREM A.2.3. — *Suppose (Z_H), (Reg_D) for $D \in S_H$, either (TR_D) or (Ex_D) for each $D \in S$ and that n is prime to p. Suppose further that the $I_{D,p}$'s for all $D \in S$ and H_p generate G^{ab}_p. Then we have $\mathrm{Im}(\alpha^{ord}) = R^{ord}_G$. In particular, for any deformation $\rho \in \mathcal{F}^{ord}_{G,H}(A)$, there is a unique extension $\pi \in \mathcal{F}^{ord}_G(A)$ such that $\pi|_H = \rho$. If further $[\rho^{\Delta,ord}_H] = 0$ in $H^2(\Delta, \widehat{\mathbb{G}}_m(B))$ for a flat extension B of $R^{ord}_{H,\Delta}$, then $R^{ord}_{H,\Delta} \cong \mathrm{Im}(\alpha^{ord}) = R^{ord}_G$, where $R^{ord}_{H,\Delta} = R^{ord}_H/\Sigma_{\sigma\in\Delta}R^{ord}_H([\sigma]-1)R^{ord}_H$.*

A.2.4. — Deformations with fixed determinant

We take a character $\chi : G \to \mathcal{O}^\times$ such that $\chi \equiv \det(\overline{\rho}) \bmod m_{\mathcal{O}}$. We then define $\mathcal{F}^{\chi,?}_X(A) = \{\rho \in \mathcal{F}^?_X(A) \mid \det(\rho) = \chi|_X\}$. Supposing the representability of $\mathcal{F}^?_X$, it is easy to check that $\mathcal{F}^{\chi,?}_X$ is representable. Since the determinant is already fixed and can be extended to G, by the argument in the previous sections shows that if n is prime to p,

$$\mathcal{F}^{\chi,?,\Delta}_H = \mathcal{F}^{\chi,?}_{G,H} = \mathcal{F}^{\chi}_G\,.$$

Write $(R^{\chi,?}_X, \rho^{\chi,?}_X)$ for the universal couple representing $\mathcal{F}^{\chi,?}_X$ and define $\alpha^{\chi,?} : R^{\chi,?}_H \to R^{\chi,?}_G$ so that $\alpha^{\chi,?}\rho^{\chi,}_H \approx \rho^{\chi,?}_G$. Then we have

PROPOSITION A.2.3. — *Suppose (Z_H), (Reg_D) for $D \in S_H$ and that n is prime to p. Then we have*

$$R^{\chi,?}_H/\Sigma_{\sigma\in\Delta}R^{\chi,?}_H([\sigma]-1)R^{\chi,?}_H = R^{\chi,?}_{G,H} \cong \mathrm{Im}(\alpha^{\chi,?}) = R^{\chi,?}_G\,,$$

where $R^{\chi,?}_G$ is either R^{χ}_G, $R^{\chi,n.o}_G$ or $R^{\chi,ord}_G$.

For each $\rho \in \mathcal{F}_H^{n.o}(A)$, we decompose $\det(\rho) = \chi\xi$ so that ξ is a p–power order. If n is prime to p, there is a unique character $\xi^{1/n} : H \to \widehat{\mathbb{G}}_m(A)$ exists. Then we define $\rho^\chi = \rho \otimes \xi^{-1/n}$, which is an element of $\mathcal{F}_H^{\chi,n.o}(A)$. Writing f_H for the deformation functor $\mathcal{F}_H$ for $\chi|H$ in place of $\overline{\rho}|_H$, we have a natural transformation : $\mathcal{F}_H^{n.o} \to \mathcal{F}_H^{\chi,n.o} \times f_H$ given by $\rho \longmapsto (\rho^\chi, \det(\rho))$. If $(\rho^\chi, \det(\rho)) = (\rho'^\chi, \det(\rho'))$, then

$$\rho = \rho^\chi \otimes (\det(\rho)/\chi)^{1/n} = \rho'^\chi \otimes (\det(\rho')/\chi)^{1/n} = \rho' .$$

Thus the transformation is a monomorphism. For a given $(\rho^\chi, \det(\rho))$, we can recover ρ as above. Thus we get $\mathcal{F}_H^{n.o} \cong \mathcal{F}_H^{\chi,n.o} \times f_H$. Since $(\mathcal{O}[[H_p^{ab}]], \kappa)$ represents f_H, we see, if n is prime to p,

$$(R_H^{n.o}, \rho_H^{n.o}) \cong (R_H^{\chi,n.o}[[H_p^{ab}]], \varepsilon_H^{n.o} \otimes \kappa^{1/n}) .$$

Similarly we get

$$(R_H, \rho_H) \cong (R_H^\chi[[H_p^{ab}]], \varepsilon_H \otimes \kappa^{1/n}) .$$

Note that, if n is prime to p,

$$\mathcal{F}_H^{?,\Delta} \cong \mathcal{F}_H^{\chi,?,\Delta} \times f_H^\Delta = \mathcal{F}_{G,H}^{\chi,?} \times f_H^\Delta \text{ and } \mathcal{F}_{G,H}^{?} \cong \mathcal{F}_{G,H}^{\chi,?} \times \mathcal{F}_{G,H} .$$

Thus $\alpha^? = \alpha^{\chi,?} \times \alpha'$ for α' as in the end of the paragraph A.2.1. This shows that

THEOREM A.2.4. — *Suppose (Z_H), (Reg_D) for $D \in S_H$ and that n is prime to p. Then if $\mathcal{O}$ contains a primitive $|\Delta_p^{ab}|$–root of unity, we have*

$$\mathrm{Im}(\alpha^?) = R_G^{\chi,?} \widehat{\otimes}_{\mathcal{O}} (\mathcal{O}[[G_p^{ab}]])^{\widehat{\Delta}(\mathcal{O})} ,$$

where $R_G^{\chi,?}$ is either R_G^χ or $R_G^{\chi,n.o}$.

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Corrections to "On Λ–adic forms of half integral weight for $SL(2)_{/\mathbb{Q}}$"

by Haruzo Hida

in "Number Theory, Paris 1992–93" Lecture Note Series **215**, 139–166

p. 145 line 15 : "$U(p^\alpha) = \left\{ s \in \mathbb{S} \mid s_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^\alpha \right\}$" should read

$$U(p^\alpha) = \left\{ s \in U \mid s_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^\alpha \right\}$$

p. 145 line 1 from the bottom : "for $\omega_{1/2}^{\otimes k} \otimes \omega^\circ \ |_{U_\alpha}$" should read "for $\omega_{1/2}^{\otimes k} \otimes \omega^\circ \,|_{U_\alpha}$, where ω° is the dualizing sheaf on $X_{\alpha/\mathbb{Z}_p}$".

p. 146 line 8 : "the first horizontal map" should read "the first vertical map".

p. 146 line 10 : "the first row" should read "the second column".

p. 146 line 14 : "second row" should read "first column".

p. 146 line 14 : "the vertical maps" should read "the horizontal maps".

p. 146 line 15 : "rows" should read "columns"

The second diagram in p. 146 should be replaced by the following :

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(U_\gamma, \omega(k+\frac{3}{2})) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \xleftarrow{E_\alpha} & H^0(U_\gamma, \omega(k+\frac{1}{2}) \otimes \mathbb{Z}/p^\beta\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^0(U_\gamma, \omega(k+2)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \xleftarrow{E_\alpha} & H^0(U_\gamma, \omega(k+1) \otimes \mathbb{Z}/p^\beta\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^0(U_\gamma, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & = & H^0(U_\gamma, \mathcal{O}(D) \otimes \mathbb{Z}/p^\beta\mathbb{Z})
\end{array}$$

The diagram in p. 147 should be replaced by the following :

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(U_\infty, \omega(k+\frac{3}{2})) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \xleftarrow{E_\alpha} & H^0(U_\infty, \omega(k+\frac{1}{2}) \otimes \mathbb{Z}/p^\beta\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^0(U_\infty, \omega(k+2)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \overset{\sim}{=\!=} & H^0(U_\infty, \omega(k+1) \otimes \mathbb{Z}/p^\beta\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^0(U_\infty, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & =\!= & H^0(U_\infty, \mathcal{O}(D) \otimes \mathbb{Z}/p^\beta\mathbb{Z})
\end{array}
$$

The second formula in (4.1) : “$a(n, f \mid T(q^2)) = a(p^2 n, f)$ if $q \mid Np^\alpha$”, should read

$$\text{“}a(n, f \mid T(q^2)) = a(q^2 n, f) \text{ if } q \mid Np^\alpha\text{”}$$

p. 149 line 9 from the bottom “$P^{r(P)-1}$” should read $p^{r(P)-1}$”.

In the formula of Theorem 3 in p. 153 : “$\psi_p(n+m)$” should read “$\psi_p(n/m)$”.

At several places in pp. 155–157, “$\mathbb{Q}_e ll$” should read “$\mathbb{Q}_\ell$”.

In the proof of Lemma 3, $(k/2)$ should read $k+(1/2)$ (thus $(k/2)-1$ is replaced by $(2k-1)/2$).

p. 157 line 5 from the bottom : “$\mu^2 \neq \alpha$” should read “$\mu^2 = \alpha$”.

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