$\mathcal{L}$-INVARIANT OF p-ADIC L-FUNCTIONS

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Contents

Introduction [1]

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the field of all algebraic numbers. We fix a prime $p > 2$ and a p-adic absolute value $|\cdot|_p$ on $\overline{\mathbb{Q}}$. Then $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}$ under $|\cdot|_p$. We write $W = \{x \in K \big| |x|_p < 1\}$ for the p-adic integer ring of sufficiently large extension $K/\mathbb{Q}_p$ inside $\mathbb{C}_p$. We write $\overline{\mathbb{Q}}_p$ for the field of all numbers in $\mathbb{C}_p$ algebraic over $\mathbb{Q}_p$. Start with a strictly compatible system $\{\rho_{\mathfrak{l}}\}$ of semi-simple Galois representations $\rho_{\mathfrak{l}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_d(T_{\mathfrak{l}})$ for primes $\mathfrak{l}$ of the coefficient field $T \subset \overline{\mathbb{Q}}$. We suppose that ρ is associated to a pure (absolute Hodge) motive in the sense of Deligne (see [D]). We assume that ρ does not contain the trivial representation as a subquotient. We write S for the finite set of ramification of ρ and $\rho_{\mathfrak{l}}$ is unramified outside $S \cup \{\infty, \ell\}$, where ℓ is the residual characteristic of $\mathfrak{l}$. We write $\mathfrak{p} = \{\xi \in O_T \big| |\xi|_p < 1\}$ and often write $W := O_{T,\mathfrak{p}}$, where O_T is the integer ring of T. Often we just write ρ for $\rho_{\mathfrak{p}}$ which acts on $V = T_{\mathfrak{p}}^d$.

For simplicity, we assume that $p \notin S$. Let $E_\ell(X) = \det(1 - \rho_{\mathfrak{q}}(Frob_\ell)|_{V_{I_\ell}} X) \in T[X]$ (assuming $\mathfrak{q} \nmid \ell$). We always assume that $\rho_{\mathfrak{p}}$ is ordinary in the following sense: ρ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is upper triangular with diagonal characters $\mathcal{N}^{a_j}$ on

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[1]This introduction is the note of the lecture delivered at the conference on L-functions at Kyushu university on February in 2006, and the following four sections are the notes of the lectures at Harvard University, Spring 2006, in their eigenvariety semester organized by Mazur and Taylor. We would like to thank the organizers of the conference at Kyushu university and the semester at Harvard university.

the inertia I_p for the p-adic cyclotomic character $\mathcal{N}$ ordered from top to bottom as $a_1 \geq a_2 \geq \cdots \geq 0 \geq \cdots \geq a_d$. Thus

$$\rho|_{I_p} = \begin{pmatrix} \mathcal{N}^{a_1} & * & \cdots & * \\ 0 & \mathcal{N}^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{N}^{a_d} \end{pmatrix}.$$

In other words, we have a decreasing filtration $\mathcal{F}^{i+1}\rho \subset \mathcal{F}^i\rho$ stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that the Tate twists $gr^i(\rho)(-i) := (\mathcal{F}^i\rho/\mathcal{F}^{i+1}\rho)(-i)$ is unramified. Define

$$H_p(X) = \prod_i \det(1 - Frob_p|_{gr^i(\rho)(-i)} p^i X) = \prod_{j=1}^{d} (1 - \alpha_j X).$$

Then it is believed to be $E_p(X) = H_p(X)$ if $p \notin S$ and $E_p(X)|H_p(X)$ otherwise. In any case, $\mathrm{ord}_p(\alpha_j) \in \mathbb{Z}$. Let us define

$$\beta_j = \begin{cases} \alpha_j & \text{if } \mathrm{ord}_p(\alpha_j) \geq 1, \\ p\alpha_j^{-1} & \text{if } \mathrm{ord}_p(\alpha_j) \leq 0 \end{cases}$$

and put $e = |\{j|\beta_j = p\}|$.

$$\mathcal{E}(\rho) = \prod_{j=1}^{d} (1 - \beta_j p^{-1}) \text{ and } \mathcal{E}^+(\rho) = \prod_{j=1, \beta_j \neq p}^{d} (1 - \beta_j p^{-1}).$$

Then the complex L-function is defined by $L(s, \rho) = \prod_\ell E_\ell(\ell^{-s})^{-1}$. We assume that the value at 1 is critical for $L(s, \rho)$ (in the motivic sense of Deligne in [D]). We suppose to have an algebraicity result (conjectured by Deligne) that for a well defined period $c^+(\rho)(1)) \in \mathbb{C}^\times$ such that $\frac{L(s,\rho\otimes\varepsilon)}{c^+(\rho(1))} \in \overline{\mathbb{Q}}$ for all finite order characters $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}(\overline{\mathbb{Q}})$. Then we should have

Conjecture 0.1. *Suppose that $s = 1$ is critical for ρ. Then there exist a power series $\Phi^{an}(X) \in W[[X]]$ and a p-adic L-function $L_p^{an}(s, \rho) = \Phi_\rho^{an}(\gamma^{1-s} - 1)$ interpolating $L(1, \rho \otimes \varepsilon)$ for p-power order character ε such that $\Phi_\rho^{an}(\varepsilon(\gamma) - 1) \sim \mathcal{E}(\rho \otimes \varepsilon) \frac{L(1,\rho\otimes\varepsilon)}{c^+(\rho(1))}$ with the modifying p-factor $\mathcal{E}(\rho)$ as above (putting $\mathcal{E}(\rho \otimes \varepsilon) = 1$ if $\varepsilon \neq 1$). The L-function $L_p^{an}(s, \rho)$ has zero of order $e + \mathrm{ord}_{s=1} L(s, \rho)$ for a nonzero constant $\mathcal{L}^{an}(\rho) \in \mathbb{C}_p^\times$ (called the analytic $\mathcal{L}$-invariant), we have*

$$\lim_{s \to 1} \frac{L_p^{an}(s, \rho)}{(s-1)^e} = \mathcal{L}^{an}(\rho)\mathcal{E}^+(\rho) \frac{L(1, \rho)}{c^+(\rho(1))},$$

where "$\lim_{s\to1}$" is the p-adic limit, $c^+(\rho(1))$ is the transcendental factor of the critical complex L-value $L(1, \rho)$, and $\mathcal{E}^+(\rho)$ is the product of nonvanishing modifying p-factors.

When $e > 0$, we call that $L_p^{an}(s, \rho)$ has an *exceptional zero* at $s = 1$. Here is an example. Start with a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \overline{\mathbb{Q}}^\times$ with $\chi(-1) = -1$. Then $c(\rho^+(1)) = (2\pi i)$. If we suppose $\chi = \left(\frac{-D}{\cdot}\right)$ for a square free positive integer D, the modifying Euler factor vanishes at $s = 1$ if the Legendre symbol $\left(\frac{-D}{p}\right) = 1 \Leftrightarrow (p) = \mathfrak{p}\overline{\mathfrak{p}}$ in $O_{\mathbb{Q}[\sqrt{-D}]}$ with $\mathfrak{p} = \left\{x \in O_{\mathbb{Q}[\sqrt{-D}]} \middle| |x|_p < 1\right\}$. By a work of Kubota–Leopoldt and Iwasawa, we have a p-adic analytic L-function $L_p^{an}(s, \chi) =$

$\Phi^{an}(\gamma^{1-s}-1)$ for a power series $\Phi^{an}(X) \in \Lambda = W[[X]]$ and $\gamma = 1+p$ such that for $\mathcal{E}(\chi\mathcal{N}^m) = (1-\chi(p)p^{m-1})$

$$L_p^{an}(m,\chi) = \Phi^{an}(\gamma^{1-m}-1) = \mathcal{E}(\chi\mathcal{N}^m)L(1-m,\chi) \sim \mathcal{E}(\chi\mathcal{N}^m)\frac{L(m,\chi)}{(2\pi i)^m}$$

for all positive integer m as long as $|n^m - n|_p < 1$ for all n prime to p. If we have an exceptional zero at 1, it appears that we lose the exact connection of the p-adic L-value and the corresponding complex L-value. However, the conjecture says we can recover the complex L-value via an appropriate derivative of the p-adic L-function as long as we can compute $\mathcal{L}^{an}(\rho)$. We may regard χ as a Galois character $\mathrm{Gal}(\mathbb{Q}[\mu_N]/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \{\pm 1\}$, and we remark that $\chi(Frob_p) = 1$ to have the exceptional zero. For our later use, for the class number h of $\mathbb{Q}[\sqrt{-D}]$, we write the generator of $\overline{\mathfrak{p}}^h$ as ϖ; so, $\overline{\mathfrak{p}}^h = (\varpi)$ for $\varpi \in \mathbb{Q}[\sqrt{-D}]$.

Though we formulated the conjecture for $p \notin S$, if $\rho_\mathfrak{p}$ is ordinary semi-stable, we have the same phenomena and can formulate a conjecture similarly. Here is such an example. Start with an elliptic curve $E_{/\mathbb{Q}}$, which yields a compatible system $\rho_E := \{T_\ell E\}$ given by the ℓ-adic Tate module $T_\ell E$. Suppose that E has split multiplicative reduction at p. In this case, $H_p(X) = (1-X)(1-pX)$ and $E_p(X) = (1-X)$, $\mathcal{E}(\rho_E) = 0$ and $\mathcal{E}^+(\rho) = 1$. Then by the solution of the Shimura-Taniyama conjecture by Wiles et al, this L-function has p-adic analogue constructed by Mazur such that we have $\Phi_E^{an}(X) \in \Lambda$ with $\Phi_E^{an}(\varepsilon(\gamma)-1) = \mathcal{E}(\rho_E \otimes \varepsilon)\frac{G(\varepsilon^{-1})L(1,E,\varepsilon)}{\Omega_E}$ for all p-power order character $\varepsilon : \mathbb{Z}_p^\times \to W^\times$; in other words, $L_p^{an}(s,E) = \Phi_E^{an}(\gamma^{1-s}-1)$. Here Ω_E is the period of the Néron differential of E. Thus if $Frob_p$ has eigenvalue 1 on $T_\ell E$, the exceptional zero appears at $s=1$ as in the case of Dirichlet character. The $Frob_p$ has eigenvalue 1 if and only if E has multiplicative reduction mod p.

The problem of $\mathcal{L}$-invariant is to compute explicitly the $\mathcal{L}$-invariant $\mathcal{L}^{an}(\rho)$. The $\mathcal{L}$-invariant in the cases where $\rho = \chi = \left(\frac{-D}{\cdot}\right)$ as above and $\rho = \rho_E$ for E with split multiplicative reduction is computed in the 1970s to 90s, and the results are

Theorem 0.2. *Let the notation and the assumption be as above.*

(1) $\mathcal{L}^{an}(\chi) = \frac{\log_p(q)}{\mathrm{ord}_p(q)} = -\frac{\log_p(q)}{h}$ *for* $q \in \mathbb{C}_p$ *given by* $q = \varpi/\overline{\varpi}$ ($\overline{\mathfrak{p}}^h = (\varpi)$) *and the class number* h *of* $\mathbb{Q}[\sqrt{-D}]$*;*

(2) *For* E *split multiplicative at* p*, writing* $E(\mathbb{C}_p) = \mathbb{C}_p^\times/q^\mathbb{Z}$ *for the Tate period* $q \in \mathbb{Q}_p^\times$*, we have* $\mathcal{L}^{an}(\rho_E) = \frac{\log_p(q)}{\mathrm{ord}_p(q)}$.

Here $\log_p$ *is the Iwasawa logarithm and* $|x|_p = p^{-\mathrm{ord}_p(x)}$.

The first assertion is due to Gross–Koblitz [GsK] and Ferrero–Greenberg [FG], and the second was conjectured by Mazur–Tate–Teitelbaum [MTT] and later proven by Greenberg–Stevens [GS] and by some others further later. In the first formula of the theorem, the sign "$-$" of $-\frac{\log_p(q)}{h}$ is correct as explained in [G] (10) (and if we evaluate the value at $s=0$, the minus sign should be removed as was done in [FG].

Starting with an ordinary p-adic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_d(W)$, there is a systematic way to create many Galois representations whose eigenvalues of $Frob_p$ contain 1. Indeed, let ρ acts on the $d \times d$ matrices $M_d(W)$ by conjugation. Since

$$M_d(W) = Ad(W) \oplus \{\text{scalar matrices}\}$$

for the trace 0 space $Ad(W)$ which is stable under the conjugation. Then the action of $Ad(\rho)(Frob_p)$ on $Ad(W)$ has eigenvalue 1 with multiplicity $\geq d-1$. However it is easy to check that the system $Ad(\rho)$ is not critical if $d > 2$. Thus we assume that $d = 2$. Now require that $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(W)$ be a Galois representation associated to a p-ordinary Hilbert Hecke eigenform (belonging to a discrete series at ∞) over a totally real field F. We make $Ad(\rho_F)$ and consider the induced representation $\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)$ whose eigenvalues for $Frob_p$ contains 1 with multiplicity e for the number e of prime factors of p in F. The system $\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)$ is critical at $s = 1$.

Arithmetic $\mathcal{L}$-invariant. Returning to a general ordinary representation $\rho = \rho_{\mathfrak{p}}$, we describe an arithmetic way of constructing p-adic L-function due to Iwasawa and others. We define Galois cohomologically the Selmer group

$$\mathrm{Sel}(\rho) \subset H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty), \rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

for the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ inside $\mathbb{Q}(\mu_{p^\infty})$ by the subgroup of cohomology classes unramified outside p whose image in $H^1(I_p, (\rho/\mathcal{F}^+\rho) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ vanishes. Here $\mathcal{F}^+\rho$ is the middle filtration $\mathcal{F}^1\rho$ and I_p is each inertia group at p. The Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ acts on $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty), \rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and hence on $\mathrm{Sel}(\rho)$, making it as a discrete module over the group algebra $W[[\Gamma]] = \varprojlim_n W[\Gamma/\Gamma^{p^n}]$. Identifying Γ with $1+p\mathbb{Z}_p$ by the cyclotomic character, we may regard $\gamma \in \Gamma$. Then $W[[\Gamma]] \cong \Lambda$ by $\gamma \mapsto 1+X$. By the classification theory of compact Λ-modules of finite type, the Pontryagin dual $\mathrm{Sel}^*(\rho)$ has a Λ-linear map into $\prod_{f\in\Omega} \Lambda/f\Lambda$ with finite kernel and cokernel for a finite set $\Omega \subset \Lambda$. The power series $\Phi_\rho = \prod_{f\in\Omega} f(X)$ is uniquely determined up to unit multiple. We then define $L_p(s,\rho) = \Phi_\rho(\gamma^{1-s}-1)$. Greenberg gave a recipe of defining $\mathcal{L}(\rho)$ for this $L_p(s,\rho)$ and verified in 1994 the conjecture for this $L_p(s,\rho)$ except for the nonvanishing of $\mathcal{L}(\rho)$ (under some restrictive conditions). For the adjoint square $Ad(\rho_F)$ for ρ_F associated to a Hilbert modular form, the conjecture (except for the nonvanishing of $\mathcal{L}(\rho)$) was again proven in my paper [H00] in the Israel journal (in 2000) under the condition that $\overline{\rho}_F = (\rho_F \mod \mathfrak{m}_W)$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/F[\mu_p])$ and the p-distinguishedness condition for $\overline{\rho}_F|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ for all $\mathfrak{p}|p$ (which we recall later). If there exists an analytic p-adic L-function $L_p^{an}(s,\rho) = \Phi_\rho^{an}(\gamma^{1-s}-1)$ interpolating complex L-values, the main conjecture of Iwasawa's theory confirms $\Phi_\rho = \Phi_\rho^{an}$ up to unit multiple.

Suppose now that ρ_F is associated to a Hilbert modular Hecke eigenform of weight $k \geq 2$ over a totally real field F. Following Greenberg's recipe, we try to compute $\mathcal{L}(Ad(\rho_F)) = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F))$. By ordinarity, we have $\rho_F|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})} \cong \left(\begin{smallmatrix} \beta_{\mathfrak{p}} & * \\ 0 & \alpha_{\mathfrak{p}} \end{smallmatrix}\right)$ with two distinct diagonal characters $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ factoring through $I_{\mathfrak{p}} \to \mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$ for the inertia group $I_{\mathfrak{p}}$ for all $\mathfrak{p}|p$. We consider the universal nearly ordinary deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(R)$ over K with the pro-Artinian local universal K-algebra R. This means that for any Artinian local K-algebra A with maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(1) unramified outside ramified primes for ρ_F;
(2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_{\mathfrak{p}} \mod \mathfrak{m}_A$ such that the diagonal characters factor through $I_{\mathfrak{p}} \to \mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$ for all $\mathfrak{p}|p$;
(3) $\det(\rho_A) = \det \rho_F$;

(4) $\rho_A \equiv \rho_F \mod \mathfrak{m}_A$,

there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. Write $\Gamma_\mathfrak{p} \cong \mathbb{Z}_p$ for the p-profinite part of $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$. Choose a generator $\gamma_\mathfrak{p}$ of $\Gamma_\mathfrak{p}$ and identify $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$ by $\gamma_\mathfrak{p} \leftrightarrow 1+X_\mathfrak{p}$. Since $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_\mathfrak{p}\end{smallmatrix}\right)$, $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1} : \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p}) \to R$ factors through $\Gamma_\mathfrak{p}$ and induces an algebra structure on R over $W[[X_\mathfrak{p}]]$. Thus R is an algebra over $K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. If we write $\varphi_\rho : R \to K$ for the morphism with $\varphi_\rho \circ \boldsymbol{\rho} \cong \rho_F$, by our construction, $\mathrm{Ker}(\varphi_\rho) \supset (X_\mathfrak{p})_{\mathfrak{p}|p} = (X)$. We state a conjecture:

Conjecture 0.3. *We have* $R \cong K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$.

By the results of Wiles, Taylor-Wiles, Fujiwara, Kisin and Skinner-Wiles, the conjecture holds at least if either $F = \mathbb{Q}$ (see [Ki], [Ki1] and [Ki2]) or the residual representation $\overline{\rho}_F = (\rho \mod \mathfrak{m}_W)$ is absolutely irreducible and $\overline{\rho}_F|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})})^{ss} \cong \overline{\alpha}_\mathfrak{p} \oplus \overline{\beta}_\mathfrak{p}$ with $\overline{\alpha}_\mathfrak{p} \neq \overline{\beta}_\mathfrak{p}$ (see [Fu] and [Fu1]). Here is a theorem:

Theorem 0.4. *Assume* Conjecture 0.3. *Then for the local Artin symbol* $[p, F_\mathfrak{p}] = Frob_\mathfrak{p}$, *we have*

$$\mathcal{L}(\mathrm{Ind}_F^\mathbb{Q} Ad(\rho_F)) = \det\left(\frac{\partial \boldsymbol{\delta}_\mathfrak{p}([p,F_\mathfrak{p}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{X=0} \prod_\mathfrak{p} \log_p(\gamma_\mathfrak{p})\alpha_\mathfrak{p}([p,F_\mathfrak{p}])^{-1},$$

where $\gamma_\mathfrak{p}$ *is the generator of* $\Gamma_\mathfrak{p}$ *by which we identify the group algebra* $W[[\Gamma_\mathfrak{p}]]$ *with* $W[[X_\mathfrak{p}]]$.

This result is proved as [HMI] Theorem 3.73 under some redundant hypothesis in order to make the book [HMI] self-contained. In this note, we will sketch the proof in the general case assuming that p totally splits in $F/\mathbb{Q}$ and that ρ_F is unramified outside p and ∞. The proof in the nonsplit case is more complicated, and we will give a full proof in [H06] along with conjectures predicting $\mathcal{L}$-invariant of symmetric powers of ρ_F.

Here are some examples showing usefulness of this theorem: Take a totally imaginary quadratic extension M/F in which all prime factors $\mathfrak{p}|p$ in F splits as $\mathfrak{P}\overline{\mathfrak{P}}$. Take a set $\Sigma = \{\mathfrak{P}|p\}$ so that $\Sigma \sqcup \overline{\Sigma}$ is the set of all prime factors of p in M. Write h for the class number of M and choose $\varpi(\mathfrak{P}) \in M$ so that $\mathfrak{P}^h = (\varpi(\mathfrak{P}))$ for $\mathfrak{P} \in \overline{\Sigma}$. For any Galois character $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to W^\times$ of M with $\psi(\sigma) \neq \psi^c(\sigma)$ for $\psi^c(\sigma) = \psi(c\sigma c^{-1})$ and a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we have $Ad(\mathrm{Ind}_M^F \psi) = \chi \oplus \mathrm{Ind}_M^F \psi^{1-c}$ for $\chi = \left(\frac{M/F}{\cdot}\right)$, and we can easily show $\mathcal{L}(\chi) = \mathcal{L}(Ad(\mathrm{Ind}_M^F \psi))$. The arithmetic p-adic L-function $L_p(s,\chi)$ for $\chi = \left(\frac{M/F}{\cdot}\right)$ constructed à la Iwasawa has an exceptional zero of order $\geq e$ for $e = |\Sigma|$. Since we can compute explicitely the universal deformation $\boldsymbol{\rho}$ of $\rho = \mathrm{Ind}_M^F \psi$, we get from the theorem

Corollary 0.5. *We have* $\mathcal{L}(\mathrm{Ind}_F^\mathbb{Q} \chi) = (-1)^e \frac{\det\left(\log_p(N_{\mathfrak{P}'}(\varpi(\mathfrak{P})^{(1-c)}))\right)_{\mathfrak{P},\mathfrak{P}'\in\overline{\Sigma}}}{\prod_{\mathfrak{P}\in\overline{\Sigma}}(h/e(\mathfrak{P}))}$, *where* $N_\mathfrak{P}$ *is the local norm* $N_{M_\mathfrak{P}/\mathbb{Q}_p}$ *and* $e(\mathfrak{P})$ *is the ramification index of* $\mathfrak{P}/p$.

A proof of this is given in [HMI] Corollary 5.39, though the sign $(-1)^e$ is erroneously omitted there. We will revisit briefly the proof of this corollary in Lecture 3 and correct the sign error.

If $E_{/F}$ is an elliptic curve with split multiplicative reduction at all $\mathfrak{p}|p$, we write $E_{/F_\mathfrak{p}}(\overline{F}_\mathfrak{p}) \cong \overline{F}_\mathfrak{p}^\times/q_\mathfrak{p}^\mathbb{Z}$ for the Tate period $q_\mathfrak{p} \in F_\mathfrak{p}^\times$. Then we have directly from the theorem the following

Corollary 0.6. *Assume that $E_{/F}$ is associated to a Hilbert modular form on $GL(2)_{/F}$. Then the $\mathcal{L}$-invariant $\mathcal{L}(\mathrm{Ind}_F^\mathbb{Q} Ad(\rho_E))$ is given by $\prod_\mathfrak{p} \frac{\log_p(N_\mathfrak{p}(q_\mathfrak{p}))}{\mathrm{ord}_p(N_\mathfrak{p}(q_\mathfrak{p}))}$, where $N_\mathfrak{p}$ is the local norm $N_{F_\mathfrak{p}/\mathbb{Q}_p}$.*

The above two corollaries are obtained by explicitly computing the universal representation $\boldsymbol{\rho}$. The case where $F = \mathbb{Q}$ is treated in [H04]. We will generalize this corollary to Theorem 2.1 dealing with elliptic curves having multiplicative reduction at some prime factors of p and ordinary good reduction at the other prime factors of p and give a proof of Theorem 2.1 limiting ourselves to E having good reduction everywhere outside p and to $F/\mathbb{Q}$ in which p totally splits. The general case will be treated in [H06].

1. Lecture 1: Galois deformation and $\mathcal{L}$-invariant

In the following three lectures delivered at the eigenvariety semester at Harvard university in Spring 2006, we give a sketch of the proof of the results mentioned in the introduction, assuming as a simplifying assumption that p completely splits in $F/\mathbb{Q}$ (the general case is treated in [HMI] Chapters 3 and 5 and also in [H06]).

Let $p > 2$ be a prime, and fix a totally real finite extension $F/\mathbb{Q}$. As we wrote, for **simplicity**, hereafter, we assume always that p splits completely in $F/\mathbb{Q}$. We start with a Galois representation $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(W)$ associated to a discrete series ($\Leftrightarrow k \geq 2$) Hilbert modular form f (over F) with coefficients in a finite extension $W/\mathbb{Z}_p$ (a DVR). We assume the ordinarity of ρ_F:

$$\rho_F|_{D_\mathfrak{p}} \cong \begin{pmatrix} \epsilon_\mathfrak{p} & * \\ 0 & \alpha_\mathfrak{p} \end{pmatrix} \text{ with } \epsilon_\mathfrak{p} \neq \alpha_\mathfrak{p},\ \epsilon_\mathfrak{p}|_{I_\mathfrak{p}} = \mathcal{N}^{k-1} \text{ and } \alpha_\mathfrak{p}(I_\mathfrak{p}) = 1$$

on the decomposition group and the inertial group $I_\mathfrak{p} \subset D_\mathfrak{p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for all prime factor $\mathfrak{p}$ of p in F. Here $\mathcal{N}(\sigma) \in \mathbb{Z}_p^\times$ is the p-adic cyclotomic character with $\exp(\frac{2\pi i}{p^n})^\sigma = \exp(\frac{\mathcal{N}(\sigma)2\pi i}{p^n})$ for all $n > 0$ and $k > 1$ is an integer. Again for **simplicity**, we assume that ρ is unramified outside p and ∞. Thus for any prime $\mathfrak{l} \nmid p$, writing $f|T(\mathfrak{l}) = a_\mathfrak{l} f$, we have $\mathrm{Tr}(\rho(Frob_\mathfrak{l})) = a_\mathfrak{l} \in W$. Let K be the quotient field of W (so, $K/\mathbb{Q}_p$ is a finite extension).

We consider the **universal** nearly ordinary deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(R)$ over K with the pro-Artinian local universal K-algebra R. This means that for any Artinian local K-algebra A with maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(K1) unramified outside p;
(K2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$;
(K3) $\det(\rho_A) = \det \rho_F$;
(K4) $\rho_A \equiv \rho_F \mod \mathfrak{m}_A$,

there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. Note that $\mathcal{N} : \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p}) \cong \mathbb{Z}_p^\times$ (by splitting of p in $F/\mathbb{Q}$). Let $\Gamma_\mathfrak{p} = 1+p\mathbb{Z}_p \subset \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$. Choose a generator $\gamma_\mathfrak{p}$ of $\Gamma_\mathfrak{p}$ and identify $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$ by $\gamma_\mathfrak{p} \leftrightarrow 1 + X_\mathfrak{p}$. Since $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_\mathfrak{p} \end{smallmatrix}\right)$, $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1} : \Gamma_\mathfrak{p} \to R$ induces an algebra structure on R over $W[[X_\mathfrak{p}]]$. Thus R is an algebra over $K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$.

If we write $\varphi : R \to K$ for the morphism with $\varphi \circ \boldsymbol{\rho} \cong \rho_F$, by our construction, $\mathrm{Ker}(\varphi) \supseteq (X_{\mathfrak{p}})_{\mathfrak{p}|p}$.

Here is the theorem we have seen in the first lecture:

Theorem 1.1. *Suppose $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then, if $\varphi \circ \boldsymbol{\rho} \cong \rho_F$, for the local Artin symbol $[p, F_{\mathfrak{p}}] = Frob_{\mathfrak{p}}$, we have*

$$\mathcal{L}(Ad(\rho_F)) = \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{X=0} \prod_{\mathfrak{p}} \log_p(\gamma_{\mathfrak{p}})\alpha_{\mathfrak{p}}([p, F_{\mathfrak{p}}])^{-1}.$$

Greenberg proposed a conjectural formula of the $\mathcal{L}$–invariant for a general p-adic p-ordinary Galois representation V with an exceptional zero. When $V = Ad(\rho_F)$, his definition goes as follows. Under some hypothesis, he found a unique subspace $\mathbb{H} \subset H^1(F, Ad(\rho_F))$ of dimension $e = |\{\mathfrak{p}|p\}|$ represented by cocycles $c : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to Ad(\rho_F)$ such that

(1) c is unramified outside p;
(2) c restricted to $D_{\mathfrak{p}}$ is upper triangular after conjugation for all $\mathfrak{p}|p$.

By the condition (2), $c|_{I_{\mathfrak{p}}}$ modulo upper nilpotent matrices factors through the cyclotomic Galois group $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p)$ because $F_{\mathfrak{p}} = \mathbb{Q}_p$, and hence $c|_{D_{\mathfrak{p}}}$ modulo upper nilpotent matrices becomes unramified everywhere over the cyclotomic $\mathbb{Z}_p$-extension F_∞/F. In other words, the cohomology class $[c]$ is in $\mathrm{Sel}_{F_\infty}(Ad(\rho_F))$ but not in $\mathrm{Sel}_F(Ad(\rho_F))$.

Take a basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$ of $\mathbb{H}$ over K. Write

$$c_{\mathfrak{p}}(\sigma) \sim \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \quad \text{for } \sigma \in D_{\mathfrak{p}'} \text{ with any } \mathfrak{p}'|p.$$

Then $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \to K$ is a homomorphism. His $\mathcal{L}$-invariant is defined by

$$\mathcal{L}(Ad(\rho_F)) = \det\Big((a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}])_{\mathfrak{p},\mathfrak{p}'|p} \left(\log_p(\gamma_{\mathfrak{p}'})^{-1} a_{\mathfrak{p}}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p})^{-1}\Big).$$

The above value is independent of the choice of the basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$. When $F = \mathbb{Q}$, by a result of Kisin [Ki] 9.10, [Ki1] and [Ki2] 3.4 (generalizing those of Wiles [W] and Taylor-Wiles [TW]), we always have $R \cong K[[X_p]]$. In general, assuming the following two conditions:

(ai) $\overline{\rho} = (\rho_F \mod \mathfrak{m}_W)$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/F[\mu_p])$;
(ds) $\overline{\rho}^{ss}$ has a non-scalar value over $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ for all prime factors $\mathfrak{p}|p$,

by using a result of Fujiwara (see [Fu] and [Fu1]), we can prove $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. The following conjecture for the arithmetic L-function is a theorem under the condition (ai) and semi-stability of ρ_F over O except for the nonvanishing $\mathcal{L}(Ad(\rho_F)) \neq 0$ (see [HMI] Theorem 5.27 combined with (5.2.6) there):

Conjecture 1.2 (Greenberg)**.** *Suppose* (ds) *and that $\overline{\rho}$ is absolutely irreducible. For $L_p^{arith}(s, Ad(\rho_F)) = \Phi^{arith}(\gamma^{1-s} - 1)$, then $L_p^{arith}(s, Ad(\rho_F))$ has zero of order equal to $d = [F : \mathbb{Q}]$ and for the constant $\mathcal{L}(Ad(\rho_F)) \in K^\times$ specified by the determinant as in the theorem, we have*

$$\lim_{s \to 1} \frac{L_p^{arith}(s, Ad(\rho_F))}{(s-1)^d} = \mathcal{L}(Ad(\rho_F))\left|\left|\mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F))\right|\right|_p^{-1/[K:\mathbb{Q}_p]}$$

up to units.

The factor $\mathcal{E}^+(Ad(\rho))$ does not show up in the above formula, because if we write $S_F(Ad(\rho_F))$ for the Bloch-Kato Selmer group $H^1_f(F, Ad(\rho)\otimes\mathbb{Q}_p/\mathbb{Z}_p)$, we have, as explained in [MFG] page 284,

$$\big||\mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F))|\big|_p^{-1/[K:\mathbb{Q}_p]} = \mathcal{E}^+(Ad(\rho_F))\big||S_F(Ad(\rho_F))|\big|_p^{-1/[K:\mathbb{Q}_p]} \text{ up to units,}$$

and the value $\big||S_F(Ad(\rho_F))|\big|_p^{-1/[K:\mathbb{Q}_p]}$ is directly related the primitive complex L-value $L(1, Ad(\rho_F))$ up to a period. In the following section, we describe the Selmer group and how to define $\mathbb{H}$.

1.1. **Selmer Groups.** We recall Greenberg's definition of Selmer groups. Write $F^{(p)}/F$ for the maximal extension unramified outside p and ∞. Put $\mathfrak{G} = \mathrm{Gal}(F^{(p)}/F)$ and $\mathfrak{G}_M = \mathrm{Gal}(F^{(p)}/M)$. Let $V = Ad(\rho_F)$ with a continuous action of $\mathfrak{G}$. We fix a W-lattice T in V stable under $\mathfrak{G}$.

Write $D = D_{\mathfrak{p}} \subset \mathfrak{G}$ for the decomposition group of each prime factor $\mathfrak{p}|p$. Choosing a basis of ρ_F so that $\rho_F|_D$ is upper triangular. We have a 3-step filtration:

$$\text{(ord)} \qquad V \supset \mathcal{F}^-_{\mathfrak{p}}V \supset \mathcal{F}^+_{\mathfrak{p}}V \supset \{0\},$$

where taking a basis so that $\rho_F|_D$ is upper triangular, $\mathcal{F}^-_{\mathfrak{p}}V$ is made up of upper triangular matrices, and $\mathcal{F}^+_{\mathfrak{p}}V$ is made up of upper nilpotent matrices, and on $\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V$, D acts trivially (getting eigenvalue 1 for $Frob_{\mathfrak{p}}$). Since V is self-dual, its dual $V^*(1) = \mathrm{Hom}_K(V,K)\otimes\mathcal{N}$ again satisfies (ord).

Let M/F be a subfield of $F^{(p)}$, and put $\mathfrak{G}_M = \mathrm{Gal}(F^{(p)}/M)$. We write $\mathfrak{p}$ for a prime of M over p and $\mathfrak{q}$ for general primes of M. We put

$$L_{\mathfrak{p}}(V) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{p}}, V) \to H^1(I_{\mathfrak{p}}, \frac{V}{\mathcal{F}^+_{\mathfrak{p}}(V)})).$$

Then for a $\mathfrak{G}_M$-stable W-lattice T of V, we define for the image $L_{\mathfrak{p}}(V/T)$ of $L_{\mathfrak{p}}(V)$ in $H^1(M_{\mathfrak{p}}, V/T)$

$$\text{(1.1)} \qquad \mathrm{Sel}_M(A) = \mathrm{Ker}(H^1(\mathfrak{G}_M, A) \to \prod_{\mathfrak{p}} \frac{H^1(M_{\mathfrak{p}}, A)}{L_{\mathfrak{p}}(A)})) \text{ for } A = V, V/T.$$

The classical Selmer group of V is given by $\mathrm{Sel}_M(V/T)$, equipped with discrete topology. Write F_∞ for the cyclotomic $\mathbb{Z}_p$–extension of F. We define "$-$" Selmer group $\mathrm{Sel}^-_M(V/T)$ replacing $L_{\mathfrak{p}}(A)$ by

$$L^-_{\mathfrak{p}}(V) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{p}}, V) \to H^1(I_{\mathfrak{p}}, \frac{V}{\mathcal{F}^-_{\mathfrak{p}}(V)})).$$

Lemma 1.3. *Suppose* $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. *Then we have* $\mathrm{Sel}^-_F(V) \cong \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$ *and* $\mathrm{Sel}_F(V) = 0$.

Proof. We consider the space $Der_K(R,K)$ of continuous K-derivations. Let $K[\varepsilon] = K[t]/(t^2)$ for the dual number $\varepsilon = (t \mod t^2)$. Then writing K-algebra homomorphism $\phi : R \to K[\varepsilon]$ as $\phi(r) = \phi_0(r)+\phi_1(r)\varepsilon$ and sending ϕ to $\phi_1 \in Der_K(R,K)$, we have $\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong Der_K(R,K) = \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$. By the universality of $(R, \boldsymbol{\rho})$, we have

$$\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong \frac{\{\rho : \mathfrak{G} \to GL_2(K[\varepsilon])|\rho \text{ satisfies the condtions (K1–4)}\}}{\cong}.$$

Pick ρ as above. Write $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$. Then $c_\rho = \rho_1\rho_F^{-1}$ can be easily checked to be a 1-cocycle having values in $M_2(K) \supset V$. Since $\det(\rho) = \det(\rho_F) \Rightarrow \mathrm{Tr}(c_\rho) = 0$, c_ρ has values in V. By the reducibility condition (K2), $[c_\rho] \in \mathrm{Sel}_F^-(V)$. We see easily that $\rho \cong \rho' \Leftrightarrow [c_\rho] = [c_{\rho'}]$. We can reverse the above argument starting a cocycle c giving an element of $\mathrm{Sel}_F^-(V)$ to construct a deformation ρ_c with values in $K[\varepsilon]$. Thus we have

$$\frac{\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K[\varepsilon]) | \rho \text{ satisfies the condtions (K1–4)}\}}{\cong} \cong \mathrm{Sel}_F^-(V).$$

Since the algebra structure of R over $W[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$ is given by $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1}$, the K-derivation $\delta : R \to K$ corresponding to a $K[\varepsilon]$-deformation ρ is a $W[[X_\mathfrak{p}]]$-derivation if and only if $\rho_1|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \sim \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$, which is equivalent to $[c_\rho] \in \mathrm{Sel}_F(V)$, because we already knew that $\mathrm{Tr}(c_\rho) = 0$. Thus we have $\mathrm{Sel}_F(V) \cong Der_{W[[X_\mathfrak{p}]]}(R, K) = 0$. $\square$

We also have

Lemma 1.4.

$$\text{(V)} \qquad \mathrm{Sel}_F(V) = 0 \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{L_\mathfrak{p}(V)}.$$

Indeed, by the Poitou-Tate exact sequence, the following sequence is exact:

$$\mathrm{Sel}_F(V) \to H^1(\mathfrak{G}_M, V) \to \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{L_\mathfrak{p}(V)} \to \mathrm{Sel}_F(V^*(1))^*.$$

It is an old theorem of Greenberg that $\dim \mathrm{Sel}_F(V) = \dim \mathrm{Sel}_F(V^*(1))^*$ (see [G] Proposition 2); so, we have the assertion (V). $\square$

1.2. **Greenberg's $\mathcal{L}$–invariant.** Here is Greenberg's definition of $\mathcal{L}(V)$: The long exact sequence of $\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V \hookrightarrow V/\mathcal{F}_\mathfrak{p}^+ V \twoheadrightarrow V/\mathcal{F}_\mathfrak{p}^- V$ gives a homomorphism, noting $F_\mathfrak{p} = \mathbb{Q}_p$,

$$H^1(F_\mathfrak{p}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) = \mathrm{Hom}(\mathfrak{G}_{\mathbb{Q}_p}^{ab}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) \xrightarrow{\iota_\mathfrak{p}} H^1(F_\mathfrak{p}, V)/L_\mathfrak{p}(V).$$

Note that

$$\mathrm{Hom}(G_{\mathbb{Q}_p}^{ab}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) \cong (\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V)^2 \cong K^2$$

canonically by $\phi \mapsto (\frac{\phi([\gamma, F_\mathfrak{p}])}{\log_p(\gamma)}, \phi([p, F_\mathfrak{p}]))$. Here $[x, F_\mathfrak{p}] = [x, \mathbb{Q}_p]$ is the local Artin symbol (suitably normalized). Since

$$L_p(\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) = \mathrm{Ker}(H^1(F_\mathfrak{p}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) \xrightarrow{\mathrm{Res}} H^1(I_\mathfrak{p}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V)),$$

the image of $\iota_\mathfrak{p}$ is isomorphic to $\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V \cong K$. By (V), we have a unique subspace $\mathbb{H}$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\prod_\mathfrak{p} \mathrm{Im}(\iota_\mathfrak{p}) \hookrightarrow \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{L_\mathfrak{p}(V)}.$$

Then by the restriction, $\mathbb{H}$ gives rise to a subspace L of

$$\prod_\mathfrak{p} \mathrm{Hom}(G_{F_\mathfrak{p}}^{ab}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) \cong \prod_\mathfrak{p} (\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V)^2$$

isomorphic to $\prod_\mathfrak{p} (\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V)$. If a cocycle c representing an element in $\mathbb{H}$ is unramified, it gives rise to an element in $\mathrm{Sel}_F(V)$. By the vanishing of $\mathrm{Sel}_F(V)$

(Lemma 1.3), this implies $c = 0$; so, the projection of L to the first factor $\prod_{\mathfrak{p}} \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$ (via $\phi \mapsto (\phi([\gamma, F_{\mathfrak{p}}])/\log_p(\gamma))_{\mathfrak{p}}$) is surjective. Thus this subspace L is a graph of a K–linear map $\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$. We then define $\mathcal{L}(V) := \det(\mathcal{L}) \in K$.

Let $\boldsymbol{\rho} : \mathfrak{G}_F \to GL_2(R)$ be the universal nearly ordinary deformation with $\boldsymbol{\rho}\big|_D = \begin{pmatrix} * & * \\ 0 & \boldsymbol{\delta} \end{pmatrix}$. Then $c_{\mathfrak{p}} = \frac{\partial \boldsymbol{\rho}}{\partial X_{\mathfrak{p}}}|_{X=0} \rho_F^{-1}$ is a 1-cocycle (by the argument proving Lemma 1.3) giving rise to a class of $\mathbb{H}$. By Lemma 1.3, $\mathbb{H} = \mathrm{Sel}_F^-(V)$, and $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ gives a basis of $\mathbb{H}$ over K. We have $\boldsymbol{\delta}([u, F_{\mathfrak{p}}]) = (1 + X_{\mathfrak{p}})^{\log_p(u)/\log_p(\gamma)}$ for $u \in O_{\mathfrak{p}}^\times = \mathbb{Z}_p^\times$. Writing

$$c_{\mathfrak{p}}(\sigma) = \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \rho_F(\sigma)^{-1},$$

we have $a_{\mathfrak{p}} = \boldsymbol{\delta}^{-1} \frac{d\boldsymbol{\delta}}{dX_{\mathfrak{p}}}|_{X=0}$, and from this we get the desired formula of $\mathcal{L}(Ad(\rho_F))$. This finishes the proof of Theorem 0.4. □

If one restricts $c \in \mathbb{H}$ to $\mathfrak{G}_\infty = \mathrm{Gal}(F^{(p)}/F_\infty)$, its ramification is exhausted by $\Gamma = \mathrm{Gal}(F_\infty/F)$ (because $F_{\mathfrak{p}} = \mathbb{Q}_p$) giving rise to a class $[c] \in \mathrm{Sel}_{F_\infty}(V)$. The kernel of the restriction map: $H^1(\mathfrak{G}, V) \to H^1(\mathfrak{G}_\infty, V)$ is given by $H^1(\Gamma, H^0(\mathfrak{G}_\infty, V)) = 0$ because $H^0(\mathfrak{G}_\infty, V) = 0$. Thus the image of $\mathbb{H}$ in $\mathrm{Sel}_{F_\infty}(V/T)$ gives rise to the order d exceptional zero of $L^{arith}(s, Ad(\rho_F))$ at $s = 1$. We have proved

Proposition 1.5. *For the number of prime factors $e = [F : \mathbb{Q}]$ of p in F, we have*

$$\mathrm{ord}_{s=1} L_p^{arith}(s, Ad(\rho_F)) \geq e.$$

2. Lecture 2: Elliptic curves with multiplicative reduction

Let p be an odd prime. Order the prime factors of p in F as $\mathfrak{p}_1, \dots, \mathfrak{p}_e$. In this lecture, we describe the computation of the $\mathcal{L}$-invariant of $Ad(T_pE)$ for a modular elliptic curve $E_{/F}$ with split multiplicative reduction at $\mathfrak{p}_j|p > 2$ for $j = 1, 2, \dots, k$ and ordinary good reduction at $\mathfrak{p}_j|p$ for $j > k$.

Theorem 2.1. *Assume that $R \cong \mathbb{Q}_p[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Suppose that the Hilbert-modular elliptic curve E has split multiplicative reduction at $\mathfrak{p}_j$ for $j = 1, 2, \dots, k$ ($k \leq e$) with Tate period q_j at $\mathfrak{p}_j$ for $j \leq k$ and has ordinary good reduction at $\mathfrak{p}_i$ with $i > k$. Then for the local Artin symbol $[p, F_{\mathfrak{p}}] = Frob_{\mathfrak{p}}$ and the norm $Q_j = N_{F_{\mathfrak{p}_j}/\mathbb{Q}_p}(q_j)$, we have for $\rho_E = T_pE$*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \left(\prod_{j=1}^{k} \frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)}\right) \cdot \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_i])}{\partial X_j}\right)_{i>k, j>k} \Big|_{X=0} \prod_{i>k} \frac{\log_p(\gamma_{\mathfrak{p}_i})}{\alpha_{\mathfrak{p}}([p, F_i])},$$

where $\gamma_{\mathfrak{p}}$ is the generator of the p-profinite part $\Gamma_{\mathfrak{p}}$ of $\mathcal{N}(\mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}}))$ by which we identify the group algebra $W[[\Gamma_{\mathfrak{p}}]]$ with $W[[X_{\mathfrak{p}}]]$.

In the proof, for simplicity, as before, we assume that p is completely split in $F/\mathbb{Q}$. Also, again for simplicity, in the following proof, we assume E has good reduction outside p and $k = 1$. We put $\Gamma_F = \prod_{\mathfrak{p}} \Gamma_{\mathfrak{p}}$.

2.1. **Hecke algebras for quaternion algebras.** We make some preparation for the proof, gathering known facts. We assume that $F \neq \mathbb{Q}$ (otherwise the theorem is known by Greenberg-Stevens). For simplicity, p splits completely in $F/\mathbb{Q}$. Take first a quaternion algebra $B_{0/F}$ central over F unramified everywhere such that $B_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{d-r}$ with $0 \leq r \leq 1$ (so $r \equiv d \mod 2$). Then we consider the automorphic variety (either a Shimura curve ($r = 1$) or a 0-dimensional point set ($r = 0$)) given by

$$X_{11}(p^n) = B_0^\times \backslash B_{0,\mathbb{A}}^\times / S_{11}(p^n) Z_{\mathbb{A}} C_\infty,$$

where $Z_{\mathbb{A}} \cong F_{\mathbb{A}}^\times$ is the center of $B_{\mathbb{A}}^\times$, C_∞ is a maximal compact subgroup of the identity component of $B_{0,\infty}^\times$ and identifying $B_{0,\mathfrak{l}}^{(\infty)} = B_0 \otimes_{\mathbb{Q}} F_{\mathfrak{l}}$ with $M_2(F_{\mathfrak{l}})$ for all primes $\mathfrak{l}$,

$$S_{11}(p^n) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}) \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod p^n \right\}$$

for $\widehat{O} = \prod_{\mathfrak{l}} O_{\mathfrak{l}}$. Consider $M_n \cong H_r(X_{11}(p^n), \mathbb{Z}_p)$ which is the Pontryagin dual of $H^r(X_{11}(p^n), \mathbb{Q}_p/\mathbb{Z}_p)$ which is a finite rank free $\mathbb{Z}_p$-module with Hecke operator action of $T(\mathfrak{n})$ for all prime ideals outside p and $U(p_{\mathfrak{p}}^n) = U(p_{\mathfrak{p}})^n$ and the diamond operator action $\langle z \rangle$ coming from $\left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $z \in O_p$. Let $e = \lim_{n\to\infty} U(p)^{n!}$ as an operator acting on M_n ($U(p) = \prod_{\mathfrak{p}} U(p_{\mathfrak{p}})$). Let M_n^{ord} be the direct summand eM_n. We have natural trace map $M_m \twoheadrightarrow M_n$ for $m > n$ compatible with all Hecke operators and all diamond operators. By the diamond operator action, $M_\infty^{ord} = \varprojlim_n M_n^{ord}$ naturally become a $W[[\Gamma_F]]$-module. Here is an old theorem of mine:

Theorem 2.2. *The $W[[\Gamma_F]]$-module M_∞^{ord} is free of finite rank over $W[[\Gamma_F]]$.*

Let $\mathbf{h}$ be the $W[[\Gamma_F]]$-algebra generated over $W[[\Gamma_F]]$ by $T(\mathfrak{n})$ for all $\mathfrak{n}$ prime to p and all $U(\mathfrak{p})$. Then we have

Corollary 2.3. $\mathbf{h}$ *is torsion free of finite type over $W[[\Gamma_F]]$ with $\mathbf{h}_F/(X_{\mathfrak{p}})_{\mathfrak{p}|p}\mathbf{h}_F$ pseudo isomorphic to the Hecke algebra of $H_r(X_{11}(p), W)$.*

Actually if $p \geq 5$, $\mathbf{h}$ is known to be free over $W[[\Gamma_F]]$ and the pseudo isomorphism as above is actually an isomorphism.

Let $\mathbb{T}$ be the local ring of the universal nearly ordinary Hecke algebra $\mathbf{h}$ acting nontrivially on the Hecke eigenform associated to E. Let $P \in \mathrm{Spf}(\mathbb{T})(\mathbb{Q}_p)$ corresponding to ρ_E, that is, $\rho_{\mathbb{T}} \mod P \sim \rho_E$. Let $\widehat{\mathbb{T}}_P = \varprojlim_n \mathbb{T}_P/P^n\mathbb{T}_P$ for the localization $\mathbb{T}_P$. Since $\rho_E = T_pE \otimes \mathbb{Q}_p$ is absolutely irreducible, by the technique of pseudo representation, we can construct the modular deformation $\rho_{\mathbb{T}} : \mathfrak{G} \to GL_2(\widehat{\mathbb{T}}_P)$ which satisfies (K1–4); in particular, $\det \rho_{\mathbb{T}} = \mathcal{N}$, because the central character is trivial. Since E is modular over F, we have the surjective $\mathbb{Q}_p$-algebra homomorphism $R \to \widehat{\mathbb{T}}_P$ for the localization-completion $\widehat{\mathbb{T}}_P$. Since $\widehat{\mathbb{T}}_P$ is integral and of dimension d, we have

Corollary 2.4. *If $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$, then $R \cong \widehat{\mathbb{T}}_P$.*

Under absolute irreducibility of $\overline{\rho}_F$ over $\mathrm{Gal}(\overline{F}/F[\mu_p])$ with non-scalar semi-simplification of $\overline{\rho}_F|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ for all $\mathfrak{p}|p$, the isomorphism $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ will be proven by showing $R \cong \widehat{\mathbb{T}}_P$ first (see Appendix).

Take a quaternion algebra $B_{1/F}$ such that $B_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^q \times \mathbb{H}^{d-q}$ with $q \leq 1$ and B is ramified only at $\mathfrak{p}_1$ (among finite places). Then at $\mathfrak{p}_1$, we have a unique maximal order R_1 in $B_{\mathfrak{p}_1}$. Then we define $U_{11}(p^n)$ to be the product of $S_{11}(p^n)^{(\mathfrak{p}_1)}$ and $R_1^\times$ and define

$$Y_{11}(p^n) = B_1^\times \backslash B_{1,\mathbb{A}}^\times / U_{11}(p^n) Z_{\mathbb{A}} C_\infty.$$

Then we define $e_1 = \lim_{n\to\infty} U(p^{(\mathfrak{p}_1)})^{n!}$ acting on the dual $N_n = H_q(Y_{11}(p^n), \mathbb{Z}_p)$ of the cohomology group $H^q(Y_{11}(p^n), \mathbb{Q}_p/\mathbb{Z}_p)$. Let $\Gamma_1 = \prod_{\mathfrak{p}\neq\mathfrak{p}_1} \Gamma_{\mathfrak{p}}$. We go through all the above process and define $\mathbf{h}_1 \subset \mathrm{End}_{W[[\Gamma_1]]}(\varprojlim_n e_1 N_n)$. Since ρ_E (or corresponding automorphic representation π_E) is Steinberg at $\mathfrak{p}_1$, by the Jacquet-Langlands correspondence, we have a Hecke eigenvector f_1 in $H^q(Y_{11}(p), \mathbb{Z}_p)$ giving rise to E. Then we define $\mathbb{T}_1$ to be the local ring of $\mathbf{h}_1$ acting nontrivially on f_1. Let $P_1 \in \mathrm{Spf}(\mathbb{T}_1)(W)$ be the point associated to ρ_E. We then have a deformation $\rho_{\mathbb{T}_1} : \mathfrak{G} \to GL_2(\widehat{\mathbb{T}}_{1,P})$ of ρ_E. Since the central character is trivial, we have $\det \rho_{\mathbb{T}_1} = \mathcal{N}$.

Theorem 2.5. *We have*

(1) $\mathbf{h}_1$ *is torsion-free of finite rank over* $W[[\Gamma_1]]$, *and* $\widehat{\mathbb{T}}_{1,P_1} \cong K[[X_{\mathfrak{p}_2}, \ldots, X_{\mathfrak{p}_d}]]$;
(2) $\rho_{\mathbb{T}_1}$ *restricted to* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}_1}/F_{\mathfrak{p}_1})$ *is isomorphic to* $\left(\begin{smallmatrix}\varepsilon\mathcal{N} & * \\ 0 & \varepsilon\end{smallmatrix}\right)$, *where* $\varepsilon = \pm 1$ *is the eigenvalue of* $Frob_{\mathfrak{p}_1}$ *on the étale quotient of* T_pE;
(3) *There is a surjective algebra homomorphism* $\mathbb{T}/X_{\mathfrak{p}_1}\mathbb{T} \twoheadrightarrow \mathbb{T}_1$ *inducing an isomorphism* $\widehat{\mathbb{T}}_P/X_{\mathfrak{p}_1}\widehat{\mathbb{T}}_P \cong \widehat{\mathbb{T}}_{1,P_1}$;
(4) *There is a surjective algebra homomorphism* $\mathbb{T}/(U(\mathfrak{p}_1)-\varepsilon)\mathbb{T} \twoheadrightarrow \mathbb{T}_1$ *sending* $T(\mathfrak{n})$ *to* $T(\mathfrak{n})$, *where* $U(\mathfrak{p}_1) = U(p_{\mathfrak{p}_1})$.

Here is a sketch of proof. The first assertion follows from construction; in other words, it can be proven by the same way as the proof of Corollary 2.3. By the Jacquet-Langlands correspondence, $\mathbb{T}$ covers $\mathbb{T}_1$. Any automorphic representation π corresponding to a point of $\mathrm{Spf}(\mathbb{T}_1)(\overline{\mathbb{Q}}_p)$ is Steinberg at $\mathfrak{p}_1$ because B_1 ramifies at $\mathfrak{p}_1$. Since points corresponding classical automorphic representation is Zariski dense in $\mathrm{Spf}(\mathbb{T}_1)$, the Galois representation has to have the form as in (2). Thus the eigenvalue of $U(\mathfrak{p}_1)$ of π is ± 1 and the corresponding Galois representation has the form as in (2). The assertion (1) implies (3). By (2), $U(\mathfrak{p}_1)$ is either ± 1. Since $U(\mathfrak{p}_1)$ is a formal function on the connected $\mathrm{Spf}(\mathbb{T}_1)$, $U(\mathfrak{p}_1) = \varepsilon$ is a constant, which implies (4). □

2.2. **Proof of Theorem 2.1.** Write for simplicity, $X_j := X_{\mathfrak{p}_j}$, $F_j = F_{\mathfrak{p}_j}$ and $p_j = p_{\mathfrak{p}_j}$. By (3) and (4) of Theorem 2.5, $U(\mathfrak{p}_1) \equiv \varepsilon \mod X_1$ is a constant independent of $X_j := X_{\mathfrak{p}_j}$ for all $j \geq 2$. Thus $\frac{\partial U(\mathfrak{p}_1)}{\partial X_j}|_{X_1=0} = 0$ for all $j \geq 2$. Thus

$$\det\left(\frac{\partial U(\mathfrak{p}_i)}{\partial X_j}\right)\Big|_{X=0} = \frac{\partial U(\mathfrak{p}_1)}{\partial X_1}\Big|_{X_1=0} \times \det\left(\frac{\partial U(\mathfrak{p}_i)}{\partial X_j}\right)_{i\geq 2, j\geq 2}\Big|_{X=0}.$$

Since $\boldsymbol{\delta}_{\mathfrak{p}_i}([p, F_i]) = U(\mathfrak{p}_i)$, we get from the formula we stated in the first lecture:

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{X=0} \prod_{\mathfrak{p}} \log_p(\gamma_{\mathfrak{p}}) \alpha_{\mathfrak{p}}([p, F_{\mathfrak{p}}])^{-1},$$

the following new formula:

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p,F_1])}{\partial X_1}\Big|_{X_1=0} \log_p(\gamma_{\mathfrak{p}_1})\alpha_{\mathfrak{p}_1}([p,F_1])^{-1} \times \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p,F_i])}{\partial X_j}\right)_{i\geq 2, j\geq 2}\Big|_{X=0} \prod_{j\geq 2} \log_p(\gamma_{\mathfrak{p}_j})\alpha_{\mathfrak{p}}([p,F_j])^{-1}. \tag{2.1}$$

Thus the result follows from the following lemma of Greenberg-Stevens:

Lemma 2.6. *Let us write $\gamma = \gamma_{\mathfrak{p}_1}$. We have*

$$\frac{\partial \boldsymbol{\delta}_1([p,F_1])}{\partial X_1}\Big|_{X_1=0} \log_p(\gamma)\alpha_{\mathfrak{p}_1}([p,F_1])^{-1} = \frac{\log_p(q_1)}{\mathrm{ord}_p(q_1)}$$

for $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_{\mathfrak{p}_1}$.

Proof. Since $\alpha_{\mathfrak{p}_1}([p,F_1]) = 1$ (split multiplicative reduction), we can forget about this factor. Since the linear operator $\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V \to \prod_{\mathfrak{p}} \mathcal{F}^-_{\mathfrak{p}} V/\mathcal{F}^+_{\mathfrak{p}} V$ induces $\mathcal{L}_1 : \mathcal{F}^-_{\mathfrak{p}_1} V/\mathcal{F}^+_{\mathfrak{p}_1} V \to \mathcal{F}^-_{\mathfrak{p}_1} V/\mathcal{F}^+_{\mathfrak{p}_1} V$ by our diagonalization of its matrix. This $\mathcal{L}_1$ comes from the subspace

$$L_1 \subset \mathrm{Hom}(D_1^{ab}, \mathcal{F}^-_{\mathfrak{p}_1} V/\mathcal{F}^+_{\mathfrak{p}_1} V) \cong \mathrm{Hom}(D_1^{ab}, \mathbb{Q}_p)$$

for $D_1 = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_1)$ has a generator $\phi_0 = \boldsymbol{\delta}_1^{-1}\frac{\partial \boldsymbol{\delta}_1}{\partial X_1}\Big|_{X_1=0} : D_1^{ab} \to \mathbb{Q}_p$. Thus by definition

$$\frac{\partial \boldsymbol{\delta}_1([p,F_1])}{\partial X_1}\Big|_{X_1=0} \log_p(\gamma) = \log_p(\gamma)\frac{\phi_0([p,F_1])}{\phi_0([\gamma,F_1])}.$$

Let $\rho_E = T_pE\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, $\widetilde{\rho}_E = (\boldsymbol{\rho} \mod (X_1^2, X_2, \dots, X_d))$, and write $\widetilde{\mathbb{Q}_p} = \mathbb{Q}_p[X_1]/(X_1^2)$. The character $(\boldsymbol{\delta}_1 \mod X_1^2)$ is an infinitesimal deformation of the trivial character fitting into the following commutative diagram of D_1-modules:

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(\boldsymbol{\epsilon}_1) & \xrightarrow{\hookrightarrow} & \widetilde{\rho}_E & \xrightarrow{\twoheadrightarrow} & \widetilde{\mathbb{Q}_p}(\boldsymbol{\delta}_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \longrightarrow & \rho_E & \longrightarrow & \mathbb{Q}_p. \end{array}$$

Twist this diagram by $\boldsymbol{\epsilon}_1^{-1}\mathcal{N} = \boldsymbol{\delta}_1$, getting a new diagram

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(1) & \xrightarrow{\hookrightarrow} & \widetilde{\rho}_E & \xrightarrow{\twoheadrightarrow} & \widetilde{\mathbb{Q}_p}(\boldsymbol{\delta}_1^2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \longrightarrow & \rho_E & \longrightarrow & \mathbb{Q}_p. \end{array}$$

Once this type of diagram is obtained (with leftmost column given by $\widetilde{\mathbb{Q}_p}(1) \twoheadrightarrow \mathbb{Q}_p(1)$), by a general result of Greenberg-Stevens in such a situation (see [GS1] (2.3.4)), we get

$$\frac{\partial \boldsymbol{\delta}_1^2}{\partial X_1}([q_1,\mathbb{Q}_p])\Big|_{X_1=0} = 0 \Rightarrow \frac{\partial \boldsymbol{\delta}_1}{\partial X_1}([q_1,\mathbb{Q}_p])\Big|_{X_1=0} = 0.$$

Write $q_1 = p^a u$ for $a = \mathrm{ord}_p(q_1)$ and $u \in \mathbb{Z}_p^\times$. Then $\log_p(u) = \log_p(q_1)$. We have

$$\boldsymbol{\delta}_1([q_1,\mathbb{Q}_p]) = \boldsymbol{\delta}_1([p,\mathbb{Q}_p])^a \boldsymbol{\delta}_1([u,\mathbb{Q}_p]) = \boldsymbol{\delta}_1([p,\mathbb{Q}_p])^a (1+X_1)^{-\log_p(u)/\log_p(\gamma)}$$

(because $\mathcal{N}([u,\mathbb{Q}_p]) = u^{-1}$). Differentiating this identity with respect to X_1, we get from $\boldsymbol{\delta}_1([u,\mathbb{Q}_p])|_{X_1=0} = \boldsymbol{\delta}_1([p,\mathbb{Q}_p])|_{X_1=0} = 1$

$$a\frac{\partial\boldsymbol{\delta}_1([p_1,\mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} - \frac{\log_p(q_1)}{\log_p(\gamma)} = a\frac{\partial\boldsymbol{\delta}_1([p_1,\mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} - \frac{\log_p(u)}{\log_p(\gamma)} = 0.$$

From this, we conclude

$$\log_p(\gamma)\frac{\partial\boldsymbol{\delta}_1([p_1,\mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} = \frac{\log_p(q_1)}{\mathrm{ord}_p(q_1)}.$$

□

The fixed field of the kernel of ϕ_0 is a $\mathbb{Z}_p$-extension $M_\infty/\mathbb{Q}_p$ ($F_1 = \mathbb{Q}_p$). Since $L_1 \ni \phi \mapsto \frac{\phi([\gamma,\mathbb{Q}_p])}{\log_p \gamma} \in \mathbb{Q}_p$ is surjective, M_∞ ramifies fully. Then by local class field theory, $\bigcap_{n=1}^{\infty} N_{M_n/\mathbb{Q}_p}(M_n^\times)$ has a rank 1 torsion-free part, which contains $q_0 = p^b v$ with $a \neq 0$ and $v \in \mathbb{Z}_p^\times$. The quantity $\frac{\log_p(q_0)}{\mathrm{ord}_p(q_0)} \in \mathbb{Q}_p$ is determined uniquely independent of the choice of q_0, and we now prove

Proposition 2.7.

$$\log_p(\gamma)\frac{\partial\boldsymbol{\delta}_1([p_1,\mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} = \frac{\log_p(q_0)}{\mathrm{ord}_p(q_0)}.$$

Proof. Let $\phi_0 = \boldsymbol{\delta}_1^{-1}\frac{\partial\boldsymbol{\delta}_1}{\partial X_1} : D_1^{ab} \to \mathbb{Q}_p$. Let $\mathbf{M}_\infty/\mathbb{Q}_p$ be the composite of all $\mathbb{Z}_p$-extensions of $\mathbb{Q}_p$; so, by local class field theory, $\mathrm{Gal}(\mathbf{M}_\infty/\mathbb{Q}_p) \cong \mathbb{Z}_p^2$. Then we have $[q_0,\mathbb{Q}_p] \in \mathrm{Gal}(\mathbf{M}_\infty/M_\infty)$ again by local class field theory, and by definition, $\phi_0([q_0,\mathbb{Q}_p]) = 0$. Since $[q_0,\mathbb{Q}_p] = [v,\mathbb{Q}_p][p,\mathbb{Q}_p]^b$ ($b = \mathrm{ord}_p(q_0)$) we have $0 = \phi_0([q_0,\mathbb{Q}_p]) = \phi_0([v,\mathbb{Q}_p]) + b\phi_0([p,\mathbb{Q}_p])$. Writing $M_\infty^{ur}/\mathbb{Q}_p$ for the unique unramified $\mathbb{Z}_p$-extension and $M_\infty^+/\mathbb{Q}_p$ for the cyclotomic $\mathbb{Z}_p$-extension, the restriction of ϕ_0 to $\Gamma^+ := \mathrm{Gal}(M_\infty^+/\mathbb{Q}_p)$ is a constant multiple of $\log_p \circ \mathcal{N}_p$ for the cyclotomic character $\mathcal{N}_p$; i.e., $\phi_0|_{\Gamma^+} = x(\log_p \circ \mathcal{N}_p)$ for $x \in \mathbb{Q}_p^\times$. Since $\log_p(\mathcal{N}_p([v,\mathbb{Q}_p])) = \log_p(v^{-1}) = -\log_p(q_0)$, we have $x\log_p(v^{-1}) + b\phi_0([p,\mathbb{Q}_p]) = 0$. Thus $\mathcal{L}(Ad(T_pE)) = \phi_0([p,\mathbb{Q}_p])/x = \frac{\log_p(q_0)}{\mathrm{ord}_p(q_0)}$. □

3. Lecture 3: $\mathcal{L}$-invariants of CM fields

Let p be an odd prime. Let M/F be a totally imaginary quadratic extension of the base totally real field F. We study the adjoint square Selmer group when the Galois representation is an induction of a Galois character of $\mathfrak{G}_M := \mathrm{Gal}(M^{(p)}/M)$. Put $\mathfrak{G}_F := \mathrm{Gal}(M^{(p)}/F)$. For simplicity, we assume that $p > 2$ totally splits in $M/\mathbb{Q}$. We relate the Selmer group with a more classical Iwasawa module of a quadratic extension of F, and from the torsion property of the Selmer group already proven, we deduce some (new) torsion property of such classical Iwasawa modules.

3.1. **Ordinary CM fields and their Iwasawa modules.** Let O_M be the integer ring of M. We consider $Z = \varprojlim_n Cl_M(p^n)$ for the ray class group $Cl_M(p^n)$ of M modulo p^n. Let Δ be the maximal torsion subgroup of Z, and put $\Gamma_M = Z/\Delta$, which has a natural action of $\mathrm{Gal}(M/F)$. We split $Z = \Delta \times \Gamma_M$. We define $\Gamma^+ = H^0(\mathrm{Gal}(M/F), \Gamma_M)$ and $\Gamma^- = \Gamma_M/\Gamma^+$. Since $p > 2$, the action of $\mathrm{Gal}(M/F)$ splits the extension $\Gamma^+ \hookrightarrow \Gamma_M \twoheadrightarrow \Gamma^-$, and we have a canonical decomposition $\Gamma_M = \Gamma^+ \times \Gamma^-$. Write $\pi^- : Z \to \Gamma^-$, $\pi^+ : \Gamma_M \to \Gamma^+$ and $\pi_\Delta : Z \to \Delta$ for the three projections. Take a character $\varphi : \Delta \to \overline{\mathbb{Q}}^\times$, and regard it as a character of Z through the projection: $Z \twoheadrightarrow \Delta$.

Let M_∞ be the composite of all $\mathbb{Z}_p$-extensions of M. Then by class field theory, M_∞ is the subfield of the ray class filed of M modulo p^∞ fixed by Δ. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension. Let M_∞^{cyc} be the composite $M\mathbb{Q}_\infty/M$. Define M_∞^- (resp. M_∞^+) for the fixed subfield of Γ^- (resp. Γ^+). Since M_∞^{cyc} is abelian over F, we have $M_\infty^{cyc} \subset M_\infty^+$ and a projection $\pi_{cyc} : \Gamma^+ \to \mathrm{Gal}(M_\infty^{cyc}/M) \subset 1 + p\mathbb{Z}_p$. The Leopoldt conjecture for F asserts that π_{cyc} is an isomorphism; in other words, $M_\infty^+ = M_\infty^{cyc}$. The extension M_∞^-/M is called the anticyclotomic tower over M. Thus if the Leopoldt conjecture holds for F, M_∞ is the composite of the cyclotomic $\mathbb{Z}_p$-extension M_∞^{cyc} and the anticyclotomic $\mathbb{Z}_p^{[F:\mathbb{Q}]}$–extension M_∞^-.

To introduce Iwasawa modules for the multiple $\mathbb{Z}_p$-extensions $M_\infty^?/M$, we fix a CM type Σ, which is a set of embeddings of M into $\overline{\mathbb{Q}}$ such that $I_M = \Sigma \sqcup \Sigma c$ for the generator c of $\mathrm{Gal}(M/F)$. Over $\mathbb{C}$, an abelian variety with complex multiplication by M has $\mathbb{C}$-points isomorphic to $\mathbb{C}^\Sigma/\Sigma(\mathfrak{a})$ for a lattice $\mathfrak{a}$ in M (see [ACM] 5.2), where $\Sigma(\mathfrak{a}) = \{(\sigma(a))_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma | a \in \mathfrak{a}\}$. By composing i_p, we write Σ_p for the set of p-adic places induced by $i_p \circ \sigma$ for $\sigma \in \Sigma$. We assume

$$\text{(spt)} \qquad \Sigma_p \cap \Sigma_p c = \emptyset.$$

This is to guarantee the abelian variety of CM type Σ to have ordinary good reduction modulo p (whose Galois representation is hence ordinary at all $\mathfrak{p}|p$).

Writing $M(p^\infty)$ for the ray class field over M modulo p^∞, we identify Z with $\mathrm{Gal}(M(p^\infty)/M)$ via the Artin reciprocity law. Fix a character φ of Δ. We then define M_Δ by the fixed field of Γ in $M(p^\infty)$; so, $\mathrm{Gal}(M_\Delta/M) = \Delta$.

Since φ is a character of Δ, φ factors through $\mathrm{Gal}(M_\infty^? M_\Delta/M)$ for ? indicating one of $+, -, cyc$ or "nothing". When nothing is attached, it refers to the object for the full multiple $\mathbb{Z}_p$-extension M_∞. Let $L_\infty^?/M_\infty^? M_\Delta$ be the maximal p–abelian extension unramified outside Σ_p. Each $\gamma \in \mathrm{Gal}(L_\infty/M)$ acts on the normal subgroup $X^? = \mathrm{Gal}(L_\infty^?/M_\infty^? M_\Delta)$ continuously by conjugation, and by the commutativity of $X^?$, this action factors through $\mathrm{Gal}(M_\Delta M_\infty^?/M)$. Then we look into the compact p-profinite $\Gamma^?$–module: $X^?[\varphi] = X^? \otimes_{\mathbb{Z}_p[\Delta],\varphi} W$, where $\Gamma^? = \mathrm{Gal}(M_\infty^?/M)$. We study when $X^?[\varphi]$ is a torsion Iwasawa module over $\Lambda^? = W[[\Gamma^?]]$. The module $X^?[\varphi]$ is generally expected to be torsion of finite type over $\Lambda^?$ for the naturally defined multiple $\mathbb{Z}_p$-extensions $M_\infty^?$.

The torsion property of $X^{cyc}[\varphi]$ over Λ^{cyc} is classically known (e.g., [HT2] Theorem 1.2.2). This implies

Theorem 3.1. *The modules $X[\varphi], X^+[\varphi]$ and $X^{cyc}[\varphi]$ are torsion modules over the corresponding Iwasawa algebra Λ, Λ^+ and Λ^{cyc}, respectively.*

We refer this result to [HT2] Theorem 1.2.2 (which was originally due to R. Greenberg). We study the anticyclotomic Iwasawa module $X^-[\varphi]$ over Λ^- from our new view point of Galois deformation theory. As is well known, $X^-[\varphi]$ is a Λ^--module of finite type, and under mild assumptions (including anticyclotomy of φ), we will prove the torsion property of $X^-[\varphi]$ in Theorem 3.3.

The Σ-Leopoldt conjecture for abelian extensions of M is almost equivalent to the torsion property of $X^-[\varphi]$ over Λ^- for all possible φ (see [HT2] Theorem 1.2.2). Here, for an abelian extension L/M with integer ring O_L, the Σ-Leopoldt conjecture asserts the closure $\overline{O_L^\times}$ of $O_L^\times$ in $L_\Sigma = \prod_{\mathfrak{p}\in\Sigma_p} L_{\mathfrak{p}}$ satisfies

$$\dim_{\mathbb{Q}}(O_L^\times \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}_p}(\overline{O_L^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

If $X^-[\varphi]$ is a torsion Λ^--module, we can think of the characteristic element $\mathcal{F}^-(\varphi) \in \Lambda^-$ of the module $X^-[\varphi]$. The anticyclotomic main conjecture (cf. [HT] Conjecture 2.2) predicts the identity (up to units) of $\mathcal{F}^-(\varphi)$ and the projection of (the φ-branch of) the Katz p-adic L-function (constructed in [K] and [HT1]) under π^-.

3.2. **Anticyclotomic Iwasawa modules.** A character ψ of Δ is called *anticyclotomic* if $\psi(c\sigma c^{-1}) = \psi^{-1}(\sigma)$ for a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Fix an algebraic closure $\overline{F}$ of F. Regarding φ as a Galois character, we define $\varphi^-(\sigma) = \varphi(c\sigma c^{-1}\sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\overline{F}/M)$. Then $\psi := \varphi^-$ is anticyclotomic.

We define a Galois character $\widetilde{\varphi} : \mathfrak{G}_F \to W[[\Gamma^-]]$ by $\widetilde{\varphi}(\sigma) = \varphi(\sigma)(\sigma|_{M_\infty^-})^{1/2}$, where $(\sigma|_{M_\infty^-})^{1/2}$ is the unique square root of $(\sigma|_{M_\infty^-})$ in Γ^- and $(\sigma|_{M_\infty^-}) \in \Gamma^-$ is regarded as a group element in $\Gamma^- \subset W[[\Gamma^-]]$. Note that $\widetilde{\varphi}^-(\sigma) = \widetilde{\varphi}(c\sigma c^{-1}\sigma^{-1}) = \psi(\sigma)\sigma|_{M_\infty^-}$. Then we consider $\mathrm{Ind}_M^F(\widetilde{\varphi}) : \mathrm{Gal}(\overline{F}/F) \to GL_2(W[[\Gamma^-]])$. We write $\alpha_{M/F}$ for the quadratic character of $\mathrm{Gal}(\overline{F}/F)$ identifying $\mathrm{Gal}(M/F)$ with $\{\pm 1\}$.

Lemma 3.2. *We have*

(1) $\det(\mathrm{Ind}_M^F \chi) = \alpha_{M/F}\chi|_{F_{\mathbb{A}}^\times}$ *and* $\mathrm{Tr}(\mathrm{Ind}_M^F \chi(Frob_{\mathfrak{l}})) = \sum_{\mathfrak{b}\subset O_M, N_{M/F}(\mathfrak{b})=\mathfrak{l}} \chi(\mathfrak{b})$ *for a prime* $\mathfrak{l}$ *of* F *unramified for* $\mathrm{Ind}_M^F \chi$*, identifying a character* χ *of* $\mathrm{Gal}(\overline{F}/M)$ *with a character of* $M_{\mathbb{A}^{(\infty)}}^\times/M^\times$ *by the Artin symbol,*

(2) $Ad(\mathrm{Ind}_M^F(\widetilde{\varphi})) \cong \alpha_{M/F} \oplus \mathrm{Ind}_M^F(\widetilde{\varphi}^-)$ *as* $\mathfrak{G}_F$*-modules.*

Since $\mathrm{Ind}_M^F(\widetilde{\varphi})|_{\mathfrak{G}_M} = \widetilde{\varphi} \oplus \widetilde{\varphi}_c$ with $\widetilde{\varphi}_c(\sigma) = \widetilde{\varphi}(c\sigma c^{-1})$, we define $\mathcal{F}_{\mathfrak{p}}^+ \mathrm{Ind}_M^F \widetilde{\varphi} = \widetilde{\varphi}$ for $\mathfrak{p} \in \Sigma_p$. In Lecture 1, we have already defined $\mathcal{F}_{\mathfrak{p}}^\pm Ad(\mathrm{Ind}_M^F \widetilde{\varphi})$ and the Selmer group $\mathrm{Sel}_F(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}) \otimes_{\mathbb{Z}_p} (W[[\Gamma^-]])^*)$. Since the image of $\mathcal{F}_{\mathfrak{p}}^+(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}))$ in $\alpha_{M/F}$ is trivial in the above decomposition in Lemma 3.2 and the image of $\mathcal{F}_{\mathfrak{p}}^+(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}))$ is given by $\mathcal{F}_{\mathfrak{p}}^+(\mathrm{Ind}_M^F(\widetilde{\varphi}^-))$, we get (cf. [HMI] Exercise 1.12 and Corollary 3.81)

$$\begin{aligned}\mathrm{Sel}_F(&Ad(\mathrm{Ind}_M^F(\widetilde{\varphi}) \otimes_{W[[\Gamma^-]]} (W[[\Gamma^-]])^*) \\ &= \mathrm{Sel}_F\left(\alpha_{M/F} \otimes_{\mathbb{Z}_p} (W[[\Gamma^-]])^*\right) \oplus \mathrm{Sel}_F(\mathrm{Ind}_M^F(\widetilde{\varphi}^-) \otimes_{W[[\Gamma^-]]} (W[[\Gamma^-]])^*) \\ &= \mathrm{Hom}(Cl_M^- \otimes_{\mathbb{Z}} W[[\Gamma^-]], \mathbb{Q}_p/\mathbb{Z}_p) \oplus \mathrm{Sel}_M((\widetilde{\varphi}^-) \otimes_{W[[\Gamma^-]]} W[[\Gamma^-]]^*),\end{aligned}$$

where Cl_M^- is the quotient of CL_M by the image of Cl_F (the order of Cl_M^- is equal to the order of the $\alpha_{M/F}$-eigenspace of Cl_M up to a power of 2). By the definition of the Selmer group, we note that

$$\mathrm{Sel}_M(\widetilde{\varphi}^- \otimes_{W[[\Gamma^-]]} (W[[\Gamma^-]])^*) \cong \mathrm{Hom}(X^-[\varphi^-], \mathbb{Q}_p/\mathbb{Z}_p), \tag{3.1}$$

which shows

Theorem 3.3. *Let the notation be as above. Then we have*

$$\mathrm{Sel}_F^*(Ad(\mathrm{Ind}_M^F(\widetilde{\varphi}))) \cong \left(Cl_M^- \otimes_{\mathbb{Z}} W[[\Gamma^-]]\right) \oplus X^-[\varphi^-]$$

as $W[[\Gamma^-]]$*-modules. Moreover* $X^-[\varphi^-]$ *is a torsion* $W[[\Gamma^-]]$*-module without exceptional zero if* $\psi := \varphi^-$ *satisfies the following conditions:*

(at1) *The character* ψ *has order prime to* p.

(at2) *The local character* $\psi_{\mathfrak{P}}$ *is non-trivial for all* $\mathfrak{P} \in \Sigma_p$.

(at3) *The restriction ψ^* of ψ to $\mathrm{Gal}(\overline{F}/M^*)$ for the composite M^* of M and the unique quadratic extension inside $F[\mu_p]$ is non-trivial.*

The first assertion follows from the argument given as above. The torsion property follows from the theorem of Taylor–Wiles and Fujiwara and the propositions in the following appendix. In [HMI] Theorem 5.33, it is checked that the assumptions milder than (at1–3) imply the assumption of the theorem of Taylor–Wiles and Fujiwara.

3.3. **The $\mathcal{L}$-invariant of CM fields.** Consider the universal couple $(\mathcal{R}_F, \varrho)$ deforming $\rho_F = \mathrm{Ind}_M^F \varphi \mod \mathfrak{m}_W$ among W-deformations ρ_A into $GL_2(A)$ for proartinian W-algebras A with residue field $W/\mathfrak{m}_W$ satisfying the following conditions

(W1) unramified outside p;
(W2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$ and $\alpha_{A,\mathfrak{p}}\alpha_\mathfrak{p}^{-1}|_{I_\mathfrak{p}}$ factoring through $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$;
(W3) $\det(\rho_A) = \det \rho$;
(W4) $\rho_A \equiv \overline{\rho} \mod \mathfrak{m}_A$,

We know $\mathcal{R}_F \cong \mathbb{T}$ by Fujiwara (see Appendix). Since $\dim \mathrm{Spf}(W[[\Gamma^-]]) = \dim \mathrm{Spf}(\mathbb{T})$, $\mathrm{Spf}(W[[\Gamma^-]])$ gives an irreducible component of $\mathrm{Spf}(\mathcal{R}_F)$. Write $\mathbb{I} = W[[\Gamma^-]]$ simply. Let $\pi_\mathbb{I} : \mathbb{T} = \mathcal{R}_F \twoheadrightarrow \mathbb{I}$ be the projection (which factors through π^{cyc}). We would like to compute the $\mathcal{L}$-invariant of the component $\mathbb{I}$. Thus we need to compute $\mathbf{a}(p_\mathfrak{p}) = \pi_\mathbb{I}(U(p_\mathfrak{p}))$. The following fact follows from the fact $\mathrm{Ind}_M^F \phi|_{\mathfrak{G}_M} = \phi \oplus \phi^c$.

Lemma 3.4. *Let the notation be as above. Then we have $\mathbf{a}(p_\mathfrak{p}) = \widetilde{\varphi}([p_\mathfrak{P}, M_\mathfrak{P}])$ for the prime factor $\mathfrak{P} \in \Sigma_p^c$ of $\mathfrak{p}$.*

Define the character $\boldsymbol{\kappa} : \mathrm{Gal}(\overline{F}/M) \to (\Lambda^-)^\times$ by $\boldsymbol{\kappa}(\sigma) = (\sigma|_{M_\infty^-})^{1/2}$. Then $\widetilde{\varphi} = \lambda\boldsymbol{\kappa}$, and we write $\boldsymbol{\kappa}_\mathbb{I} = \pi_\mathbb{I} \circ \boldsymbol{\kappa} : \mathrm{Gal}(\overline{F}/M) \to \mathbb{I}^\times$. Then, $\boldsymbol{\kappa}_\mathbb{I}$ restricted to the inertia group $I_\mathfrak{P}$ at $\mathfrak{P}$ factors through the projection: $I_\mathfrak{P} \to \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$. Since the $W[[\Gamma_F]]$-algebra structure of $\mathbb{I}$ is induced by the nearly ordinary character of $\mathrm{Ind}_M^F \boldsymbol{\kappa}_\mathbb{I}$ (restricted to the inertia group $I_\mathfrak{P}$), for $u_\mathfrak{P} \in O_{M,\mathfrak{P}}^\times$ ($\mathfrak{P} \in \Sigma_p^c$), we have

$$\boldsymbol{\kappa}_\mathbb{I}([u_\mathfrak{P}, M_\mathfrak{P}]) = (1 + X_\mathfrak{p})^{\log_p(N_\mathfrak{p}(u_\mathfrak{P}))/\log_p(\gamma_\mathfrak{p})}, \tag{3.2}$$

where $\mathfrak{p} = \mathfrak{P} \cap O$, $N_\mathfrak{p} : M_\mathfrak{P} = F_\mathfrak{p} \to \mathbb{Q}_p$ is the norm map and $\gamma_\mathfrak{p}$ is the generator of $\Gamma_\mathfrak{p} := (1 + p\mathbb{Z}_p) \cap N_\mathfrak{p}(O_\mathfrak{p}^\times)$. Choose an element $\varpi(\mathfrak{P}) \in M$ so that $\mathfrak{P}^h = (\varpi(\mathfrak{P}))$ for each $\mathfrak{P} \in \Sigma_p^c$, where $h = |Cl_M|$ (the class number of M). Then $p_\mathfrak{P}^h = u_\mathfrak{P}\varpi(\mathfrak{P})_\mathfrak{P}^{e(\mathfrak{p})}$ with $u_\mathfrak{P} \in O_{M,\mathfrak{P}}^\times$ for the absolute ramification index $e(\mathfrak{p})$ of $\mathfrak{p}$ (which is the absolute ramification index of $\mathfrak{P}$ also). Regarding $\boldsymbol{\kappa}_\mathbb{I}$ as a character of $M_{\mathbb{A}^{(\infty)}}^\times/M^\times$ by class field theory, we have $\boldsymbol{\kappa}_\mathbb{I}(\varpi(\mathfrak{P})) = 1 = \boldsymbol{\kappa}_\mathbb{I}(\varpi(\mathfrak{P})_\mathfrak{l})$ with the $\mathfrak{l}$-component $\varpi(\mathfrak{P})_\mathfrak{l} \in M_\mathfrak{l}^\times$ for any prime $\mathfrak{l}$ outside p, because $\boldsymbol{\kappa}_\mathbb{I}(O_\mathfrak{l}^\times) = 1$ and $\varpi(\mathfrak{P}) \in M^\times$. Then we have

$$\boldsymbol{\kappa}_\mathbb{I}(p_\mathfrak{P}^h) = \boldsymbol{\kappa}_\mathbb{I}(p_\mathfrak{P}^h\varpi(\mathfrak{P})^{-e(\mathfrak{p})}) = \boldsymbol{\kappa}_\mathbb{I}(u_\mathfrak{P}) \prod_{\mathfrak{P}'|p, \mathfrak{P}' \neq \mathfrak{P}} \boldsymbol{\kappa}_\mathbb{I}(\varpi(\mathfrak{P})_{\mathfrak{P}'}^{-e(\mathfrak{p})}),$$

where $\varpi(\mathfrak{P})_{\mathfrak{P}'}$ is the $\mathfrak{P}'$-component of $\varpi(\mathfrak{P}) \in M^\times \subset M_\mathbb{A}^\times$. By (3.2), we get

$$\boldsymbol{\kappa}_\mathbb{I}(p_\mathfrak{P}^h) = (1 + X_\mathfrak{p})^{\frac{\log_p(N_\mathfrak{p}(\varpi(\mathfrak{P})^{e(\mathfrak{p})c}u_\mathfrak{p}))}{\log_p(\gamma_\mathfrak{p})}} \prod_{\mathfrak{P}' \in \Sigma_p^c - \{\mathfrak{P}\}} (1 + X_{\mathfrak{p}'})^{\frac{e(\mathfrak{p})\log_p(N_{\mathfrak{p}'}(\varpi(\mathfrak{P})_{\mathfrak{P}'}^{c-1}))}{\log_p(\gamma_{\mathfrak{p}'})}},$$

where $\mathfrak{p}' = \mathfrak{P}' \cap O$. Here $\log_p$ is the Iwasawa p-adic logarithm defined over $\overline{\mathbb{Q}}_p^\times$ characterized by $\log_p(p) = 0$. In particular, we have

$$\log_p(N_\mathfrak{p}(u_\mathfrak{P})) = \log(N_\mathfrak{p}(p_\mathfrak{P}^h \varpi(\mathfrak{P})_\mathfrak{P}^{-e(\mathfrak{p})})) = -e(\mathfrak{p}) \log_p(N_\mathfrak{p}(\varpi(\mathfrak{P})_\mathfrak{P})).$$

Thus we have

Lemma 3.5. *Let the notation be as above. Then we have, for primes $\mathfrak{P}' \in \Sigma_p$ and $\mathfrak{p}' = \mathfrak{P}' \cap O$,*

$$\frac{\partial \boldsymbol{\kappa}(p_\mathfrak{P})}{\partial X_{\mathfrak{p}'}} = \frac{e(\mathfrak{p}) \log_p(N_{\mathfrak{p}'}(\varpi(\mathfrak{P})_{\mathfrak{P}'}^{(c-1)}))}{h \log_p(\gamma_{\mathfrak{p}'})} \boldsymbol{\kappa}(p_\mathfrak{p})(1 + X_{\mathfrak{p}'})^{-1}.$$

We have $\mathbf{a}(p_\mathfrak{p}) = c_\mathfrak{p} \boldsymbol{\kappa}(p_\mathfrak{p})$ for a nonzero constant $c_\mathfrak{p} \in W^\times$, because the nearly ordinary character of $\mathrm{Ind}_M^F \widetilde{\varphi}$ is $\boldsymbol{\kappa}$ times a character of $D_\mathfrak{P}$ with values in $W^\times$. We do not need to pay much attention to the constant $c_\mathfrak{p}$, because the formula of the $\mathcal{L}$-invariant only involve

$$(\prod_{\mathfrak{p}|p} \mathbf{a}(p_\mathfrak{p})^{-1} \delta_\mathfrak{p}([\gamma_\mathfrak{p}, F_\mathfrak{p}])) \det\left(\left(\frac{\partial \mathbf{a}(p_\mathfrak{p})}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\right)$$

in which the constant $c_\mathfrak{p}$ cancels out. Specializing the above formula to the locally cyclotomic point P, we get

Theorem 3.6. *Let the notation and the assumption be as above and as in* Theorem 3.3, *including* (at1–4). *Then we have, for any specialization $\widetilde{\varphi}_P$ of $\widetilde{\varphi}$ modulo a locally cyclotomic point $P \in \mathrm{Spf}(\mathbb{I})(W)$,*

$$\mathcal{L}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}_P)) = (-1)^e \det\left(\left(\log_p(N_{\mathfrak{p}'}(\varpi(\mathfrak{P})_{\mathfrak{P}'}^{(1-c)}))\right)_{\mathfrak{P},\mathfrak{P}' \in \Sigma_p^c}\right) \prod_{\mathfrak{p}|p} \frac{e(\mathfrak{p})}{h},$$

where $\mathfrak{p} = O \cap \mathfrak{P}$ and $\mathfrak{p}' = O \cap \mathfrak{P}'$

Note here $\mathrm{ord}_p(\varpi(\mathfrak{P})_{\mathfrak{P}'}^{(1-c)}) = -h/e(\mathfrak{p})$, taking the valuation ord_p associated to $\overline{\mathfrak{P}} \in \Sigma$. By Lemma 3.2 (2) and Theorem 3.3, we see

$$\mathcal{L}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}_P)) = \mathcal{L}(\alpha_{M/F}),$$

and this is the reason for the independence of $\mathcal{L}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}_P))$ on the choice of the locally cyclotomic points P. If $F = \mathbb{Q}$, we have $\varpi(\mathfrak{P})\varpi(\mathfrak{P})^c = p^h$ and hence $\log_p(\varpi(\mathfrak{P})) = -\log_p(\varpi(\mathfrak{P})^c)$. Thus $\log_p(\varpi(\mathfrak{P})^{1-c}) = 2\log_p(\varpi(\mathfrak{P}))$, and therefore the above formula coincides with the classical analytic $\mathcal{L}$-invariant formula for $\alpha_{M/F}$ of Ferrero–Greenberg.

For a given ordinary CM type (M, Σ_p), we can choose ψ satisfying the assumptions of Theorems 3.3 and 3.6. Then through the above process, we can compute $\mathcal{L}(\alpha_{M/F})$ as follows:

Corollary 3.7. *Suppose that M/F is an ordinary CM-quadratic extension of M satisfying* (spt). *Choose a p-ordinary CM-type Σ of M. Then the $\mathcal{L}$-invariant $\mathcal{L}(\alpha_{M/F})$ of Greenberg for the quadratic Galois character $\alpha_{M/F} = \left(\frac{M/F}{\cdot}\right)$ is given by*

$$(-1)^e \det\left(\left(\log_p(N_{\mathfrak{p}'}(\varpi(\mathfrak{P})_{\mathfrak{P}'}^{(1-c)}))\right)_{\mathfrak{P},\mathfrak{P}' \in \Sigma_p^c}\right) \prod_{\mathfrak{p}|p} \frac{e(\mathfrak{p})}{h},$$

where h is the class number of M, $\mathfrak{p}' = \mathfrak{P}' \cap O$ and $\varpi(\mathfrak{P})$ is a generator of $\mathfrak{P} \in \Sigma_p^c$. If the prime p does not split in $F/\mathbb{Q}$, the $\mathcal{L}$-invariant of $\alpha_{M/F}$ does not vanish.

A regulator similar to the above determinant was introduced long ago in [FeG] (3.8) in the context of (classical) cyclotomic Iwasawa's theory.

Suppose that p does not split in $F/\mathbb{Q}$. Then for $\varpi = \varpi(\mathfrak{P})$, $N_{\mathfrak{P}}(\varpi) = \prod_{\sigma\in\Sigma} \varpi^\sigma = \varpi^\Sigma$. Then we have $|\varpi^\Sigma|_p = 1$ and $|\varpi^{c\Sigma}|_p < 1$ and hence $N_{\mathfrak{P}}(\varpi^{1-c})$ cannot be of the form ζp^α for $\alpha \in \mathbb{Q}$ and a root of unity ζ. Thus $\log_p(N_{\mathfrak{P}}(\varpi^{1-c})) \neq 0$ as claimed in the corollary. Of course, by Baker's argument (exploited by Brumer in the p-adic case), if $M/\mathbb{Q}$ is abelian, we can also confirm the nonvanishing of the determinant in the corollary.

4. Appendix: Differential and adjoint square Selmer group

Recall the **universal** nearly ordinary deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(R)$ over K with the pro-Artinian local universal K-algebra R. This means that for any Artinian local K-algebra A with maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(K1) unramified outside p;
(K2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_{\mathfrak{p}} \mod \mathfrak{m}_A$;
(K3) $\det(\rho_A) = \det \rho$;
(K4) $\rho_A \equiv \rho \mod \mathfrak{m}_A$,

there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. We write $\Phi_K(A)$ the collection of the isomorphism classes of the deformations ρ_A.

Let $\overline{\rho} = (\rho \mod \mathfrak{m}_W)$, and consider a similar deformation changing base ring from K to W. Then we have a universal couple $(\mathcal{R}, \varrho)$ as long as (ai_F) $\overline{\rho}$ is absolutely irreducible and (ds) $\overline{\rho}^{ss}$ is not scalar-values over $D_{\mathfrak{p}}$ for all $\mathfrak{p}|p$ (these assumptions we always assume). This means that for any pro-Artinian local W-algebra A with $A/\mathfrak{m}_A = W/\mathfrak{m}_W = \mathbb{F}$ for the maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(W1) unramified outside p;
(W2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_{\mathfrak{p}} \mod \mathfrak{m}_A$;
(W3) $\det(\rho_A) = \det \rho$;
(W4) $\rho_A \equiv \overline{\rho} \mod \mathfrak{m}_A$,

there exists a unique W-algebra homomorphism $\varphi : \mathcal{R} \to A$ such that $\varphi \circ \varrho \cong \rho_A$. We write $\Phi(A)$ the collection of the isomorphism classes of this finer deformations ρ_A. Thus $\Phi(A) \cong \mathrm{Hom}_{W-alg}(R, A)$.

Let $\widetilde{\rho} \in \Phi(A)$ acting on $\widetilde{L}$. Define

$$\widetilde{T} = \left\{ \phi \in \mathrm{End}_A(\widetilde{L}) \middle| \mathrm{Tr}(\phi) = 0 \right\}. \tag{4.1}$$

We let $\sigma \in \mathfrak{G}_F$ act on $v \in \widetilde{T}$ by conjugation $v \mapsto \widetilde{\rho}(\sigma) v \widetilde{\rho}(\sigma)^{-1}$. As before, $\widetilde{T}$ has the following three step filtration stable under $D_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}|p$ of F:

$$\widetilde{T} \supset \mathcal{F}_{\mathfrak{p}}^- \widetilde{T} \supset \mathcal{F}_{\mathfrak{p}}^+ \widetilde{T} \supset \{0\}. \tag{4.2}$$

Let $\mathbb{Z}_p^* = \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{Hom}(\mathbb{Z}_p.\mathbb{Q}_p/\mathbb{Z}_p)$ and $A^* = \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$. We thus have $\mathrm{Sel}_F(V/T) = \mathrm{Sel}_F(T \otimes \mathbb{Z}_p^*)$ for $V/T := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$, and we also have

$$\mathrm{Sel}_F(Ad(\widetilde{\rho})) = \mathrm{Sel}_F(\widetilde{T} \otimes_A A^*) = \mathrm{Ker}(H^1(\mathfrak{G}, \widetilde{T} \otimes_A A^*) \to \prod_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, \frac{\widetilde{T} \otimes_A A^*}{\mathcal{F}_{\mathfrak{p}}^+ \widetilde{T} \otimes_A A^*}))$$

for $\widetilde{\rho} \in \Phi(A)$ and the inertia subgroup $I_{\mathfrak{p}} \subset \mathfrak{G}$. Note that $D_{\mathfrak{p}}$ acts trivially on $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$. We often indicate this fact by writing $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \cong K$ as $D_{\mathfrak{p}}$-modules.

Proposition 4.1 (B. Mazur)**.** *Suppose that Φ has a universal couple $(\mathcal{R}_F, \varrho_F)$. Then the Pontryagin dual $\mathrm{Sel}_F^*(V/T)$ is canonically isomorphic to the module of 1-differentials $\Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F,\varphi} W$, where $\varphi : \mathcal{R}_F \to W$ is the W-algebra homomorphism such that $\rho \cong \varphi \circ \varrho_F$. More generally, for any $\widetilde{T} \in \Phi(A)$, we have*

$$\mathrm{Sel}_F^*(\widetilde{T} \otimes_A A^*) = \mathrm{Hom}(\mathrm{Sel}_F(\widetilde{T} \otimes_A A^*), \mathbb{Z}_p^*) \cong \Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F,\phi} A,$$

where $\phi : \mathcal{R}_F \to A$ is the W-algebra homomorphism such that $\widetilde{\rho} \cong \phi \circ \varrho_F$.

This proposition is from [MFG] Theorem 5.14. Here *Kähler* 1*-differentials* are supposed to be continuous with respect to the profinite topology.

Here is a sketch of a **proof** due to Mazur: Write simply $(\mathcal{R}, \varrho)$ for $(\mathcal{R}_F, \varrho_F)$. Let $\Phi = \Phi^{n.ord,\nu}$; so, $\Phi(A) \cong \mathrm{Hom}_{W\text{-alg}}(\mathcal{R}, A)$. For simplicity, we assume that X be a profinite $\mathcal{R}$-module, (in general, we take an inductive limit of such modules). Then $\mathcal{R}[X]$ is an object in CL_W. We consider the W-algebra homomorphism $\xi : \mathcal{R} \to \mathcal{R}[X]$ with $\xi \mod X = \mathrm{id}$. Then we can write $\xi(r) = r \oplus d_\xi(r)$ with $d_\xi(r) \in X$. By the above definition of the product, we get $d_\xi(rr') = rd_\xi(r') + r'd_\xi(r)$ and $d_\xi(W) = 0$. Thus d_ξ is a W-derivation, i.e., $d_\xi \in Der_W(\mathcal{R}, X)$. For any derivation $d : \mathcal{R} \to X$ over W, $r \mapsto r \oplus d(r)$ is obviously a W-algebra homomorphism, and we get

$$\begin{aligned}(4.3)\quad \left\{\widetilde{\rho} \in \Phi(\mathcal{R}[X]) \middle| \widetilde{\rho} \mod X = \varrho\right\} / \approx_X \\ \cong \left\{\widetilde{\rho} \in \Phi(\mathcal{R}[X]) \middle| \widetilde{\rho} \mod X \approx \varrho\right\} / \approx \\ \cong \left\{\xi \in \mathrm{Hom}_{W\text{-alg}}(\mathcal{R}, \mathcal{R}[X]) \middle| \xi \mod X = \mathrm{id}\right\} \\ \cong Der_W(\mathcal{R}, X) \cong \mathrm{Hom}_{\mathcal{R}}(\Omega_{\mathcal{R}/W}, X),\end{aligned}$$

where "$\approx_X$" is conjugation under $(1 \oplus M_n(X)) \cap GL_2(\mathcal{R}[X])$, and "$\approx$" is conjugation by elements in $GL_2(\mathcal{R}[X])$.

Let $\widetilde{\rho}$ be the deformation in the left-hand side of (4.3). Then we may write $\widetilde{\rho}(\sigma) = \varrho(\sigma) \oplus u'(\sigma)$ (here $u'(\sigma)$ is a "derivative" of $\widetilde{\rho}(\sigma)$). We see

$$\varrho(\sigma\tau) \oplus u'(\sigma\tau) = (\varrho(\sigma) \oplus u'(\sigma))(\varrho(\tau) \oplus u'(\tau)) = \varrho(\sigma\tau) \oplus (\varrho(\sigma)u'(\tau) + u'(\sigma)\varrho(\tau)).$$

Define $u(\sigma) = u'(\sigma)\varrho(\sigma)^{-1}$, which is a cocycle with values on $M_2(X)$ by the above formula. Since $\det \widetilde{\rho} = \det \rho = \det \varrho$, $x(\sigma) = \widetilde{\rho}(\sigma)\varrho(\sigma)^{-1}$ has values in $SL_2(\mathcal{R}[X])$, u has values in $Ad(X) = L(Ad(\varrho)) \otimes_{\mathcal{R}} X$. Hence $u : \mathfrak{G}_F^S \to Ad(X)$ is a 1-cocycle. It is a straightforward computation to see the injectivity of the map:

$$\left\{\widetilde{\rho} \in \Phi(\mathcal{R}[X]) \middle| \widetilde{\rho} \mod X \approx \varrho\right\} / \approx_X \hookrightarrow H^1(\mathfrak{G}_F^S, Ad(X))$$

given by $\widetilde{\rho} \mapsto [u]$. We put $\mathcal{F}_{\mathfrak{p}}^{\pm}(Ad(X)) = \mathcal{F}_{\mathfrak{p}}^{\pm} L(Ad(\varrho)) \otimes_{\mathcal{R}} X$. Since $\varrho|_{I_{\mathfrak{p}}}$ is upper-triangular (up to conjugation), we have $u|_{I_{\mathfrak{p}}}$ has values in $\mathcal{F}_{\mathfrak{p}}^- Ad(X)$.

If further we insist on $d_\xi(W[[\Gamma_F]]) = 0$, since $W[[\Gamma_F]]$-algebra structure is given by $\boldsymbol{\delta}_{\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1}$ which is the character of lower right corner of ϱ (restricted to $I_{\mathfrak{p}}$) this means the corresponding cocycle $u|_{I_\mathfrak{p}}$ has values in $\{\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)\}$. Since $\mathrm{Tr}(u) = 0$, we conclude $u|_{I_\mathfrak{p}} \in \mathcal{F}^+_\mathfrak{p} Ad(X)$. If $\widetilde{L}$ is finite, we may tale $X = \widetilde{L}$, and this gives the desired isomorphism, because $\widetilde{T} \otimes_A A^* = Ad(\widetilde{L})$. If $\widetilde{L}$ is not finite, then $\widetilde{L} \otimes_A A^*$ can be written as a union of $Ad(X)$ for finite X, and by taking the inductive limit, we get the assertion. □

If we replace $\mathcal{F}^+\widetilde{?}$ in the definition of the Selmer group by $\mathcal{F}^-\widetilde{?}$, we get the "minus" Selmer group $\mathrm{Sel}^-_F(?)$, and by the same argument

$$\mathrm{Sel}^-_F(\widetilde{T} \otimes_A A^*)^* \cong \Omega_{\mathcal{R}_F/W} \otimes_\mathcal{R} A.$$

We can apply the above argument to $(R, \boldsymbol{\rho})$. If $\rho \in \Phi(W)$, we have a unique $P \in \mathrm{Spf}(\mathcal{R}_F)(W)$ such that $\varrho \mod P = \rho$. Then R is canonically isomorphic to the P-adic completion-localization $\widehat{\mathcal{R}}_P$ of $\mathcal{R}$ at P and $\boldsymbol{\rho} : \mathfrak{G}_F \xrightarrow{\varrho} GL_2(\mathcal{R}) \to GL_2(\widehat{\mathcal{R}}_P) = GL_2(R)$. Thus we get

Corollary 4.2. *Assume $R \cong K[[X_\mathfrak{p}]]$. Then we have $\Omega_{R/K} \otimes_{R,\varphi_\rho} K \cong \mathrm{Sel}^-_F(V)$ which is isomorphic to $\bigoplus_{\mathfrak{p}|p} K dX_\mathfrak{p}$.*

Under the conjecture, the Selmer group $\mathrm{Sel}^-_F(V)$ is exactly $\mathbb{H} \subset H^1(\mathfrak{G}, V)$ discussed in the second lecture, and the restriction map takes $\mathbb{H} = \mathrm{Sel}^-_F(V)$ into $\mathrm{Sel}_{F_\infty}(V)$ as we have seen. Recall Greenberg's formula for a base $a_\mathfrak{p}$ of $\mathbb{H}$:

$$\mathcal{L}(Ad(\rho)) = \det\left((a_\mathfrak{p}([p, F_{\mathfrak{p}'}])_{\mathfrak{p},\mathfrak{p}'|p} \left(\log_p(\gamma_\mathfrak{p})^{-1} a_\mathfrak{p}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}\right)^{-1} \right).$$

Then by the above corollary, putting $c_\mathfrak{p} = \frac{\partial \boldsymbol{\rho}}{\partial X_\mathfrak{p}} \boldsymbol{\rho}^{-1}\Big|_{X=0}$, $\{c_\mathfrak{p}\}_{\mathfrak{p}|p}$ is a basis of $\mathbb{H}$. Then writing $c_\mathfrak{p} \sim \left(\begin{smallmatrix} -a_\mathfrak{p} & * \\ 0 & a_\mathfrak{p} \end{smallmatrix}\right)$, we can compute the above formula. Note that $a_\mathfrak{p} = \boldsymbol{\delta}_\mathfrak{p}^{-1} \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\delta}_\mathfrak{p}}\Big|_{X=0}$, and $\boldsymbol{\delta}_\mathfrak{p}([\gamma_\mathfrak{p}, F_\mathfrak{p}]) = (1 + X_\mathfrak{p})$ and $\boldsymbol{\delta}_\mathfrak{p}([p_\mathfrak{p}, F_\mathfrak{p}]) = U(p_\mathfrak{p})$. From this we get the formula we stated in the first lecture.

We add the following condition to the deformations $\widetilde{L}$ satisfying (W1–4) to make the universal ring small enough to prove $\mathrm{Sel}_F(V/T)$ is finite (and $\mathrm{Sel}_F(V) = 0$). Let Σ_p be the set of all prime factors of p in O. Fix a pair of integers $(\kappa_{1,\mathfrak{p}}, \kappa_{2,\mathfrak{p}})$ for each $\mathfrak{p} \in \Sigma_p$, and write κ for the tuple $(\kappa_{1,\mathfrak{p}}, \kappa_{2,\mathfrak{p}})_\mathfrak{p}$. We assume that $[\kappa] = \kappa_{1,\mathfrak{p}} + \kappa_{2,\mathfrak{p}}$ is independent of $\mathfrak{p} \in \Sigma_p$. As an extra condition, we now consider

(W5) On $\widetilde{T}/\mathcal{F}^+_\mathfrak{p}\widetilde{T}$, $\mathrm{Gal}(F^{ur}_\mathfrak{p}[\mu_{p^\infty}]/F^{ur}_\mathfrak{p})$ acts by the character $\mathcal{N}^{\kappa_{1,\mathfrak{p}}}$ for all $\mathfrak{p}|p$, and $\det(\widetilde{T}) = \mathcal{N}^{[\kappa]}$ on an open subgroup of $I_\mathfrak{p}$.

We write $\Phi_\kappa(A)$ for the set of isomorphism classes of deformations $\widetilde{\rho} : \mathfrak{G}_F \to GL_2(A)$ of $\overline{\rho}$ satisfying (W1–5). Under (ai_F) or (ds), we have the universal couple $(R_{\kappa,F}, \varrho_{\kappa,F})$ among the deformations satisfying (W1–5). We call $c \in \mathfrak{G}_F$ a complex conjugation, if c is in the conjugacy class of a complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Conjecture 4.3. *Suppose* (ds) *and* (ai_F) *for $\overline{\rho}$ and that F is totally real. If $\det(\rho)(c) = -1$ for any complex conjugation c, the universal ring $R_{\kappa,F}$ is free of finite rank over W, and $R_{\kappa,F}$ is a reduced local complete intersection if $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ for all $\mathfrak{p} \in \Sigma_p$.*

Here a reduced algebra A free of finite rank over $W[[x_1, \ldots, x_t]]$ is a *local complete intersection* over $R = W[[x_1, \ldots, x_t]]$ if $A \cong R[[T_1, \ldots, T_r]]/(f_1(T), \ldots, f_r(T))$ for r power series $f_i(T)$, where r is the number of variables in $R[[T_1, \ldots, T_r]]$. Though the assertion of $R_{\kappa,F}$ being a local complete intersection is technical, as we will see later, this claim is a key to relating the size of the Selmer group with the corresponding L-value. In the classical setting of Galois representations associated to elliptic modular forms of weight k (in $S_k(\Gamma_1(N))$), we have $\kappa = (0, k-1)$. Thus the condition $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ is equivalent to requiring $k \geq 2$.

Theorem 4.4 (Wiles, Taylor, Fujiwara). *Suppose that the initial representation ρ is associated to a Hilbert modular form of p-power level (in this case, we call ρ modular). If* (ai_M) *holds for $M = F[\mu_p]$,* Conjecture 4.3 *holds.*

See Fujiwara's papers [Fu] and [Fu1]. A more general version of this theorem is proven as Theorem 3.67 and Corollary 3.42 in [HMI].

Proposition 4.5. *Assume* Conjecture 4.3. *Then*

(1) $\mathrm{Sel}_F(Ad(\widetilde{\rho}) \otimes_W W^*)$ *is finite for any $\widetilde{\rho} \in \Phi_k(W)$ and $R \cong K[[X_\mathfrak{p}]]_{\mathfrak{p}\in\Sigma_p}$ if $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ for all $\mathfrak{p} \in \Sigma_p$,*
(2) *$\mathcal{R}_F$ is a reduced local complete intersection free of finite rank over $W[[\Gamma_F]]$,*
(3) $\mathrm{Sel}^*_F(Ad(\varrho_F) \otimes_{\mathcal{R}_F} \mathcal{R}^*_F)$ *is a torsion $\mathcal{R}_F$-module,*
(4) *For an irreducible component* $\mathrm{Spf}(\mathbb{I})$ *of* $\mathrm{Spf}(\mathcal{R}_F)$*, write $\rho_\mathbb{I} = \pi \circ \varrho_F$ for the projection $\pi : \mathcal{R}_F \twoheadrightarrow \mathbb{I}$. Then* $\mathrm{Sel}^*_F(Ad(\rho_\mathbb{I}) \otimes_\mathbb{I} \mathbb{I}^*)$ *is a torsion $\mathbb{I}$-module.*

Proof. If R is reduced and free of finite rank over W, $\Omega_{R/W}$ is a finite module. Thus the first assertion follows. Note that $P_\kappa = \mathrm{Ker}(\kappa : W[[\Gamma_F]] \to W)$ is generated by $((1+x_\mathfrak{p}) - \mathcal{N}(\gamma_\mathfrak{p})^{\kappa_{1,\mathfrak{p}}})$ for $\mathfrak{p} \in S$. Thus $\cap_\kappa P_\kappa = \{0\}$. Since $\mathcal{R}_F/P_\kappa\mathcal{R}_F \cong R_{\kappa,F}$ which is free of finite rank s over W, by Nakayama's lemma, $\mathcal{R}_F$ is generated by s elements $r_1, \ldots, r_s$ over $W[[\Gamma_F]]$ which give a basis of $R_{\kappa,F}$ over W. Thus we have a surjective $W[[\Gamma_F]]$-linear map $\iota : W[[\Gamma_F]]^s \to \mathcal{R}_F$ sending $(a_1, \ldots, a_s)$ to $\sum_j a_j r_j$. Taking another κ', we find that $\mathcal{R}_F/P_{\kappa'}\mathcal{R}_F \cong R_{\kappa',F}$ which is free over W; so, it has to be free of rank s over W. Thus $\mathrm{Ker}(\iota) \subset P^s_{\kappa'}$ for all κ'; so, ι has to be an isomorphism. This shows the freeness in the second assertion.

Let C be the set of all $\kappa = (\kappa_\mathfrak{p})_\mathfrak{p}$ such that $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ for all $\mathfrak{p}$. Then we still have $\bigcap_{\kappa\in C} P_\kappa = \{0\}$. Thus the natural W-algebra homomorphism $\mathcal{R}_F \to \prod_{\kappa\in C} R_{\kappa,F}$ is an injection. The right-hand side is reduced (i.e., no nilpotent radical), and $\mathcal{R}_F$ is reduced.

We write

$$R_\kappa = \mathcal{R}/P_\kappa = \frac{W[[T_1, \ldots, T_r]]}{(\overline{f}_1(T), \ldots, \overline{f}_r(T))}.$$

Write $\overline{t}_j \in \mathfrak{m}_{R_\kappa}$ for the image of T_j in R_κ. Take a lift t_j in $\mathfrak{m}_\mathcal{R}$ of $\overline{t}_j$ so that $\overline{t}_j = (t_j \mod P_\kappa\mathcal{R})$. Define $\varphi : W[[\Gamma_F]][[T_1, \ldots, T_r]] \twoheadrightarrow \mathcal{R}_F$ by $\varphi(f(T_1, \ldots, T_r)) = f(t_1, \ldots, t_r)$. Since $\mathcal{R}_F$ is $W[[\Gamma_F]]$-free, $\mathrm{Ker}(\varphi) \otimes_{W[[\Gamma_F]],\kappa} W = (\overline{f}_1, \ldots, \overline{f}_r)$; so, taking a lift $f_j \in \mathrm{Ker}(\varphi)$ of $\overline{f}_j$, we find $\mathrm{Ker}(\varphi) = (f_1, \ldots, f_r)$ by Nakayama's lemma, and hence $\mathcal{R}_F$ is a local complete intersection over $W[[\Gamma_F]]$.

Since $\mathcal{R}_F$ is reduced and finite over $W[[\Gamma_F]]$, $\Omega_{\mathcal{R}_F/W[[\Gamma_F]]}$ is a torsion $\mathcal{R}_F$-module. From this, the last two assertions follow. Since $R_{\kappa,F} = \mathcal{R}_F/P_\kappa\mathcal{R}_F$ is reduced, $\mathrm{Spf}(\mathcal{R}_F)$ is étale over $\mathrm{Spf}(W[[\Gamma_F]])$ around $\rho = P$; so, $R = \widehat{\mathcal{R}}_P \cong K[[X_\mathfrak{p}]]$. This finishes the proof. $\square$

Since $\mathcal{R}_F$ is reduced and free of finite rank over $W[[\Gamma_F]]$, its total quotient ring Q is a product of fields of finite dimension over the field $\mathcal{K}$ of fractions of $W[[\Gamma_F]]$. For simplicity, we assume that $\mathbb{I} = W[[\Gamma_F]]$. In particular, writing $\mathbb{K}$ for the field of fractions of $\mathbb{I}$, we have $Q = \mathbb{K} \oplus X$ for a complementary ring direct summand

X. Let $\mathbb{I}'$ be the projection of $\mathcal{R}_F$ to X. Then $\mathrm{Spf}(\mathcal{R}_F) = \mathrm{Spf}(\mathbb{I}) \cup \mathrm{Spf}(\mathbb{I}')$ (and $\mathrm{Spf}(\mathbb{I}')$ is the union of irreducible components other than $\mathrm{Spf}(\mathbb{I})$). We take the intersection $\mathrm{Spf}(C_0) = \mathrm{Spf}(\mathbb{I}) \cap \mathrm{Spf}(\mathbb{I}')$; so, $C_0 = \mathbb{I} \otimes_{\mathcal{R}_F} \mathbb{I}'$, which is a torsion $\mathbb{I}$-module called the *congruence module* of $\mathbb{I}$ (or of $\mathrm{Spf}(\mathbb{I})$). It is easy to see that $\mathbb{I} \otimes_{\mathcal{R}_F} \mathbb{I}' \cong \mathbb{I}/\left((\mathbb{K} \oplus 0) \cap \mathcal{R}_F\right)$ (cf. [H88] 6.3). By the above expression, the $W[[\Gamma_F]]$-freeness tells us that $\mathrm{char}_{\mathbb{I}}(C_0) = (\mathbb{K} \oplus 0) \cap \mathcal{R}_F$ is an intersection of a power of prime divisors (cf. [BCM] 7.4.2). Since $\mathbb{I} = W[[\Gamma_F]]$ is regular, and hence $\mathrm{char}(C_0)$ is a principal ideal generated by $h \in \mathbb{I}$. For this conclusion, we do not need the isomorphism $\mathbb{I} \cong W[[\Gamma_F]] = W[[x_{\mathfrak{p}}]]_{\mathfrak{p}}$ but a milder condition that $\mathbb{I}$ is a Gorenstein ring over $W[[\Gamma_F]]$ is enough (that is, $\mathrm{Hom}_{W[[\Gamma_F]]}(\mathbb{I}, W[[\Gamma_F]]) \cong \mathbb{I}$ as $\mathbb{I}$-modules; see [H88] Theorem 6.8). Note that a local complete intersection over $W[[\Gamma_F]]$ is a Gorenstein ring (e.g., [CRT] Theorem 21.3). Now by a theorem of Tate (e.g., [MFG] 5.3.4),

$$\mathrm{char}(\Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F} \mathbb{I}) = \mathrm{char}(C_0) = (h).$$

We have for any prime ideal $P \in \mathrm{Spf}(\mathbb{I})$ with $\iota : \mathbb{I}/P \cong W$, writing $\rho_P = \iota \circ \rho_{\mathbb{I}} : \mathfrak{G}_F \to GL_2(W)$

$$\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*) \cong \Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F, P} W \cong \mathrm{Sel}^*_F(Ad(\rho_{\mathbb{I}}) \otimes_{\mathbb{I}} \mathbb{I}^*) \otimes_{\mathbb{I}} \mathbb{I}/P.$$

This shows that if $\mathrm{char}(\mathrm{Sel}^*_F(Ad(\rho_{\mathbb{I}}) \otimes_{\mathbb{I}} \mathbb{I}^*)) = (h)$ for $h \in \mathbb{I}$, we have

$$\mathrm{char}(\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*)) = (h(P)),$$

where $h(P) = (h \mod P) \in W$. Thus we get

Corollary 4.6. *We have* $|\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*)| = |h(P)|_p^{-[K:\mathbb{Q}_p]}$ *for all* $P \in \mathrm{Spf}(\mathbb{I})(W)$.

In this corollary, we do not preclude the case where $\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*)$ is infinite. In such an extreme case, simply $h(P) = 0$ and, hence, $|h(P)|_p^{-[K:\mathbb{Q}_p]} = \infty$.

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