3. Lecture 3: Non-vanishing modulo p of L–values

We construct an $\mathbb{F}$–valued measure ($\mathbb{F}=\overline{\mathbb{F}}_p$) over the anti-cyclotomic class group $Cl_\infty = \varprojlim_n Cl_n$ modulo $\mathfrak{l}^\infty$ whose integral against a character χ is the Hecke L–value $L(0,\chi^{-1}\lambda)$ (up to a period). The idea is to use Hecke relation of the Eisenstein series translated into a distribution relation on the profinite group Cl_∞ and density of CM points.

3.1. **Arithmetic Hecke characters.** Let M be an imaginary quadratic field. Each integral linear combination $k\cdot i_\infty + j\cdot c \in \mathbb{Z}[\mathrm{Gal}(M/\mathbb{Q})]$ is regarded as a character of T by $x \mapsto i_\infty(x)^k c(x)^j$. We fix an arithmetic Hecke character λ of infinity type $(k\cdot i_\infty + \kappa(i_\infty - c))$ for integers k and κ. Originally λ is defined as an ideal character with $\lambda(\alpha) = \alpha^{-k-\kappa(1-c)}$ if $\alpha \equiv 1 \mod \mathfrak{C}$ for its conductor ideal $\mathfrak{C}$.

For each prime $\mathfrak{L}$ of M, we choose its local generator $\varpi_{\mathfrak{L}} \in M$ so that $\mathfrak{L}\mathfrak{O}_{(\mathfrak{L})} = (\varpi_{\mathfrak{L}})$, where

$$\mathfrak{O}_{(\mathfrak{L})} = \left\{ \frac{b}{a} \middle| a, b \in \mathfrak{O},\ a\mathfrak{O} + \mathfrak{L} = \mathfrak{O} \right\} \quad \text{(the localization at } \mathfrak{L}).$$

We write $\mathfrak{O}_{\mathfrak{L}}$ for the completion $\mathfrak{O}_{\mathfrak{L}} = \varprojlim_n \mathfrak{O}/\mathfrak{L}^n$ and $M_{\mathfrak{L}} = \mathfrak{O}_{\mathfrak{L}} \otimes_{\mathfrak{O}} M$. Recall the adele ring of M as an M-subalgebra in the product $\prod_{\mathfrak{L}} M_{\mathfrak{L}}$ made up of $x^{(\infty)} = (x_{\mathfrak{L}})_{\mathfrak{L}}$ with $x_{\mathfrak{L}} \in \mathfrak{O}_{\mathfrak{L}}$ except for finitely many prime ideals $\mathfrak{L}$, and $M_{\mathbb{A}} = M_{\mathbb{A}}^{(\infty)} \times \mathbb{C}$. Embedding $M \subset M_{\mathfrak{L}}$ naturally and in $\mathbb{C}$ by i_∞, we regard $M_{\mathbb{A}}$ as an M-algebra by the diagonal embedding

$$M \ni \xi \mapsto (\xi, \dots, \overset{\mathfrak{L}}{\xi}, \dots, \xi) \in M_{\mathbb{A}}.$$

The infinity component of $x \in M_{\mathbb{A}}$ is written as x_∞. Put

$$M_{\mathbb{A}}^{(\mathfrak{C}\infty)} = \{x \in M_{\mathbb{A}}^{(\infty)} | x_{\mathfrak{L}} = 1 \ \text{ if } \mathfrak{L} \supset \mathfrak{C}\},$$

and $U(\mathfrak{C}) = \{x \in \widehat{\mathfrak{O}}^\times | x \equiv 1 \mod \widehat{\mathfrak{C}}\}$ for an ideal $0 \neq \mathfrak{C} \subset \mathfrak{O}$. Write $U(\mathfrak{C})^{(\mathfrak{C})} = U(\mathfrak{C}) \cap (M_{\mathbb{A}}^{(\mathfrak{C}\infty)})^\times$ and $U(\mathfrak{C}) = U(\mathfrak{C})_{\mathfrak{C}} \times U(\mathfrak{C})^{(\mathfrak{C})}$.

Exercise 3.1. *Let $\mathcal{I}(\mathfrak{C})$ be group of fractional ideals of M prime to $\mathfrak{C}$. Prove an isomorphism $\mathcal{I}(\mathfrak{C}) \cong (M_{\mathbb{A}}^{(\mathfrak{C}\infty)})^\times / U(\mathfrak{C})^{(\mathfrak{C})}$ of groups sending each prime ideal $\mathfrak{L}$ to the element in $(M_{\mathbb{A}}^{(\mathfrak{C}\infty)})^\times$, written still $\varpi_{\mathfrak{L}}$, whose $\mathfrak{L}$-component is equal to $\varpi_{\mathfrak{L}}$ and all other components are* 1.

By the above exercise, identifying $(M_{\mathbb{A}}^{(\mathfrak{C}\infty)})^\times / U(\mathfrak{C})^{(\mathfrak{C})}$ with $\mathcal{I}(\mathfrak{C})$, we may regard λ as a character of $(M_{\mathbb{A}}^{(\mathfrak{C}\infty)})^\times / U(\mathfrak{C})^{(\mathfrak{C})}$. We can then extend λ to a character of $M^\times \backslash M_{\mathbb{A}}^\times$ in the following way for a place v.

Definition 3.2. *For a plave v, put $\lambda(\alpha x u) = u_v^{k\cdot i_v + \kappa(i_v - c i_v)} \lambda(x) \in \mathbb{C}_v^\times$ for $\alpha \in M^\times$, $u \in U(\mathfrak{C}) M_\infty^\times$ and $x \in (M_{\mathbb{A}}^{(v\mathfrak{C})})^\times$, where $\mathbb{C}_v = \mathbb{C}$ or $\overline{\mathbb{Q}}_p$*

according as $v = \infty$ or a prime p. If $v = p$, we write the character obtained as $\widehat{\lambda} : M^\times \backslash M_{\mathbb{A}}^\times \to \overline{\mathbb{Q}}_p^\times$ and if $v = \infty$, we simply write it as λ.

Because of this definition, $k \cdot i_v + \kappa(i_v - c)$ appears as the exponent at v; so, it is called ∞-type of λ and p-type of $\widehat{\lambda}$. The p-adic character $\widehat{\lambda}$ is called the *p-adic avatar* of λ (and its construction was originally due to Weil).

Exercise 3.3. *Let $Cl(\mathfrak{C}p^\infty) = \varprojlim_n Cl(\mathfrak{C}p^n)$ as profinite compact group and $U(\mathfrak{C}p^\infty) = \bigcap_n U(\mathfrak{C}p^n)$. Then prove that (i) $M^\times \backslash M_{\mathbb{A}^{(\infty)}}^\times / U(\mathfrak{C}p^n) \cong Cl(\mathfrak{C}p^n)$ as compact group if $n \leq \infty$ and (ii) $\widehat{\lambda}$ factors through $Cl(\mathfrak{C}p^\infty)$.*

Regarding λ as an idele character of $M_{\mathbb{A}}^\times$, we assume $\lambda(x_\infty) = x_\infty^{ki_\infty + \kappa(i_\infty - ci_\infty)}$ for $x_\infty \in T(\mathbb{R})$. As before fixing two odd primes $p \neq l$, we assume the following three conditions for simplicity (more general cases are treated in [H07]):

(ct) *$k \geq 2$ and $\kappa \geq 0$ ($\Rightarrow$ criticality at $s = 0$ for $L(s, \lambda)$).*

(ol) *The conductor $\mathfrak{C}$ of λ is trivial; i.e., $\mathfrak{C} = 1$ and $p \geq 5$.*

3.2. **Degeneration operators.** Let $\mathfrak{l} = (l)$ $(l > 0)$ be a prime ideal in $\mathbb{Z}$. Consider the covariant classification functors defined over the category of $\mathbb{Z}[\frac{1}{6l}]$-algebras:

$$\mathcal{P}_{\Gamma_0(\mathfrak{l})}(A) = \left[(E, C, \omega)_{/A}\right] \quad \text{and} \quad \mathcal{P}(A) = \left[(E, \omega)_{/A}\right],$$

where $[\cdot] = \{\cdot\}/\cong$ and C is a cyclic subgroup in E of order $\mathfrak{l}$. Since $\mathfrak{l}A = A$, we may consider the following morphism of functors $[\mathfrak{l}] : \mathcal{P}_{\Gamma_0(\mathfrak{l})}(A) \to \mathcal{P}(A)$ sending $(E, C, \omega)_{/A}$ to $(E/C, (\pi^*)^{-1}\omega)_{/A}$ for the projection $\pi : E \to E/C$. Plainly $[\mathfrak{l}]$ is a morphism of functors; so, by pull back, we get the degeneration morphism $[\mathfrak{l}] : G_k(1; A) \to G_k(\Gamma_0(\mathfrak{l}); A)$ given by $f|[\mathfrak{l}](E, C, \omega) = f(E/C, (\pi^*)^{-1}\omega)$. Adding level p^∞-structure $\phi_p : \mu_{p^\infty} \hookrightarrow E$, we get the corresponding map $[\mathfrak{l}] : V(1; B) \to V_{\Gamma_0(\mathfrak{l})/B}$.

Exercise 3.4. *Prove $f|[\mathfrak{l}](q) = f(\mathrm{Tate}(q^l), \frac{1}{l}\omega_{can}) = l^k \cdot f(q^l)$.*

3.3. **Hecke operators.** We define an operator $T(\mathfrak{l}) : G_k(1; B) \to G_k(1; B)$ for a prime $\mathfrak{l} = (l)$ with $l > 0$ invertible in B by

$$f|T(\mathfrak{l})(E, \omega) = \frac{1}{l} \sum_C (E/C, (\pi^*)^{-1}\omega),$$

where C runs over all cyclic subgroup of order l. Similarly we define $U(\mathfrak{l}) : G_k(\Gamma_0(\mathfrak{l}); B) \to G_k(\Gamma_0(\mathfrak{l}); B)$ by

$$f|U(\mathfrak{l})(E, C', \omega) = \frac{1}{l} \sum_C (E/C, \pi(C') = (C + C')/C', (\pi^*)^{-1}\omega),$$

where C runs over all cyclic subgroup of order l different from C'.

Exercise 3.5. *Write q-expansion of modular forms f at the infinity, i.e., at $(\mathrm{Tate}(q), C_{can}, \omega_{can})$ as $f(q) = \sum_n a(n,f)q^n$. Then prove*

$$a(n, f|T(\mathfrak{l})) = a(nl, f) + l^{k-1}a(\frac{n}{l}, f) \quad \text{and} \quad a(n, f|U(\mathfrak{l})) = a(nl, f).$$

3.4. **Eisenstein series.** We are going to define an optimal Eisenstein series whose special values at CM points interpolate the values $L(0, \lambda\chi)$ for anticyclotomic characters χ of finite order.

For any even positive integer $k > 0$, we can now define the Eisenstein series E_k. We define the value $E_k(L)$ for $L \in Lat = \{\mathbb{Z}w_1 + \mathbb{Z}w_2 \subset \mathbb{C} | \operatorname{Im}(w_1/w_2) > 0\}$ by

$$E_k(\underline{L}) = (-1)^k\Gamma(k+s) \sum_{w \in L/\mathbb{Z}^\times}{}' \frac{1}{w^k|w|^{2s}}\Big|_{s=0}. \tag{3.1}$$

Here "$\sum'$" indicates that we are excluding $w = 0$ from the summation. This type of series is convergent when the real part of s is sufficiently large and can be continued to a meromorphic function well defined at $s = 0$ (as long as either $k \geq 4$; see [LFE] §2.5 for analytic continuation).

Lemma 3.6. *If $4 \leq k \in 2\mathbb{Z}$, the function E_k gives an element in $G_k(1;\mathbb{C})$, whose q–expansion at the cusp ∞ is given by*

$$E_k(q) = 2^{-1}\zeta(1-k) + \sum_{0<n\in\mathbb{Z}} \sum_{\substack{(a,b)\in(\mathbb{Z}\times\mathbb{Z})/\mathbb{Z}^\times \\ ab=n}} \frac{a}{|a|}a^{k-1}q^n. \tag{3.2}$$

When $k = 2$, $E_k(z)$ is non-holomorphic and its Fourier expansion contains an extra term $\frac{c}{2i\operatorname{Im}(z)}$ for a constant c in addition to the above holomorphic q-expansion.

From the effect of $T(\mathfrak{l})$ and $U(\mathfrak{l})$ on q-expansion, we verify easily the following lemma.

Lemma 3.7. *For a prime $\mathfrak{l} = (l)$, we have*

$$E_k|T(\mathfrak{l}) = (1 + l^{k-1})E_k, \tag{3.3}$$

On the elliptic curve side, $(\mathfrak{l})(E,\omega) = (E \otimes_{\mathbb{Z}} \mathfrak{l}, \omega')$, where as an fppf abelian sheaf, $E \otimes_{\mathbb{Z}} \mathfrak{l}$ is the sheafication of $A \mapsto E \otimes_{\mathbb{Z}} \mathfrak{l}(R) = E(R) \otimes_{\mathbb{Z}} \mathfrak{l}$. Tensoring E with the exact sequence $0 \to \mathfrak{l} \to \mathbb{Z} \to \mathbb{Z}/\mathfrak{l} \to 0$ (of constant abelian fppf sheaves), since E is a divisible fppf abelian sheaf, $E \otimes_{\mathbb{Z}} \mathbb{Z}/\mathfrak{l} = 0$, and we get

$$0 \to \mathrm{Tor}_1(E, \mathbb{Z}/\mathfrak{l}) \to E \otimes_{\mathbb{Z}} \mathfrak{l} \to E \to 0$$

is exact. Since $\mathbb{Z}$ is a principal ideal domain, $\mathrm{Tor}_1(E, \mathbb{Z}/\mathfrak{l}) \cong E[\mathfrak{l}] = E[l]$; so, we have a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Tor}_1(E, \mathbb{Z}/\mathfrak{l}) & \xrightarrow{\hookrightarrow} & E \otimes_{\mathbb{Z}} \mathfrak{l} & \xrightarrow{\twoheadrightarrow} & E \\ \wr\downarrow & & \wr\downarrow & & \Vert\downarrow \\ E[l] & \xrightarrow[\hookrightarrow]{} & E & \xrightarrow[l]{} & E. \end{array}$$

Multiplication by l induces $E \cong E \otimes_{\mathbb{Z}} \mathfrak{l}$, which acts on ω by $\omega \mapsto l\omega =: \omega'$. On the lattice side, $\mathbb{C}/L \otimes_{\mathbb{Z}} \mathfrak{l} = \mathbb{C}/\mathfrak{l}L$; so, it is given by multiplication $L \mapsto lL$. For modular form f, we define $f|(\mathfrak{l})(E, \omega) = f((\mathfrak{l})(E, \omega))$.

Exercise 3.8. *Let $\mathfrak{l}$ be a prime outside $\mathfrak{f}$. Suppose that $\mathfrak{l} = (l)$ for a positive $l \in \mathbb{Q}$. Let $\mathbb{E}'_k = E_k - l^{-1} E_k|[\mathfrak{l}]$ and $\mathbb{E}_k = E_k - E_k|(\mathfrak{l})|[\mathfrak{l}]$. Then prove*

(1) $\mathbb{E}'_k|U(\mathfrak{l}) = \mathbb{E}'_k$,
(2) $\mathbb{E}_k|U(\mathfrak{l}) = l^{k-1}\mathbb{E}_k$,
(3) *Even if E_2 is non-holomorphic, $\mathbb{E}_2$ is holomorphic.*

Remark 3.9. By $a(n, df) = n \cdot a(n, f)$, the Hecke operator $T(\mathfrak{l})$ and $U(\mathfrak{l})$ satisfies $T(\mathfrak{l}) \circ d = l \cdot d \circ T(\mathfrak{l})$ and $T(\mathfrak{l}) \circ d = l \cdot d \circ T(\mathfrak{l})$ for the Katz differential operator d. Thus for $\mathbb{E}(\lambda) = d^\kappa \mathbb{E}_k$ and $\mathbb{E}'(\lambda) = d^\kappa \mathbb{E}'_k$, we have under the notation of Lemma 3.8

$$\mathbb{E}'(\lambda)|U(\mathfrak{l}) = l^\kappa \mathbb{E}'(\lambda), \quad \mathbb{E}(\lambda)|U(\mathfrak{l}) = l^{k-1+\kappa}\mathbb{E}(\lambda). \tag{3.4}$$

3.5. **Anticyclotomic Hecke L–functions.** Pick a prime $\mathfrak{l}$ of $\mathfrak{O}$. Define the order $\mathfrak{O}_n = \mathbb{Z} + \mathfrak{l}^n\mathfrak{O}$ of conductor $\mathfrak{l}^n$. We determine the type of Hecke L–function obtained by values of Eisenstein series at CM points. The result (equivalent to the one presented here) is explained well in H. Yoshida [LAP] V.3.2.

Exercise 3.10. *Prove the identity:*

$$\{\text{non-proper } \mathfrak{O}_{n+1}\text{–ideals}\} = \{\mathfrak{l}\mathfrak{a} | \mathfrak{a} \text{ is an } \mathfrak{O}_n\text{–ideal}\}.$$

We admit

Proposition 3.11. *Let I_n be the group of all proper fractional $\mathfrak{O}_n$–ideals. Associating to each $\mathfrak{O}_{n+1}$–ideal $\mathfrak{a}$ the $\mathfrak{O}_n$–ideal $\mathfrak{O}_n\mathfrak{a}$, we get the following homomorphism of groups $\pi_n : I_{n+1} \to I_n$. The homomorphism π is surjective, and the kernel of π is isomorphic to $\mathfrak{O}_{n,\mathfrak{l}}^\times/\mathfrak{O}_{n+1,\mathfrak{l}}^\times$. We have the following exact sequence:*

$$1 \to \mathfrak{O}_{n,\mathfrak{l}}^\times/\mathfrak{O}_{n+1,\mathfrak{l}}^\times\mathfrak{O}_n^\times \to Cl_{n+1} \to Cl_n \to 1.$$

Let χ be a character of the group of fractional proper ideals of $\mathfrak{O}_n$. By the above proposition, χ gives rise to a unique character of the full group of fractional ideals of M. Put $N(\mathfrak{a}) = [\mathfrak{O}_n : \mathfrak{a}] = [\mathfrak{O} : \mathfrak{O}\mathfrak{a}]$. We then define a formal L–function:

$$L^n(s,\chi) = \sum_{\mathfrak{a}\subset\mathfrak{O}_n} \chi(\mathfrak{a})N(\mathfrak{a})^{-s}, \tag{3.5}$$

where $\mathfrak{a}$ runs over all proper $\mathfrak{O}_n$–ideals. We write $L(s,\chi)$ for $L^0(s,\chi)$, which is the classical Hecke L–function. This L–function depends on n, because the set of proper $\mathfrak{O}_n$–ideals depends on n. However L^0 and L^n are only different at Euler $\mathfrak{l}$-factor.

Since Cl_∞ is almost pro-l group, all finite order characters of Cl_∞ has values in $\mathcal{W}[\mu_{p^m}]$ if every element of the finite p-Sylow subgroup of Cl_∞ is killed by p^m. Replacing W be $W[\mu_{p^m}] = W(\overline{\mathbb{F}}_p)[\mu_{p^m}]$, we write hereafter $\mathcal{W}$ for $i_p^{-1}(W(\overline{\mathbb{F}}_p)[\mu_{p^m}])$. We will prove, assuming $\mathfrak{C} = 1$, the following theorem at the very end of this chapter after long preparation (a proof in the general case where $\mathfrak{C} \neq 1$ can be found in [H07]):

Theorem 3.12. *Let p be an odd prime splitting in $M/\mathbb{Q}$. Let λ be a Hecke character of M of conductor 1 and of infinity type $k+\kappa(1-c)$ with $k \geq 2$. Suppose* (ct) *and* (ol) *in* §3.1. *Then $\frac{\pi^\kappa\Gamma_\Sigma(k+\kappa)L^{(\mathfrak{l})}(0,\chi^{-1}\lambda)}{\Omega_\infty^{k+2\kappa}} \in \mathcal{W}$ for all finite order characters $\chi : Cl_\infty \to \mathcal{W}^\times$ with nontrivial conductor. Moreover, except for finitely many characters χ in Cl_∞, we have*

$$\frac{\pi^\kappa\Gamma(k+\kappa)L^{(\mathfrak{l})}(0,\nu^{-1}\chi^{-1}\lambda)}{\Omega_\infty^{k+2\kappa}} \not\equiv 0 \mod \mathfrak{m}_{\mathcal{W}}.$$

3.6. **Values at CM points.** We take a proper $\mathfrak{O}_{n+1}$–ideal $\mathfrak{a}$ for $n \geq 0$, and regard it as a lattice in $\mathbb{C}$ by $a \mapsto i_\infty(a)$. We suppose that $\mathfrak{a}_p = \mathfrak{O}_p = \mathfrak{O}_\mathfrak{p} \oplus \mathfrak{O}_{\overline{\mathfrak{p}}}$. This implies $H^0(E(\mathfrak{O}),\Omega_{E(\mathfrak{O})/\mathcal{W}}) = H^0(E(\mathfrak{a}),\Omega_{E(\mathfrak{a})/\mathcal{W}})$ as $E(\mathfrak{a})$ and $E(\mathfrak{O})$ are isogenous by an isogeny of degree prime to p. Then a generator $\omega(\mathfrak{O})$ of $H^0(E(\mathfrak{O}),\Omega_{E(\mathfrak{O})/\mathcal{W}})$ gives us $\omega(\mathfrak{a})$ which generates $H^0(E(\mathfrak{a}),\Omega_{E(\mathfrak{a})/\mathcal{W}})$. We have fixed $\phi_p(\mathfrak{O}) : \mu_{p^\infty} \cong E(\mathfrak{O})[\mathfrak{p}^\infty]$, and this identification $\mathfrak{a}_p = \mathfrak{O}_p = \mathfrak{O}_\mathfrak{p} \oplus \mathfrak{O}_{\overline{\mathfrak{p}}}$ induces $\phi_p(\mathfrak{a}) : \mu_{p^\infty} \cong E(\mathfrak{a})[\mathfrak{p}^\infty]$. Then $\mathfrak{a}_\mathfrak{l} \cong \mathfrak{O}_{n+1,\mathfrak{l}} = \mathbb{Z}_l + \mathfrak{l}^{n+1}\mathfrak{O}_\mathfrak{l}$, and hence $\mathfrak{a}\mathfrak{O}_n \supset \mathfrak{a}$. The subgroup $C(\mathfrak{a}) = \mathfrak{a}\mathfrak{O}_n/\mathfrak{a}$ in $E(\mathfrak{a})(\mathbb{C}) = \mathbb{C}/i_\infty(\mathfrak{a})$ gives a canonical cyclic subgroup $C(\mathfrak{a}) \subset E(\mathfrak{a})$ of order l (defined over $\mathcal{W}$). Write $\omega_\infty(\mathfrak{a}) = du$ for the variable $u \in \mathbb{C}$. For a p–adic modular form f of the form $d^\kappa g$ for classical $g \in G_k(\Gamma_0(\mathfrak{l});\mathcal{W})$, we have by Theorem 2.12

$$\frac{d^\kappa f(x(\mathfrak{a}),\phi_p(\mathfrak{a}))}{\Omega_p^{k+2\kappa}} = \delta_k^\kappa f(x(\mathfrak{a}),\omega(\mathfrak{a})) = \frac{\delta_k^\kappa f(x(\mathfrak{a}),\omega_\infty(\mathfrak{a}))}{\Omega_\infty^{k+2\kappa}}.$$

Here $x(\mathfrak{a})$ is the test object: $x(\mathfrak{a}) = (E(\mathfrak{a}),C(\mathfrak{a}))_{/\mathcal{W}} \in X_0(\mathfrak{l})(\mathcal{W})$.

We write $c_0 = (-1)^k \frac{\pi^\kappa \Gamma(k+\kappa)}{\mathrm{Im}(\delta)^\kappa \sqrt{D}\Omega_\infty^{k+2\kappa}}$ with $2\delta = \sqrt{-D}$. Here $\Gamma(s)$ is the Euler's Gamma function. By definition, we find, for $e = [\mathfrak{O}_{n+1}^\times : \mathbb{Z}^\times]$ (which is equal to 1 if $n > 0$),

$$
\begin{aligned}
&(c_0 e)^{-1}\delta_k^\kappa E_k(x(\mathfrak{a}), \omega(\mathfrak{a}))\\
&= \sum_{w \in \mathfrak{a}/\mathfrak{O}_{n+1}^\times}{}' \frac{1}{w^{k+\kappa(1-c)} N_{M/\mathbb{Q}}(w)^s}\Big|_{s=0}\\
&= \sum_{w \in \mathfrak{a}/\mathfrak{O}_{n+1}^\times}{}' \frac{\lambda(w^{(\infty)})^{-1}}{N_{M/\mathbb{Q}}(w)^s}\Big|_{s=0}\\
&= \sum_{w \in \mathfrak{a}/\mathfrak{O}_{n+1}^\times}{}' \frac{\lambda(w)}{N_{M/\mathbb{Q}}(w)^s}\Big|_{s=0}\\
&= \lambda(\mathfrak{a}) N_{M/\mathbb{Q}}(\mathfrak{a}^{-1})^s \sum_{w\mathfrak{a}^{-1} \subset \mathfrak{O}_{n+1}}{}' \frac{\lambda(w\mathfrak{a}^{-1})}{N_{M/\mathbb{Q}}(w\mathfrak{a}^{-1})^s}\Big|_{s=0}\\
&= \lambda(\mathfrak{a}) L_{[\mathfrak{a}^{-1}]}^{n+1}(0, \lambda),
\end{aligned}
\tag{3.6}
$$

where for the ideal class $[\mathfrak{a}^{-1}] \in Cl_{n+1}$ represented by a proper $\mathfrak{O}_{n+1}$–ideal $\mathfrak{a}^{-1}$,

$$L_{[\mathfrak{a}^{-1}]}^{n+1}(s, \lambda) = \sum_{\mathfrak{b} \in [\mathfrak{a}^{-1}]} \lambda(\mathfrak{b}) N_{M/\mathbb{Q}}(\mathfrak{b})^{-s}$$

is the partial L–function of the class $[\mathfrak{a}^{-1}]$ for $\mathfrak{b}$ running over all $\mathfrak{O}_{n+1}$–proper integral ideals prime to $\mathfrak{C}$ in the class $[\mathfrak{a}^{-1}]$. In the second line of (3.6), we regard λ as an idele character and in the other lines as an ideal character. For an idele a with $a\widehat{\mathfrak{O}} = \mathfrak{a}\widehat{\mathfrak{O}}$, we have $\lambda(a^{(\infty)}) = \lambda(\mathfrak{a})$ and $\widehat{\lambda}(a^{(p)}) = \widehat{\lambda}(\mathfrak{a})$.

We put $\mathbb{E}(\lambda) = d^\kappa \mathbb{E}_k$ and $\mathbb{E}'(\lambda) = d^\kappa \mathbb{E}'_k$ as in Remark 3.9. We want to evaluate $\mathbb{E}(\lambda)$ and $\mathbb{E}'(\lambda)$ at $x = (x(\mathfrak{a}), \omega(\mathfrak{a}))$. Thus we write, for example, $\mathbb{E}(\lambda)$ and $\mathbb{E}'(\lambda)$ for $\mathbb{E}(\lambda)$ and $\mathbb{E}'(\lambda)$. Then by definition and Theorem 2.12, we have for $x = (x(\mathfrak{a}), \omega(\mathfrak{a}))$

$$
\begin{aligned}
\mathbb{E}'(\lambda)(x) &= \delta_k^\kappa E_k(x) - l^{-1}\delta_k^\kappa E_k(x(\mathfrak{a}\mathfrak{O}_n), \omega(\mathfrak{a}\mathfrak{O}_n))\\
\mathbb{E}(\lambda)(x) &= \delta_k^\kappa E_k(x) - \delta_k^\kappa E_k(x(\mathfrak{a}\mathfrak{O}_n), \omega(\mathfrak{a}\mathfrak{O}_n)).
\end{aligned}
\tag{3.7}
$$

because $C(\mathfrak{a}) = \mathfrak{a}\mathfrak{O}_n/\mathfrak{a}$ and hence $[\mathfrak{l}](x(\mathfrak{a})) = x(\mathfrak{a}\mathfrak{O}_n)$.

To simplify the notation, write $\phi([\mathfrak{a}]) = \widehat{\lambda}(\mathfrak{a})^{-1}\phi(x(\mathfrak{a}), \omega(\mathfrak{a}))$. By Exercise 2.4, for $\phi = \mathbb{E}(\lambda)$ and $\mathbb{E}'(\lambda)$, the value $\phi([\mathfrak{a}])$ only depends on the ideal class $[\mathfrak{a}]$ but not the individual $\mathfrak{a}$. The formula (3.7) combined with (3.6) shows, for a proper $\mathfrak{O}_{n+1}$–ideal $\mathfrak{a}$,

$$
\begin{aligned}
e^{-1}\mathbb{E}'(\lambda)([\mathfrak{a}]) &= c_0\left(L^{n+1}_{[\mathfrak{a}^{-1}]}(0,\lambda) - l^{-1}L^{n}_{[\mathfrak{a}^{-1}\mathfrak{O}_n]}(0,\lambda)\right)\\
e^{-1}\mathbb{E}(\lambda)([\mathfrak{a}]) &= c_0\left(L^{n+1}_{[\mathfrak{a}^{-1}]}(0,\lambda) - \lambda(\mathfrak{l})L^{n}_{[\mathfrak{l}^{-1}\mathfrak{a}^{-1}\mathfrak{O}_n]}(0,\lambda)\right)
\end{aligned}
\tag{3.8}
$$

where $e = [\mathfrak{O}^\times : \mathbb{Z}^\times]$. For a primitive character $\chi : Cl_f \to \mathcal{W}^\times$

$$
L^n(s,\lambda\chi) = \sum_{\mathfrak{a}} \lambda\chi(\mathfrak{a})N_{M/\mathbb{Q}}(\mathfrak{a})^{-s},
\tag{3.9}
$$

where $\mathfrak{a}$ runs over all proper ideals in $\mathfrak{O}_n$ prime to $\mathfrak{l}^f$ and $N_{M/\mathbb{Q}}(\mathfrak{a}) = [\mathfrak{O}_n : \mathfrak{a}]$. For each primitive character $\chi : Cl_f \to \overline{\mathbb{Q}}^\times$, taking $n = f$, by a similar but more involved computation using (3.7), we have

(3.10)

$$
\begin{aligned}
e^{-1}\sum_{[\mathfrak{a}]\in Cl_{n+1}} \chi(\mathfrak{a})\mathbb{E}'(\lambda)([\mathfrak{a}]) &= c_0\cdot\left(L^{n+1}(0,\lambda\chi^{-1}) - L^n(0,\lambda\chi^{-1})\right)\\
e^{-1}\sum_{[\mathfrak{a}]\in Cl_{n+1}} \chi(\mathfrak{a})\mathbb{E}(\lambda)([\mathfrak{a}]) &= c_0\cdot\left(L^{n+1}(0,\lambda\chi^{-1}) - \lambda\chi^{-1}(\mathfrak{l})l\cdot L^n(0,\lambda\chi^{-1})\right).
\end{aligned}
$$

We define a possibly imprimitive L–function

$$
L^{(\mathfrak{l})}(s,\chi^{-1}\lambda) = L_{\mathfrak{l}}(s,\chi^{-1}\lambda)L^0(s,\chi^{-1}\lambda)
$$

removing the $\mathfrak{l}$–Euler factor. Combining all these formulas with the computation of $L^n(s,\chi^{-1}\lambda)$ in [LAP] V.3.2, we find

$$
e^{-1}\sum_{[\mathfrak{a}]\in Cl_{n+1}} \chi(\mathfrak{a})\mathbb{E}(\lambda)([\mathfrak{a}]) = c_0\cdot L^{(\mathfrak{l})}(0,\chi^{-1}\lambda),
\tag{3.11}
$$

and up to p-units,

$$
e^{-1}\sum_{[\mathfrak{a}]\in Cl_{n+1}} \chi(\mathfrak{a})\mathbb{E}'(\lambda)([\mathfrak{a}]) = c_0\cdot L^{(\mathfrak{l})}(0,\chi^{-1}\lambda) \quad \text{if } f>0.
\tag{3.12}
$$

By Theorems 2.5 and 2.12, we have proven that all these values are algebraic in $\overline{\mathbb{Q}}$ and actually integral over $\mathcal{W}$:

Theorem 3.13. *Let* $c_0 = (-1)^k\frac{\pi^\kappa\Gamma(k+\kappa)}{\mathrm{Im}(\delta)^\kappa\sqrt{D}\Omega_\infty^{k+2\kappa}}$ *for integers* $k>0$ *and* $\kappa\ge 0$*. Then the value of* (3.11) *is in* $\mathcal{W}$ *if* $f>0$ *and* $p\ge 5$ *or* $\kappa>0$*.*

This follows from the fact that $\mathbb{E}(\lambda)$ has $\mathcal{W}$-integral q-expansion (i.e., no constant term) if either $k\neq 2$ or $\kappa>0$. If $\kappa=0$ and $k=2$, the constant term $2^{-1}(1-l)\zeta(-1)$ of $\mathbb{E}'(\lambda)$ is p-integral under the condition: $p\ge 5$. so, the result is clear from the formula (3.12) and Theorems 2.5 and 2.12.

3.7. **Construction of a modular measure.** Let $R = W$ or $\mathbb{F} = \overline{\mathbb{F}}_p$. Let $f \in V_{\Gamma_0(\mathfrak{l})/R}$ be a normalized Hecke eigenform (here normalization means that $f|T(n) = a(n,f)f$, $f|U(\mathfrak{l}) = a(l,f)f$ and $f(q) = \sum_{n=0}^{\infty} a(n,f)q^n$). A typical example of f can be given as follows: Take a modular form g in $G_k(\Gamma_0(\mathfrak{l}); R)$ for $R = \mathbb{F}$. Put $f = d^\kappa g$ for the differential operator d^κ in 2.2. We write $f(x(\mathfrak{a}))$ for the value of at $x(\mathfrak{a})$. The Hecke operator $U(\mathfrak{l})$ takes the space $V(\Gamma_0(\mathfrak{l}); R)$ into $V(\Gamma_0(\mathfrak{l}); R)$. We regard $U(\mathfrak{l})$ as an operator acting on $V(\Gamma_0(\mathfrak{l}); W)$. Suppose that $g|U(\mathfrak{l}) = a \cdot g$ with $a \in W^\times$; so, $f|U(\mathfrak{l}) = l^\kappa a \cdot f$ for the positive generator l of $\mathfrak{l}$ (see Remark 3.9). The Eisenstein series $\mathbb{E}(\lambda)$ satisfies this condition by Lemma 3.8.

Choosing a basis $w = (w_1, w_2)$ of $\widehat{\mathfrak{O}} = \mathfrak{O} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, identify the full Tate module $T(E(\mathfrak{O})_{/\overline{\mathbb{Q}}}) = \varprojlim_N E(\mathfrak{O})[N] = \widehat{\mathfrak{O}}$ with $\widehat{\mathbb{Z}}^2$ by $\widehat{\mathbb{Z}} \ni (a,b) \mapsto aw_1 + bw_2 \in T(E(\mathfrak{O}))$, getting a level structure: $\mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(E(\mathfrak{O})) := T(E(\mathfrak{O}) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined over $\mathcal{W}$. Elliptic curves $E_{/A}$ with such level structute $\eta^{(p)} : (\mathbb{A}^{(p\infty)})^2 \cong V^{(p)}(E)$ is classified by $Sh^{(p)}(A) = \varprojlim_{p\nmid N} Y(N)_{/\mathbb{Q}}$ up to prime-to-p isogenies; i.e., if $\phi : E \to E'_{/A}$ is an isogeny with degree prime-to-p with $\phi \circ \eta^{(p)} = \eta'^{(p)}$ gives a unique point $x \in Sh^{(p)}(A)$ such that $(\mathbf{E}, \boldsymbol{\eta}^{(p)}) \times_{Sh^{(p)}} x = (E, \eta^{(p)})$ for the universal couple $(\mathbf{E}, \boldsymbol{\eta}^{(p)})_{/Sh^{(p)}}$. The Shimura curve $Sh^{(p)}_{/\mathcal{W}}$ has a right action of $GL_2(\mathbb{A}^{(p\infty)})$ by $\eta^{(p)} \mapsto \eta^{(p)} \circ g$ (Shimura's global reciprocity). Choose the basis w satisfying the following two conditions:

(B) $w_{1,\mathfrak{l}} = 1$ and $\mathfrak{O}_\mathfrak{l} = \mathbb{Z}_\mathfrak{l}[w_{2,\mathfrak{l}}]$.

Let $\mathfrak{a}$ be a proper $\mathfrak{O}_n$–ideal (for $\mathfrak{O}_n = \mathbb{Z} + \mathfrak{l}^n\mathfrak{O}$) prime to $\mathfrak{f}$. Write $l_\mathfrak{l} = (1, \dots, 1, \overset{\mathfrak{l}}{l}, 1 \dots, 1) \in \mathbb{A}^\times$. Then $(w_1, l_\mathfrak{l}^n w_2)$ is a base of $\widehat{\mathfrak{O}}_n$ and gives a level structure $\eta^{(p)}(\mathfrak{O}_n) : \mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(E(\mathfrak{O}_n))$. We also write $l_\mathfrak{l}$ for $l \in \mathbb{Z}_\mathfrak{l}$ (if we want to avoid confusion). We choose a complete representative set $A = \{a_1, \dots, a_H\} \subset M_\mathbb{A}^\times$ so that $M_\mathbb{A}^\times = \bigsqcup_{j=1}^H M^\times a_j \widehat{\mathfrak{O}}_n^\times M_\infty^\times$. Then $\mathfrak{a}\widehat{\mathfrak{O}}_n = \alpha a_j \widehat{\mathfrak{O}}_n$ for $\alpha \in M^\times$ for some index j. We then define $\eta^{(p)}(\mathfrak{a}) = \alpha a_j \eta^{(p)}(\mathfrak{O}_n)$. The small ambiguity of the choice of α does not cause any trouble.

Write $x(\mathfrak{a}) = (E(\mathfrak{a}), C(\mathfrak{a}), \omega(\mathfrak{a}))_{/\mathcal{W}}$ (for $C(\mathfrak{a}) = \eta^{(p)}(\mathfrak{a}\widehat{\mathfrak{O}}_{n-1}/\widehat{\mathfrak{a}}) = \mathfrak{a}\mathfrak{O}_{n-1}/\mathfrak{a} \subset E(\mathfrak{a})$). This is a test object of level $\Gamma_0(\mathfrak{l})$ and is the image of $x(\mathfrak{a})$ in $X_0(\mathfrak{l})$. We pick a subgroup $C \subset E(\mathfrak{O}_n)$ such that $C \cong \mathbb{Z}/\mathfrak{l}^m$ $(m > 0)$ but $C \cap C(\mathfrak{O}_n) = \{0\}$. Since $\mathcal{W}$ is strictly henselian (i.e., $\mathcal{W}/\mathfrak{m}_\mathcal{W} = \overline{\mathbb{F}}_p = \mathbb{F}$) and $\mathfrak{l} \nmid p$, $E(\mathfrak{O}_n)[\mathfrak{l}^m]$ is a constant étale group scheme isomorphic to $(\mathbb{Z}/\mathfrak{l})^2$; so, making the quotient $E(\mathfrak{O}_n)/C$ is easy (see [GME] §1.8.3). Then we define $x(\mathfrak{O}_n)/C$ by

$$\left(\frac{E(\mathfrak{O}_n)}{C}, \frac{C + C(\mathfrak{O}_n)}{C}, (\pi^*)^{-1}\omega(\mathfrak{O}_n)\right)$$

for the projection map $\pi : E(\mathfrak{O}_n) \twoheadrightarrow E(\mathfrak{O}_n)/C$.

Lemma 3.14. *We have*

$$x(\mathfrak{O}_n)/C = x(\mathfrak{a}_C) \in \mathcal{M}_{\Gamma_0(\mathfrak{l})}(\mathcal{W})$$

for a proper $\mathfrak{O}_{n+m}$–ideal $\mathfrak{a} = \mathfrak{a}_C \supset \mathfrak{O}_n$ with $(\mathfrak{a}\mathfrak{a}^c) = \mathfrak{l}^{-2m}$, and for $u \in \mathbb{Z}_{\mathfrak{l}}^\times$ we have

$$x(\mathfrak{a}_C) = x(\mathfrak{O}_n)/C = x(\mathfrak{O}_{m+n})| \begin{pmatrix} 1 & \frac{u}{l_{\mathfrak{l}}^m} \\ 0 & 1 \end{pmatrix}. \tag{3.13}$$

Proof. Write simply η for $\eta^{(p)}$. The base of $\mathfrak{O}_{n,\mathfrak{l}}$ is given by $\alpha_n{}^t(1, w_2)$ for $\alpha_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & l_{\mathfrak{l}}^n \end{smallmatrix}\right)$ with a prime element $l_{\mathfrak{l}}$ of $\mathbb{Z}_{\mathfrak{l}}$. The action on level structure $\eta \mapsto \eta \circ g$ induces the action $\widehat{L}^{(p)} \mapsto g^{-1}\widehat{L}^{(p)}$ for $\widehat{\mathbb{Z}}$-lattices, as $\widehat{L}^{(p)} = \eta^{-1}(TE^{(p)}) \mapsto (\eta \circ g)^{-1}(TE^{(p)}) = g^{-1}\widehat{L}^{(p)}$. Thus we find that $\alpha_n^{-1}(x(\mathfrak{O})) = x(\mathfrak{O}_n)$ and $\alpha_1^{-1}(x(\mathfrak{O}_{n-1})) = x(\mathfrak{O}_n)$. Since the general case of $m > 1$ follows by iteration of the formula in the case of $m = 1$, we suppose $m = 1$. Then the formula becomes, for a suitable $u \in \mathbb{Z}_{\mathfrak{l}}^\times$

$$l_{\mathfrak{l}}^{-1}(x(\mathfrak{a})) = x(l\mathfrak{a}) = x(\mathfrak{O}_{n+1})| \begin{pmatrix} 1 & \frac{u}{l_{\mathfrak{l}}} \\ 0 & 1 \end{pmatrix} \tag{3.14}$$

if $x(\mathfrak{a}) = x(\mathfrak{O}_n)/C$ for C as above. To see this, note that the base of $l_{\mathfrak{l}}\mathfrak{a}_{\mathfrak{l}}$ is given by

$$\begin{pmatrix} 1-l_{\mathfrak{l}}^n u w_2 \\ l_{\mathfrak{l}}^{n+1} w_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-u}{l_{\mathfrak{l}}} \\ 0 & 1 \end{pmatrix} \alpha_{n+1} \begin{pmatrix} 1 \\ w_2 \end{pmatrix}.$$

Thus $\mathfrak{a}_{\mathfrak{l}}/\mathfrak{O}_n$ is generated by $\frac{1-ul_{\mathfrak{l}}^n w_2}{l_{\mathfrak{l}}} \mod \mathfrak{O}_{n,\mathfrak{l}}$ which gives the subgroup C for a suitable choice of u. Since $l^{(l)} \in \widehat{\mathbb{Z}}^\times$, the acton of $l_{\mathfrak{l}}$ is equivalent to the action of $l \in Z(\mathbb{Q})$ which is trivial; so, we forget $l_{\mathfrak{l}}$ in (3.14). $\square$

For each proper $\mathfrak{O}_n$–ideal $\mathfrak{a}$, we have an embedding $\rho_{\mathfrak{a}} : M^\times_{\mathbb{A}^{(p\infty)}} \to GL_2(\mathbb{A}^{(p\infty)})$ given by $\alpha\eta^{(p)}(\mathfrak{a}) = \eta^{(p)}(\mathfrak{a}) \circ \rho_{\mathfrak{a}}(\alpha)$. Since $\det(\rho_{\mathfrak{a}}(\alpha)) = \alpha\alpha^c \gg 0$, $\alpha \in \mathfrak{O}_{(p)}^\times$ acts on $Sh^{(p)}$ through $\rho_{\mathfrak{a}}(\alpha) \in G(\mathbb{A})$. We have

$$\rho_{\mathfrak{a}}(\alpha)(x(\mathfrak{a})) = (E(\mathfrak{a}), \eta^{(p)}(\mathfrak{a})\rho_{\mathfrak{a}}(\alpha)) = (E(\alpha\mathfrak{a}), \eta^{(p)}(\alpha\mathfrak{a}))$$

for the prime-to–p isogeny $\alpha \in \mathrm{End}_{\mathbb{Z}}(E(\mathfrak{a})) = \mathfrak{O}_{(p)}$. Thus $\mathfrak{O}_{(p)}^\times$ acts on $Sh^{(p)}$ fixing the point $x(\mathfrak{a})$. We find $\rho(\alpha)^*\omega(\mathfrak{a}) = \alpha\omega(\mathfrak{a})$ and

$$g(x(\alpha\mathfrak{a}), \alpha\omega(\mathfrak{a})) = g(\rho(\alpha)(x(\mathfrak{a}), \omega(\mathfrak{a}))) = \alpha^{-k} g(x(\mathfrak{a}), \omega(\mathfrak{a})).$$

From this, we conclude

$$f(x(\alpha\mathfrak{a}), \alpha\omega(\mathfrak{a})) = f(\rho(\alpha)(x(\mathfrak{a}), \omega(\mathfrak{a}))) = \alpha^{-k-\kappa(1-c)} f(x(\mathfrak{a}), \omega(\mathfrak{a})),$$

because the effect of the differential operator d is identical with that of δ at the CM point $x(\mathfrak{a})$ by Theorem 2.12. Since

$$\widehat{\lambda}(\alpha\mathfrak{a}) = \alpha^{-k-\kappa(1-c)}\widehat{\lambda}(\mathfrak{a}),$$

the value $\widehat{\lambda}(\mathfrak{a})^{-1}f(x(\mathfrak{a}),\omega(\mathfrak{a}))$ is independent of the representative set $A = \{a_j\}$ for Cl_n. Defining, for a proper $\mathfrak{O}_n$–ideal $\mathfrak{a}$ prime to p,

$$f([\mathfrak{a}]) = \widehat{\lambda}(\mathfrak{a})^{-1}f(x(\mathfrak{a}),\omega(\mathfrak{a})), \tag{3.15}$$

we find that $f([\mathfrak{a}])$ only depends on the proper ideal class $[\mathfrak{a}] \in Cl_n$.

We write $x(\mathfrak{a}_u) = x(\mathfrak{a})|\alpha_1^{-1}\left(\begin{smallmatrix}1 & \frac{u}{l}\\ 0 & 1\end{smallmatrix}\right)$. Then $\mathfrak{a}_u$ depends only on u mod $\mathfrak{l}$, and $\{\mathfrak{a}_u\}_{u \bmod \mathfrak{l}}$ gives a complete representative set for proper $\mathfrak{O}_{n+1}$–ideal classes which project down to the ideal class $[\mathfrak{a}] \in Cl_n$. Since $\mathfrak{a}_u\mathfrak{O}_n = l^{-1}\mathfrak{a}$, we find $\widehat{\lambda}(\mathfrak{a}_u) = \widehat{\lambda}(\mathfrak{l})^{-1}\widehat{\lambda}(\mathfrak{a})$. Recalling $f|U(\mathfrak{l}) = l^{\kappa}a\cdot f$, we have

$$l^{\kappa}a\cdot f([\mathfrak{a}]) = \widehat{\lambda}(\mathfrak{a})^{-1}f|U(\mathfrak{l})(x(\mathfrak{a})) = \frac{1}{\widehat{\lambda}(\mathfrak{l})l}\sum_{u \bmod \mathfrak{l}} f([\mathfrak{a}_u]). \tag{3.16}$$

Definition 3.15. *For a continuous function $\phi : Cl_\infty \to \mathbb{F}$, taking $n > 0$ so that ϕ factors through Cl_n, we define a measure φ_f on Cl_∞ with values in $\mathbb{F}$ by*

$$\int_{Cl_\infty}\phi\, d\varphi_f = b^{-n}\sum_{\mathfrak{a}\in Cl_n}\phi(\mathfrak{a}^{-1})f([\mathfrak{a}]) \quad \textit{(for } b = l^{\kappa+1}a\widehat{\lambda}(\mathfrak{l})\textit{)}. \tag{3.17}$$

3.8. **Non-triviality of the modular measure.** The non-triviality of the measure φ_f is proven in [H04] Theorems 3.2 and 3.3. To recall the result in [H04], we recall a functorial action (introduced earlier) on p–adic modular forms, commuting with $U(\mathfrak{l})$. Let $\mathfrak{q}$ be a prime ideal of $\mathbb{Q}$ different from $\mathfrak{l}$. For a test object (E,η) of level $\Gamma_0(\mathfrak{l}\mathfrak{q})$, the $\mathfrak{q}$-part $\eta_\mathfrak{q}$ of η is a subgroup $C \cong \mathbb{Z}/\mathfrak{q}$ in E. Then we can construct canonically $[\mathfrak{q}](E,\eta) = (E',\eta')$ with $E' = E/C$. If $\mathfrak{q}$ splits into $\mathfrak{Q}\overline{\mathfrak{Q}}$ in $M/\mathbb{Q}$, choosing $\eta_\mathfrak{q}$ induced by $E(\mathfrak{a})[\mathfrak{q}^\infty] \cong M_\mathfrak{Q}/\mathfrak{O}_\mathfrak{Q} \times M_{\overline{\mathfrak{Q}}}/\mathfrak{O}_{\overline{\mathfrak{Q}}} \cong \mathbb{Q}_\mathfrak{q}/\mathbb{Z}_\mathfrak{q} \times \mathbb{Q}_\mathfrak{q}/\mathbb{Z}_\mathfrak{q}$, we always have a level $\Gamma_0(\mathfrak{q})$–structure $C = E(\mathfrak{a})[\mathfrak{Q}_n]$ for $\mathfrak{Q}_n = \mathfrak{Q}\cap\mathfrak{O}_n$ on $E(\mathfrak{a})$ induced by the choice of the factor $\mathfrak{Q}$. Then $[\mathfrak{q}](E(\mathfrak{a})) = E(\mathfrak{a}\mathfrak{Q}_n^{-1})$ for a proper $\mathfrak{O}_n$–ideal $\mathfrak{a}$, as $\mathfrak{Q}_n^{-1}\mathfrak{a}/\mathfrak{a} \cong C$ by $\eta_\mathfrak{q}$ (so, $E(\mathfrak{a})/C = E(\mathfrak{a})/(\mathfrak{a}\mathfrak{Q}_n^{-1}/\mathfrak{a}) = E(\mathfrak{a}\mathfrak{Q}_n^{-1})$). When $\mathfrak{q}$ ramifies in $M/\mathbb{Q}$ as $\mathfrak{q} = \mathfrak{Q}^2$, $E(\mathfrak{a})$ has a subgroup $C = E(\mathfrak{a})[\mathfrak{Q}_n]$ isomorphic to $\mathbb{Z}/\mathfrak{q}$; so, we can still define $[\mathfrak{q}](E(\mathfrak{a})) = E(\mathfrak{a}\mathfrak{Q}_n^{-1})$. The effect of $[\mathfrak{q}]$ on the q–expansion at the infinity cusp is computed in §3.2 and is given by a unit multiple of the q–expansion of f at the Tate curve Tate(q^{ϖ}) for a positive generator ϖ of $\mathfrak{q}$. The operator $[\mathfrak{q}]$ corresponds to the action of $g = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}}^{-1} \end{pmatrix} \in GL_2(\mathbb{Q}_{\mathfrak{q}})$. In §3.2, we saw that $[\mathfrak{q}]$ induces a linear map well defined on $V_{\Gamma_0(\mathfrak{l})/R}$ into $V_{\Gamma_0(\mathfrak{l}\mathfrak{q})/R}$.

We fix a decomposition $Cl_\infty = \Gamma \times \Delta$ for a finite group Δ and a torsion-free subgroup Γ. Since each fractional $\mathfrak{O}$–ideal $\mathfrak{A}$ prime to $\mathfrak{l}$ defines a class $[\mathfrak{A}]$ in Cl_∞, we can embed the ideal group of fractional ideals prime to $\mathfrak{l}$ into Cl_∞. We write Cl^{alg} for its image.

Exercise 3.16. (1) *Complex conjugation acts on $z \in Cl_\infty$ by $z \mapsto z^{-1}$.*
(2) *The intersection $\Delta^{alg} = \Delta \cap Cl^{alg}$ is represented by square-free products of prime ideals of M ramified over $\mathbb{Q}$. In other words, Δ^{alg} is isomorphic to the ambiguous class group of M.*
(3) *The quotient $Cl_\infty/\Gamma\Delta^{alg}$ has a complete representative set in the set of prime ideals split over $\mathbb{Q}$ (prime to $\mathfrak{l}$).*
(4) *Write $[\mathfrak{Q}]_\Gamma$ (resp. $[\mathfrak{Q}]_\Delta$) for the projection of $[\mathfrak{Q}] \in Cl^{alg}$ to Γ (resp. to Δ). If $[\mathfrak{Q}]_\Delta \notin [\mathfrak{Q}']_\Delta \Delta^{alg}$, then $[\mathfrak{Q}]_\Gamma/[\mathfrak{Q}']_\Gamma \notin Cl^{alg}$.*

We choose a complete representative set $\{\mathfrak{R}^{-1} | \mathfrak{r} \in \mathcal{R}\}$ for Δ^{alg} such that the set $\mathcal{R}$ is a subset of the set of all square-free product of primes in $\mathbb{Q}$ ramifying in $M/\mathbb{Q}$, and $\mathfrak{R}$ is a unique ideal in M with $\mathfrak{R}^2 = \mathfrak{r}$. The set $\{\mathfrak{R} | \mathfrak{r} \in \mathcal{R}\}$ is a complete representative set for 2–torsion elements in the class group Cl_0 of $\mathfrak{O}$ (i.e., the ambiguous classes). We fix a character $\nu : \Delta \to \mathbb{F}^\times$, and define

$$f_\nu = \sum_{\mathfrak{r} \in \mathcal{R}} \widehat{\lambda}\nu^{-1}(\mathfrak{R}) f | [\mathfrak{r}]. \tag{3.18}$$

Choose a complete representative set $\mathcal{Q}$ for $Cl_\infty/\Gamma\Delta^{alg}$ made of primes of M split over $\mathbb{Q}$ outside $p\mathfrak{l}$. Since Cl^{alg} is dense in Cl_∞, we can choose $\mathfrak{Q} \in \mathcal{Q}$ whose projection to Γ is whatever close to 1 under the profinite topology (this remark will be useful later). We choose $\eta_n^{(p)}$ out of the base (w_1, w_2) of $\widehat{\mathfrak{O}}_n$ so that at $\mathfrak{q} = \mathfrak{Q} \cap \mathbb{Q}$, $w_1 = (1,0) \in \mathfrak{O}_{\mathfrak{Q}} \times \mathfrak{O}_{\mathfrak{Q}^c} = \mathfrak{O}_{\mathfrak{q}}$ and $w_2 = (0,1) \in \mathfrak{O}_{\mathfrak{Q}} \times \mathfrak{O}_{\mathfrak{Q}^c} = \mathfrak{O}_{\mathfrak{q}}$. Since all operators $[\mathfrak{q}]$ and $[\mathfrak{r}]$ involved in this definition commutes with $U(\mathfrak{l})$, $f_\nu|[\mathfrak{q}]$ is still an eigenform of $U(\mathfrak{l})$ with the same eigenvalue as f. Thus in particular, we have a measure φ_{f_ν}. We project it to Γ along ν which produces a measure φ_f^ν on Γ explicitly given by

$$\int_\Gamma \phi d\varphi_f^\nu = \sum_{\mathfrak{Q} \in \mathcal{Q}} \widehat{\lambda}\nu^{-1}(\mathfrak{Q}) \int_\Gamma \phi | \mathfrak{Q} d\varphi_{f_\nu | [\mathfrak{q}]},$$

where $\phi|\mathfrak{Q}(y) = \phi(y[\mathfrak{Q}]_\Gamma^{-1})$ for the projection $[\mathfrak{Q}]_\Gamma$ in Γ of $[\mathfrak{Q}] \in Cl_\infty$.

Lemma 3.17. *If $\chi : Cl_\infty \to \mathbb{F}^\times$ is a character with $\chi|_\Delta = \nu$, we have*
$$\int_\Gamma \chi d\varphi_f^\nu = \int_{Cl_\infty} \chi d\varphi_f.$$

Proof. Write $\Gamma_{f,n}$ for the image of Γ in Cl_n. For proper $\mathfrak{O}_n$–ideal $\mathfrak{a}$, by the above definition of these operators,
$$f|[\mathfrak{r}]|[\mathfrak{q}]([\mathfrak{a}]) = \widehat{\lambda}(\mathfrak{a})^{-1} f(x(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{a}), \omega(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{a})).$$
For sufficiently large n, χ factors through Cl_n. Since $\chi = \nu$ on Δ, we have
$$\int_\Gamma \chi d\varphi_f^\nu = \sum_{\mathfrak{Q}\in\mathcal{Q}} \sum_{\mathfrak{r}\in\mathcal{R}} \sum_{\mathfrak{a}\in\Gamma_{f,n}} \widehat{\lambda}\chi^{-1}(\mathfrak{Q}\mathfrak{R}\mathfrak{a}) f|[\mathfrak{r}]|[\mathfrak{q}]([\mathfrak{a}])$$
$$= \sum_{\mathfrak{a},\mathfrak{Q},\mathfrak{r}} \chi(\mathfrak{Q}\mathfrak{R}\mathfrak{a}) f([\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{a}]) = \int_{Cl_\infty} \chi d\varphi_f,$$
because $Cl_\infty = \bigsqcup_{\mathfrak{Q},\mathfrak{R}} [\mathfrak{Q}^{-1}\mathfrak{R}^{-1}]\Gamma$. □

In the next couple of sections, we prove the following result (given in [H04] as Theorems 3.2 and 3.3):

Theorem 3.18. *Fix a character $\nu : \Delta \to \mathbb{F}^\times$, and define f_ν as in* (3.18)*. If f satisfies the following condition:*

(H) for any given integer $r > 0$ and any congruence class $u \in (\mathbb{Z}/l^r\mathbb{Z})^\times$, there exists $0 \leq \xi \in u$ such that $a(\xi, f_\nu) \neq 0$,

then non-vanishing $\int_{Cl_\infty} \nu\chi d\varphi_f \neq 0$ holds except for finitely many characters $\chi : \Gamma \to \mu_{l^\infty}(\mathbb{F})$.

3.9. **Preliminary to the proof of Theorem 3.18.** We regard f as a function of $Cl^{(\infty)} = \bigsqcup_n Cl_n$ (embedded into $Sh^{(p)}$ over $X_0(\mathfrak{l})$ by $\mathfrak{a} \mapsto x(\mathfrak{a})$). By (3.16), we have, for an integer $n > m$,
$$\sum_{[\mathfrak{a}]\in Cl_n,\ \mathfrak{a}\mapsto[\mathfrak{A}]\in Cl_m} f([\mathfrak{a}]) = (\widehat{\lambda}(\mathfrak{l})l)^{n-m} f|U(\mathfrak{l}^{n-m})([\mathfrak{A}]), \tag{3.19}$$
where $[\mathfrak{a}]$ runs over all classes in Cl_n which project down to $[\mathfrak{A}] \in Cl_m$.

We suppose that $f|U(\mathfrak{l}) = (a/\widehat{\lambda}(\mathfrak{l})l)f$ with a unit $a \in A$. For each function $\phi : Cl_\infty \to A$ factoring through Cl_m, we define
$$\int_{Cl_\infty} \phi d\varphi_f = a^{-m} \sum_{\mathfrak{a}\in Cl_m} \phi(\mathfrak{a}^{-1}) f([\mathfrak{a}]). \tag{3.20}$$

Classical modular forms are actually defined over a number field; so, we assume that f is defined over the localization $\mathcal{V}$ of the integer ring in a number field K containing M over which $E(\mathfrak{a})$ for each class $[\mathfrak{a}] \in Cl_0$ is defined. We write $\mathcal{P}|p$ for the prime ideal of the p–integral closure $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ in $\overline{\mathbb{Q}}$ corresponding to $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. More generally, if $f = d^\kappa g$ for a classical modular form g integral over $\mathcal{V}$, the value $f([\mathfrak{a}])$ is algebraic, abelian over M and $\mathcal{P}$–integral over $\mathcal{V}$ by a result of Shimura and Katz (see Theorem 2.5 and Theorem 2.12).

Let $f = d^\kappa g$ for $g \in G_k(\Gamma_0(\mathfrak{l}); \mathcal{V})$. Suppose $f|U(\mathfrak{l}) = (a/l)f$ for a giving a unit of $\widetilde{\mathcal{V}}/\mathcal{P}$. For the moment, let φ be the measure associated to f with values in $R = W$ for a finite extension W of $W(\overline{\mathbb{F}}_p)$ containing $\mathcal{V}$. We have a well defined measure $\varphi \mod \mathcal{P}$. Let K_f be the field generated by $f([\mathfrak{a}])$ over $K[\mu_{l^\infty}]$. Then K_f/K is an abelian extension unramified outside l, and we have the Frobenius element $\sigma_{\mathfrak{b}} \in \mathrm{Gal}(K_f/K)$ (that is, the image of $\mathfrak{b}$ under the Artin reciprocity map) for each ideal $\mathfrak{b}$ of K prime to l. By Shimura's CM reciprocity law, we find for $\sigma = \sigma_{\mathfrak{b}}$, $x(\mathfrak{a})^\sigma = x(N(\mathfrak{b})^{-1}\mathfrak{a})$ for the norm $N : K \to M$. From this, if we extend K further if necessary, we see $f([\mathfrak{a}])^\sigma = f([N(\mathfrak{b})^{-1}\mathfrak{a}])$ for any ideal $\mathfrak{b}$. We then have

$$\sigma \cdot \left(\int_{Cl_\infty} \phi(x) d\varphi_f(x) \right) = \int_{Cl_\infty} \sigma \circ \phi(N(\mathfrak{b})x) d\varphi_f(x), \tag{3.21}$$

where $N(\mathfrak{b})$ is the norm of $\mathfrak{b}$ over M.

We now assume that $R = \mathbb{F} = W/\mathfrak{m}_W = \widetilde{\mathcal{V}}/\mathcal{P}$ and regard the measure φ_f as having values in $\mathbb{F}$. Then (3.21) shows that if ϕ is a character χ of Cl_∞, for $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^r})$ ($\mathbb{F}_{p^r} = \mathcal{V}/\mathcal{P} \cap \mathcal{V}$),

$$\int_{Cl_\infty} \chi(x) d\varphi_f(x) = 0 \iff \int_{Cl_\infty} \sigma \circ \chi(x) d\varphi_f(x) = 0. \tag{3.22}$$

Decompose Cl_∞ into a product of the maximal torsion-free l–profinite subgroup Γ and a finite group Δ.

Let $\mathbb{F}_q$ be the finite subfield of $\mathbb{F}$ generated by all $l|\Delta|$–th roots of unity over the field $\mathbb{F}_{p^r}$ of rationality of f and λ. For any finite extension $\mathbb{F}'/\mathbb{F}_q$, we consider the trace map: $\mathrm{Tr}_{\mathbb{F}'/\mathbb{F}_q}(\xi) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{F}'/\mathbb{F}_q)} \sigma(\xi)$ for $\xi \in \mathbb{F}'$. If $\chi : Cl_n \to \mathbb{F}^\times$ is a character, $d := [\mathrm{Im}(\chi) : \mathrm{Im}(\chi) \cap {\mathbb{F}_q}^\times]$ is not divisible by p (as $|\mathbb{F}^\times_{p^m}| = p^m - 1 \not\equiv 0 \mod p$). Thus $d \in \mathbb{F}^\times$, and

$$\int_{Cl_\infty} \mathrm{Tr}_{\mathbb{F}_q(\chi)/\mathbb{F}_q} \circ \chi(y^{-1}x) d\varphi_f(x) = \frac{d}{a^n} \sum_{\mathfrak{a} \in Cl_n : \chi(\mathfrak{a}y^{-1}) \in \mathbb{F}_q} \chi(y^{-1}\mathfrak{a}) f([\mathfrak{a}]), \tag{3.23}$$

because, by Exercise 1.5, for an l–power root of unity $\zeta \in \mu_{l^n} - \mu_l$,

$$\mathrm{Tr}_{\mathbb{F}_q(\mu_{l^n})/\mathbb{F}_q}(\zeta^s) = \begin{cases} l^{n-m}\zeta^s & \text{if } \zeta^s \in \mathbb{F}_q \text{ and } \mathbb{F}_q \cap \mu_{l^\infty}(\mathbb{F}) = \mu_{l^m}(\mathbb{F}) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\int_{Cl_\infty} \chi(x) d\varphi_f(x) = 0$ for an infinite set $\mathcal{X}$ of characters χ. For sufficiently large m, we always find a character $\chi \in \mathcal{X}$ such that $\mathrm{Ker}(\chi) \subset \Gamma^{l^m}$. Then writing $\mathrm{Ker}(\chi) = \Gamma^{l^n}$ for $n \geq m$, we have the vanishing from (3.22)

$$\int_{Cl_\infty} \sigma \circ \chi d\varphi_f = 0 \quad \text{for all } \sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q).$$

This combined with (3.23), we find $\sum_{y \in \chi^{-1}(\mathbb{F}_q^\times)} \chi(y\mathfrak{a}) f(y[\mathfrak{a}])) = 0$ for all $\mathfrak{a} \in \Gamma_n$, where Γ_n is the image of Γ in Cl_n.

3.10. **Proof of Theorem 3.18.** We write $\mathbb{F}_p[f]$ for the minimal field of definition of $f \in V(\mathbb{F})$ (i.e., the field generated by $a(\xi, f) \in \mathbb{F}$ for all $0 \leq \xi \in \mathbb{Z}$). Similarly $\mathbb{F}_p[\lambda]$ (resp. $\mathbb{F}_p[\nu]$) is the subfield of $\mathbb{F}$ generated by the values $\widehat{\lambda}([\mathfrak{a}]) \mod \mathcal{P}$ (resp. $\nu([\mathfrak{a}])$) for all $[\mathfrak{a}] \in C^{alg}$. Define $\mathbb{F}_p[f, \lambda, \nu]$ by the composite of these fields and $\mathbb{F}_p[\mu_l]$. Note that $\mathbb{F}_p[f, \lambda, \nu]$ is a finite extension of $\mathbb{F}_p$ as f is mod p reduction of some classical modular form of some weight ≥ 2. Define $1 \leq r = r(\nu) \in \mathbb{Z}$ by $|\mu_{l^\infty}(\mathbb{F}_p[f, \lambda, \nu])| = l^r$.

By definition, the projection $\{[\mathfrak{Q}]_\Gamma\}_{\mathfrak{Q} \in \mathcal{Q}}$ of $[\mathfrak{Q}]$ in Γ are all distinct in Cl_∞/C^{alg}. By Lemma 3.17, we need to prove that the integral $\int_\Gamma \chi d\varphi_f^\nu$ vanishes only for finitely many characters χ of Γ. Suppose by absurdity that the integral vanishes for characters χ in an infinite set $\mathcal{X}$.

Let $\Gamma(n) = \Gamma^{l^{n-r}}/\Gamma^{l^n}$ for $r = r(\nu)$. By applying (3.23) to a character in $\mathcal{X}$ with $\mathrm{Ker}(\chi) = \Gamma^{l^n}$, we find

$$\sum_{\mathfrak{Q} \in \mathcal{Q}} \nu(\mathfrak{Q})^{-1} \sum_{\mathfrak{a} \in y\chi^{-1}(\mu_{l^r})} \chi(\mathfrak{a}) f_\nu([\mathfrak{a}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma) = 0. \tag{3.24}$$

Fix $\mathfrak{Q} \in \mathcal{Q}$. By Lemma 3.14, $\{x(\mathfrak{a}) | [\mathfrak{a}] \in y\chi^{-1}(\mu_{l^r})\}$ is given by $\alpha(\frac{u}{\mathfrak{l}^r})(x(\mathfrak{a}_0))$ for any member $\mathfrak{a}_0 \in y\chi^{-1}(\mu_{p^r})$, where

$$\alpha(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right). \tag{3.25}$$

Actually $\mathfrak{a} \mapsto u \mod \mathfrak{l}^r$ gives a bijection of $y\chi^{-1}(\mu_{l^r})$ onto $O/\mathfrak{l}^r$. We write the element $\mathfrak{a}$ corresponding to u as $\alpha(\frac{u}{\mathfrak{l}^r})\mathfrak{a}_0$. This shows, choosing a primitive l^r–th root of unity $\zeta = \exp(2\pi i/l^r)$ and $\mathfrak{a}_y \in y\chi^{-1}(\mu_{l^r})$ so that $\chi(\alpha(\frac{u}{\mathfrak{l}^r})\mathfrak{a}_y) = \zeta^{uv}$ for an integer $0 < v < l^r$ prime to l (independent of y), the inner sum of (3.24) is equal to

$$\sum_{u \mod \mathfrak{l}^r} \zeta^{uv} (f_\nu | \alpha(\frac{u}{\mathfrak{l}^r}))([\mathfrak{a}_y \mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma).$$

The choice of v depends on χ. Since $\mathcal{X}$ is infinite, we can choose an infinite subset $\mathcal{X}'$ of $\mathcal{X}$ for which v is independent of the element in $\mathcal{X}'$. Then write n_j for the integers given by $\Gamma^{l^{n_j}} = \mathrm{Ker}(\chi)$ for $\chi \in \mathcal{X}'$ (in increasing order), and define Ξ to be the set of points $x(\mathfrak{a})$ for $\mathfrak{a} \in Cl_{n_j}$ with $[\mathfrak{a}\mathfrak{O}_{n_1}] = [\mathfrak{O}_{n_1}]$ in Cl_{n_1}. Define also $h_{\mathfrak{Q}} = \sum_{u \mod \mathfrak{l}^r} \zeta^{uv} f_\nu | \alpha(\frac{u}{l^r_{\mathfrak{l}}})$, because (3.24) is now the sum:

$$\sum_{\mathfrak{Q}\in\mathcal{Q}} \nu(\mathfrak{Q})^{-1} h_{\mathfrak{Q}}|[\mathfrak{q}]([\mathfrak{a}][\mathfrak{Q}]_\Gamma) = 0,$$

where $\mathfrak{q} = \mathfrak{Q} \cap F$. If necessary, as we remarked already, we reselect the representative set $\mathcal{Q}$ so that $[\mathfrak{Q}]_\Gamma \in \mathrm{Ker}(Cl_\infty \to Cl_{n_1})$. This is possible because $\{[\mathfrak{Q}_f] \in \Gamma | \mathfrak{Q} \sim \mathfrak{A}\}$ for all split primes is dense by Chebotarev-density, where $\mathfrak{Q} \sim \mathfrak{A}$ means the class of $\mathfrak{Q}$ is equal to the class of $\mathfrak{A}$ in $Cl_\infty/\Gamma\Delta^{alg}$. Take $\mathfrak{Q}, \mathfrak{Q}'$ in $\mathcal{Q}$. Then by Exercise 3.16 (4), $[\mathfrak{Q}]_\Gamma/[\mathfrak{Q}']_\Gamma \in Cl^{alg} \Leftrightarrow \mathfrak{Q} = \mathfrak{Q}'$. Thus we may apply Corollary 3.21 in the following section to the following set of functions: $\{[\mathfrak{a}] \mapsto h_{\mathfrak{Q}}|[\mathfrak{q}]([\mathfrak{a}][\mathfrak{Q}]_\Gamma)\}$. By the corollary, if $h_{\mathfrak{Q}}|[\mathfrak{q}] \neq 0$ for one $\mathfrak{Q}$, the above sum is nonzero as a function of $[\mathfrak{a}]$; so, this implies that $h_{\mathfrak{Q}}[\mathfrak{q}] = 0$. By q-expansion principle, we conclude $h_{\mathfrak{Q}} = 0$ (as $h|[\mathfrak{q}](q) = h(q^\varpi)$ for the positive generator ϖ of $\mathfrak{q}$).

However, since we have $f_\nu | \left(\begin{smallmatrix} 1 & \frac{u}{l^r} \\ 0 & 1 \end{smallmatrix}\right) = \sum_{0 \le \xi \in \mathbb{Z}} a(\xi, f_\nu)\zeta^u q^\xi$ for $\zeta = \exp(\frac{2\pi i}{l^r})$, the q–expansion coefficient $a(\xi, h_{\mathfrak{Q}})$ of $h_{\mathfrak{Q}}$ is given by $a(\xi, f_\nu)$ if $\xi \equiv -v \mod \mathfrak{l}^r$ and vanishes otherwise. This is a contradiction against the assumption (H).

3.11. **Linear independence.** Fix a positive integer $n_1 > 0$. We create complete representative set R_n for $\mathrm{Ker}(Cl_n \to Cl_{n_1})$ by $\alpha(\frac{u}{l^n})(x(\mathfrak{O}_{n_1}))$ (for $\alpha(t)$ as in (3.25)) by choosing suitable integers $0 < u < l^n$. Choose an infinite sequence $\underline{n} := 0 < n_1 < n_2 < \cdots < n_m < \cdots$ of positive integers. Take a geometrically irreducible component $V_{/\mathbb{F}} \subset Sh^{(p)}_{/\mathbb{F}}$ containing $x(\mathfrak{O}_n)$, where $Sh^{(p)}_{/\mathbb{F}} = Sh^{(p)} \times_{\mathcal{W}} \mathbb{F}$. Since V is affine, we can write $V = \mathrm{Spec}(O_V)$ for $O_V = H^0(V, \mathcal{O}_V)$. Sometimes we just write $O = O_V$ if confusion is unlikely. Define

$$\Xi_{\underline{n}} = \Xi = \bigcup_{j=1}^{\infty} \{x(\mathfrak{a}) \in V | \mathfrak{a} \in R_{n_j}\} \subset V.$$

Since $SL_2(\mathbb{A}^{(p\infty)})$ keeps V (by Shimura's global reciprocity), $x(\mathfrak{a})$ as above always resides in one component V.

Let $\mathcal{F} = \mathcal{F}_\Xi$ denote the $\mathbb{F}$-algebra of functions $\phi : \Xi \to \mathbf{P}^1(\mathbb{F}) = \mathbb{F} \sqcup \{\infty\}$ with $|\phi^{-1}(0)| < \infty$ and $|\phi^{-1}(\infty)| < \infty$. The profinite class group $C = C_{n_1} := \mathrm{Ker}(Cl_\infty \to Cl_{n_1})$ acts on $\mathcal{F}$ by translation: $f(x) \mapsto f(xy)$ ($y \in Cl_\infty$). In particular, $\alpha \in \mathfrak{O}_{(p)}$ with trivial $[(\alpha)] \in Cl_{n_1}$ acts on Ξ and such α is p-adically dense in $\mathfrak{O}_p^\times$. For $f \in \mathbb{F}(V)^\times$, for each

$x = x(\mathfrak{a}) \in \Xi$, expanding f into a Laurent series $f(t) = \sum_n a_n t^n \in \mathbb{F}[[t]][t^{-1}]$ with leading nonzero term $a_m t^m$ ($m \in \mathbb{Z}$), we may define

$$f(x(\mathfrak{a})) = \begin{cases} \infty & \text{if } m < 0, \\ a_0 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

By Zariski density of Ξ in $V = V^{(p)}_{/\mathbb{F}}$, we can embed into $\mathcal{F}$ the function field $\mathbb{F}(V)$ of V.

Exercise 3.19. *Why is Ξ Zariski dense in V? Why does density imply injectivisty of $\mathbb{F}(V)$ into $\mathcal{F}$?*

We will prove the following analogue of Sinnott's theorem later if time allows.

Proposition 3.20. *Take a finite set $\Delta = \{\gamma_1, \dots, \gamma_m\} \subset C_{n_1}$ injecting into Cl_∞/Cl^{alg}. Then the subset $\widetilde{\Xi} := \{(x(\delta(\mathfrak{a}))_{\delta\in\Delta} | x(\mathfrak{a}) \in \Xi\}$ is Zariski dense in the product $V^{\Delta}_{/\mathbb{F}}$ of Δ copies of $V_{/\mathbb{F}}$. This implies that the fields $\gamma_1(\mathbb{F}(V)), \dots, \gamma_m(\mathbb{F}(V))$ are linearly disjoint over $\mathbb{F}$ in $\mathcal{F}_\Xi$, where $\gamma(\mathbb{F}(V))$ is the image of $\mathbb{F}(V) \subset \mathcal{F}$ under the action of $\gamma \in C_{n_1}$. In other words, we have injectivity of the map $\gamma_1 \otimes \cdots \otimes \gamma_m : \overbrace{O_V \otimes_{\mathbb{F}} O_V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} O_V}^{m} \to \mathcal{F}$ sending $f_1 \otimes \cdots \otimes f_m$ to an element in $\mathcal{F}$ given by $x(\mathfrak{a}) \mapsto \prod_j f_j(x(\gamma_j \mathfrak{a}))$.*

The linear independence applied to the global sections of a modular line bundle (regarded as sitting inside the function field) yields the following result:

Corollary 3.21. *Let the notation and the assumption be as in Proposition 3.20. Let $\underline{\omega}^k$ be a modular line bundle over the Igusa tower $Ig_{/\mathbb{F}}$ over $V_{/\mathbb{F}}$. Then for a finite set $\Delta \subset Cl_\infty$ injectiong into Cl_∞/Cl^{alg} and a set $\{s_\delta \in H^0(I, \underline{\omega}^k)\}_{\delta\in\Delta}$ of non-constant global sections s_δ of $\underline{\omega}^k$ finite at Ξ, the functions $s_\delta \circ \delta$ ($\delta \in \Delta$) are linearly independent in $\mathcal{F}_\Xi$.*

Choosing one nonzero section s (different from constant multiple of any of s_δs) and replacing s_δ by s_δ/s, which is a modular function, we can bring the situation in the case of modular functions which is taken care of by the above theorem.

3.12. **$\mathfrak{l}$–Adic Eisenstein measure modulo p.** We apply Theorem 3.18 to the Eisenstein series $\mathbb{E}(\lambda)$ in (3.4) for the Hecke character λ fixed in 3.1. We can easily check (H) in Theorem 3.18 for $f = \mathbb{E}(\lambda) \mod \mathfrak{m}_W$ and get Theorem 3.12.