A FINITENESS PROPERTY OF ABELIAN VARIETIES WITH POTENTIALLY ORDINARY GOOD REDUCTION

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Contents

A g-dimensional abelian variety $A_{/F}$ over a number field F is of $GL(2)$-type if $\mathrm{End}^0(A_{/F}) := \mathrm{End}(A_{/F}) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a field E of degree g. We call such a field E an endomorphism field of A. We say that an F-simple abelian variety $A_{/F}$ over a number field F of dimension g is non-CM if $\mathrm{End}^0(A_{/F} \times_F \overline{\mathbb{Q}})$ does not contain any semi-simple commutative algebra of degree $2g$ over $\mathbb{Q}$ (cf. [ACM]). If an abelian variety $A_{/F}$ of GL(2)-type is F-simple, $D = \mathrm{End}^0(A_{/F})$ is a division algebra with a positive involution $\alpha \mapsto \alpha^*$. Since D has a maximal commutative subfield stable under $*$, we may assume that its endomorphism field E is totally real or a CM field. The Galois action on the Tate module of an F-simple abelian variety $A_{/F}$ of $GL(2)$-type with endomorphism field E produces a two-dimensional strictly compatible system of Galois representations $\rho_A = \{\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(E_\lambda)\}_\lambda$ indexed by primes λ of E. Thus we have its L-function $L(s, \rho_A)$. Two F-simple abelian varieties A and $B = A_\chi$ are twist equivalent if $L(s, \rho_B) = L(s, \rho_A \otimes \chi)$ for a finite order character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{\mathbb{Q}}^\times$. Note that the dimension is possibly unbounded over a twist equivalent class. Since Tate's conjecture has been proven by Faltings for abelian varieties, one could formulate this equivalence by insisting that the two abelian varieties share a simple component over an abelian extension of F. In this more geometric context, $\mathbb{Q}$-simple abelian varieties have been studied in depth as $\mathbb{Q}$-simple factors of modular Jacobians by Ribet (for example, see [R] and his papers quoted there). However we adopt an analytic definition of twist-equivalence using their L-function as it can also be applied to rank 2 $\mathbb{Q}$-motives (for which the Tate conjecture is still unknown). Here the identity of two L-functions

means the identity of the coefficients of the Dirichlet series after suitably choosing embeddings of the endomorphism fields of A and B, respectively, into $\overline{\mathbb{Q}}$. We recall that an abelian scheme A of dimension g over a field κ of characteristic p is called *ordinary* if we can embed μ_p^g into A over an algebraic closure of κ. In this paper, the expression *number field* means a finite extension of $\mathbb{Q}$. We make the following

Conjecture. *For a given base number field F, fixing a prime ideal $\mathfrak{p}$ of F over a rational prime p, there are only finitely many twist equivalence classes of non-CM F-simple abelian varieties of $GL(2)$-type with good reduction everywhere outside $\mathfrak{p}$ and potentially ordinary good reduction modulo $\mathfrak{p}$.*

Assuming $F \neq \mathbb{Q}$, if $F_\mathfrak{p} = \mathbb{Q}_p$ and F is not an imaginary quadratic field, there are only finitely many characters of $\mathfrak{p}$-power conductor; so, one may replace twist-equivalence by isomorphism in the conjecture. If $F_\mathfrak{p} \neq \mathbb{Q}_p$, there could be infinitely many characters of $\mathfrak{p}$-power conductor (this is a problem related to the Leopoldt conjecture). Anyway, in the extreme case of $[F_\mathfrak{p} : \mathbb{Q}_p] = [F : \mathbb{Q}]$, we have infinitely many characters of p-power conductor, and starting from an F-simple A, the twist A_χ by a character χ with a p-power conductor satisfies an inequality $\dim A_\chi \geq [\mathbb{Q}(\chi) : \mathbb{Q}]$ if the order of χ is sufficiently large (since $\mathrm{End}^0(A) \supset \mathbb{Q}(\chi)$). Thus without bounding the dimension, we cannot replace twist-equivalence by isomorphism. We have formulated the conjecture in the above way since by doing this, there is no need to bound the dimension (nor the $\mathfrak{p}$-conductor). If we delve into Ribet's theory of absolutely simple factors of modular Jacobians (cf. [R1]), one might be able to formulate the conjecture in terms of absolutely simple factors (but we do not touch this point in this paper).

For some primes p, it is easy to see that there are infinitely many twist equivalence classes of super-singular (i.e., no nontrivial p-torsion $\overline{\mathbb{F}}_p$-points) $\mathbb{Q}$-simple abelian varieties of $GL(2)$-type having potentially good reduction at p and good reduction everywhere else (see Remark 3.2). If we fix a $\mathbb{Q}$-simple non-CM abelian variety A with $\dim A \leq 2$ and vary primes, A is ordinary at each prime (of a well-chosen number field possibly different from $\mathbb{Q}$) in a set with Kronecker density one (see [O], 2.7 and [H12], Section 7). The density 1 result is only known up to abelian surfaces but is expected to be true in general. On the other hand, if we vary such abelian varieties, even allowing potential ordinarity at p (but requiring good reduction outside p), we believe that its twist equivalence classes are finitely many. What we prove in this paper is

Main Theorem. *If $p > 2$ and $F = \mathbb{Q}$, then the above conjecture holds.*

Actually, for small primes $p = 3, 5$ and 7, there is no such abelian variety defined over $\mathbb{Q}$ (see the argument after Question 2.3). There is also a motivic version of this theorem (see Remark 4.2). If we take for granted a well-believed principle that the L-function of the compatible system ρ_A of an F-simple abelian variety A of $GL(2)$-type should be given by the L-function of a cohomological automorphic representation π of $GL_2(F_\mathbb{A})$ of weight corresponding to Hodge weight $(1, 0)$, one would expect that there are finitely many isogeny classes of F-simple abelian varieties with everywhere good reduction, since such a π must have level 1 and there are only finitely many such π of level 1. Since modularity of ρ_A has been proved at least potentially if F is totally real by R. Taylor (cf. [T]), this finiteness (under bounding conductor) could now be accessible, but some serious problems remain. More precisely, for each ρ_A over a totally real field F, Taylor finds a totally real

extension $F_{A/F}$ dependent on A over which ρ_A is modular. A conceivable idea is to bound the number of isogeny classes of modular abelian varieties A' of $GL(2)$-type of conductor 1 over F_A which descends to F. If we find such a finite extension $F' := F_A$ independent of A, the desired finiteness follows, as there are finitely many automorphic forms of level 1 on $GL(2)_{/F'}$ of a given weight. The problem of finding F' independent of A is difficult without assuming the desired finiteness outright. Thus, plainly there is some serious work to be done for making explicit the field F_A (at present, we do not even know the finiteness of isomorphism classes of 2-dimensional mod p odd Galois representations of a given prime-to-p conductor for a general $F \neq \mathbb{Q}$). More geometrically, it is a celebrated theorem of Fontaine that there is no abelian scheme over $\mathbb{Z}$ (see [F]). Fontaine's theorem is a starting point of the induction process found by Khare–Wintenberger for proving Serre's mod p modularity conjecture (see [KW]). If we ease the reduction property to potentially good ordinary reduction, we have to allow all π of p-power level; so, even for twist-equivalence classes, it is not evident that the number of classes is finite. By [KW], I, Theorem 10.1, if $F = \mathbb{Q}$, ρ_A is known to be associated to an elliptic Hecke eigenform f and A is isogenous to Shimura's abelian factor A_f of the Jacobian of $X_1(N)$ for the conductor N of ρ_A. We use this fact and a characterization of CM p-adic analytic families of cusp forms given in [H11] to prove in the text a stronger version of the main theorem (see Theorem 4.1).

We assume $p > 2$ in the main theorem because of the reliance of this result on one of the main results in [H11] (see Theorem 1.1 in the text) whose proof was given assuming $p > 2$, though we probably would be able to remove this assumption at least for the result we use here. Since one of the referees pointed out the importance of including the prime $p = 2$ in our scope in view of (hopefully) plausible applications towards generalization (in Hilbert modular cases) of the proof of the mod p modularity conjecture, the author hopes to include $p = 2$ and also try to generalize the result of this paper to totally real base fields in a forthcoming paper (see Remark 4.3 for some more details of possible generalization).

Throughout this paper, we fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and field embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. The p-adic completion of $\overline{\mathbb{Q}}_p$ is denoted by $\mathbb{C}_p$.

1. Analytic families of Hecke eigenforms

Fix a positive integer N prime to p. A p-adic analytic family $\mathcal{F}$ of modular forms is defined with respect to the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We write $|\alpha|_p$ for the p-adic absolute value (with $|p|_p = 1/p$) induced by i_p. We also fix a field embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ inducing the inclusion on $\overline{\mathbb{Q}} \subset \mathbb{C}$. As a base ring, we take a (sufficiently large) discrete valuation ring $W \subset \overline{\mathbb{Q}}_p$ that is finite flat over the p-adic integer ring $\mathbb{Z}_p$. Put $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ otherwise. Take a Dirichlet character $\psi : (\mathbb{Z}/Np^r\mathbf{p}\mathbb{Z})^\times \to W^\times$ with $(p \nmid N, r \geq 0)$, and consider the space of elliptic cusp forms $S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$ with character ψ as defined in [IAT] (3.5.4). Let the ring $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively. The Hecke algebra over $\mathbb{Z}[\psi]$ is the subalgebra of the linear endomorphism algebra of $S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$ generated by the Hecke operators $T(n)$:

$$h = \mathbb{Z}[\psi][T(n)|n = 1, 2, \cdots] \subset \mathrm{End}(S_k(\Gamma_0(Np^r\mathbf{p}), \psi)),$$

where $T(n)$ is the Hecke operator as in [IAT], §3.5. We put $h_{k,\psi} = h_{k,\psi/W} = h \otimes_{\mathbb{Z}[\psi]} W$. When we need to indicate that our $T(l)$ is the Hecke operator of a prime factor l of $Np^r\mathbf{p}$, we write it as $U(l)$, since $T(l)$ acting on a subspace $S_k(\Gamma_0(N'),\psi) \subset S_k(\Gamma_0(Np^r\mathbf{p}),\psi)$ of level N' prime to l does not coincide with $U(l)$ on $S_k(\Gamma_0(Np^r\mathbf{p}),\psi)$. The ordinary part $\mathbf{h}_{k,\psi/W} \subset h_{k,\psi/W}$ is the maximal ring direct summand on which $U(p)$ is invertible. We write e for the idempotent of $\mathbf{h}_{k,\psi/W}$, and hence $e = \lim_{n\to\infty} U(p)^{n!}$ under the p-adic topology of $h_{k,\psi/W}$. By the fixed embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, the idempotent e not only acts on the space of modular forms with coefficients in W but also on the classical space $S_k(\Gamma_0(Np^r\mathbf{p}),\psi)$. We write the image of the idempotent as S_k^{ord}.

Fix ψ, and assume now that $\psi_p = \psi|_{\mathbb{Z}_p^\times}$ has conductor at most $\mathbf{p}$ and $\psi(-1) = 1$. Let ω be the modulo $\mathbf{p}$ Teichmüller character (so, if $p = 2$, ω is the unique nontrivial character of $(\mathbb{Z}/4\mathbb{Z})^\times$). Recall the multiplicative group $\Gamma := 1 + \mathbf{p}\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ and its topological generator $\gamma = 1+\mathbf{p}$. The Iwasawa algebra $\Lambda = W[[\Gamma]] = \varprojlim_n W[\Gamma/\Gamma^{p^n}]$ is identified with the power series ring $W[[x]]$ by a W-algebra isomorphism sending $\gamma \in \Gamma$ to $1+x$. As constructed in [H86a], [H86b] and [GME], we have a unique 'big' ordinary Hecke algebra $\mathbf{h} = \mathbf{h}(\psi)$. The algebra $\mathbf{h}$ is characterized by the following two properties (called Control theorems; see [H86a], Theorem 3.1, Corollary 3.2 and [H86b], Theorem 1.2 for $p \geq 5$ and [GME], Theorem 3.2.15 and Corollary 3.2.18 for general p):

(C1) $\mathbf{h}$ is free of finite rank over Λ equipped with $T(n) \in \mathbf{h}$ for all $1 \leq n \in \mathbb{Z}$ (so $U(l)$ for $l|Np$),

(C2) if $k \geq 2$ and $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a character,

$$\mathbf{h}/(1 + x - \varepsilon(\gamma)\gamma^k)\mathbf{h} \cong \mathbf{h}_{k,\varepsilon\psi_k} \ (\gamma = 1 + p) \text{ for } \psi_k := \psi\omega^{-k},$$

sending $T(n)$ to $T(n)$ (and $U(l)$ to $U(l)$ for $l|Np$).

In the sequel, we sometimes make use of another variable $X = \gamma^{-1}(1+x) - 1$. We still have $\Lambda = W[[X]]$. The prime ideal $((1 + x) - \gamma)$ is equal to (X); so, if (C2) were valid for $k = 1$, $\mathbf{h}/X\mathbf{h}$ would have been the Hecke algebra of weight 1; so, we call the variable X of Λ the variable *centered* at weight 1.

Let $\mathrm{Spec}(\mathbb{I})$ be a reduced irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. Write $a(n)$ for the image of $T(n)$ in $\mathbb{I}$ (so, $a(p)$ is the image of $U(p)$). If a point P of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ (regarded as a W-algebra homomorphism $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$) kills $(1 + x - \varepsilon(\gamma)\gamma^k) = (1 + X - \varepsilon(\gamma)\gamma^{k-1})$ with $2 \leq k \in \mathbb{Z}$ (i.e., $P(1 + x - \varepsilon(\gamma)\gamma^k) = 0$), we call it an *arithmetic* point, and we write $\varepsilon_P = \varepsilon$, $k(P) = k \geq 2$ and $p^{r(P)}$ for the order of ε_P. If P is arithmetic, by (C2), we have a Hecke eigenform $f_P \in S_k(\Gamma_0(Np^{r(P)}\mathbf{p}), \varepsilon\psi_k)$ such that its eigenvalue for $T(n)$ is given by $a_P(n) := P(a(n)) \in \overline{\mathbb{Q}}_p$ for all n. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \{f_P | \text{arithmetic } P \in \mathrm{Spec}(\mathbb{I})\}$ of Hecke eigenforms. We define a *p-adic analytic family of slope* 0 (with coefficients in $\mathbb{I}$) to be the family as above of Hecke eigenforms associated to an irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. We call this family slope 0 because $|a_P(p)|_p = 1 = p^0$ for the p-adic absolute value $|\cdot|_p$ of $\overline{\mathbb{Q}}_p$ (it is also often called an ordinary family). We call this family analytic because the Hecke eigenvalue $P \mapsto a_P(n)$ at P for $T(n)$ is given by an analytic function $a(n)$ on (the rigid analytic space associated to) the p-profinite formal spectrum $\mathrm{Spf}(\mathbb{I})$. Identify $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}_{W\text{-alg}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$ so that each element $a \in \mathbb{I}$ gives rise to a "function" $a : \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ whose value at $(P : \mathbb{I} \to \overline{\mathbb{Q}}_p) \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$

is $a_P := P(a) \in \overline{\mathbb{Q}}_p$. Then a is an analytic function of the rigid analytic space associated to $\mathrm{Spf}(\mathbb{I})$ (in the sense of Berthelot as in [dJ], Section 7).

Each (reduced) irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$ has a 2-dimensional absolutely irreducible continuous representation $\rho_{\mathbb{I}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients in the quotient field of $\mathbb{I}$ (see [H86b]). The representation $\rho_{\mathbb{I}}$ restricted to the p-decomposition group D_p is reducible with unramified *quotient* character (e.g., [GME], §4.2 and §4.3). We write $\rho_{\mathbb{I}}^{ss}$ for its semi-simplification over D_p. As is now well known (e.g., [GME], §4.2), $\rho_{\mathbb{I}}$ is unramified outside Np and satisfies
(Gal)
$$\mathrm{Tr}(\rho_{\mathbb{I}}(\mathrm{Frob}_l)) = a(l) \ \ (l \nmid Np), \ \rho_{\mathbb{I}}^{ss}([\gamma^s, \mathbb{Q}_p]) \sim \left(\begin{smallmatrix} (1+X)^s & 0 \\ 0 & 1 \end{smallmatrix} \right) \text{ and } \rho_{\mathbb{I}}^{ss}([p, \mathbb{Q}_p]) \sim \left(\begin{smallmatrix} * & 0 \\ 0 & a(p) \end{smallmatrix} \right),$$

where $\gamma^s = (1+\mathbf{p})^s = \sum_{n=0}^{\infty} \binom{s}{n} \mathbf{p}^n \in \mathbb{Z}_p^{\times}$ for $s \in \mathbb{Z}_p$ and $[x, \mathbb{Q}_p]$ is the local Artin symbol.

By (Gal) and Chebotarev density, $\mathrm{Tr}(\rho_{\mathbb{I}})$ has values in $\mathbb{I}$; so, $P \circ \mathrm{Tr}(\rho_{\mathbb{I}}) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}_p$ ($P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$) gives rise to a pseudo-representation of Wiles (e.g., [MFG], §2.2). Then by a theorem of Wiles, we can make a unique 2-dimensional semi-simple continuous representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$ unramified outside Np with $\mathrm{Tr}(\rho_P(\mathrm{Frob}_l)) = a_P(l)$ for all primes l outside Np (though the construction of ρ_P does not require the technique of pseudo representation and was known before the invention of the technique; see [MW], §9, Proposition 1). When P is arithmetic, this is the Galois representation associated to the Hecke eigenform f_P (constructed earlier by Eichler–Shimura and Deligne; e.g., [GME], §4.2).

A component $\mathbb{I}$ is called a *CM component* if there exists a nontrivial character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{I}^{\times}$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$. We also say that $\mathbb{I}$ has *complex multiplication* if $\mathbb{I}$ is a CM component. In this case, we call the corresponding family $\mathcal{F}$ a CM family (or we say that $\mathcal{F}$ has complex multiplication). If $\mathcal{F}$ is a CM family associated to $\mathbb{I}$ with $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$, then χ is a quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which cuts out an imaginary quadratic field M, i.e., $\chi = \left(\frac{M/\mathbb{Q}}{} \right)$. Write $\widetilde{\mathbb{I}}$ for the integral closure of Λ inside the quotient field of $\mathbb{I}$. The following three conditions are known to be equivalent:

(CM1) $\mathcal{F}$ *has CM with* $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \left(\frac{M/\mathbb{Q}}{} \right)$ *($\Leftrightarrow \rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \widehat{\lambda}$ for a character $\widehat{\lambda} :$* $\mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^{\times}$*).*

(CM2) *For all arithmetic P of* $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$*, f_P is a binary theta series of the norm form of $M/\mathbb{Q}$.*

(CM3) *For some arithmetic P of* $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$*, f_P is a binary theta series of the norm form of $M/\mathbb{Q}$.*

Indeed, (CM1) is equivalent to $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \widehat{\lambda}$ for a character $\widehat{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^{\times}$ unramified outside Np (e.g., [MFG], Lemma 2.15). Since the characteristic polynomial of $\rho_{\mathbb{I}}(\sigma)$ has coefficients in $\mathbb{I}$, its eigenvalues fall in $\widetilde{\mathbb{I}}$; so, the character $\widehat{\lambda}$ has values in $\widetilde{\mathbb{I}}^{\times}$ (see [H86c], Corollary 4.2). Then by (Gal), $\widehat{\lambda}_P = P \circ \widehat{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^{\times}$ for an arithmetic $P \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ is a locally algebraic p-adic character, which is the p-adic avatar of a Hecke character $\lambda_P : M_{\mathbb{A}}^{\times}/M^{\times} \to \mathbb{C}^{\times}$ of type A_0 of the quadratic field $M_{/\mathbb{Q}}$. Again by (Gal), f_P is the theta series with q-expansion $\sum_{\mathfrak{a}} \lambda_P(\mathfrak{a}) q^{N(\mathfrak{a})}$, where $\mathfrak{a}$ runs over all integral ideals of M. By $k(P) \geq 2$ (and (Gal)), M has to be an imaginary quadratic field in which p is split (as holomorphic binary theta series of real quadratic fields are limited to

weight 1; cf. [MFM], §4.8). This shows that (CM1)⇒(CM2)⇒(CM3). If (CM2) is satisfied, we have an identity $\mathrm{Tr}(\rho_{\mathbb{I}}(\mathrm{Frob}_l)) = a(l) = \chi(l)a(l) = \mathrm{Tr}(\rho_{\mathbb{I}} \otimes \chi(\mathrm{Frob}_l))$ with $\chi = \left(\frac{M/\mathbb{Q}}{}\right)$ for all primes l outside Np. By Chebotarev density, we have $\mathrm{Tr}(\rho_{\mathbb{I}}) = \mathrm{Tr}(\rho_{\mathbb{I}} \otimes \chi)$, and we get (CM1) from (CM2) as $\rho_{\mathbb{I}}$ is semi-simple. If a component $\mathrm{Spec}(\mathbb{I})$ contains an arithmetic point P with theta series f_P as above of $M/\mathbb{Q}$, either $\mathbb{I}$ is a CM component or otherwise P is in the intersection in $\mathrm{Spec}(\mathbf{h})$ of a component $\mathrm{Spec}(\mathbb{I})$ not having CM by M and another component having CM by M (as all families with CM by M are made up of theta series of M by the construction of CM components in [H86a], §7). The latter case cannot happen as two distinct components never cross at an arithmetic point in $\mathrm{Spec}(\mathbf{h})$ (i.e., the reduced part of the localization $\mathbf{h}_P$ is étale over Λ_P for any arithmetic point $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$; see [HMI], Proposition 3.78). Thus (CM3) implies (CM2). We call a binary theta series of the norm form of an imaginary quadratic field a *CM theta series*.

We quote the following key result from [H11], Theorem 3.3 and Corollary 6.3 combined.

Theorem 1.1. *Suppose $p > 2$. Let $K := \mathbb{Q}[\mu_{p^\infty}]$ and $\mathcal{A} \subset \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$ be an infinite set of arithmetic points P with fixed weight $k(P) = k \geq 2$. The family $\mathcal{F}$ has complex multiplication if and only if $\limsup_{P \in \mathcal{A}}[K(a(p, f_P)) : K] < +\infty$.*

2. ARCHIMEDEAN AND p-ADIC CM TYPES

We define $\mathrm{Inf}_{L/F} \Sigma = \left\{\sigma : L \hookrightarrow \overline{\mathbb{Q}} \,\middle|\, \sigma|_F \in \Sigma\right\}$ for a subset Σ of the set of embeddings of a number field F into $\overline{\mathbb{Q}}$ and a finite extension L/F. Assume that F and L are CM fields. Then $\mathrm{Inf}_{L/F} \Sigma$ is again a CM type of L if Σ is a CM type of F. For a CM type Σ of F, write Σ_p for the p-adic places of F induced by $i_p \circ \sigma$ for $\sigma \in \Sigma$. Write c for the complex conjugation induced by i_∞. If $\Sigma_p \cup \Sigma_p^c$ is the set of all p-adic places of F with $\Sigma_p \cap \Sigma_p^c = \emptyset$, we call Σ_p a *p-adic CM type* (of F). We prove

Theorem 2.1. *Let $\mathcal{A}$ be an infinite set of arithmetic points in $\mathrm{Spec}(\mathbb{I})$ outside $((1+x) - \gamma^2)$. Suppose $p > 2$ and that $k(P)$ for all $P \in \mathcal{A}$ is a constant $k \geq 2$. We then have*

   (1) *The family $\mathcal{F}$ has CM if and only if*
$$\Sigma_P = \left\{\sigma : \mathbb{Q}(a(p, f_P)) \hookrightarrow \overline{\mathbb{Q}} \,\middle|\, |i_p(\sigma(a(p, f_P)))|_p = 1\right\}$$
   is a CM type of $\mathbb{Q}(a(p, f_P))$ for all $P \in \mathcal{A}$.
   (2) *The family $\mathcal{F}$ has CM if and only if the ratio $[\mathbb{Q}(a(p, f_P)) : \mathbb{Q}]/|\Sigma_P|$ is bounded independent of $P \in \mathcal{A}$.*

We avoid primes over $((1 + x) - \gamma^2)$ because f_P for such a prime could have its abelian variety A_P with potentially multiplicative reduction and in that case, Σ_P is the complete set of embeddings of $\mathbb{Q}(a(p, f_P))$ into $\overline{\mathbb{Q}}_p$ (not a half or less).

Proof. Actually the second assertion essentially implies the first, because we have $[\mathbb{Q}(a(p, f_P)) : \mathbb{Q}]/|\Sigma_P| = 2$ for all $P \in \mathcal{A}$ if Σ_P is a CM type. However for expository reasons, we first prove (1).

A CM abelian variety of type (F, Σ) over a number field has potentially *ordinary* good reduction at the place induced by i_p if and only if Σ_p is a p-adic CM type. Let $F_P = \mathbb{Q}(a(p, f_P))$. If $\mathcal{F}$ has CM under an imaginary quadratic field M, then

F_P contains M, and $\Sigma_P = \text{Inf}_{F_P/M} \Sigma_0$ for the CM type Σ_0 given by $\Sigma_{0,p} = \{i_p|_M\}$. This shows the "only if" part.

To prove the "if" part, let $K_P = \mathbb{Q}(\varepsilon_P)$ and $L_P = F_P(\varepsilon_P)$. Then $\text{Inf}_{L_P/F_P} \Sigma_P$ is a CM type of L_P. Thus we have $2|\text{Inf}_{L_P/F_P} \Sigma_P| = [L_P : \mathbb{Q}]$. Let $K = \mathbb{Q}[\mu_{p^\infty}]$. By the definition of Σ_P, we have $\Sigma_{P,p} \cap \Sigma_{P,p}^c = \emptyset$; i.e., Σ_P gives rise to a p-adic CM type. Since K has only one p-adic place and $[K_P : \mathbb{Q}] = p^{r(P)-1}(p-1)$, for a fixed embedding $\sigma_0 : K \hookrightarrow \overline{\mathbb{Q}}$, we have

(2.1)
$$\left|\text{Inf}_{L_P/F_P} \Sigma_P\right| = \left|\{\sigma \in \text{Inf}_{L_P/F_P} \Sigma_P : \sigma|_{K_P} = \sigma_0|_{K_P}\}\right| \cdot [K_P : \mathbb{Q}] = Cp^{r(P)}[L_P : K_P]$$

for the constant $C = (p-1)/p$ independent of P. Take W sufficiently large so that all characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$ have values in $W^\times$.

Write $H = \bigoplus_\psi \mathbf{h}(\psi)$, where ψ runs over all (even) characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. Since $\left|\text{Inf}_{L_P/F_P} \Sigma_P\right|$ is at most the number of conjugate slope 0 forms f_P^σ indexed by $\sigma \in \Sigma_P$ (which is bounded by the rank of the Hecke algebra $H/((1+x)^{p^r} - \gamma^k)H$ acting on them), we have, for $r = r(P)$,

(2.2) $|\text{Inf}_{L_P/F_P} \Sigma_P| \leq \text{rank}_W H/((1+x)^{p^r} - \gamma^k)H = p^{r(P)} \cdot \text{rank}_{W[[x]]} H.$

Since $[K(a(p, f_P)) : K] = [L_P : L_P \cap K] \leq [L_P : K_P]$ as $L_P \cap K \supset K_P$, (2.1) and (2.2) combined tells us that

$$[K(a(p, f_P)) : K] \leq C^{-1} \text{rank}_{W[[x]]} H.$$

This is impossible if $\mathcal{F}$ does not have CM, since $\sup_{P \in \mathcal{A}}[K(a(p, f_P)) : K] = \infty$ by Theorem 1.1.

We now prove (2). Write the bound as B; so, $[F_P : \mathbb{Q}] \leq B|\Sigma_P|$ for all $P \in \mathcal{A}$. Thus $[L_P : \mathbb{Q}] \leq B|\text{Inf}_{L_P/F_P} \Sigma_P|$. Since the equality (2.1) and the estimate (2.2) still hold, we have, for $C = (p-1)/p$,

$$[K(a(p, f_P)) : K]p^{r(P)}C = [L_P : L_P \cap K]p^{r(P)}C \leq [L_P : K_P]p^{r(P)}C = [L_P : \mathbb{Q}]$$
$$= B\left|\text{Inf}_{L_P/F_P} \Sigma_P\right| \leq Bp^{r(P)} \text{rank}_{W[[x]]} H,$$

which implies that

$$[K(a(p, f_P)) : K] \leq BC^{-1} \text{rank}_{W[[x]]} H.$$

Again by Theorem 1.1, $\mathcal{F}$ has CM. $\square$

If $k(P) = 2$ and $\psi_2\varepsilon_P \neq 1$, Σ_P is a CM type if and only if the abelian variety A_P associated to f_P in [IAT] has ordinary good reduction over $\mathbb{Z}_p[\mu_{p^{r(P)+1}}]$. To see this, as is well known, the Frobenius endomorphism Φ of $\widetilde{A}_P := A_P \otimes_{\mathbb{Z}_p[\mu_{p^{r(P)+1}}]} \mathbb{F}_p$ coincides with $U(p)$ on the étale Barsotti–Tate group $\widetilde{A}_P[p^\infty]^{et}$ (e.g., [GME], Theorem 4.2.6 (1)); so, the characteristic polynomial $\mathcal{P}(X)$ of Φ over $\mathbb{Q}$ coincides with a power of the characteristic polynomial of $\alpha := a(p, f_P)$. Then decomposing the total set of field embeddings of $\mathbb{Q}(f_P)$ into

$$\text{Inf}_{\mathbb{Q}(f_P)/F_P} \Sigma_P \sqcup \Xi_P \sqcup \text{Inf}_{\mathbb{Q}(f_P)/F_P} \Sigma_P^c,$$

the set Ξ_P is characterized by

$$\left\{\sigma : \mathbb{Q}(f_P) \hookrightarrow \overline{\mathbb{Q}}_p \big| p^{-1} < |i_p(\alpha^\sigma)|_p < 1\right\}.$$

Thus $\Xi_P \neq \emptyset$ ($\Leftrightarrow \Sigma_P$ is not a CM type) implies that the Newton polygon of $\mathcal{P}(X)$ has middle positive slope between 0 and 1; so, $\widetilde{A}_P$ is ordinary if and only if Σ_P is a CM type. Thus we get from the above theorem the following fact.

Corollary 2.2. *Let $\mathcal{A}$ be an infinite set of arithmetic points in $\mathrm{Spec}(\mathbb{I})$ outside $((1+x)-\gamma^2)$. Suppose $p > 2$ and $k(P) = 2$ for all $P \in \mathcal{A}$. The family $\mathcal{F}$ has CM if and only if the abelian variety A_P associated to f_P by Shimura has ordinary good reduction over $\mathbb{Z}_p[\mu_{p^{r(P)+1}}]$ for all $P \in \mathcal{A}$. In particular, if $\mathcal{F}$ does not have CM, the set made up of arithmetic points P with A_P having ordinary good reduction over $\mathbb{Z}_p[\mu_{p^{r(P)+1}}]$ is a finite set.*

Question 2.3. For a given slope 0 family $\mathcal{F}$ without CM indexed by $\mathrm{Spec}(\mathbb{I})$, let

$$Ord_2(\mathbb{I}) = \{P | A_P \text{ has ordinary good reduction over } \mathbb{Z}_p[\mu_{p^{r(P)+1}}]\}.$$

By the corollary above, this set is a finite set. What are the numbers

$$|Ord_2(\mathbb{I})| \quad \text{and} \quad R = \sup_{P \in Ord_2(\mathbb{I})} r(P)?$$

Could $Ord_2(\mathbb{I})$ be empty?

For $N = 1$ and for very small primes $p = 3, 5$ and 7, because of the nonexistence of slope 0 analytic families of prime-to-p level 1, we conclude the nonexistence of non-CM $\mathbb{Q}$-simple abelian varieties of $GL(2)$-type defined over $\mathbb{Z}[\frac{1}{p}]$ with potentially ordinary good reduction at p. Similarly, it is known that $X_1(Np)$ has genus 0 if $1 \leq Np \leq 10$ and $Np = 12$; so, we can extend a bit our list of (N, p) without a potentially p-ordinary abelian variety of $GL(2)$-type of prime-to-p conductor N. On the other hand, the prime $p = 11$ is the smallest to have a nontrivial family (of prime-to-p level 1), but even this simplest family at $p = 11$ containing Ramanujan's Δ-function, we do not know if $Ord_2(\mathbb{I}) = \emptyset$. Thus it would be interesting to make specific computations targeted to finding potentially ordinary factors of the modular Jacobians for a fixed small level N.

Eknath Ghate informed me of the following example of A_P with nonordinary good reduction at $p = 13$: Take weight 2, level $221 = 13 \cdot 17$ with quadratic nebentypus. Then the Hecke algebra breaks into 4 Galois orbits (of dimension each $4+4+6+6$ dimensional over $\mathbb{Q}$). The two six-dimensional factors have equal Hecke polynomial for $T(13)$ given by

$$x^6 + 8x^5 + 31x^4 + 104x^3 + 403x^2 + 1352x + 2197 \equiv x^4(x^2 + 8x + 5) \mod 13.$$

Thus we have slopes 0, 1/2, and 1, each of multiplicity 2 for the above polynomial. These abelian varieties are associated to slope 0 forms but do not have ordinary good reduction over $\mathbb{Z}_{13}$. See [BG], Table I for some other information of this example. Numerically, we can find A_P with nonordinary reduction modulo p, but they are rather rare for small level primes p; so, $Ord_2(\mathbb{I})$ could be large though it is a finite set.

The same question can be asked, fixing $k(P) = k > 2$ and moving arithmetic P, for

$$Ord_k(\mathbb{I}) = \{P | \text{the motive } M_P \text{ is potentially ordinary crystalline over } \mathbb{Z}_p[\mu_{p^{r(P)+1}}]\}.$$

Here M_P is the rank 2 motive attached to f_P with coefficients in $\mathbb{Q}(f_P)$. However the ordinarity here means that its Newton polygon of M_P as a motive with coefficients in $\mathbb{Q}$ coincides with the Hodge polygon of M_P. Again for (N, p) listed above

for the nonexistence of such abelian schemes, we conclude also the nonexistence of such motives *independent* of weight k.

We add here a p-adic version of Theorem 2.1. Though we do not use the following result in the proof of the main theorem, the result might have some value on its own, as there is some hope of determining (asymptotically) the number of primes over p and their ramification in the Hecke field via this type of results.

Theorem 2.4. *Let the notation be as in* Theorem 2.1. *Let $\mathcal{A}$ be an infinite set of arithmetic points in* $\mathrm{Spec}(\mathbb{I})$ *outside* $((1+x)-\gamma^2)$. *Suppose $p > 2$ and that $k(P)$ for $P \in \mathcal{A}$ is a constant $k \geq 2$. We then have:*

  (1) *The family $\mathcal{F}$ has CM if and only if $\Sigma_{P,p}$ is a p-adic CM type of $\mathbb{Q}(a(p, f_P))$ for all $P \in \mathcal{A}$.*
  (2) *The family $\mathcal{F}$ has CM if and only if the ratio $[\mathbb{Q}(a(p, f_P)) : \mathbb{Q}]/|\Sigma_{P,p}|^{r(P)}$ is bounded independent of $P \in \mathcal{A}$.*

Proof. By the definition of Σ_P, we have $\Sigma_{P,p} \cap \Sigma_{P,p}^c = \emptyset$. If $\Sigma_{P,p}$ is a p-adic CM type of $\mathbb{Q}(a(p, f_P))$, then plainly Σ_P is a CM type of $\mathbb{Q}(a(p, f_P))$; so, the first assertion follows from Theorem 2.1 (1).

We prove (2). We use the notation defined in the proof of Theorem 2.1; so, $F_P = \mathbb{Q}(a(p, f_P))$, $K_P = \mathbb{Q}(\varepsilon_P)$ and $L_P = F_P(\varepsilon_P)$. Let $K = \mathbb{Q}[\mu_{p^\infty}]$. Since K has only one p-adic place and $[K_P : \mathbb{Q}] = p^{r(P)-1}(p-1)$, for a fixed embedding $\sigma_0 : K \hookrightarrow \overline{\mathbb{Q}}$, we have

$$(2.3) \qquad \left|\mathrm{Inf}_{L_P/F_P} \Sigma_{P,p}\right| \leq \left|\{\sigma \in \mathrm{Inf}_{L_P/F_P} \Sigma_P : \sigma|_{K_P} = \sigma_0|_{K_P}\}\right| = [L_P : K_P].$$

Take W sufficiently large so that all characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$ have values in $W^\times$. Write $H = \bigoplus_\psi \mathbf{h}(\psi)$, where ψ runs over all characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. On the other hand, since $\mathrm{Inf}_{L_P/F_P} \Sigma_P$ is at most the number of conjugate slope 0 forms f_P^σ (which is bounded by the rank of the Hecke algebra $H/((1+x)^{p^r} - \gamma^k)H$ acting on them), we have, for $r = r(P)$,

$$|\mathrm{Inf}_{L_P/F_P} \Sigma_{P,p}|^{r(P)-1}(p-1) \leq \mathrm{rank}_W H/((1+x)^{p^r} - \gamma^k)H = p^{r(P)} \mathrm{rank}_{W[[x]]} H.$$

Thus for $C = \frac{p}{p-1}$, we have

$$(2.4) \qquad\qquad |\mathrm{Inf}_{L_P/F_P} \Sigma_{P,p}| \leq C \, \mathrm{rank}_{W[[x]]} H.$$

Write the bound as B; so, $[F_P : \mathbb{Q}]/p^{r(P)} \leq B|\Sigma_{P,p}|$ for all $P \in \mathcal{A}$. Thus $[L_P : K_P] \leq C^{-1}B|\mathrm{Inf}_{L_P/F_P} \Sigma_P|$. Since $[K(a(p, f_P)) : K] = [L_P : L_P \cap K] \leq [L_P : K_P]$ as $L_P \cap K \supset K_P$, by the equality (2.3) and the estimate (2.4), we have, for $C = (p-1)/p$,

$$[K(a(p, f_P)) : K] = [L_P : L_P \cap K] \leq [L_P : K_P]$$
$$= C^{-1}B \left|\mathrm{Inf}_{L_P/F_P} \Sigma_{P,p}\right| \leq C^{-1}B \, \mathrm{rank}_{W[[x]]} H,$$

which implies that

$$[K(a(p, f_P)) : K] \leq BC^{-1} \mathrm{rank}_{W[[x]]} H.$$

Again by Theorem 1.1, $\mathcal{F}$ has CM.   □

3. Super-cuspidality implies super-singularity

Let f be an elliptic Hecke eigenform of weight 2 which generates an auto-morphic representation $\pi = \bigotimes_l \pi_l$ with local representation π_l at primes l. Let $\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_l)$ be the λ-adic Galois representation associated to f. Then we know by [C], Théorème A and [W], if π_p is super-cuspidal and $p > 2$, for $\lambda \nmid p$, there exists a quadratic extension $M/\mathbb{Q}_p$ and an infinite order character $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/M) \to \overline{\mathbb{Q}}_l^\times$ such that $\rho_\lambda|_{D_p}$ is irreducible, $\rho_\lambda|_{D_p} \cong \mathrm{Ind}_M^{\mathbb{Q}} \varphi$ and $\varphi|_{I_p}$ has finite order (for the decomposition group $D_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and its inertia subgroup I_p). Even if $p = 2$, again by Weil [W] (see also [K], 5.1), there exists a finite extension K of $\mathbb{Q}_2$ and a quadratic extension M/K such that $\rho_\lambda|_{D'_p}$ is irre-ducible, $\rho_\lambda|_{D'_p} \cong \mathrm{Ind}_M^K \varphi$ and $\varphi|_{I'_p}$ has finite order (now for the decomposition group $D'_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ and its inertia subgroup I'_p). Since any Galois conjugate f^σ has its automorphic representation super-cuspidal at p, writing l for the rational prime below λ, the l-adic Tate module of Shimura's abelian variety A_f attached to f (cf. [IAT], Theorem 7.14) becomes unramified at p over a finite extension of $\mathbb{Q}_p$; so, A_f has potentially good reduction at p by the theorem of Néron–Ogg–Shafarevich in [ST].

Proposition 3.1. *Let the notation and the assumption be as above. If π is super-cuspidal at p, A_f has potentially good reduction at p but can never be ordinary.*

Proof. We use the notation introduced above the proposition. In particular, take $K = \mathbb{Q}_p$ if $p > 2$ and $K_{/\mathbb{Q}_2}$ to be as defined above if $p = 2$, and $M_{/K}$ is the quadratic extension specified above the proposition. Let E be the subfield of $\overline{\mathbb{Q}}$ generated over $\mathbb{Q}$ by the Hecke eigenvalues of f; so, we identify E with the E-linear endomorphism algebra $\mathrm{End}_E^0(A_{f/\mathbb{Q}})$.

We have already proven that A_f has potentially good reduction modulo p. Take a finite extension $L \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ such that A_f extends to an abelian scheme A over the integer ring of L. By extending L, we may assume that L contains the field M and is a Galois extension of K. Let $\widetilde{A}$ be the special fiber of A. Define another character $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/M) \to \overline{\mathbb{Q}}_l^\times$ by $\psi(g) = \varphi(\tau g \tau^{-1})$ for an element $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ inducing a nontrivial automorphism on M. Write ξ_L for the restriction of ξ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L) \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/M)$ for any character ξ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/M)$. Then the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ on the λ-adic Tate module $T_\lambda \widetilde{A}$ is isomorphic to $\varphi_L \oplus \psi_L$. The two characters φ_L and ψ_L are unramified. Thus for the prime element ϖ of L and its Artin symbol $\phi := [\varpi, L]$, $\varphi_L(\phi) + \psi_L(\phi) \in E$, and $\varphi_L(\phi)\psi_L(\phi)$ is equal to the value at ϕ of the l-adic cyclotomic character (after extending L further if necessary); so, we have $\varphi(\phi)\psi(\phi) = |N_{L/\mathbb{Q}_p}(\varpi)|_p^{-1}$. Thus $P(X) = (X - \varphi(\phi))(X - \psi(\phi))$ is in $E[X]$. Then the characteristic polynomial over $\mathbb{Q}$ of the Frobenius endomorphism Φ of $\widetilde{A}$ over the residue field $\mathbb{F}$ of L is given by $\prod_{\sigma:E \hookrightarrow \overline{\mathbb{Q}}} P^\sigma(X)$ for conjugates $P^\sigma(X)$ of $P(X)$ under field embeddings $\sigma : E \to \overline{\mathbb{Q}}$. If $(\varpi)^e = (p)$ in the integer ring of L, we get

$$\psi^\sigma(\phi^e) = \psi^\sigma([p, L]) = \varphi^\sigma(\tau[p, L]\tau^{-1}) = \varphi^\sigma([p^\tau, L]) = \varphi^\sigma([p, L]) = \varphi^\sigma(\phi^e).$$

This implies that $|\varphi^\sigma(\phi)|_p = |\psi^\sigma(\phi)|_p$, and by $\varphi(\phi)\psi(\phi) = |N_{L/\mathbb{Q}_p}(\varpi)|_p^{-1}$ already mentioned, we have

$$|\varphi^\sigma(\phi^e)\psi^\sigma(\phi^e)|_p = |\varphi(\phi)|_p^{2e} = |\psi(\phi)|_p^{2e} = |N_{L/\mathbb{Q}_p}(p)|_p.$$

Thus we get $|\varphi_L^\sigma(\phi)|_p = |\psi^\sigma(\phi)|_p = (N_{L/\mathbb{Q}}(p))^{-1/(2e)}$. Therefore $0 < |\psi^\sigma(\phi)|_p < 1$, and hence, all eigenvalues of Φ have positive slope between 0 and 1; so, $\widetilde{A}$ cannot be ordinary, and moreover $\widetilde{A}(\overline{\mathbb{F}}_p) = \{0\}$. $\qquad\qquad\square$

Though it might be well known to specialists, the fact that $|\varphi([p, L])|_p < 1$ (used in the above proof) was once pointed out to the author by A. Yamagami. In the following remark, we would like to show that there are infinitely many weight 2 Hecke new eigen forms f of Haupt-typus of p-power level super cuspidal at p for some choice of $p > 2$ (i.e., infinity of twist equivalence classes of $\mathbb{Q}$-simple abelian varieties of $GL(2)$-type with potentially super-singular good reduction at p and good reduction everywhere else). This may again be well known, but it is added as the author was explicitly asked this question by some geometers far from automorphic theory.

Remark 3.2. Pick a prime $p \not\equiv -1 \mod 4$ and a quaternion algebra D over $\mathbb{Q}$ exactly ramified at p and ∞. Consider the linear algebraic group $G_{/\mathbb{Q}}$ of adjoint type associated to $D^\times$ (i.e., $G(\mathbb{Q}) = D^\times/\mathbb{Q}^\times$ identifying $\mathbb{Q}^\times$ with the center of $D^\times$). Pick a maximal order O_D and put $G(\widehat{\mathbb{Z}}^{(p)}) = (O_D \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^{(p)})^\times/(\widehat{\mathbb{Z}}^{(p)})^\times \subset G(\mathbb{A}^{(\infty)})$, where $\widehat{\mathbb{Z}}^{(p)} = \prod_{l \neq p} \mathbb{Z}_l$. Then we consider the L^2-space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}^{(p)})G(\mathbb{R}))$ of square integrable functions on the double coset space $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}^{(p)})G(\mathbb{R})$ with respect to an invariant measure under right translation by elements of $G(\mathbb{A}^{(\infty)})$. The right translation gives rise to a unitary representation which has an infinite discrete spectrum of irreducible automorphic representations π^D with multiplicity 1 (see [AAG]). Such a π^D appearing in this spectrum which does not have nonzero $G(\widehat{\mathbb{Z}})$-fixed vectors is transferred, by the Jacquet-Langlands correspondence (e.g. [AAG]), to a holomorphic automorphic representation π of $GL(2)$ of weight 2 of p-power conductor super-cuspidal at p. Among such πs, there is no automorphic induction from an imaginary quadratic field $M = \mathbb{Q}(\sqrt{-D})$ of discriminant D, since such an automorphic induction has conductor divisible by D (as there is no imaginary quadratic field only ramified at p because of $p \not\equiv -1 \mod 4$). Thus π as above corresponds to a $\mathbb{Q}$-simple abelian variety of $GL(2)$-type with potentially good super-singular reduction modulo p and having good reduction everywhere else. The twist classes of π_D are finitely many as π^D has trivial central character. Thus in this case, plainly, there are infinitely many twist equivalence classes of $\mathbb{Q}$-simple abelian varieties of $GL(2)$-type without CM having potentially super-singular good reduction at p and good reduction everywhere else. Without assuming $p \not\equiv -1 \mod 4$, by counting CM automorphic representations appearing in this spectrum, it is plausible to get in general infinity of twist equivalence classes in the potentially super-singular case. Thus the finiteness of twist equivalence classes should be limited to potentially ordinary ones.

Proposition 3.3. *If A_f has potentially ordinary good reduction modulo p, then π_p is in principal series of the form $\pi(\alpha, \beta)$ for two characters $\alpha, \beta : \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}_p^\times$, and one of $\alpha(p)$ and $\beta(p)$ is a p-adic unit (i.e., $\max(|\alpha(p)|_p, |\beta(p)|_p) = 1$); so, f is nearly p-ordinary, and its twist by a character modulo p-power has slope 0.*

Proof. By the above proposition, π_p has to be either in principal series or Steinberg. By [C], Théorème A, Steinberg cases correspond to potentially multiplicative reduction at p; so, π_p has to be in principal series; so, $\pi_p = \pi(\alpha, \beta)$. Since

$|\alpha\beta(p)|_p = |p|_p$, we may assume that $1 \geq |\alpha(p)|_p \geq |\beta(p)|_p > 0$. We have canonical isomorphisms $\mathbb{Z}_p^\times \cong \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q})$. Thus regarding α, β as Galois characters, we can lift $\alpha|_{I_p}$ and $\beta|_{I_p}$ to a unique global character $\widetilde{\alpha}, \widetilde{\beta} : \mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \to \overline{\mathbb{Q}}^\times$ of finite order. Regarding these characters as adelic Hecke characters, we can think of $\pi \otimes \widetilde{\alpha}^{-1}$ and $\pi \otimes \widetilde{\beta}^{-1}$. We write A' for Shimura's abelian variety associated to $\pi \otimes \widetilde{\alpha}^{-1}$. Again by [C], Théorème A or [AME], §14.5 or [L], A' has good reduction modulo p over the fixed field of $\mathrm{Ker}(\widetilde{\alpha}^{-1}\beta)$, and the Frobenius endomorphism of A' shares the same characteristic polynomial with the $U(p)$ operator (e.g., [GME], Theorem 4.2.6) and is given by $\widetilde{\alpha}^{-1}\alpha(p)$ (up to Galois conjugation). Thus if A' has ordinary good reduction at p, we get $|\alpha(p)| = |\widetilde{\alpha}^{-1}\alpha(p)|_p = 1$, and the new form $f \otimes \widetilde{\alpha}^{-1}$ associated to $\pi \otimes \widetilde{\alpha}^{-1}$ has slope 0. $\qquad\square$

4. Twist classes of ordinary abelian varieties of $GL(2)$-type

We prove the following theorem, which implies the main theorem in the introduction:

Theorem 4.1. *Suppose $p > 2$. For a given positive integer N prime to p, there are only finitely many twist classes of non-CM $\mathbb{Q}$-simple abelian varieties of $GL(2)$-type with potentially good ordinary reduction modulo p and with prime-to-p conductor N.*

As remarked already after Corollary 2.2, if we assume $1 < Np \leq 10$ or $Np = 12$, there is no such abelian variety over $\mathbb{Q}$ as in the above theorem.

Proof. If A is such an abelian variety as in the theorem, by the theorem of Khare–Wintenberger ([KW], Theorem 10.1) combined with a theorem of Faltings (e.g., [ARG]), A is isogenous to A_f for an elliptic Hecke eigenform f. Twisting f by a Dirichlet character of p-power conductor, we may assume that f is a p-ordinary form of slope 0 (see Proposition 3.1 and Proposition 3.3 and their proofs). The prime-to-p conductor of f remains N after the twist by a character of the p-power conductor. Then f belongs to a p-adic analytic family of slope 0 forms of prime-to-p conductor N. Since there are only finitely many such families and each family can contain only finitely many A_f of potentially ordinary good reduction by Corollary 2.2, the desired result follows. $\qquad\square$

Remark 4.2. Fix an integer $k \geq 2$. We look into a rank 2 $\mathbb{Q}$-simple motive of weight $(k-1, 0)$, which has its compatible system of Galois representations. Though we do not know if the identity of L-functions of two Grothendieck motives implies that the two motives are isomorphic (assuming their coefficient fields are the same), we can anyway think of twist equivalence classes of such motives in an obvious sense. Again by the theorem of Khare–Wintenberger, such a motive is twist equivalent to a modular motive (constructed in [S]) belonging to a slope 0 family. Then, basically by the same argument, we can prove finiteness of twist classes of potentially ordinary crystalline $\mathbb{Q}$-simple motives of rank 2 of a given prime-to-p conductor. Again for the same small pairs (N, p) we listed above, there are no such motives (independent of weight k).

Remark 4.3. Our proof of finiteness of twist classes is based on the result in [H11] (i.e., Theorem 1.1 in the text) and the solution of Serre's modulo p modularity

conjecture. It is plausible that our conjecture follows from the totally real version of Serre's modularity conjecture once the result of [H11] is generalized to the Hilbert modular case. In the proof of Theorem 1.1 given in [H11], we used at many places the fact that the base Iwasawa algebra has a single variable. However, the number of variables of the $\mathfrak{p}$-ordinary Hecke algebra of level $\mathfrak{p}^\infty$ (under the notation in the conjecture) is greater than or equal to $[F_\mathfrak{p} : \mathbb{Q}_p]$. Thus more work needs to be done to generalize Theorem 1.1 to Hilbert modular cases.

References

[AAG] S. S. Gelbart, *Automorphic Forms on Adele Groups*, Annals of Math. Studies **83**, Princeton University Press, Princeton, NJ, 1975. MR0379375 (52:280)

[ABV] D. Mumford, *Abelian Varieties*, TIFR Studies in Mathematics, Oxford University Press, New York, 1994. MR2514037 (2010e:14040)

[ACM] G. Shimura, *Abelian Varieties with Complex Multiplication and Modular Functions*, Princeton University Press, Princeton, NJ, 1998. MR1492449 (99e:11076)

[AME] N. M. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Math. Studies **108**, Princeton University Press, Princeton, NJ, 1985. MR772569 (86i:11024)

[ARG] G. Cornell and J. H. Silverman, editors, *Arithmetic Geometry*, Springer, New York, 1986. MR861969 (89b:14029)

[BG] A. Brown and E. Ghate, Dihedral congruence primes and class fields of real quadratic fields. J. Number Theory **95** (2002), 14–37. MR1916078 (2003g:11130)

[C] H. Carayol, Sur les représentations l-adiques associées aux formes modulaires de Hilbert, Ann. Sci. École Norm. Sup. (4) **19** (1986), 409–468. MR870690 (89c:11083)

[dJ] A. J. De Jong, Crystalline Dieudonné module theory via formal and rigid geometry. Inst. Hautes Études Sci. Publ. Math. No. **82** (1995), 5–96. MR1383213 (97f:14047)

[DR] P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, in "Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)", pp. 143–316. Lecture Notes in Math., Vol. **349**, Springer, Berlin, 1973. MR0337993 (49:2762)

[F] J.-M. Fontaine, Il n'y a pas de variété abélienne sur **Z**. Invent. Math. **81** (1985), 515–538. MR807070 (87g:11073)

[GME] H. Hida, *Geometric Modular Forms and Elliptic Curves*, second edition, World Scientific, Singapore, 2011.

[H86a] H. Hida, Iwasawa modules attached to congruences of cusp forms, Ann. Sci. Ecole Norm. Sup. 4th series **19** (1986), 231–273. MR868300 (88i:11023)

[H86b] H. Hida, Galois representations into $GL_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms, Inventiones Math. **85** (1986), 545–613. MR848685 (87k:11049)

[H86c] H. Hida, Hecke algebras for GL_1 and GL_2, Sém. de Théorie des Nombres, Paris 1984-85, Progress in Math. **63** (1986), 131–163. MR897346 (88i:11078)

[H11] H. Hida, Hecke fields of analytic families of modular forms, J. Amer. Math. Soc. **24** (2011), 51–80. MR2726599 (2011m:11085)

[H12] H. Hida, Local indecomposability of Tate modules of non CM abelian varieties with real multiplication, preprint, 2011, 18 pages

[HMI] H. Hida, *Hilbert modular forms and Iwasawa theory*, Oxford University Press, 2006. MR2243770 (2007h:11055)

[KW] C. Khare and J.-P. Wintenberger, Serre's modularity conjecture. I, II. I: Invent. Math. **178** (2009), 485–504; II. Invent. Math. **178** (2009), 505–586. MR2551763 (2010k:11087); MR2551764 (2010k:11088)

[IAT] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, Princeton, NJ, and Iwanami Shoten, Tokyo, 1971. MR0314766 (47:3318)

[K] P. Kutzko, The Langlands conjecture for Gl_2 of a local field. Ann. of Math. (2) **112** (1980), 381–412. MR592296 (82e:12019)

[L] R. P. Langlands, Modular forms and ℓ-adic representations, in "Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)", pp. 361–500. Lecture Notes in Math., Vol. **349**, Springer, Berlin, 1973. MR0354617 (50:7095)

[MFG]   H. Hida, *Modular Forms and Galois Cohomology*, Cambridge Studies in Advanced Mathematics **69**, Cambridge University Press, Cambridge, England, 2000. MR1779182 (2002b:11071)

[MFM]   T. Miyake, *Modular Forms*, Springer, New York-Tokyo, 1989. MR1021004 (90m:11062)

[MW]   B. Mazur and A. Wiles, On *p*-adic analytic families of Galois representations, Compositio Math. **59** (1986), 231–264. MR860140 (88e:11048)

[O]   A. Ogus, Hodge cycles and crystalline cohomology, in: *Hodge Cycles, Motives, and Shimura Varieties*, Chapter VI, Lecture Notes in Math. **900** (1982) 357–414. MR0654325 (84m:14046)

[R]   K. Ribet, Abelian varieties over $\mathbb{Q}$ and modular forms. Algebra and topology 1992 (Taejŏn), 53–79, Korea Adv. Inst. Sci. Tech., Taejŏn, 1992. MR1212980 (94g:11042)

[R1]   K. A. Ribet, Fields of definition of abelian varieties with real multiplication, Contemporary Math. **174** (1997), 107–118. MR1299737 (95i:11057)

[S]   A. J. Scholl, Motives for modular forms, Inventiones Math. **100** (1990), 419–430. MR1047142 (91e:11054)

[ST]   J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) **88** (1968), 452–517. MR0236190 (38:4488)

[T]   R. Taylor, On the meromorphic continuation of degree two *L*-functions, Documenta Math., Extra Volume: John Coates' Sixtieth Birthday (2006), 729-779. MR2290604 (2008c:11154)

[W]   A. Weil, Exercices dyadiques, Inventiones Math. **27** (1974), 1–22 (Œuvres III, [1974e]). MR0379445 (52:350)

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