A few years ago in 2023, Pol van Hoften pointed me out an incompleteness of the proof of Lemma 2.4 in the Academia Sinica paper. In the proof, the following fact is used without a proof: For a closed subgroup H of $1+pM_n(\mathbb{Z}_p)$, its image in $M_n(\mathbb{Q}_p)$ under the p-adic logarithm map is stable under the Lie bracket. We give a proof of Lemma 2.4 without using this claim (which may not be known).

1. Group theory

Fix a local field $K/\mathbb{Q}_p$ (a finite extension) with p-adic integer ring O_K and residue field κ. Write $\mathfrak{m}$ for the maximal ideal of O_K; so, $O_K/\mathfrak{m} = \kappa$. Let $\mathcal{B}$ be a complete representative set for the isomorphism classes of all commutative semi-simple algebras of dimension d over K. Fix a basis over O_K of the integral closure O_B of O_K in $B \in \mathcal{B}$ and embed O_B into $M_d(O_K)$ by the regular representation ρ_B with respect to a basis of O_B over O_K. We write $\overline{B} := O_B/\mathfrak{m} = O_B \otimes_{O_K} \kappa$.

Let k be either K or κ. Let $F_{/k}$ be a field extension of degree d and $N = k[X]/(X^d)$. By regular representation, we embed F and N into $M_d(k)$ and write them as ρ_F and ρ_N. Let P be the maximal parabolic subgroup of $\mathrm{GL}_d(k)$ such that $P = \left\{\left(\begin{smallmatrix} y & u \\ 0 & v \end{smallmatrix}\right) \middle| u \in k^{d-1}, y \in \mathrm{GL}_{d-1}(k), v \in k^\times\right\}$; so, the Levi subgroup M of P is isomorphic to $\mathrm{GL}_{d-1}(k) \times \mathrm{GL}_1(k)$, and the unipotent radical U of P is isomorpgic to the additive group k^{d-1}.

Lemma 1.1. *The group $\mathrm{GL}_d(k)$ is generated by $\rho_F(F^\times)$, $\rho_N(N^\times)$ and M, and the group $\mathrm{SL}_d(k)$ is generated by $\rho_F(F^\times)_1$, $\rho_N(N^\times)_1$ and M_1, where adding subscript "1" to a subgroup X of $\mathrm{GL}_d(k)$, we indicate the intersection of a group X with $\mathrm{SL}_d(k)$.*

Proof. By conjugation, we may assume that $\rho_N(X) = (x_{ij})$ with $x_{ij} = \delta_{i,j-1}$ for Kronecker symbol $\delta_{i,j}$. Then for $u \in k^{d-1} = {}^t(u_j)_{j=1,\dots,d-1}$, the last column of $n := 1_d + \sum u_j X^j$ is given by $\left[\begin{smallmatrix} u \\ 1 \end{smallmatrix}\right]$. Thus for $\mathrm{diag}[y,v]^{-1}\left(\begin{smallmatrix} y & * \\ 0 & v \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1_{d-1} & u \\ 0 & 1 \end{smallmatrix}\right)$ for $u \in k^{d-1}$. Therefor $\mathrm{diag}[y,v]^{-1}\left(\begin{smallmatrix} y & * \\ 0 & v \end{smallmatrix}\right) n^{-1} \in M$. Thus P is generated by M and N. We let F acts on the row vector space k_d of dimension d by right multiplication of ρ_F. Then $k_d = (0,0,\dots,0,1)\rho_F(F)$. Thus for any non-zero vector $v \in k_d$, we can find $f \in F^\times$ such that $v = (0,0,\dots,0,1)\rho_F(f)$. Taking v to be the last row vector of $g \in \mathrm{GL}_d(k)$, we find $g\rho_F(f)^{-1} \in P$. Therefore, the desired assertion follows, and indeed $\mathrm{GL}_d(k) = M\rho_N(N^\times)\rho_F(F^\times)$.

Now we deal with $\mathrm{SL}_d(k)$. In the above argument dealing with GL_d, we need to impose the relation $\det(y)v = 1$. Since $\mathrm{SL}_d(k)$ acts transitively on k_d, the rest of the argument is the same, and we get $\mathrm{SL}_d(k) = M_1\rho_N(N_1^\times)\rho_F(F_1^\times)$. $\square$

Let B_δ/K be an unramified field extension of degree δ and B'_δ/K be a fully ramified field extension of equal degree δ. For a partition $d = \delta+\beta$ with $\delta\beta \neq 0$, we define $\overline{B}_{\delta,\beta} := \overline{B}_\delta \oplus \overbrace{\kappa \oplus \cdots \oplus \kappa}^{\beta}$ and $\overline{B}'_{\delta,\beta} := \overline{B}'_\delta \oplus \overbrace{\kappa \oplus \cdots \oplus \kappa}^{\beta}$. We put $\overline{B}_{\delta,0} = \overline{B}_\delta$, $\overline{B}'_{\delta,0} = \overline{B}'_\delta$, $\overline{B}_{0,\beta} = \overbrace{\kappa \oplus \cdots \oplus \kappa}^{\beta}$. Let $\overline{\mathcal{B}} := \{\overline{B}_{\delta,\beta}, \overline{B}'_{\delta,\beta}\}$ for all partitions $d = \delta+\beta$ with $\delta \geq 0$ and $\beta \geq 0$. Not that $\{X \otimes_{O_K} \kappa | X \in \mathcal{B}\}$ contains $\overline{\mathcal{B}}$ (up to isomorphisms). Regard elements of $\overline{\mathcal{B}}$ as κ-subalgebras of $M_d(\kappa)$ by the regular representation. Then, by Lemma 1.1, the subgroup $\mathrm{GL}_\delta(\kappa) \times \overbrace{\kappa^\times \times \cdots \times \kappa^\times}^{\beta}$ diagonally embedded in $\mathrm{GL}_d(\kappa)$ and $\{\overline{B}_{\delta',\beta}, \overline{B}'_{\delta',\beta}\}_{\delta'\geq\delta}$ generate $\mathrm{GL}_d(\kappa)$ by induction on β up to $d-\delta$. This shows the following corollary for $\mathrm{GL}_d(\kappa)$.

Corollary 1.2. *The groups* $\mathrm{GL}_d(\kappa)$ *(resp.* $\mathrm{SL}_d(\kappa)$*) are generated by subgroups in* $\overline{\mathcal{B}}$ *(resp.* $\overline{\mathcal{B}}_1 := \{\overline{B} \cap \mathrm{SL}_d(\kappa) | \overline{B} \in \overline{\mathcal{B}}\}$*).*

Proof. We need to prove the corollary for $\mathrm{SL}(d)$. Note that $M_1 \cong \mathrm{GL}_{d-1}(\kappa)$ for $k = \kappa$. Then the assertion follows from the above argument applied to first to $M_1 \cong \mathrm{GL}_{d-1}(\kappa)$ and Lemma 1.1 then to $\mathrm{SL}_d(\kappa)$ with the above argument. □

There is an obvious additive version of Lemma 1.1 and Corollary 1.2.

Lemma 1.3. *The Lie algebra* $\mathfrak{gl}_d(k)$ *(resp.* $\mathfrak{sl}_d(k)$*) is generated by* $\rho_F(F)$*,* $\rho_N(N)$ *and the Lie algebra* $\mathfrak{M}$ *of* M *(resp.* $\rho_F(F)_1$*,* $\rho_N(N)_1$ *and the Lie algebra* $\mathfrak{M}_1$ *of* M_1*), and hence* $\mathfrak{gl}_d(k)$ *(resp.* $\mathfrak{sl}_d(k)$*) is generated by* $\{\overline{\rho}_B(\overline{B})\}_{\overline{B}\in\overline{\mathcal{B}}}$ *(resp.* $\overline{\mathcal{B}}_1 := \{\overline{B}_1 | \overline{B} \in \overline{\mathcal{B}}\}$*). In particular, the closed subgroup of* $\mathrm{GL}_d(O_K)$ *(resp.* $\mathrm{SL}_d(O_K)$*) generated by* $\{\rho_B(O_B)^\times | B \in \mathcal{B}\}$ *is a p-adic open subgroup of* $\mathrm{GL}_d(O_K)$ *(resp.* $\mathrm{SL}_d(O_K)$*).*

Lemma 1.4. *The groups* $\mathrm{GL}_d(O_K)$ *(resp.* $\mathrm{SL}_d(O_K)$*) is generated topologically by the subgroups in* $\mathcal{B}$ *(resp.* $\mathcal{B}_1$*).*

Proof. Let $H \subset \mathrm{GL}_d(O_K)$ be the closed subgroup topologically generated by $\rho_B(B)$ for groups B in $\mathcal{B}$. By Corollary 1.2, $\mathrm{GL}_n(\kappa)$ is generated by $\overline{\rho}_B(\overline{B}^\times)$ for $\overline{B}$ running through $\overline{\mathcal{B}}$. As remarked already $\overline{\mathcal{B}}$ is a subset of $\{O_B \otimes_{O_K} \kappa | B \in \mathcal{B}\}$. Thus $\overline{H} = \mathrm{GL}_d(\kappa)$.

By Lemma 1.3, H is an open subgroup of $\mathrm{GL}_d(O_K)$. Then H contains $\Gamma(\mathfrak{m}^n) := \mathrm{Ker}(\mathrm{GL}_d(O_K) \ni g \mapsto (g \bmod \mathfrak{m}^n) \in \mathrm{GL}_d(O_K/\mathfrak{m}^n))$ for some $n > 0$. We now show that as long as $n \geq 2$, H contains $\Gamma(\mathfrak{m}^{n-1})$. Let $G_n := \Gamma(\mathfrak{m}^{n-1})/\Gamma(\mathfrak{m}^n)$. Note that

$$\iota : G_n \ni x\Gamma(\mathfrak{m}^n) \mapsto (x-1)/\varpi^n \in \mathfrak{gl}_d(O_K/\mathfrak{m}) = \mathfrak{gl}_d(\kappa)$$

for a generator ϖ of $\mathfrak{m}$ is an isomorphism from the multiplicative group to the additive group. The group H acts on X_n by conjugation and on $\mathfrak{gl}_d(O_K/\mathfrak{m})$ by the adjoint action. The map ι is equivariant under the H-action. Let $\overline{X}_n := (H \cap \Gamma(\mathfrak{m}^{n-1}))/\Gamma(\mathfrak{m}^n)$. Since for the unramified field extension B/K,

$$0 \neq (1 + \mathfrak{m}^{n-1}O_B)/(1 + \mathfrak{m}^n O_B) \hookrightarrow \overline{X}_n,$$

we have $\overline{X}_n \neq 0$. Since $\mathrm{GL}_d(O_K) \rhd \Gamma(\mathfrak{m}^{n-1})$, $\overline{X}_n$ is stable under the action of H. The group $\overline{X}_n$ identified with an additive subgroup of $\mathfrak{gl}_d(O_K/\mathfrak{m})$ which intersects with any conjugacy class under $\mathrm{GL}_d(\kappa)$ of any semi-simple elements and nilpotent elements. Since $\overline{H} = \mathrm{GL}_d(\kappa)$ by Corollary 1.2, this shows that $\overline{X}_n = X_n$ and hence $H \supset \Gamma(\mathfrak{m}^{n-1})$. By induction on n, we find $H \supset \Gamma(\mathfrak{m})$. Since we have a commutative diagram with exact rows:

$$\begin{array}{ccccc}
\Gamma(\mathfrak{m}) & \xrightarrow{\hookrightarrow} & H & \xrightarrow{\twoheadrightarrow} & \overline{H} \\
\| \downarrow & & \cap \downarrow & & \| \downarrow \\
\Gamma(\mathfrak{m}) & \xrightarrow[\hookrightarrow]{} & \mathrm{GL}_d(O_K) & \xrightarrow[\twoheadrightarrow]{} & \mathrm{GL}_d(\kappa),
\end{array}$$

we conclude $H = \mathrm{GL}_d(O_K)$.

Plainly the above argument works for $\mathrm{PGL}(d)$ in place of $\mathrm{GL}(d)$ as $\mathrm{PGL}(d)$ is a quotient of $\mathrm{GL}(d)$. Let $\overline{\Gamma}(\mathfrak{m}^n)$ be the image of $\Gamma(\mathfrak{m}^n)$ in $\mathrm{PGL}_n(O_K)$ and $\Gamma^1(\mathfrak{M}^n) = \mathrm{SL}_n(O_K) \cap \Gamma(\mathfrak{m}^n)$. If $p \nmid d$ and $n \geq 1$, the natural morphism π_n :

$\Gamma^1(\mathfrak{m}^n) \to \overline{\Gamma}(\mathfrak{m}^n)$ is an onto isomorphism. Indeed, since $\det(g) \equiv 1 \mod \mathfrak{m}^n$ for $g \in \Gamma(\mathfrak{m}^n)$, we have a unique $\det(g)^{1/d} \in 1 + \mathfrak{m}^d$ and $g \mapsto \det(g)^{-1/d} g$ gives rise to the inverse of π. Thus the result follows for SL_d under $p \nmid d$. If $p|d$, we consider the pairing $(x, y) = \mathrm{Tr}(x^t y)$ on $\mathfrak{gl}_d(K)$, which gives rise to a perfect pairing on $\mathfrak{gl}_d(O_K)$. We have a decomposition $\mathfrak{gl}_d(K) = \mathfrak{sl}_d(K) \oplus \mathfrak{z}(K)$ for scalar matrices $\mathfrak{z}(A)$ in $\mathfrak{gl}_d(A)$ by $X \mapsto (X - \frac{1}{d}\mathrm{Tr}(X), \frac{1}{d}\mathrm{Tr}(X))$. Thus $\mathfrak{sl}_d(O_K) = \mathfrak{sl}_d(K) \cap \mathfrak{gl}_d(O_K)$ and the projected image $\mathfrak{sl}_d^*(O_K)$ of $\mathfrak{gl}_d(O_K)$ in $\mathfrak{sl}_d(K)$ is the dual lattice of $\mathfrak{sl}_d(O_K)$ in $\mathfrak{sl}_d(K)$. Since $\mathrm{SL}(d)$ is simply connected and $\mathrm{PGL}(d)$ is of adjoint type, $\Gamma^1(\mathfrak{m}^{n-1})/\Gamma^1(\mathfrak{m}^n) \cong \mathfrak{sl}_d(\mathfrak{m}^{n-1})/\mathfrak{sl}_d(\mathfrak{m}^n)$ and $\overline{\Gamma}(\mathfrak{m}^{n-1})/\overline{\Gamma}(\mathfrak{m}^n) \cong \mathfrak{sl}_d^*(\mathfrak{m}^{n-1})/\mathfrak{sl}_d^*(\mathfrak{m}^n)$ as $\mathrm{SL}_d(\kappa)$-modules and also $\mathrm{GL}_d(\kappa)$-modules. They are contragredient each other. The set of irreducible $\mathrm{GL}_d(\kappa)$-subquotients and $\mathrm{SL}_d(\kappa)$-subquotients are in bijection (although the order of the appearance of the factors can be different). Therefore by using the $\mathrm{SL}_d(\kappa)$ Jordan-Hölder sequence, the proof for PGL implies the results for SL. $\square$

Let K now be a semi-simple commutative algebra over $\mathbb{Q}_p$. We take a complete set $\mathcal{B}$ of representatives of isomorphism classes of semi-simple commutative K-algebras B free of rank d over K. We fix a regular representation $\rho_B : O_B \to M_d(O_K)$ for each $B \in \mathcal{B}$. Since $\mathrm{GL}_d(O_K) = \prod_i \mathrm{GL}_d(O_{K_i})$ and $\mathrm{SL}_d(O_K) = \prod_i \mathrm{SL}_d(O_{K_i})$ for simple components K_i of K, we have the following obvious version of the above lemma for the semi-simple base ring K:

Corollary 1.5. *The group $\mathrm{GL}_d(O_K)$ (resp. $\mathrm{SL}_d(O_K)$) is generated topologically by $\{\rho_B(O_B^\times)\}_{B\in\mathcal{B}}$ (resp. $\{\rho_B(\mathrm{Ker}(N_{K/B}))\}_{B\in\mathcal{B}}$), where $N_{B/K} : O_B^\times \to O_K^\times$ is the norm map.*