*Proof of non-vanishing theorem

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*We give a sketch of the proof of the non-vanishing theorem. To study Zariski density in $G_m^n(\mathbb{Q}_\ell)$, we consider a discrete valuation ring $W_\ell \subset \mathbb{C}_\ell$ finite over $W(\mathbb{F}_\ell)$. Recall $\mathcal{X} = \mathcal{X}_v := \{\chi | \int_{\text{Cl}_n} \chi \psi d\varphi_f \neq 0 \text{ for some } n, \ v(\chi) = v\}$ and the sequence $n := \{\min(m) | \chi \text{ factors through } \Gamma_m \text{ for } \chi \notin \mathcal{X}\}$ defines $\Xi = \{s(A) | [A] \in \bigcup_{n \in \mathbb{N}} \text{Ker}(\Gamma_n \to \Gamma_j)\}$ for some $j \geq r$. We show that $n$ contains an arithmetic progression if $\dim \overline{\mathcal{X}} < d$ for the Zariski closure $\overline{\mathcal{X}}$ in $G_m^n$. 
§0. Case $\dim \overline{X} = 0$ (and $\Gamma \cong \mathbb{Z}_\ell$). We can take $j := r$ for $r$ given by $\ell^r \mid |\mathbb{F}_p[f, \lambda, \psi, \mu]^{\times}|$. If the Zariski closure $\overline{X}$ has dimension 0, it is a finite set stable under $t \mapsto t^P$ ($\mathbb{F}_P = \mathbb{F}_p[f, \lambda, \psi, \mu]$); so, there exists an integer $N$ such that if the order of $\chi$ is larger than $\ell^N$, we have $\int_{\mathcal{O}l} \chi \psi d\varphi_f = 0$ for all $n$. Let $0 < m_0 \in \mathbb{Z}$ be the minimal integer such that $\ell^{m_0} = (\varphi \varphi^c)$ with $\varphi \in \mathcal{R}$. Thus there exists a positive integer such that $n \supset \{n \in m_0 \mathbb{Z} | n \geq N'\}$, which is an arithmetic progression; so, the assertion follows from Density theorem if $\Gamma \cong \mathbb{Z}_\ell$.

Assume $0 < \dim \overline{X} < d$; so, $\mathcal{O}_l \neq \mathbb{Z}_\ell$ and $\dim \overline{X} < d$. We have $\chi^{\sigma^m} = \chi^{P^m} \in \mathcal{X}$ if $\chi \in \mathcal{X}$; i.e., $\mathcal{X}$ is $\sigma$-stable.

1. we may not have lower bound $\ell^N$ of the order of $\chi \in \mathcal{X}$,

2. the Galois action $\chi \mapsto \chi^{P^m} \quad m \in \mathbb{Z}$ only covers 1-dimensional segment starting with $\chi$. In other words, $\text{Tr}_{\mathbb{F}_P[\chi]/\mathbb{F}_P \circ \chi}$ only factor through $\text{Ker}(\chi|_{\Gamma_n}) \Gamma_n[\ell^r]$ whose order grows dependent on $n$ (not a finite bounded sum of $f|_{\alpha(u/\varpi_1^r)}([\mathcal{A}])$ over $u \in \mathcal{O}/\varpi^r$.

These are two points of difficulty which have to be addressed.
§1. Rigidity of torus.
Let $W_\ell$ be a discrete valuation ring finite flat over $W(\overline{\mathbb{F}}_\ell)$. We state the following theorem, which is a key to show that $\mathbb{Z}$ contains arithmetic progression.

Rigidity Theorem. Let $X = \text{Spf}(T)$ be a closed formal subscheme of $\hat{G} = \hat{G}_m^n/W_\ell$ flat geometrically irreducible over $W_\ell$ (i.e., $T \cap \overline{Q}_\ell = W_\ell$). Suppose there exists an open subgroup $U$ of $\mathbb{Z}_\ell^\times$ such that $X$ is stable under the action $\hat{G} \ni t \mapsto t^u \in \hat{G}$ for all $u \in U$. If $X$ contains a Zariski dense subset $\Omega \subset X(\mathbb{C}_\ell) \cap \mu_\ell^n(\mathbb{C}_\ell)$, then there exist $\omega \in \Omega$ and a formal subtorus $T$ such that $X = T\omega$.

If time permits, we describe a sketch of the proof at the end (see my papers: JAMS 24 (2011) Rigidity lemma and Contemporary Math. 614 (2014) Lemma 4.1).
§2. Use of rigidity. Let, for a class \( v \in O/l^j \) for \( j \geq r \),
\[
\mathcal{X} = \mathcal{X}_v := \{ \chi \in \text{Hom}(\Gamma, \mu_{l\infty}) | \int_{\mathcal{C}_{l^n}} \chi \psi d\varphi_f \neq 0 \ \exists n, v(\chi) = v \}
\]
\[
\mathcal{Z} = \mathcal{Z}_v := \{ \chi \in \text{Hom}(\Gamma, \mu_{l\infty}) | \int_{\mathcal{C}_{l^n}} \chi \psi d\varphi_f = 0 \ \forall n, v(\chi) = v \}
\]
Write \( \hat{\mathcal{X}} \) for the formal Zariski closure of \( \mathcal{X} \) in \( \hat{G}_{m/W_{\ell}}^d \). Assume \( \dim_{W_{\ell}} \hat{\mathcal{X}} < d \) which leads to absurdity. Note \( \dim_{W_{\ell}} \hat{\mathcal{X}} = \dim \overline{\mathcal{X}} \).

Overcoming reducibility: Since \( \chi \in \mathcal{X} \Rightarrow \chi^\sigma \in \mathcal{X} \) for \( \sigma = \text{Frob} \in \text{Gal}(\mathbb{F}/\mathbb{F}_{p^r}) \), \( \sigma \) permutes irreducible components of \( \hat{\mathcal{X}} \). Thus each irreducible component is fixed by some \( \tau := \sigma^m \) for \( m > 0 \). Note \( \sigma(x) = x^P \) for \( P = p^r \), and put \( P = P^m \). Since \( P \equiv 1 \mod \ell^r \) with \( r > 0 \), we have \( P \equiv 1 \mod \ell \). Since each irreducible component is formal, it is stable under \( \tau^{\mathbb{Z}_{\ell}} \) which is an open subgroup of \( \mathbb{Z}_{\ell}^\times \). By Rigidity Theorem, each irreducible component of \( \hat{\mathcal{X}} \) is of the form \( \omega T \) for a subtorus \( T \). Then \( \hat{\mathcal{X}} = \bigcup_{j \in J} \omega_j T_j \) for a finite index set \( J \) with \( \omega_j \in \Omega \) and subtori \( T_j \). The argument has been given if \( \dim_{W_{\ell}} \hat{\mathcal{X}} = \dim \overline{\mathcal{X}} = 0 \); so, assume \( 0 < \dim_{W_{\ell}} \hat{\mathcal{X}} < d \).
§3. Tubular neighborhood. Replacing $\widehat{G}^d_m$ by $\widehat{G}^d_m/\langle \omega_j \rangle_j$, we may assume that $\omega_j = 1$ for all $j$. Let $V_j$ be the $\mathbb{Q}_\ell$-span of the Tate module of $T_j$ as a subspace of $V := \mathbb{Q}_\ell(1)^d$. Since $0 < \dim V_j < d$, we claim to find a basis $B := \{e_1, \ldots, e_d\}$ of $\mathbb{Z}_\ell(1)^d$ such that $B' := B \cup \{e := \sum_j e_j\}$ is outside $\bigcup_j V_j$. Since $d > 1$, the set \(\{B \in \text{GL}_d(\mathbb{Z}_\ell)|B' \cap \bigcup_j V_j \neq \emptyset\}\) is a proper closed subset of $\text{GL}_d(\mathbb{Z}_\ell)$ of dimension $d^2 - \max_j (\dim V_j) < d^2 = \dim \text{GL}(d)$. This shows a plenty of the choice $B$.

Let $\Gamma_P = P^{\mathbb{Z}_\ell}$. Then $U := \Gamma_P e_1 + \cdots + \Gamma_P e_d$ is an open tubular neighborhood of the line $\mathbb{Z}_\ell \cdot e$. By replacing $P$ by its power (i.e., shrinking $U$), the image $\text{Cone}(U)$ of $\bigcup_{u \in U} \mathbb{Q}_\ell \cdot u$ in $(\mathbb{Q}_\ell(1)/\mathbb{Z}_\ell(1))^d$ is disjoint from $\mathcal{X}[\ell^{N'}]^\times = \mathcal{X}[\ell^{N'}] - \mathcal{X}[\ell^{N'-1}]$ for all sufficiently large $N' > 0$. 

Let $\text{Cone}(U)[\ell^M]^{\times}$ be the set of order $\ell^M$ elements in $\text{Cone}(U)$ and $\chi_i$ be the order $\ell^M$ element corresponding to $\frac{1}{\ell^M} e_i$. Write $P = p^j \ (j \geq r)$ and define $Z = Z_v$ for $v \in O/\ell^j$. Then for $M \geq N'$, writing $m := \dim_{\mathbb{F}_P} \mathbb{F}_P[\mu_{\ell^M}]$

$$\text{Cone}(U)[\ell^M]^{\times} = \left\{ \prod_{i=1}^{d} \chi_i^{u_i} | u_i \in \Gamma_P \right\} = \left\{ \prod_{i=1}^{d} \chi_i^{P_{m_i}} | 0 \leq m_i \leq m-1 \right\} \subset Z.$$

Thus if $\chi|_{\Gamma_n[\ell^j]} = \chi_v$ for $v \in O/\ell^j$,

$$\sum_{\chi \in \text{Cone}(U)[\ell^M]^{\times}} \chi = \prod_{i=1}^{d} \text{Tr}_{\mathbb{F}_P[\mu_{\ell^M}]/\mathbb{F}_P}(\chi_i)^{\text{Trace rel.}} \equiv [\mathbb{F}_P[\mu_{\ell^M}] : \mathbb{F}_P]^{d} \chi_v,$$

where $\chi_v = 0$ outside $\Gamma_n[\ell^j]$. This $j$ depends on $l$ and $\Xi$ contains $n = \{ n \in \mathbb{Z} | n \geq N' \}$. Thus if $a(\xi, f) \neq 0$ for $(\xi \mod \ell^j) = -v$, we get the contradiction. \qed
§5. Preliminary to the proof of Rigidity Theorem. The regular locus of $X^\circ$ is open dense in the generic fiber $\text{Spec}(T)/K$ (for the field $K = \text{Frac}(W)$ for $W = W_\ell$). Then $\Omega^\circ := X^\circ \cap \Omega$ is Zariski dense in $\text{Spec}(T)/K$. Write $X^s := \text{Spec}(T)/K - X^\circ$ (the singular locus). The stabilizer $U_\zeta$ of $\zeta \in \Omega$ in $U$ is an open subgroup of $U$. By $t \mapsto t\zeta - 1$, we assume that the identity $1 \in \Omega^\circ$.

By adding subscript $an$, $X_{an}$ denotes the rigid analytic spaces associated to $X$. Then $X_{an}^\circ = X_{an} - X_{an}^s$ is an open rigid analytic subspace of $X_{an}$. Apply the logarithm $\log : \hat{G}_{an}^\times(\mathbb{C}_\ell) \to \mathbb{C}_\ell^n = \text{Lie}(\hat{G}_{an})$ sending $(t_j)_j \in \hat{G}_{an}^\times(\mathbb{C}_\ell)$ to $(\log_\ell(t_j))_j \in \mathbb{C}_\ell^n$ for the $\ell$-adic logarithm map $\log_\ell : \mathbb{C}_\ell^\times \to \mathbb{C}_\ell$. Then for each smooth point $x \in X^\circ(W)$, taking a small analytic open ball $G_x$ centered at $x$ in $\hat{G}_{an}$ so that $V_x = G_x \cap X^\circ(W)$ for a $d$-dimensional open ball in $X^\circ(W)$ centered at $x \in X^\circ(W)$. Then $\log(X^\circ(W))$ contains the origin $0 \in \mathbb{C}_\ell^n$. Take $\zeta \in \Omega^\circ$. Write $T_\zeta$ for the Tangent space at $\zeta$ of $X$. Then $X_\zeta \cong W^d$ for $d = \dim_W X$. The space $T_\zeta \otimes_W \mathbb{C}_\ell$ is canonically isomorphic to the tangent space $T_0$ of $\log(V_\zeta)$ at $0$. 
§6. **Proof in case:** \( \dim_W X = 1 \). If \( \dim_W X = 1 \), there exists an infinite order element \( t_1 \in X(W) \). We write \( U = (1 + \ell^m \mathbb{Z}_\ell) \) for \( 0 < m \in \mathbb{Z} \). Then \( X \) is the (formal) Zariski closure \( t_1^U \) of

\[
    t_1^U = \{ t_1^{1+\ell^m z} | z \in \mathbb{Z}_\ell \} = t_1 \{ t_1^{\ell^m z} | z \in \mathbb{Z}_\ell \},
\]

which is a coset of a formal subgroup \( Z \). Since \( t_1^U \) is an infinite set, we have \( \dim_W Z > 0 \). From irreducibility and \( \dim_W X = 1 \), we conclude \( X = t_1 Z \) and \( Z \cong \hat{\mathbb{G}}_m \). Since \( X \) contains roots of unity \( \zeta \in \Omega \subset \mu_{n_\infty}^n(W) \), we confirm that \( X = \zeta Z \) for \( \zeta \in \Omega \cap \mu_{n}^{m'} \) for \( m' \gg 0 \). Replacing \( t_1 \) by \( t_1^{\ell^m} \) for \( m \) as above if necessary, we have the translation \( \mathbb{Z}_\ell \ni s \mapsto \zeta t_1^s \in Z \) of one parameter subgroup \( \mathbb{Z}_\ell \ni s \mapsto t_1^s \). Thus we have \( \log(t_1) = \frac{dt^s_1}{ds} \big|_{s=0} \in T_\zeta \), which is sent by \( \log : \hat{\mathbb{G}} \to \mathbb{C}_\ell^n \) to \( \log(t_1) \in T_0 \). This implies that \( \log(t_1) \in T_0 \) and hence \( \log(t_1) \in T_\zeta \) for any \( \zeta \in \Omega^\circ \) (under the identification of the tangent space at any \( x \in \hat{\mathbb{G}} \) with \( \text{Lie}(\hat{\mathbb{G}}) \)). Therefore \( T_\zeta \)'s over \( \zeta \in \Omega^\circ \) can be identified canonically.
§7. **Proof in case** $\dim_W X > 1$. Consider the Zariski closure $Y$ of $t^U$ for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^o$). Since $U$ permutes finitely many geometrically irreducible components, each component of $Y$ is stable under an open subgroup of $U$. Therefore $Y = \bigcup_j \zeta_j T_j$ is a union of formal subtori $T_j$ of dimension $\leq 1$, where $\zeta_j$ runs over a finite set inside $\mu_{\ell,\infty}(\mathbb{C}_\ell) \cap X(\mathbb{C}_\ell)$. Since $\dim_W Y = 1$, we can pick $T_j$ of dimension 1 which we denote simply by $T$. Then $T$ contains $t^u$ for some $u \in U$. Applying the argument in the case of $\dim_W X = 1$ to $T$, we find $u \log(t) = \log(t^u) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^o$ and $t \in V_\zeta$.

Summarizing our argument, we have found

- (T) The Zariski closure of $t^U$ in $X$ for an element $t \in V_\zeta$ of infinite order contains a coset $\xi T$ of one dimensional subtorus $T$, $\xi^{\ell m'} = 1$ and $t^{\ell m'} \in T$ for some $m' > 0$;

- (D) Under the notation as above, we have $\log(t) \in T_\zeta$.

Moreover, the image $\overline{V}_\zeta$ of $V_\zeta$ in $\hat{G}/T$ is isomorphic to $(d - 1)$-dimensional open ball.
§8. Induction on $d$. If $d > 1$, therefore, we can find $\bar{t}' \in \bar{V}_\zeta$ of infinite order. Pulling back $\bar{t}'$ to $t' \in V_\zeta$, we find $\log(t), \log(t') \in T_\zeta$, and $\log(t)$ and $\log(t')$ are linearly independent in $T_\zeta$. Inductively arguing this way, we find infinite order elements $t_1, \ldots, t_d$ in $V_\zeta$ such that $\log(t_i)$ span over the quotient field $K$ of $W$ the tangent space $T_{\zeta/K} = T_\zeta \otimes_W K \hookrightarrow T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{1/K} \subset T_0$ with $T_{\zeta/K} \subset T_0$. Thus the tangent bundle over $X_{/K}^\circ$ is constant as it is constant over the Zariski dense subset $\Omega^\circ$. Therefore $X^\circ$ is close to an open dense subscheme of a coset of a formal subgroup. See Contemporary Math. 614 (2014) Lemma 4.1 for more details to conclude that $X^\circ$ is indeed an open dense subscheme of a coset of a formal subgroup.