* Characters of vanishing integral
and the thin point set $\Xi$

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*Assume that $Q \cong \Delta^-$ with $\Omega$ split over $F$. We describe the set $Z = \{\chi \in \text{Hom}(\Gamma, \mu_{\ell^\infty}) \text{ with } \int_{\text{Cl}_n} \chi \psi d\varphi_f = 0\}$ and relate it to the point set $\Xi \subset Sh^Q$. Recall $\int_{\Gamma_n} \chi d\varphi_{f^{Q}_\psi} = \sum_{\Omega, A \in \Gamma_n} \lambda^{\psi^{-1}}(\Omega)\chi(A)f([A\Omega^{-1}]_n[\Omega]_\Gamma)$. The action of $[\Omega]_\Gamma$ is transcendental and is incorporated into the embedding $C \hookrightarrow Sh^Q$. So we write down $f^{Q}_{\psi}$ as a value of a single modular form $f_{\psi} := \sum_{\Omega \in Q} \lambda^{\psi^{-1}}(\Omega)f|\langle \Omega \rangle$ for $f|\langle \Omega \rangle(x(A)) = f(x(A\Omega^{-1}))$. 
§0. The idea when $O_l = \mathbb{Z}_l$. Descend $\Omega \in Q$ to $q = \Omega \cap O$. Then $C_q := (\omega_q^{-1} \hat{Q}_w + \hat{Q}_w)/TX \subset X$ is a $O$-cyclic subgroup. Define for $\langle \Omega \rangle$ by the action of diag$[\omega_q^{-1}, 1]$:

$$f|\langle \Omega \rangle(X, \Lambda, w, \omega) := f(X/C_q, \Lambda_q, \langle \Omega \rangle w, \omega_q) (f|\langle \Omega \rangle([A]) = f([A\Omega^{-1}]),$$

where $\Lambda_q$ and $\omega_q$ are the push-down of $\Lambda$ and $\omega$ to the quotient $X/C_q$. Define $f_\psi = \sum_\Omega \lambda\psi^{-1}(\Omega)f|\langle \Omega \rangle$. Recall $v = v(\chi)$ such that $\chi([A_u]) = \zeta_j^{Tr(vu)}$ identifying $\Gamma_n[\hat{j}] \cong O/\hat{j}$ by $x(A_u) = \alpha(u/\omega^j_1)(x(R_n))$. We regard $(f|\langle \Omega \rangle)_\Omega$ a modular form on $Sh^Q$ and evaluate it at $\Xi = \Xi^n_{n,j}$ defined by the following sequence $n$.

$$n := \{n | \int_{C_n} \chi_w d\varphi f = 0 \text{ for } n > j \text{ with } \text{cond}(\chi) = l^n \text{ and } v(\chi) = v\}.$$  

Modifying further $f_\psi = \sum_\Omega \lambda\psi^{-1}(\Omega)f_v|\langle \Omega \rangle$ with

$$f_v = \sum_{u \in O/\hat{j}} \zeta_j^{Tr(vu)} f|\alpha(u/\omega^j_1),$$

we show $f_v([A]) = 0$ for all $s([A]) \in \Xi$; so, $f_v = 0$ if $\Xi$ is Zariski dense in $V^Q$. Note $N(l)^ja(\xi, f) = a(\xi, f_v)$ as long as $\xi \equiv -v$ mod $l^jO_l$. We suppose $j \geq r > 0$ for $r$ with $\ell^r \parallel \mathbb{F}_p[f, \lambda, \psi, \mu_\ell]^\times$. 


§1. Geometric modular forms.

Geometric modular forms classify quadruples \((X, \Lambda, w, \omega)\) with \((X, \Lambda, w, \omega)_A \in Sh(A)\), where \(\omega\) is a generator over \(O \otimes \mathbb{Z} A\) of \(H^0(X, \Omega_{X/A})\). A geometric modular form \(f/B\) \((B = W, \mathbb{F})\) is a functorial rule of assigning a value to triples \((X, \Lambda, w, \omega)\) to satisfy the following three axioms:

(G1) For a \(B\)-algebra homomorphism \(\phi: A \to A'\), we have

\[
f(((X, \Lambda, w, \omega) \times_A \phi A')) = \phi(f(X, \Lambda, w, \omega)).
\]

(G2) \(f\) is finite at all cusps, that is, the \(q\)-expansion of \(f\) at every Tate test object does not have a pole at \(q = 0\).

(G3) \(f(X, \Lambda, w, \xi \omega) = \xi^{-k\Sigma} f(X, \Lambda, w, \omega)\) for \(\xi \in T(A)\) for \(T = \text{Res}_{O/\mathbb{Z}}(\mathbb{G}_m)\).

Note \(k\Sigma \in \text{Hom}_{\text{alg. gp}}(T, \mathbb{G}_m)\) sending \(\xi\) to \(\xi^k \Sigma = \prod_{\sigma \in \Sigma} (\xi^\sigma)^k\).

Only important point about polarization is its ideal \(c\) such that \(\Lambda : X \otimes c \cong \text{Pic}^0(X)\), and \([c] \in Cl_F^+\) parameterizes geometrically irreducible components of \(Sh_K\) if \(\det(K) = \hat{O}(p), \times\). The differential operator \(d^\kappa\) changes \(k\) to \(k\Sigma + \kappa(1 - c)\). For simplicity, we assume \(\kappa = 0\).
§2. Choice of $\lambda$. For simplicity, assume that $f$ has trivial Neben types. Choose $\lambda$ so that $\lambda((\xi)) = \xi^{-k\Sigma}$ and $\lambda|_{F_A^\times(\infty)}$ is the central character of $f$. Fix $\omega$ on $X(R)$. Then by the isogeny $\iota : X(R) \to X(A)$ induced by $A = aR_n$ for $a \in M_{\hat{A}p(\infty)}^\times$, we have $\omega_A = \iota_\ast\omega$ for all $A$. Since $\xi : X(A) \cong X(\xi A)$ for $\xi \in M^\times \cap R_n^\times$, induces $\omega \mapsto \xi_\ast \omega = \xi \omega$, we find

$$f(x(\xi A)) = f(X(\xi A), \xi w, \xi \omega_A) = \xi^{-k\Sigma} f(x(A)),$$

and by $\lambda((\xi))\xi^{k\Sigma}$ is the Neben character of $f$, we find

$$f([A]) := \lambda(A)^{-1} f(x(A))$$

only depends on the class $Cl_{n}^- = M_{\hat{A}}^\times / \hat{R}_n^\times(F_{\hat{A}}(\infty)) \times M^\times M_{\infty}^\times$.

The action $\langle \Omega \rangle = \text{diag}[\omega_q^{-1}, 1]$ is at the place $q = \Omega \cap O$ and the action $\alpha(u/\omega_l^r)$ is at $l \neq q$; so, they commute. Thus

$$f|\langle \Omega \rangle([A]) \text{ and } f_v([A]) \text{ are well defined for } [A] \in Cl_{n}^-.$$
§3. Shimura’s reciprocity law.

Let $(M', \Sigma')$ be the reflex of $(M, \Sigma)$. We suppose that $f/F$ is the reduction modulo $p$ of $f/W$ and write $E$ over $M'$ be the field of rationality of $\psi$, $f/W$ and $\lambda$. Let $E_f$ be the field of rationality over $E[\mu_{\ell\infty}]$ of $x(A) \in Sh$ for all $[A] \in \text{Cl}_{\text{alg}}$. Then $E_f$ is an abelian extension over $E$. Then for an idele $b$ of $M'_{\mathbb{A}}$, we have $b^{\Sigma'} = \prod_{\sigma' \in \Sigma'} b^{\sigma'} \in M_{\mathbb{A}}^\times$, and hence we have an Artin symbol $[N(b)\Sigma', E]$ acting on $E_f$ for the norm map $N := N_{E/M'}$, whose ideal version, we write as $\sigma = \sigma_b = [N(b)\Sigma', E]$.

Here is a reciprocity law of Shimura:

$$f([A])^\sigma = f([N(b)^{-\Sigma'} A]), \quad (\text{R})$$

which implies

$$\left(\int_{\Gamma_n} \chi d\varphi^{O}_{f, \psi}\right)^\sigma = \chi^\sigma(N(b)^{\Sigma}) \int_{\Gamma_n} \chi^\sigma d\varphi^{O}_{f, \psi}. $$
§4. Trace relation. Let $\mathbb{F}_P = \mathbb{F}_p[f/F, \psi, \lambda/F, \mu_\ell]$ (the field of rationality of $f/F, \psi, \lambda/F$ and $\mu_\ell$). Define $r > 0$ by $\ell^r || |\mathbb{F}_P^\times|$. 

Lemma. For a generator $\zeta_n \in \mu_\ell^n$, if $\mathbb{F}_P[\chi] = \mathbb{F}_P[\zeta_n]$ with $n > j \geq r$, we have

$$\text{Tr}_{\mathbb{F}_P[\chi]/\mathbb{F}_P[\mu_\ell^j]}(\zeta_n) = \begin{cases} [\mathbb{F}_P[\zeta_n] : \mathbb{F}_P[\zeta_j]] \zeta_n^s & \text{if } \zeta_n^s \in \mu_\ell^j, \\ 0 & \text{otherwise.} \end{cases}$$

Note $[\mathbb{F}_P[\zeta_n] : \mathbb{F}_P[\zeta_j]] = \ell^{n-j} \neq 0$ in $\mathbb{F}$.

Proof. By our assumption, $j > 0$. Then the minimal equation of $\mathbb{F}_P[\chi]$ of $\zeta_n^s$ over $\mathbb{F}_P[\mu_\ell^j]$ is, if $\zeta_n^s \not\in \mu_\ell^j$, for $m = n - j$

$$X^{\ell^m(\ell-1)} + X^{\ell^m(\ell-2)} + \cdots + 1 = X^{\ell^m(\ell-1)} - \text{Tr}_{\mathbb{F}_P[\zeta_n^s]/\mathbb{F}_P[\mu_\ell^j]}(\zeta_n^s)X^{\ell^m(\ell-1)-1} + \cdots.$$

So, we get the above formula. $\square$
§5. $f_\psi$ to $f_v$. Recall
\[
\left( \int_{\Gamma_n[B]} \chi([A]) d\varphi_{f_\psi}([A][B]) \right)^\sigma = \chi^\sigma([N(b)\Sigma']) \int_{\Gamma_n} \chi([A]) d\varphi_{f_\psi}([A][B])
\]
by Shimura’s reciprocity law (R), and
\[
\int_{\Gamma_n} \chi([A]) d\varphi_{f_\psi}([A][B]) = 0 \iff \int_{\Gamma_n} \sigma(\chi([A])) d\varphi_{f_\psi}([A][B]) = 0.
\]
Thus for $n \in \mathbb{N}$ and any $[B] \in \Gamma_n$, we find for $\text{Tr} := \text{Tr}_{F_P[\chi]/F_P[\mu_{\ell^j}]}$
\[
0 = \sum_{\sigma \in \text{Gal}(F_P[\chi]/F_P[\mu_{\ell^j}])} \sum_{A \in \Gamma_n} \sum_{\Omega \in \mathcal{Q}} \lambda \psi^{-1}(\Omega) \chi^\sigma(A) f_{\langle \Omega \rangle}([AB][\Omega]_\Gamma)
\]
\[
= \sum_{A} \sum_{\Omega} \lambda \psi^{-1}(\Omega) \text{Tr}(\chi(A)) f_{\langle \Omega \rangle}([AB][\Omega]_\Gamma)
\]
\[
\text{Trace rel } \ell^{n-j} \sum_{\Omega \in \mathcal{Q}} \lambda \psi^{-1}(\Omega) \zeta_j \text{Tr}(vu) \sum_{u \mod \ell^j} f_{\langle \Omega \rangle} |\alpha(u/\omega^j_i)([B][\Omega]_\Gamma)\rangle
\]
\[
= \ell^{n-j} \sum_{\Omega \in \mathcal{Q}} \lambda \psi^{-1}(\Omega) f_v_{\langle \Omega \rangle}([B][\Omega]_\Gamma).
\]
§6. Conclusion.

Let $\tilde{f} := \sum_{\mathfrak{O} \in \mathcal{O}_1 \otimes \cdots \otimes \lambda \psi^{-1}(\mathfrak{O})} f_v|\langle \mathfrak{O} \rangle \otimes \cdots \otimes 1$ as a function on $V^Q$. Then for the embedding $s : C \cap V^Q \to V^Q$ given by $s(x(A)) = s(A) = (x(A[\mathfrak{O}_\Gamma]))_{\mathfrak{O} \in \mathcal{O}_1}$,

$$\sum_{\mathfrak{O} \in \mathcal{O}_1} \lambda \psi^{-1}(\mathfrak{O}) f_v|\langle \mathfrak{O} \rangle ([B][\mathfrak{O}_\Gamma]) = \lambda(B)^{-1} \tilde{f}(s(B)).$$

Thus if $\Xi$ is Zariski-dense in $V^Q$, we conclude $f_v = 0$. By computation, $a(\xi, f) \neq 0$ for $\xi \in -v$ is equivalent to $a(\xi, f_v) \neq 0$, a contradiction.

The sequence

$$n := \{ n \mid \text{cond}(\chi) = l^n \text{ and } \chi \in \mathcal{Z} \}$$

defines $\Xi = \{ s(A)[A] \in \bigcup_{n \in n} \ker(\Gamma_n \to \Gamma_j) \}$ as we took the trace to $\mathbb{F}_p[\mu_{l^j}]$. Therefore if $n$ contains an arithmetic progression, then $f_v = 0$ by the density theorem.
§7. Rigidity of torus. On the contrary to the assertion of the non-vanishing theorem, we assume that

\[ \mathcal{X} := \{ \chi \in \text{Hom}(\Gamma, \mu_{\ell\infty}) \mid \int_{\text{Cl}_n} \chi \psi d\varphi_f \neq 0, \ v(\chi) = v \} \]

has Zariski closure \( \overline{\mathcal{X}} \) with \( \dim \overline{\mathcal{X}} < d \). Since \( \mathcal{X} \) is stable by \( p \)-Frobenius \( t \mapsto t^P \) for a \( p \)-power \( P \), \( \overline{\mathcal{X}} \) is stable under \( t \mapsto t^{P^m} \) for all \( m \). Let \( W_\ell \) be a discrete valuation ring finite flat over \( W(F_\ell) \). We apply to the formal completion \( \widehat{\mathcal{X}} \) of \( \overline{\mathcal{X}} \) the following

**Rigidity Theorem.** Let \( X = \text{Spf}(T) \) be a closed formal sub-scheme of \( \widehat{G} = \widehat{G}^n / W_\ell \) flat geometrically irreducible over \( W_\ell \) (i.e., \( T \cap \overline{Q}_\ell = W_\ell \)). Suppose there exists an open subgroup \( U \) of \( \mathbb{Z}_\ell \times \) such that \( X \) is stable under the action \( \widehat{G} \ni t \mapsto t^u \in \widehat{G} \) for all \( u \in U \). If \( X \) contains a Zariski dense subset \( \Omega \subset X(\mathbb{C}_\ell) \cap \mu_{\ell\infty}^n(\mathbb{C}_\ell) \), then there exist \( \omega \in \Omega \) and a formal subtorus \( T \) such that \( X = T_\omega \).
§8. The strategy.
A key point is the use of a rigidity theorem asserting a formal subscheme of $\hat{\mathbb{G}}_m/W_\ell$ stable under $t \mapsto t^P$ for a $p$-power $P$ is a union of formal subtorus up to making finite quotient. Define $X := \{ \chi \in \text{Hom}(\Gamma, \mu_{\ell\infty}) | \int_{\text{Cl}_\infty} \chi \psi d\varphi_f \neq 0 \}$, and regard $Z$ and $X$ as a subset of $\hat{\mathbb{G}}_m/W_\ell$ for a sufficiently large $W_\ell$. Stability of $\hat{X} \subset \hat{\mathbb{G}}_m^d$ under a suitable power of $p$-Frobenius implies stability of $\hat{X}$ under an open subgroup $U \subset \mathbb{Z}_\ell^X$ generated by $P$. Assume $\dim \hat{X} < d$ for $d = [F : \mathbb{Q}]$. By the rigidity theorem applied to $\hat{X}$, we find an arithmetic progression $n$ such that $\chi$ with conductor $l^n$ for all $n \in n$ is in $\mathbb{G}_m^d - \hat{X}$ to conclude $f_v = 0$, a contradiction against $a(\xi, f_v) = N(l)^j a(\xi, f) \neq 0$ for $\xi \in -v$. Thus the non-vanishing theorem follows. The details will be discussed in the last lecture.