* Assume that $Q \cong \Delta^-$ with $\Delta$ split over $F$. We describe the set $\mathcal{Z} = \{\chi \in \text{Hom}(\Gamma, \mu_{\ell}) \text{ with } \int_{\text{Cl}^{-}_n} \chi \psi d\varphi_f = 0\}$ and relate it to the point set $\Xi \subset Sh^Q$. Recall $\int_{\Gamma_n} \chi d\varphi_{f_{\psi}} = \sum_{Q, \mathcal{A} \in \Gamma_n} \lambda \psi^{-1}(Q) \chi(\mathcal{A}) f([A\Omega^{-1}]_n[\Omega]_\Gamma)$. The action of $[\Omega]_\Gamma$ is transcendental and is incorporated into the embedding $C \hookrightarrow Sh^Q$. So we write down $f_{\psi}^Q$ as a value of a single modular form $f_{\psi} := \sum_{Q \in Q} \lambda \psi^{-1}(Q) f|\langle \Omega \rangle$ first for $f|\langle \Omega \rangle(x(\mathcal{A})) = f(x(A\Omega^{-1})).
The idea when $O_l = \mathbb{Z}_\ell$. Descend $\Omega \in \mathcal{Q}$ to $q = \Omega \cap O$. Then $C_q := (\omega_q^{-1} \hat{\omega}_1 + \hat{\omega}_2) / TX \subset X$ is a $O$-cyclic subgroup. Define for $\langle \Omega \rangle$ by the action of $\text{diag} [\omega_q^{-1}, 1]$:

$$f|_{\langle \Omega \rangle}(X, \overline{\Lambda}, w, \omega) := f(X/C_q, \overline{\Lambda}_\Omega, \langle \Omega \rangle w, \omega_\Omega)$$

$$f|_{\langle \Omega \rangle}(\mathfrak{A}) = f(\mathfrak{A}_q^{-1})$$

where $\overline{\Lambda}_\Omega$ and $\omega_\Omega$ are the push-down of $\overline{\Lambda}$ and $\omega$ to the quotient $X/C_\Omega$. Define $f_\psi = \sum_{\Omega} \lambda_\psi^{-1}(\Omega) f|_{\langle \Omega \rangle}$. Recall $v = v(\chi)$ such that $\chi([\mathfrak{A}_u]) = \zeta_j \text{Tr}(vu)$ identifying $\Gamma_n[l^j] \cong O/l^j$ by $x(\mathfrak{A}_u) = \alpha(u/\omega_l^j)(x(R_n))$. We regard $(f|_{\langle \Omega \rangle})_\Omega$ a modular form on $Sh^\mathcal{Q}$ and evaluate it at $\Xi = \Xi_{n,j}$ defined by the following sequence $n$.

$$n := \{n \mid \int_{Cl_n} \chi \psi d\varphi f = 0 \text{ for } n > j \text{ with cond}(\chi) = l^n \text{ and } v(\chi) = v \}.$$  

Modifying further $f_\psi = \sum_{\Omega} \lambda_\psi^{-1}(\Omega) f_v|_{\langle \Omega \rangle}$ with

$$f_v = \sum_{u \in O/l^j} \zeta_j^{\text{Tr}(vu)} f|_\alpha(u/\omega_l^j),$$

we show $f_v([\mathfrak{A}]) = 0$ for all $s([\mathfrak{A}]) \in \Xi$; so, $f_v = 0$ if $\Xi$ is Zariski dense in $V^\mathcal{Q}$. Note $N(l)^ja(\xi, f) = a(\xi, f_v)$ as long as $\xi \equiv -v \mod [l^jO_l]$. We suppose $j \geq r > 0$ for $r$ with $\ell^r || |F_p[f, \lambda, \psi, \mu_\ell]|^\times|.$
§1. Geometric modular forms.

Geometric modular forms classify quadruples \((X, \Lambda, w, \omega)\) with \((X, \Lambda, w)_A \in Sh(A)\), where \(\omega\) is a generator over \(O \otimes_{\mathbb{Z}} A\) of \(H^0(X, \Omega_{X/A})\). A geometric modular form \(f/B\) \((B = W, \mathbb{F})\) is a functorial rule of assigning a value to triples \((X, \Lambda, w, \omega)\) to satisfy the following three axioms:

- (G1) For a \(B\)-algebra homomorphism \(\phi : A \rightarrow A'\), we have
  \[ f((X, \Lambda, w, \omega) \times_A \phi A') = \phi(f(X, \Lambda, w, \omega)). \]

- (G2) \(f\) is finite at all cusps, that is, the \(q\)-expansion of \(f\) at every Tate test object does not have a pole at \(q = 0\).

- (G3) \(f(X, \Lambda, w, \xi \omega) = \xi^{-k\Sigma}f(X, \Lambda, w, \omega)\) for \(\xi \in T(A)\) for \(T = \text{Res}_{O/\mathbb{Z}}(\mathbb{G}_m)\).

Note \(k\Sigma \in \text{Hom}_{\text{alg. gp}}(T, \mathbb{G}_m)\) sending \(\xi\) to \(\xi^{k\Sigma} = \prod_{\sigma \in \Sigma}(\xi^\sigma)^k\).

Only important point about polarization is its ideal \(\mathfrak{c}\) such that \(\Lambda : X \otimes \mathfrak{c} \cong \text{Pic}^0(X)\), and \([\mathfrak{c}] \in \text{Cl}_F^+\) parameterizes geometrically irreducible components of \(Sh_K\) if \(\det(K) = \hat{O}^{(p)}, x\). The differential operator \(d^\kappa\) changes \(k\) to \(k\Sigma + \kappa(1 - c)\). For simplicity, we assume \(\kappa = 0\).
\section{Choice of $\lambda$.}

For simplicity, assume that $f$ has trivial Neben types. Choose $\lambda$ so that $\lambda((\xi)) = \xi^{-k\Sigma}$ and $\lambda|_{F_A^\times}$ is the central character of $f$. Fix $\omega$ on $X(R)$. Then by the isogeny $\iota : X(R) \to X(A)$ induced by $A = aR_n$ for $a \in M_{\mathbb{A}}^{\times}$, we have $\omega_A = \iota_* \omega$ for all $A$. Since $\xi : X(A) \cong X(\xi A)$ for $\xi \in M^\times \cap R_n^{\times}$, $\iota$ induces $\omega \mapsto \xi_* \omega = \xi \omega$, we find

\[ f(x(\xi A)) = f(X(\xi A), \xi w, \xi \omega_A) = \xi^{-k\Sigma} f(x(A)), \]

and by $\lambda((\xi))\xi^{k\Sigma}$ is the Neben character of $f$, we find

\[ f([A]) := \lambda(A)^{-1} f(x(A)) \]

only depends on the class $Cl_{n}^- = M_{\mathbb{A}}^{\times} / \hat{R}_{n}^{\times} (F_{\mathbb{A}}(\infty)) \times M^\times M_{\infty}^\times$.

The action $\langle \Omega \rangle = \text{diag}[\omega_q^{-1}, 1]$ is at the place $q = \Omega \cap O$ and the action $\alpha(u/\omega_l^r)$ is at $l \neq q$; so, they commute. Thus

\[ f|\langle \Omega \rangle ([A]) \text{ and } f_v([A]) \text{ are well defined for } [A] \in Cl_{n}^- .\]
§3. Shimura's reciprocity law.

Let \((M', \Sigma')\) be the reflex of \((M, \Sigma)\). We suppose that \(f/\mathbb{F}\) is the reduction modulo \(p\) of \(f/\mathcal{W}\) and write \(E\) over \(M'\) be the field of rationality of \(\psi, f/\mathcal{W}\) and \(\lambda\). Let \(E_f\) be the field of rationality over \(E[\mu_{\ell\infty}]\) of \(x(\mathcal{A}) \in Sh\) for all \([\mathcal{A}] \in Cl_{alg}\). Then \(E_f\) is an abelian extension over \(E\). Then for an idele \(b\) of \(M'_\mathbb{A}_\times\), we have \(b^{\Sigma'} = \prod_{\sigma' \in \Sigma'} b^{\sigma'} \in M'_\mathbb{A}_\times\), and hence we have an Artin symbol \([N(b)\Sigma', E]\) acting on \(E_f\) for the norm map \(N := N_{E/M'}\), whose ideal version, we write as \(\sigma = \sigma_b = [N(b)\Sigma', E]\).

Here is a reciprocity law of Shimura:

\[
f([\mathcal{A}])^{\sigma} = f([N(b)^{-\Sigma'} \mathcal{A}]), \tag{R}\]

which implies

\[
\left(\int_{\Gamma_n} \chi d\varphi_{f_\psi}^O\right)^{\sigma} = \chi^{\sigma}(N(b)^{\Sigma}) \int_{\Gamma_n} \chi^{\sigma} d\varphi_{f_\psi}^O.
\]
§4. Trace relation. Let $F_{P} = F_{p}[f/\mathbb{F}, \psi, \lambda/\mathbb{F}, \mu_{\ell}]$ (the field of rationality of $f/\mathbb{F}, \psi, \lambda/\mathbb{F}$ and $\mu_{\ell}$). Define $r > 0$ by $\ell^{r} \parallel |F_{P}^{\times}|$.

**Lemma.** For a generator $\zeta_{n} \in \mu_{\ell^{n}}$, if $F_{P}[\chi] = F_{P}[\zeta_{n}]$ with $n > j \geq r$, we have

$$\text{Tr}_{F_{P}[\chi]/F_{P}[\mu_{\ell^{j}}]}(\zeta_{n}^{s}) = \begin{cases} [F_{P}[\zeta_{n}]: F_{P}[\zeta_{j}]]\zeta_{n}^{s} & \text{if } \zeta_{n}^{s} \in \mu_{\ell^{j}}, \\ 0 & \text{otherwise.} \end{cases}$$

Note $[F_{P}[\zeta_{n}]: F_{P}[\zeta_{j}]] = \ell^{n-j} \neq 0$ in $\mathbb{F}$.

**Proof.** By our assumption, $j > 0$. Then the minimal equation of $F_{P}[\chi]$ of $\zeta_{n}^{s}$ over $F_{P}[\mu_{\ell^{j}}]$ is, if $\zeta_{n}^{s} \not\in \mu_{\ell^{j}}$, for $m = n - j$

$$X^{\ell^{m}(\ell-1)} + X^{\ell^{m}(\ell-2)} + \cdots + 1 = X^{\ell^{m}(\ell-1)} - \text{Tr}_{F_{P}[\zeta_{n}^{s}]/F_{P}[\mu_{\ell^{j}}]}(\zeta_{n}^{s})X^{\ell^{m}(\ell-1)-1} + \cdots.$$ 

So, we get the above formula. $\square$
§5. $f_{\psi}$ to $f_v$. For a while, assume that $\Gamma \cong \mathbb{Z}_\ell$. Recall
\[
\left( \int_{\Gamma_n[B]} \chi([A])d\varphi f_{\psi}([A][B]) \right)^\sigma = \chi^\sigma([N(b)^{\Sigma'}]) \int_{\Gamma_n} \chi([A])d\varphi f_{\psi}([A][B])
\]
by Shimura’s reciprocity law (R), and
\[
\int_{\Gamma_n} \chi([A])d\varphi f_{\psi}([A][B]) = 0 \iff \int_{\Gamma_n} \sigma(\chi([A]))d\varphi f_{\psi}([A][B]) = 0.
\]
Thus for $n \in \mathbb{N}$ and any $[B] \in \Gamma_n$, we find for $\text{Tr} := \text{Tr}_{\mathbb{F}_P[\chi]/\mathbb{F}_P[\mu_{\ell^j}]}$,
\[
0 = \sum_{\sigma \in \text{Gal}(\mathbb{F}_P[\chi]/\mathbb{F}_P[\mu_{\ell^j}])} \sum_{A \in \Gamma_n} \sum_{\Omega \in \mathcal{Q}} \lambda_{\psi^{-1}}(\Omega) \chi^\sigma(A)f|\langle \Omega \rangle([AB][\Omega]_{\Gamma})
\]
\[
= \sum_{A} \sum_{\Omega} \lambda_{\psi^{-1}}(\Omega) \text{Tr}(\chi(A))f|\langle \Omega \rangle([AB][\Omega]_{\Gamma})
\]
\[
\text{Trace rel } \ell^n-j \sum_{\Omega \in \mathcal{Q}} \lambda_{\psi^{-1}}(\Omega) \zeta_j \text{Tr}(vu) \sum_{u \mod \ell^j} f|\langle \Omega \rangle|\alpha(u/\omega_j^i)([B][\Omega]_{\Gamma})
\]
\[
= \ell^{n-j} \sum_{\Omega \in \mathcal{Q}} \lambda_{\psi^{-1}}(\Omega) f_v|\langle \Omega \rangle([B][\Omega]_{\Gamma}).
\]
§6. Conclusion.

Let \( \tilde{f} := \sum_{\Omega \in \mathcal{Q}} 1 \otimes \cdots \otimes \lambda \psi^{-1}(\Omega)f_v|\langle \Omega \rangle \otimes \cdots \otimes 1 \) as a function on \( V^Q \). Then for the embedding \( s : C \cap V^Q \to V^Q \) given by \( s(x(A)) = s(A) = (x(A[\mathfrak{Q}_\Gamma]))_{\Omega \in \mathcal{Q}}, \)

\[
\sum_{\Omega \in \mathcal{Q}} \lambda \psi^{-1}(\Omega)f_v|\langle \Omega \rangle([B][\mathfrak{Q}]_{\Gamma}) = \lambda(B)^{-1}\tilde{f}(s(B)).
\]

Thus if \( \Xi \) is Zariski-dense in \( V^Q \), we conclude \( f_v = 0 \). By computation, \( a(\xi, f) \neq 0 \) for \( \xi \in -v \) is equivalent to \( a(\xi, f_v) \neq 0 \), a contradiction.

The sequence

\[
n := \{n|cond(\chi) = 1^n \text{ and } \chi \in \mathcal{Z}\}
\]
defines \( \Xi = \{s(A)|A] \in \bigcup_{n \in \mathbb{N}} \text{Ker}(\Gamma_n \to \Gamma_j)\} \) as we took the trace to \( \mathbb{F}_P[\mu_{\ell^j}] \). Therefore if \( n \) contains an arithmetic progression, then \( f_v = 0 \) by the density theorem.
§7. Rigidity of torus. On the contrary to the assertion of the non-vanishing theorem, we assume that

\[ \mathcal{X} := \{ \chi \in \text{Hom}(\Gamma, \mu_{\ell^\infty}) \mid \int_{\text{Cl}_n} \chi \psi d\varphi_f \neq 0, \, v(\chi) = v \} \]

has Zariski closure \( \overline{\mathcal{X}} \) with \( \dim \overline{\mathcal{X}} < d \). Since \( \mathcal{X} \) is stable by \( p \)-Frobenius \( t \mapsto t^P \) for a \( p \)-power \( P \), \( \overline{\mathcal{X}} \) is stable under \( t \mapsto t^{Pm} \) for all \( m \). Let \( W_\ell \) be a discrete valuation ring finite flat over \( W(\overline{\mathbb{F}_\ell}) \). We apply to the formal completion \( \widehat{\mathcal{X}} \) of \( \overline{\mathcal{X}} \) the following

**Rigidity Theorem.** Let \( X = \text{Spf}(\mathcal{T}) \) be a closed formal subscheme of \( \widehat{G} = \widehat{G}^n_{m/W_\ell} \) flat geometrically irreducible over \( W_\ell \) (i.e., \( \mathcal{T} \cap \mathcal{O}_\ell = W_\ell \)). Suppose there exists an open subgroup \( U \) of \( \mathbb{Z}_\ell^\times \) such that \( X \) is stable under the action \( \widehat{G} \ni t \mapsto t^u \in \widehat{G} \) for all \( u \in U \). If \( X \) contains a Zariski dense subset \( \Omega \subset X(\mathbb{C}_\ell) \cap \mu_{\ell^\infty}(\mathbb{C}_\ell) \), then there exist \( \omega \in \Omega \) and a formal subtorus \( T \) such that \( X = T\omega \).
§8. The strategy.
A key point is the use of a rigidity theorem asserting a formal subscheme of \( \hat{G}_m/W_\ell \) stable under \( t \mapsto t^P \) for a \( p \)-power \( P \) is a union of formal subtori up to making finite quotient. Define \( \mathcal{X} := \{ \chi \in \text{Hom}(\Gamma, \mu_{\ell\infty}) | \int_{\mathbb{O}_{\ell\infty}} \chi \psi d\varphi_f \neq 0 \} \), and regard \( \mathcal{Z} \) and \( \mathcal{X} \) as a subset of \( \hat{G}_m/W_\ell \) for a sufficiently large \( W_\ell \). Stability of \( \hat{\mathcal{X}} \subset \hat{G}_m^d \) under a suitable power of \( p \)-Frobenius implies stability of \( \hat{\mathcal{X}} \) under an open subgroup \( U \subset \mathbb{Z}_\ell^X \) generated by \( P \). Assume \( \dim \hat{\mathcal{X}} < d \) for \( d = [F : \mathbb{Q}] \). By the rigidity theorem applied to \( \hat{\mathcal{X}} \), we find an arithmetic progression \( n \) such that \( \chi \) with conductor \( l^n \) for all \( n \in \mathbb{n} \) is in \( \mathbb{G}_m^d - \hat{\mathcal{X}} \) to conclude \( f_v = 0 \), a contradiction against \( a(\xi, f_v) = N(l)^j a(\xi, f) \neq 0 \) for \( \xi \in -v \). Thus the non-vanishing theorem follows. The details will be discussed in the last lecture.