Lectures on the anti-cyclotomic main conjecture, 2

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In this lecture, we sketch the proof of $H(\psi)_P|(h(M/F))L_p^-(\chi)_P$ for an arithmetic specialization at P, which finish the proof of the anti-cyclotomic main conjecture. We keep the assumptions and the notation in the first lecture and Tilouine's lecture. In particular, (M,Σ,Σ_p) is a fixed ordinary CM-type, and for simplicity, we assumed

(S) ψ has conductor $\mathfrak{C}\mathfrak{P}^e$ and order prime to p,

where $\mathfrak{C}+\mathfrak{C}^c = O_M$ and $e = (e(\mathfrak{P}))_{\mathfrak{P}\in\Sigma_p}$ with $\mathfrak{P}^e = \prod_{\mathfrak{P}\in\Sigma_p}\mathfrak{P}^{e(\mathfrak{P})}$.

§**1. q-Expansion of $f \in S(\mathbb{C})$.** Each f has Fourier expansion:

$$f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_{\mathbb{A}} \sum_{0 \ll \xi \in F} a(\xi dy, f) q^{\xi} \quad \text{with } q^{\xi} := \mathbf{e}_F(\sqrt{-1}\xi y_{\infty})\mathbf{e}_F(\xi x)$$

where $\mathbf{e}_F(x_\infty) = \exp(2\pi\sqrt{-1}\sum_{\sigma\in I} x_\sigma)$ and $y \mapsto a(y, f)$ is a function on $F^{\times}_{\mathbb{A}^{(\infty)}}$ supported on $\widehat{O}$. Here d is differental idele. Set $S(W) := \{f \in S(\mathbb{C}) | a(y, f) \in W, \forall y\}$. Define a pairings $\langle\cdot,\cdot\rangle : \mathbf{h}_P \times S(W) \to W$ and $(\cdot,\cdot) : S_B(W) \times \overline{S_B(W)} \to W$ by $\langle h, f\rangle = a(1, f|h)$ and $(f, g) = \sum_{a\in\mathcal{A}} e_a^{-1} f(a)g(a)$ for $e_a = |\overline{\Gamma}_a|$ with projected image $\overline{\Gamma}_a$ of $a\widehat{\Gamma}_0(N)a^{-1} \cap B^{\times}$ in $B^{\times}/F^{\times}$. The pairings are perfect (if $p \geq 5$); in particular,

$$\mathrm{Hom}_W(S, W) \cong \mathbf{h}_P \quad \text{and} \quad \mathrm{Hom}_W(\mathbf{h}_P, W) \cong S.$$

The latter map is induced by $\phi \mapsto |y|_{\mathbb{A}} \sum_{0\ll\xi\in\mathfrak{d}^{-1}} \phi(T(\xi dy))q^{\xi} \in S(W)$. Thus we obtain

Proposition 1. *The q–expansion for $f \in S_B(W)$ and $g \in \overline{S_B(W)}$*

$$\theta(f \otimes g)\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_{\mathbb{A}} \sum_{0\ll\xi\in F} (f|T(\xi y d), g)q^{\xi}$$

is an element of $S(W)$ (rigidifying JL).

§2. Theta series of B span $S(W)$. We find

$$S_B^{\mathbb{T}} := S_B(W)_{\mathbf{h}_P} \otimes \mathbb{T}_P \cong \mathbb{T}_P \cong \mathbb{T}_P^* \quad \text{as } \mathbb{T}\text{-modules}$$

for $\mathbb{T}_B^* = \mathrm{Hom}_W(\mathbb{T}_B, W)$, (a bi-product of the "$R = \mathbb{T}$"-theorem). By JL, $\theta : S_B^{\mathbb{T}} \otimes_W S_B^{\mathbb{T}} \to S^{\mathbb{T}}$ for $S^{\mathbb{T}} = S(W) \otimes_{\mathbf{h}_P} \mathbb{T}_P$ is onto. By computation, $\theta(f \otimes g) := \sum_{a,b \in \mathcal{A}} e_a^{-1} e_b^{-1} f(a) g(b) \widetilde{\theta}_{a,b}$, where $\widetilde{\theta}(x)$ is the adelic lift of classical theta series

$$\theta_{a,b} = \sum_{\xi \in a\widehat{\Delta}b^{-1} \cap B} \varepsilon^{-1}(a^{-1}\xi b)\mathbf{e}_F(N(\xi)z).$$

Here $\mathbf{e}_F(z) = \exp(2\pi\sqrt{-1}\sum_\sigma z_\sigma)$ for $z = (z_\sigma)_{\sigma \in I} \in \mathfrak{H}^I$ and $N : B \to F$ is the reduced norm map. Thus

Corollary 1. *Quaternionic theta series for the Schwartz–Bruhat function supported on $a\widehat{\Delta}b^{-1} \subset B_{\mathbb{A}}^{(\infty)}$ whose value is given by $a\widehat{\Delta}b^{-1} \ni axb^{-1} \mapsto \varepsilon^{-1}(x)$ for $a, b \in \mathcal{A}$ span $S(W)^{\mathbb{T}}$ over W.*

§3. Quaternionic theta is a product of two CM theta series. Consider $\theta(\phi)(z) := \sum_{\xi \in B} \phi(\xi) \mathbf{e}_F(N(\xi)z)$ for a Bruhat function $\phi : B_{\mathbb{A}}^{(\infty)} \to W$. We choose two embedding $l, r : O_M \hookrightarrow O_B$ and extend it to $M \hookrightarrow B$. Let $M \otimes_F M$ act on B by $(a \otimes b)v = l(a)vr(b^c)$. Since $M \otimes_F M \cong M \oplus M$ by $(a \otimes b) \mapsto (ab, a^c b)$, $B = (1,0)B \oplus (0,1)B$ with quadratic space $L = (1,0)B \cong M$ and $R = (0,1)B \cong M$ with norm form $N_{M/F} : L \to F$ and $-N_{M/F} : R \to F$ and $L \perp R$. By this decomposition, $\phi = \sum_{\phi_l, \phi_r} \phi_l \otimes \phi_r$, and $\theta(\phi) = \sum_{\phi_l, \phi_r} \theta(\phi_l)\theta(\phi_r)$. Thus we need to show the W-integrality of $\phi_?$ (i.e., $\phi_?$ has values in W).

To show integrality of $(\Theta, \theta(\phi_l)\theta(\phi_r))/\Omega^{2I}$, we need to use a Shimura series.

§**4. Idea of showing** $(\theta(\psi_P),\theta(\phi_l)\theta(\phi_r))/\Omega^{2I} \in W$**.** By means of Shimura series (defined in his 1981 Annals paper), we create a Hilbert modular form $\Psi(z,w)$ on $\mathrm{GL}(2)\times\mathrm{GL}(2)_{/F}$ and two lattices $\mathfrak{L},\mathfrak{R}$ in M, such that for the CM point (z_0,w_0) corresponding abelian varieties $(X(\mathfrak{L}),X(\mathfrak{R}))_{/W}$ of CM type Σ, $\Psi(z_0,w_0) = (\theta(\psi_P),\theta(\phi_l)\theta(\phi_r))/\Omega^{2I}$, where $X(\mathfrak{A})(\mathbb{C}) = \mathbb{C}^\Sigma/\mathfrak{A}^\Sigma$. We compute q-expansion to show Ψ is W-integral; hence,

$$\boxed{\frac{(\theta(\psi_P),\theta(\phi_l)\theta(\phi_r))}{\Omega^{2I}} = \Psi(z_0,w_0) \in W.}$$

To define Shimura series, put for $z,w \in \mathfrak{H}$ and $v \in M_2(\mathbb{C})$,

$$p(z,w) := -\begin{pmatrix} z \\ 1 \end{pmatrix}(w,1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \ [v;z,w] := \mathrm{Tr}(p(z,w)v^\iota).$$

By computation, $[v;z,w]$ never vanishes for $0 \neq v \in M_2(\mathbb{R})$, and for $\alpha,\beta \in \mathrm{GL}_2(\mathbb{R})$, we have

$$\begin{aligned} &\alpha p(z,w)\beta^\iota = p(\alpha(z),\beta(w))j(\alpha,z)j(\beta,z), \\ &[\alpha v\beta^\iota;z,w] = j(\alpha^\iota,z)j(\beta^\iota,w)[v;\alpha^\iota(z),\beta^\iota(w)]. \end{aligned} \tag{A}$$

§**5. Shimura series.** Shimura defined the following series $H(z,w;s)$ on $\mathrm{GL}(2)\times\mathrm{GL}(2)_{/F}$ for weight I Hilbert modular form $f=\sum_{\xi\in F}a(\xi,f)q^{\xi}$ on Γ and $[v;z,w]^{sI}:=\prod_{\sigma\in I}[v^{\sigma};z_{\sigma},w_{\sigma}]^{s}$:

$$\sum_{0\neq\alpha\in M_2(F)/U}\phi^{(\infty)}(\alpha)a(\det(\epsilon\alpha),f)[\alpha;z,w]^{-I}\left|[\alpha;z,w]^{-2sI}\right|$$

for $(z,w)\in\mathfrak{H}^I\times\mathfrak{H}^I$ with the stabilizer U in $O^{\times}$ of all the terms of H. This H converges over a right half plane and has meromorphic continuation. The residue $\Psi(z,w)$ at $s=1$ has the q-expansion for the partial Fourier transform ϕ^*,

$$\mathrm{Im}(z)^{-I}\,\mathrm{Im}(w)^{-I}\sum_{\alpha\in\Gamma\backslash M_2(F);\det(\alpha)\gg0}\phi^*(\epsilon\alpha)\mathbf{e}_F(\det(\alpha)z)f|_I\alpha(w),$$

where $\epsilon=\mathrm{diag}[-1,1]$ [AMC, Cor. 3.4]. When $f=\theta_1=\theta(\phi_1)$ and $\theta_2=\theta(\phi_R)$, writing $\Theta=\sum_{\mathfrak{A}}\Theta_{\mathfrak{A}}$ for classical theta series associated to the ideal class $\mathfrak{A}$ and choosing $\phi=\phi_L\otimes\phi_R$ and ϕ_1 well, by Shimura's evaluation, $\Psi(z_0,w_0)\doteqdot(\Theta_{\mathfrak{A}},\theta_1\theta_2)$ for a CM point $(z_0,w_0)\in\mathfrak{H}^I$ [AMC, Thm.4.1]. Suppose $M=F[z_0]=F[w_0]$ and as a point of $\mathfrak{H}^I$, $z_0=(z_0^{\sigma})_{\sigma\in\Sigma}$ and $w_0=(w_0^{\sigma})_{\sigma\in\Sigma}$.

§6. Evaluation of Shimura series.

Let $Y = M \otimes_F M = L \oplus R$ with $L \cong R \cong M$ so that $S(x,y) = \mathrm{Tr}(xy^\iota)$ on $(M_2(F), \det)$ is positive definite on L and negative on R. We let Y act on $M_2(F)$ by $(a \otimes b)v = \rho(a)vr(a^c)$ with

$$\binom{z_0 a}{a} = \rho(a)\binom{z_0}{1} \quad \text{and} \quad \binom{w_0 b}{b} = r(b)\binom{w_0}{1}.$$

The vector $\mathbf{p}_\sigma := p(z_{0,\sigma}, w_{0,\sigma}) \in M_2(\mathbb{C})$ is an Y-eigenvector such that $(a \otimes b)\mathbf{p}_\sigma = a^\sigma b^\sigma \mathbf{p}_\sigma$. Choose a generator $v \in M_2(F)$ well such that $M_2(O_p) = O_{Y,p}v$. Note that $\mathbf{p}_\sigma$ is orthogonal to Rv. So, writing $V \ni \alpha = av \oplus bv$ with $a \in L$ and $b \in R$

$$[\alpha^\sigma; z_{0,\sigma}, w_{0,\sigma}] = S(\alpha^\sigma, p(z_{0,\sigma}, w_{0,\sigma})) = a^{\sigma c}[v_\sigma; z_{0,\sigma}, w_{0,\sigma}]$$
$$[\alpha; z_0, w_0]^{-I}\,|[\alpha; z_0, w_0]|^{-2sI} = C^{-\Sigma}|C^{-2s\Sigma}|a^{-\Sigma c}N_{M/\mathbb{Q}}(a)^{-s},$$

where $C = [v_\sigma; z_{0,\sigma}, w_{0,\sigma}]$. If $\phi = \phi_L \otimes \phi_R$ with Bruhat function $\phi_?$ on ?, $C^\Sigma|C^{2s\Sigma}|H(z_0, w_0; s)$ is

$$\sum_{\alpha \in L/U} \phi_L(\alpha) \sum_{\beta \in R/U} \phi_R(\beta) a(\alpha\alpha^c - \beta\beta^c, f)\alpha^{-\Sigma c}N(\alpha)^{-s}, \qquad \text{(R1)}$$

as $\det(\epsilon(\alpha \oplus \beta)) = \alpha\alpha^c - \beta\beta^c$.

§7. Rankin convolution.

We show that $H(z_0, w_0; s)$ is the Rankin convolution of a cusp form f' and the theta series $\theta(\phi'_L)$ of the norm form of M for $\phi'_L(\alpha) = \alpha^{\Sigma}\phi_L(\alpha v)$. Choose ϕ_L so that $\theta(\phi'_L) = \Theta_{\mathfrak{A}}$. Writing $f' = f\theta(\phi_R) \in S_{(I,0)}(\Gamma')$,

$$H(z_0, w_0; s) = \sum_{\alpha \in R/U} \phi_L(\alpha v) a(\alpha\alpha^c, f') N(\alpha)^{1-s} = D(s-1; \theta(\phi'_L), f')$$

up to an explicit non-zero constant in $W^{\times}$, where $D(s; \theta(\phi'_L), f')$ is the Rankin convolution of $\theta(\phi'_L)$ and f' normalized to have a pole at $s = 0$ [AMC, (4.5)]. It is well known that the residue $\mathrm{Res}_{s=1} D(s-1; \theta(\phi'_L), f')$ is $(\theta(\phi'_L), f')$ up to an explicit constant. Taking residue at $s = 1$ and $f = \theta_1 = \theta(\phi_1)$ with $\theta_2 := \theta(\phi_R)$, we get $(\theta(\phi'_L), \theta_1\theta_2)/\Omega^{2I} \in W$ if θ_1 and θ_2 are W-integral. We have freedom to choose pairs (ϕ_1, ϕ_R) so that $S(W)$ is spanned by $\theta_1\theta_2 = \theta(\phi_1)\theta(\phi_R)$ over W.

§8. Preparation for a proof of integrality of θ_1, θ_2. Write $O_M = \mathfrak{y} z_1 + O z_2$ for the polarization O-ideal $\mathfrak{y}$ and $z_j = w_j \in M$, and start with $z_0 = (\sigma(z_1/z_2))_{\sigma \in \Sigma} = w_0 \in \mathfrak{H}^I$ as a starting CM point (so, $r = \rho$). We start modifying $z_?$ and $w_?$. Then ρ has values in $\begin{pmatrix} O & \mathfrak{y} \\ \mathfrak{y}^{-1} & O \end{pmatrix}$ by $\rho(\xi) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 \xi \\ z_2 \xi \end{bmatrix}$ and

$$\Gamma_0(\mathfrak{N}; \mathfrak{y}) = \{\gamma \in \begin{pmatrix} O & \mathfrak{y} \\ \mathfrak{N}\mathfrak{y}^{-1} & O \end{pmatrix} | 0 \ll \det(\gamma) \in O^\times\}.$$

If $M_{/F}$ ramifies at some finite places, the norm map $N_{M/F} : Cl_M \to CL_F$ is onto. Otherwise, Coker$(N_{M/F})$ is isomorphic to Gal(M/F). For $\overline{Sh}_B := B^\times \backslash B^\times_{\mathbb{A}} / \widehat{\Gamma}_0(N) B^\times_\infty F^\times_{\mathbb{A}}$, fix a complete representative set $\mathcal{A} := \{a_1 = 1, \dots, a_h\}$ with $a_{i,Np} = a_{i,\infty} = 1$. Thus

$$Sh_B \cong \{a_j\}_j \quad \text{i.e., } B^\times_{\mathbb{A}} = \bigsqcup_j B^\times a_j \widehat{\Gamma}_0(N) F^\times_{\mathbb{A}} B^\times_\infty.$$

§9. Reduction steps of the choice of a_j. We consider

$$\Delta_{ij} = a_i^{-\iota} \cdot \widehat{\Delta} a_j^{\iota} \cap B, \widehat{O}_{ijz} = a_i^{-\iota} \cdot \widehat{O}_0(N) a_j^{\iota} \cap B \quad \text{and}$$
$$\Gamma_0^i(N) = B^1 \cap a_i \widehat{\Gamma}_0(N) a_i^{-1} B_\infty^\times \quad \text{for } B^1(F) = \mathrm{Ker}(N : B^\times \to F^\times).$$

Put $\widehat{O}_i := \widehat{O}_{ii}$ and $O_i := \widehat{O}_i \cap B$. Replacing a_j by $\xi a_j u$ with $\xi \in B^\times$ and $u \in \widehat{\Gamma}_0(N) M_{\mathbb{A}}^\times$, we may assume that $N(a_j) = 1$. Pick a prime $\mathfrak{q}$ of F outside Np, by strong approximation theorem, $B^1(F)B^1(F_\mathfrak{q})$ is dense in $B^1(F_{\mathbb{A}}^{(\infty)})$. Thus we may assume that $a_j \in B^1(F_\mathfrak{q})$. Approximate a_j by $\gamma_j \in B^\times$ modulo small open compact subgroup $U \subset \widehat{O}_B^\times$ (i.e., $a_j = \gamma_j u$ with $u \in U$). We choose an embedding $i = i_1 = \rho : O_M \hookrightarrow O_1$, and conjugating i_1 by γ_j, we may assume that $i_j : O_{M,m} := O + \mathfrak{q}^m O_M \hookrightarrow O_j$ for $m \gg 0$. We have a decomposition $B = L \oplus R$ so that for projections L_i and R_i of O_i, $(L_i \oplus R_i)/O_i$ is killed by $\mathfrak{q}^m D_{M/F}$ prime to p. Choosing v well, the theta series of L_i and R_i has extra level $D_{M/F}^2 \mathfrak{q}^m$ for a sufficiently large m.

§10. Expressing ϕ supported on $\widehat{O}_i$ into a sum of $\phi_1 \otimes \phi_2$.

If ϕ is a characteristic function of $\widehat{O}_i$, by the Fourier transform of the finite additive group $Q := (L_i \oplus R_i)/(O_i \cap (L_i \oplus R_i))$, ϕ is a linear combination of additive characters of A with coefficients in $W[\frac{1}{NpN(\mathfrak{q})^m}] = W$. Additive characters are of the form $\phi_l \otimes \phi_r$. Our choice of Bruhat function is $\varepsilon\phi$, and we can check ε factors through L_i choosing v well. Thus $\varepsilon\phi = \sum_{(\phi_1,\phi_2)} \phi_1 \otimes \phi_2$ for $\phi_1 : L \to W$ and $\phi_2 : R \to W$. Note that individual $\theta_j = \theta(\phi_j)$ is on $\Gamma' := \Gamma_0(D^2_{M/F}\mathfrak{q}^m; \mathfrak{y}) \cap \Gamma_0(N \cdot N_{M/F}\mathfrak{P}^e); \mathfrak{y})$. We have the identity $(\Theta_{\mathfrak{A}}, \theta_1\theta_2)_{\Gamma'} = [\Gamma : \Gamma'](\Theta_{\mathfrak{A}}, \theta_1\theta_2)_{\Gamma}$ for $\Gamma := \Gamma_0(N; \mathfrak{y})$; so, we have

$$[\Gamma : \Gamma']\frac{(\Theta_{\mathfrak{A}}, \theta_1\theta_2)_{\Gamma}}{\Omega^{2I}} \in W.$$

We need to remove the factor $[\Gamma : \Gamma']$.

The choice of ϕ_L is determined by $\Theta_{\mathfrak{A}}$ and $H(z, w; s)$ is determined by (ϕ_L, ϕ_R, f), the choice of integral $\phi_R = \phi_2$ and $f = \theta(\phi_1)$ is arbitrary as long as the W-span of $\{\theta(\phi_1)\theta(\phi_R)\}_{\phi_1,\phi_R}$ contains $S(W)$.

$\S$**11. The factor** $[\Gamma : \Gamma']$ **is prime to** p**.** If $p|[\Gamma : \Gamma']$, then $p|(N_{F/\mathbb{Q}}(\mathfrak{q}) \pm 1)$. Since $F_{/\mathbb{Q}}$ is unramified, $N_{F/\mathbb{Q}} : O_p^\times \to \mathbb{Z}_p^\times$ is onto. Thus for a principal prime ideal $\mathfrak{q} = (\varpi)$ outside Np, if $p \geq 5$, we can choose $N_{F/\mathbb{Q}}(\varpi) \not\equiv \mp 1 \mod p$.

In [AMC], $(N(\mathfrak{l}) \pm 1)$ for prime factors $\mathfrak{l}$ of $D_{M/F}$ is also considered, but $D_{M/F}|N$, such factor does not appear in the index $[\Gamma : \Gamma']$.

Strictly speaking, there is one more Gauss sum factor appears in front of (Θ, Θ) as mentioned in the first lecture. This factor is compensated by basically the samefator appearing when we make the partial Fourier transform $\phi \mapsto \phi^*$. See details in [AMC, page 525–526].