* Irreducible components of Zariski closure

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*We prove Density theorem out of Black Box theorem a general theory of Zariski closure in a pro-étale variety of an infinite set of close points, and we prepare some notation and geometric lemmas to prove the density theorem. As before, let $\alpha_l = \text{diag}[1, \varpi_l]$. Choose a large power $l^m$ which is generated by $\varpi = \varphi \varphi^c$ ($\varphi \in R$) and put $\alpha = \text{diag}[1, \varpi]$. We may assume $\alpha/\alpha_l^m \in K$ and hence pretend $\alpha = \alpha_l^m$ to have the same action on $Sh_K = Sh/K$. $\alpha$ preserves each irreducible component of $Sh_{K/F}$ as long as $\det(K) = \hat{O}^\times$. Set $N := \{\alpha(u)|u \in O_l\}$, and assume $K \supset N$. For a variety $Y/F$, write $\text{Irr}_d(Y)$ for the set of irreducible components of dimension $d$ of $Y$. Set $\text{Irr}(Y) := \bigsqcup_d \text{Irr}_d(Y)$ and $\text{Irr}_+(Y) := \bigsqcup_{d>0} \text{Irr}_d(Y)$. 
§0. The idea.
For a proper $R_n$-ideal $A_n$ prime to $pI$, define a proper $R_n'$-ideal by $\hat{A}_n' := \hat{A}_n^{(1)} \times R_n',I$. We have three identities:

$$\alpha_I(x(A_n)) = x(A_{n+1}) \quad \text{for} \quad \alpha_I = \text{diag}[1, \varpi_I],$$

$$O_l/I^i \cong \Gamma_n[l^i] \quad \text{by} \quad u \mapsto \alpha(u/\varpi_I^i)x(R_n) =: x(A_{u,n}) \quad \text{if} \quad n \geq i,$$

For $\xi \in T_j := (M^x \cap R^x_{j,I}) \subset \text{GL}_2(F_A^{(p^{[\infty]})})$, $\xi(x(A)) = x((\xi)A)$.

We show, if $n$ is an arithmetic progression, the semi-group $\alpha^N$ generated by $\alpha^i \ i = 1,2,\ldots$ for the $m$-th power $\alpha$ of $\alpha_I$ acts on $\text{Irr}_0(X_K)$ for $X_K = \Xi_{n,j,K} \subset V^Q_K$. Since each orbit of $\alpha^N$ is infinite, for the image $X_K$ in $Sh/K$,

$$\text{Irr}_0(X_K) \neq \emptyset \Rightarrow |\text{Irr}_0(X_K)| = \infty,$$

a contradiction against noetherian property of $X_K \subset V_K$, and hence $\text{Irr}_0(X) = \varinjlim_K \text{Irr}_0(X_K) = \emptyset$. Since $\alpha \in \text{Aut}(V)$ (automorphisms) and by the above identity, $\alpha^{-1}(\Xi) - \Xi$ is a finite set, which we need later. We may replace $\alpha^N$ by $S = \alpha^N \cdot T_j$ to make the action on $\Xi$ transitive.
§1. Basics. Write $\mathcal{V} \coloneqq V^Q$, and adding the subscript $K$ implies the image in $(Sh/K)^Q$. Take $K$ sufficiently small so that $\mathcal{V}/\mathcal{V}_K$ is étale. Since for proper $R_n$-ideals $A$ and $A'$

$$X(A) \cong X(A') \iff [A]_n = [A']_n \in Cl_n,$$

a. $\pi_* : C \cap \mathcal{V} \cong C_K \cap \mathcal{V}_K$ by projection $\pi : \mathcal{V} \to \mathcal{V}_K$.

b. For the chosen infinite subset $\Xi \subset C \cap \mathcal{V}$, let $X$ (resp. $X_K$) be the Zariski closure of $\Xi$ in $\mathcal{V}$ (resp. $\Xi_K$ in $\mathcal{V}_K$). Then $X_K$ is the reduced image of $X$ in $\mathcal{V}_K$.

c. For the image $[\delta_n] \in Cl^-_n$ of $\delta \in Q$ with $\delta_n$ prime to $pl$, $Cl_n \ni [A]_n \mapsto [A\delta_n]_n \in Cl^-_n$ commutes with the action of $S$ and $\alpha(u)$ for $u \in F_l$, since the action of $S$ and $\alpha(u)$ is concentrated at $l$ and the multiplication by $[\delta_n]_n$ is outside $l$-action. So the diagonal action of $S$ on $\mathcal{V}$ preserves $C \cap \mathcal{V}$. 
§2. \text{Irr}(X). \text{Let } \pi_*(\text{Irr}(X_U)) := \{\pi(Z')|Z' \in \text{Irr}(X_U)\} \text{ for a closed subgroup } U \text{ of } K \text{ and the projection } \pi : \mathcal{V}_U \to \mathcal{V}_K \text{ for the projection, where } \pi(Z') \subset X_K \text{ is the reduced image. Then}

1 (Going up theorem). \text{For } Y \in \text{Irr}(X_K), \text{if } Z \in \text{Irr}(\pi^{-1}(Y)) \text{ is contained in } X_U, \text{we have } Z \in \text{Irr}(X_U), \text{where } \pi^{-1}(Y) = Y \times \mathcal{V}_K \mathcal{V}_U.

2. \text{The image } \pi_*(\text{Irr}(X_U)) \text{ contains } \text{Irr}(X_K); \text{so, for } Y \in \text{Irr}(X_K), \text{we have } Z' \in \text{Irr}(X_U) \text{ such that } \pi_*(Z') = Y, \text{because any closed irreducible subvariety is contained in an irreducible component.}

3. \text{We have a unique section } \text{Irr}_0(X_K) \hookrightarrow \text{Irr}_0(X_U) \text{ of } \text{Irr}_0(X_U) \twoheadrightarrow \pi_*(\text{Irr}_0(X_U)) \subset X \text{ and } \text{Irr}_0(X_U) \subset \Xi_U. \text{Moreover}

\[
\text{Irr}_0(X_U) = \varprojlim_{U'} \text{Irr}_0(X_{U'}) \subset \Xi
\]

for \( U' \) running over all open subgroups of \( K \) containing \( U \).

I will give a proof of some of these assertions later if time allows.
\section{Correspondence action of $\alpha^N$ on $\text{Irr}_d(X_K)$.}

Let $\beta = \alpha^i$. For an irreducible component $Y_K \in \text{Irr}_d(X_K)$, let $Y_U = \bigcup_{Z \in \text{Irr}_d(\pi_{U,K}^{-1}(Y_K)) \cap \text{Irr}_d(X_U)} Z$. Consider the diagram for $U = K \cap K^\beta$ for $K^\beta := \beta^{-1}K\beta$ (so, $UU^\beta{-1} \subset K$):

\[\begin{array}{ccc}
X_U \supset Y_U & \xrightarrow{v \mapsto \beta(v)} & \beta(Y_U) \subset X_{U^\beta{-1}} \\
\pi_{U,K} \downarrow & & \downarrow \pi = \pi_{U^\beta{-1},K} \\
X_K \supset Y_K & \rightarrow & \pi(\beta(Y_U)) \subset X_K.
\end{array}\]

We define the correspondence action of $\beta$ by

\[\beta(Y_K) := \{\pi\beta(Z) | Z \in \text{Irr}_d(Y_U)\}.\]

This set $\beta(Y_K)$ can be shown to be a subset of $\text{Irr}_d(X_K)$. As we only need the case of $d = 0$, we prove this fact assuming $d = 0$. Then $\beta(Y_K)$ is a singleton made of $\beta(Y_K)$. 

4. \(\beta\)-action on \(\text{Irr}_0(X_K)\). Suppose \(d = 0\), and write \(U' := U^{\beta^{-1}}\). By Property 3, \(x_K = Y_K \in \text{Irr}_0(X_K)\) falls in the image \(\Xi_K\) in \(V_K\) of \(\Xi\). Since \(\Xi \cong \Xi_U \cong \Xi_K\), \(p_{U,K}^{-1}(x_K) \cap X_U = \{x_U\} \subset \Xi_U\) is a singleton. Therefore \(Y_U = \{Z := x_U\}\) is a singleton.

Take an irreducible component \(Y'_K\) of \(X_K\) containing \(\beta(x)_K = \beta(x_K\beta)\) such that \(\beta(Z) \subset Z'\) for an irreducible component \(Z'\) of \(Y'_U\), (so, \(\beta(x_U) \in Z'\)). Such a \(Y'_K\) exists by Property 2. So \(\dim Z' = \dim Y'_K \geq 0\). We want to prove \(\dim Y'_K = 0\). Since \(\text{Irr}_+(\beta^{-1}(X)_U) = \text{Irr}_+(X_U)\) by \(|\beta^{-1}(\Xi) - \Xi| < \infty\), if \(\dim Z' > 0\), we have \(\dim \beta^{-1}(Z') > 0\) and \(\beta^{-1}(Z')\) is an irreducible component of \(X_U\). Since \(\beta^{-1}(Z') \supset Z = x_U\) by construction and the two are irreducible components of \(X_U\) (by going-up), we find that \(\beta^{-1}(Z') = Z = x_U\), a contradiction against \(\dim Z' > 0\). Hence \(\dim Z' = 0\) and \(Z' = \beta(Z) = \beta(x_U)\), and \(Y'_K = p_{U',K}(Z') = \beta(x)_K\). This implies that \([\beta]\) brings \(\text{Irr}_0(X_K)\) into \(\text{Irr}_0(X_K)\), and \(x_K \mapsto [\beta](x_K) = \beta(x)_K\) is really an action (not a correspondence action) of \(\alpha^N\) on \(\text{Irr}_0(X_K)\), and the action is compatible with the action of \(\alpha^N\) on \(\Xi\) as \(\text{Irr}_0(X_K) \subset \Xi_K \cong \Xi\).
§5. Proof of density theorem.

**Density Theorem.** Assume $Q \hookrightarrow Cl^{-\infty}_{-}/Cl^{alg}$. Let $\underline{n} \subset \mathbb{Z}_+$ be the sequence defining $\Xi$. If $\underline{n}$ contains an arithmetic progression, then $X \cap \Xi \neq \emptyset$ and $\Xi$ is Zariski dense in $V^Q$.

Proof. We can replace $\underline{n}$ by an arithmetic progression of suitable difference so that $\alpha^N$ preserve $\Xi_{\underline{n},r}$. Then $S = \alpha^N \cdot \mathcal{T}$ acts transitively on $\Xi$ with all orbits are infinite. If $\text{Irr}_0(X_K) \neq \emptyset$, by the action of $S$ on $\text{Irr}_0(X_K)$ described in §4, $\text{Irr}_0(X_K)$ is infinite. This is a contradiction, as $X_K$ is a noetherian scheme.

Thus $\text{Irr}_0(X) = \lim_{\to K} \text{Irr}_0(X_K) = \emptyset$, therefore all irreducible components of $X$ has positive dimension; so, we have an irreducible component $Z$ of $X$ with $x \in Z \cap \Xi$, By Black Box Theorem, $Z = X = V^Q$ as desired. □
Proof of Property 3. Since $\mathcal{V}_U \to \mathcal{V}_K$ is étale, it is affine; so, we may assume that $\mathcal{V}_U = \text{Spec}(A')$ and $\mathcal{V}_K = \text{Spec}(A)$ with $A'/A$ finite. Write $\text{Irr}_0(A) = \text{Irr}_0(\text{Spec}(A))$ and regard it as a set of minimal primes. Then $X_U = \text{Spec}(B')$ and $X_K = \text{Spec}(B)$ for $B' = A'/\cap_{P \in \Xi_U} P$ and $B = A/\cap_{P \in \Xi_U} (A \cap P)$ regarding $\Xi_U$ a set of maximal $A'$-ideals. Pick $m \in \text{Irr}_0(B)$. Then $B = B'(m) \oplus B/m$ for a subring $B'(m) \subset B$ as $\text{Spec}(B/m)$ is a connected component of $\text{Spec}(B)$. Since $B' \supset B$, the above decomposition induces an algebra direct sum $B' = B'(m) \oplus B'/mB'$. Since $B'$ is finite over $B$, $B'/mB'$ has dimension 0. By reducedness of $B'$, the direct summand $B'/mB'$ of $B'$ is a direct sum of fields. Then $\pi$ induces a surjection $\text{Irr}_0(B') \supset \pi_0(\text{Spec}(B'/mB')) \xrightarrow{\pi^*} \{m\}$ for each $m$. Therefore $\pi^*(\text{Irr}_0(B')) \supset \text{Irr}_0(B)$. If $m \notin \Xi_K, \Xi_K \subset \text{Spec}(B'(m))$ as $\text{Spec}(B) = \text{Spec}(B/m) \sqcup \text{Spec}(B'(m))$. This implies $B = A/\cap_{P \in \Xi_K} P$ is equal to $B'(m)$, a contradiction. Thus $m \in \Xi_K$, and $\text{Irr}_0(B) \subset \Xi_K$. Since $\Xi \cong \Xi_K$, $\pi^* : \text{Irr}_0(B') \to \pi^*(\text{Irr}_0(B'))$ has a unique section $\pi^* : \text{Irr}_0(B) \to \text{Irr}_0(B')$. 


As \( V \rightarrow V_K \) is étale, \( \pi^{-1}(Y) \) is étale over \( Y \); so, equi-dimensional.
Suppose that \( Z \subset X' \) for \( Z \in \text{Irr}(\pi^{-1}(Y)) \). Then we find \( Z' \in \text{Irr}(X') \) such that \( Z' \supset Z \); so, \( \pi(Z') \subset X \). We are going to show \( Z' = Z \). We have \( X \supset \pi(Z') \supset Y \). Since \( \pi(Z') \) is irreducible, \( \pi(Z') \) containing \( Y \in \text{Irr}(X) \) implies \( \pi(Z') = Y \). Thus \( Z' \rightarrow Y \) is a integral dominant; so, \( \dim Z' = \dim Z = \dim Y \). This shows \( Z = Z' \in \text{Irr}(X') \), as desired. Thus Property 1 follows.
Pick \( p \in \text{Irr}(B) \) giving \( Y \in \text{Irr} \left( \text{Spec}(B) \right) \). Since \( B'/B \) is integral, we find a prime \( P' \in \text{Spec}(B') \) such that \( P' \cap B = p \) by going-up theorem. For each \( P' \in \text{Spec}(B') \) with \( P' \cap B = p \) (i.e., \( P' \in \pi^{-1}(Y) = \text{Spec}(B'/pB') \)), take a minimal prime \( p' \subset P' \) (i.e., \( p' \in \text{Irr}(B') \)). Then \( p' \cap B \) is a prime ideal of \( B \) and \( p \supset p' \cap B \); so, by minimality of \( p \), we have \( p = p' \cap B \). Thus \( p \) is in the image of \( \text{Irr}(B') \). This proves Property 2.