Lectures on the anti-cyclotomic main conjecture, 1

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Under the assumptions and notation Tilouine mentioned, we first prove, for a prime $p \geq 5$ unramified in M,

$$h(M/F)L_p^-(\chi)|H(\psi)|h(M/F)\mathcal{F}(\chi) \quad (h(M/F) = h(M)/h(F))$$

in $\Lambda = W[[\Gamma_M]]$ for a CM field M with maximal real subfield F and the congruence power serie $H(\psi)$ of ψ, built on the lectures by Tilouine proving this over $\Lambda[1/p]$. In the second lecture, we give a sketch of the proof of the reverse divisibility: $H(\psi)|h(M/F)L_p^-(\chi)$ resulting the main conjecture, as $H(\psi) = h(M/F)\mathcal{F}(\chi)$ for the anticyclotomic Iwasawa power series $\mathcal{F}(\chi)$ by the $R = \mathbb{T}$-theorem. We fix an anti-cyclotomic character χ and a Hecke character ψ (both of order prime to p) such that $\chi = \psi^- = \psi^{c-1}$ for $\psi^c = \psi(c\sigma c)$. For simplicity, we actually assume that

(S) ψ has prime-to-p conductor $\mathfrak{C}$ made of split primes over F.

§**1. The factorization of the p-adic Rankin product.** We write (M, Σ, Σ_p) for a fixed ordinary CM-type and O (resp. O_M) for the integer ring of F (resp. M), and put $I = \mathrm{Isom}_{\mathsf{field}}(F, \overline{\mathbb{Q}})$. Up to finitely many Euler factors in $W[[\Gamma_M \times \Gamma_M]]$, we have

$$\frac{\mathcal{R}}{H(\psi)} = \frac{\mathcal{L}_p(\psi^{-1}\varphi)\mathcal{L}_p(\psi^{-1}\varphi_c)}{h(M/F)L_p^-(\chi)}, \tag{RK0}$$

Here $\mathcal{R} \in W[[\Gamma_M]]$ is the p-adic Rankin product of the p-adic nearly ordinary families of the automorphic inductions $\pi(\psi)$ and $\pi(\varphi)$ for a chosen Hecke character φ of M of order prime to p with split conductor, $L_p^-(\chi) \in W[[\Gamma_M^-]] \subset W[[\Gamma_M]]$ interpolating the algebraic part of $L(1, \psi_P^-)$ for members ψ_P of the family of ψ (so, $P \in \mathrm{Spec}(W[[\Gamma_M^-]])$), and $\mathcal{L}_p(?)$ is the Katz p-adic L of branch character ?. We prove that the numerator $\mathcal{L}_p(\psi^{-1}\varphi)\mathcal{L}_p(\psi^{-1}\varphi_c) \in W[[\Gamma_M \times \Gamma_M]]$ is prime to p; i.e., it has vanishing μ-invariant. The removed Euler factors have vanishing μ-invariant as $\mathrm{Frob}_{\mathfrak{q}}$ has infinite order in Γ_M. We ignore such Euler factors.

§**2.** $\mu = 0$ **Theorem.** The vanishing of the μ-invariant of the Katz p-adic L requires (S). In my Annals paper of 2010, under split conductor assumption for ?, it is proven that for $\mathcal{L}_p = \mathcal{L}_p(?)$ the restriction $\mathcal{L}_p^- = \mathcal{L}_p|_{\Gamma_M^-}$ of $\mathcal{L}_p$ to Γ_M^- according to the splitting $\Gamma_M = \Gamma_M^+ \times \Gamma_M^-$ has vanishing μ-invariant if M/F ramifies. In my 2011 Compositio paper, for each point $P \in \mathrm{Spec}(W[[\Gamma_M^+]])(W)$, restricting $\mathcal{L}_p$ to $W[[P \times \Gamma_M^-]] \cong W[[\Gamma_M^-]]$, I found that

$$\mathrm{Inf}_P\, \mu(\mathcal{L}_p|_{P \times \Gamma_M^-}) = 0.$$

So, $\mu(\mathcal{L}_p) = 0$ in $W[[\Gamma_M]]$ always, and hence $\mathcal{L}_p(\psi^{-1}\varphi)$ and $\mathcal{L}_p(\psi^{-1}\varphi_c)$ both have vanishing μ-invariant.

These results requires hard modulo p arithmetic geometry of Hilbert modular variety, we admit it.

§**3. The assumption (S).** Though we can always find ψ with $\psi^- = \chi$, we do not know if ψ satisfy (S). To cover the case without (S), we need a generalization of (RK0) to a quadratic totally real extension $F'_{/F}$. We choose F' such that all non-split factors of $\mathfrak{c} := \mathfrak{C} \cap F$ to split in $K_{/F'}$ for $K = MF'$. The CM field K contains two CM fields M' and M over F. Choose a Hecke character φ of M' with split conductor. Writing $\widehat{\psi} = \psi \circ N_{K/M}$ and $\widehat{\varphi} = \varphi \circ N_{K/M'}$. Then by the same argument proving (RK0),

$$\frac{\mathcal{R}}{H(\psi)} = \frac{\mathcal{L}_p(\widehat{\psi}^{-1}\widehat{\varphi})}{h(M/F)L_p^-(\chi)}, \tag{RK1}$$

where $\mathcal{R}$ is the Rankin product of the p-adic θ families for $\widehat{\psi}$ and $\widehat{\varphi}$ of the Hilbert modular forms over F'. Then applying the vanishing of μ for Katz p-adic L with respect to K/F', we get the divisibility $h(M/F)L_p^-(\chi)|H(\psi)$.* We are going to show

$$h(M/F)L_p^-(\chi) = H(\psi) \overset{R=\mathbb{T}}{\equiv} h(M/F)\mathcal{F}(\chi). \tag{I}$$

*§6 of my preprint [ICTS]: "Non-vanishing of integrals of a mod p modular form" (on my web) gives a detailed account of this divisibility.

§**4. Hecke algebra $\mathbb{T}$ is universal.** To avoid complications, we assume that (S). Consider $\rho_0 := \mathrm{Ind}_M^F \psi : G \to GL_2(W)$ with its reduction $\overline{\rho} = \mathrm{Ind}_M^F \overline{\psi}$ modulo $\mathfrak{m}_W$, whose Artin conductor is $N = D_{M/F}N_{M/F}(\mathfrak{C})$. Write $F(\overline{\rho})$ for the splitting field of a Galois representation $\overline{\rho}$ and let $G := \mathrm{Gal}(F^{(p)}(\overline{\rho})/F(\overline{\rho}))$, where $F^{(p)}(\overline{\rho})$ is the maximal p-profinite extension of $F(\overline{\rho})$ unramified outside p. We have a local ring $\mathbb{T}$ of an appropriate Hecke algebra and a Galois representation $\rho_{\mathbb{T}} : G \to GL_2(\mathbb{T})$ such that $(\mathbb{T}, \rho_{\mathbb{T}})$ is universal among nearly ordinary Galois deformations by the "$R = \mathbb{T}$"-theorem as described in Tilouine's lecture. For the decomposition group $D_{\mathfrak{q}} \subset G$ at each prime $\mathfrak{q}|Np$, by (S), $\rho_{\mathbb{T}}|_{D_{\mathfrak{q}}} \cong \begin{pmatrix} \epsilon_{\mathfrak{q}} & * \\ 0 & \delta_{\mathfrak{q}} \end{pmatrix}$ with $\delta_{\mathfrak{q}} \equiv \psi_c|_{D_{\mathfrak{q}}} \bmod \mathfrak{m}_W$. Regarding $O_{\mathfrak{q}}^{\times}$ as the inertia group at $\mathfrak{q}$ and making the product of $\delta_{\mathfrak{q}}$ and $\epsilon_{\mathfrak{q}}$ over $\mathfrak{q}|Np$, we have two characters $\epsilon_N, \delta_N : O_p^{\times} \times (O/N)^{\times} \to \mathbb{T}^{\times}$. Since $\det(\rho_{\mathbb{T}}) = \epsilon_N \delta_N$ is a global character, as a character of the diagonal torus $T(O_p \times O/N)$ $\epsilon_N \delta_N$ factors through $\mathcal{T} := T(O_p \times O/N)/Z(O)$ for the central torus Z $(Z(O) = O^{\times} \hookrightarrow T(O_p \times O/N)$ diagonally embedded).

§5. $\mathbb{T}$ is an algebra over an Iwasawa algebra.

We have an exact sequence $Z(O_p)/Z(O) \hookrightarrow Cl_F(p^\infty) \twoheadrightarrow Cl_F$ for the ray class group $Cl_F(?)$ modulo ?. Thus the maximal torsion-free quotient $\Gamma_F^+ \cong \Gamma_M^+$ of $Cl_F(p^\infty)$ contains the image under $\epsilon_p\delta_p|_{O_p^\times} = \det(\rho_{\mathbb{T}})$ of the maximal torsion-free quotient Γ' of $Z(O_p)/Z(O)$. Since $Z(O_p)/Z(O)$ is fixed by the main involution ι given by $x + x^\iota = \mathrm{Tr}(x)$ for 2×2 matrices x, taking "$-$"-eigenspace Γ_F^- of the maximal torsion-free quotient of $T(O_p)$, the character $\epsilon_p\delta_p$ induces a character of $\mathbf{\Gamma} := \Gamma_F^- \times \Gamma_F^+ \ni (\gamma, z) \mapsto \epsilon_p^{-1}(\gamma)\delta_p(\gamma)\det(\rho_{\mathbb{T}})(z) \in \mathbb{T}^\times$. Therefore $\mathbb{T}$ is an algebra over $W[[\mathbf{\Gamma}]]$. More precisely, for the level ideal $N := D_{M/F}N_{M/F}(\mathfrak{C})$, writing Δ_F^+ (resp. Δ_F^-) for the maximal torsion subgroup of $Cl_F(Np^\infty)$ (resp. $\mathcal{T}$), $\mathbb{T}$ is an algebra over $W[[\widetilde{\mathbf{\Gamma}}]] = W[\mathbf{\Delta}_F][[\mathbf{\Gamma}]]$ for $\widetilde{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Delta}_F$ with $\Delta_F = \Delta_F^+ \times \Delta_F^-$.

§**6. Control theorem.** Regard $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$ ($\kappa_j = \sum_\sigma \kappa_{j,\sigma}\sigma$) as a character of T by $(F^\times)^2 \ni (x,y) \mapsto x^{\kappa_1}y^{\kappa_2} \in \overline{\mathbb{Q}}^\times$ with $x^{\kappa_j} = \prod_\sigma x^{\sigma\kappa_{j,\sigma}\sigma}$. A character $\omega : \widetilde{\mathbf{\Gamma}} \to W^\times$ has weight κ if it coincides with κ over an open subgroup of $T(O_p)$.

An algebra homomorphism $P : W[[\widetilde{\mathbf{\Gamma}}]] \to W$ is called arithmetic if $P|_{\mathbf{\Gamma}}$ has weight κ and $\kappa_{1,\sigma} - \kappa_{2,\sigma} \geq 1$ for all $\sigma \in I$. We put $k = \kappa_1 - \kappa_2 + I$ for $I = \sum_\sigma \sigma$; so, the above condition means $k \geq 2I$ (i.e., the weight is bigger than or equal to 2 as in elliptic modular case). Since $\mathfrak{q}|N$ prime to p ramifies only in $F(\overline{\rho})/F$, $P|_{\Delta_F} = \det(\rho_0)$ and $\det(\rho_{\mathbb{T}})|_{\Delta_F} = \det(\rho_0)$ with $\det(\rho_{\mathbb{T}})$ having values in $W[[\mathbf{\Gamma}]]^\times$. Note $\det(\rho_0) = \psi|_{F_{\mathbb{A}}^\times}\left(\frac{M/F}{}\right)$ as a Hecke character. Then $\mathbb{T}_P := \mathbb{T} \otimes_{W[[\mathbf{\Gamma}]],P} W$ is the universal ring among nearly ordinary deformations ρ over G with $\det(\rho) = P \circ \det(\rho_{\mathbb{T}})$. The ring $\mathbb{T}_P$ is W-free of finite rank and is a local factor of a Hecke algebra of weight κ with an appropriate Neben type.

§7. Reduction steps of the proof of the main conjecture.

(0) As seen in §3, $H(\psi) = h(M/F)L_p^-(\chi)U$ for $0 \neq U \in W[[\Gamma_M]]$.

(1) $R = \mathbb{T}$-theorem implies $H(\psi) = h(M/F)\mathcal{F}(\chi)$. We need to prove $h(M/F)L_p^-(\chi) = H(\psi)$.

(2) For a weight 2-specialization $\mathbb{T}_P$, in the second lecture, we will prove

$$H(\psi)_P | h(M/F)L_p^-(\chi)_P. \tag{I'}$$

Combined with (0), $U_P = (U \mod P)$ is a unit in W, and Nakayama's lemma tells us U is a unit in $W[[\Gamma_M]]$. So,

$$h(M/F)L_p^-(\chi) = H(\psi) \overset{R=\mathbb{T}}{=} h(M/F)\mathcal{F}(\chi).$$

Since $W[[\Gamma_M]]$ is a unique factorization domain, we have

$$L_p^-(\chi) = H(\psi) \overset{R=\mathbb{T}}{=} \mathcal{F}(\chi)$$

as desired.

§8. **A direct definition of $H(\psi)_P$.** The ring $\mathbb{T}_P$ is a local ring of the Hecke algebra of $S = S_{(I,0)}(N, \varepsilon; W)$ with Petersson inner product $(\cdot,\cdot)$. Write $\Theta \in S$ for the theta series with Galois representation $\mathrm{Ind}_M^F \psi_P$. For $f \in S$, write $f = c_f\Theta + f^\perp$ for $(\Theta, f^\perp) = 0$ with $c_f = \frac{(\Theta,f)}{(\Theta,\Theta)}$. Then $(H(\psi)_P) = \{\xi \in W | \xi c_f \in W\}$ $\forall f \in S$ (the maximal denominator of c_f). We need to show $\frac{(\Theta,S)}{\Omega^{2I}} \subset W$, since $\frac{(\Theta,\Theta)}{\Omega^{2I}} = h(M/F)L_p(\psi_P^-)$ (up to a Gauss sum: (3.5) 2009 [IMRN09]).

For a product of two weight 1 (integral) theta series $\theta_1\theta_2$ of M, we can write $(\Theta, \theta_1\theta_2)$ as a special value at a CM abelian variety with O_M-multiplication of a "Shimura series" on $\mathrm{GL}(2) \times \mathrm{GL}(2)$ associated to (θ_1, θ_2) whose q-expansion is integral as I computed it in my 2006 Documenta paper [AMC, §3]. Thus the q-expansion principle tells us the integrality. To write f as a linear combination of $\theta_1\theta_2$, we use the integral Jacquet-Langlands correspondence in my paper [IMRN05].

§**9. Weight of automorphic forms.** We sketch weight $\kappa = (I, 0)$ Jacquet Langlands correspondence. Take a definite quaternion algebra B over F everywhere unramified at finite places, whose existence forces us to assume $[F : \mathbb{Q}]$ is even. Fix a maximal order O_B of B and identify $\widehat{O}_B = M_2(\widehat{O})$. Then we have Γ_0-type level subgroup $\widehat{\Gamma}_0(N)$ and the Eichler order $\widehat{O}_0(N)$ common for $B_{\mathbb{A}}^\times$ and $\mathrm{GL}_2(F_{\mathbb{A}})$. The holomorphic discrete series π_∞ of $\mathrm{GL}_2(F_\infty)$ ($F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$) is described by a weight $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]$ of the diagonal torus $T = \mathbb{G}_m \times \mathbb{G}_m \subset \mathrm{GL}(2)_{/F}$ in the following way: We have $\kappa_1 + \kappa_2 = [\kappa]I$ for $I = \sum_{\sigma \in I} \sigma$ with an integer $[\kappa]$, since the central character ω has the following form $\omega_\infty(z) = z^{(1-[\kappa])I} = \prod_\nu z_\nu^{1-[\kappa]}$ for totally positive $z \in F_\infty^\times$. The weight of a holomorphic vector under $\mathrm{SO}_2(F_\infty) = (S^1)^I$ is given by $(S^1)^I \ni \exp(\theta) := (\exp(\sqrt{-1}\theta_\sigma))_\sigma \mapsto \exp_F(k\theta) := \exp(-\sqrt{-1}\sum_\sigma k_\sigma \theta_\sigma)$ for $k = \kappa_1 - \kappa_2 + I$. The automorphic factor is $J_\kappa(g, z) = \det(g)^{\kappa_2 - I} j(g, z)^k \overset{\kappa=(I,0)}{=} \det(g)^{-I} j(g, z)^2$ for $j(g, z) = cz + d$. The corresponding representation of weight $(I, 0)$ of $B_\infty^\times$ is the trivial representation.

$\S$**10. Spaces of automorphic forms.** Note $p|N$ by the non-triviality of $\psi_{\mathfrak{P}}$ for $\mathfrak{P} \in \Sigma_p$. Write $S_B(A) = S_B(N, \varepsilon; A)$ (resp. $S(A) = S(N, \varepsilon; A)$) for the space of cusp forms of weight $(I, 0)$ on $\widehat{\Gamma}_0(N)$ with Neben-type character ε. To avoid complicity of ε, we assume that the conductor of ψ is concentrated to $\Sigma_p\mathfrak{C}$ for $\mathfrak{C}$ outside p with $\mathfrak{C} + \mathfrak{C}^c = O_M$. Then $N = D_{M/F} N_{M/F}(\mathfrak{C})$ and $\varepsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\frac{M/F}{d}\right)\psi(d)$ and $\varepsilon(\mathrm{diag}[z, z]) = \omega(z)$. For $X = B, M_2(F)$, let $Sh_X = X^\times \backslash X^\times_{\mathbb{A}} / \widehat{\Gamma}_0(N) Z_X(F_{\mathbb{A}}) C_X$ for the center Z_X of $X^\times$ and the maximal compact subgroup C_X of $X^\times_\infty$ with totally positive determinant. Sh_B is a finite set, and $Sh_{M_2(F)}$ is a Hilbert modular Shimura variety. Take a complete representative set $\mathcal{A}$ so that $B^\times_{\mathbb{A}} = \bigsqcup_{a \in \mathcal{A}} B^\times a \widehat{\Gamma}_0(N) F^\times_{\mathbb{A}} C_B$ with $a_\infty = 1$. Then $S_B(A)$ is made of functions $f : B^\times_{\mathbb{A}} \to A$ satisfying $f(\gamma x z u) = \varepsilon(u)\omega(z) f(x)$ for $\gamma \in B^\times$, $u \in \widehat{\Gamma}_0(N) C_B$ and $z \in Z_B(F_{\mathbb{A}}) = F^\times_{\mathbb{A}}$. Similarly $f \in S(\mathbb{C})$ is made of cusp forms $f : \mathrm{GL}_2(F_{\mathbb{A}}) \to \mathbb{C}$ such that $f|_{\mathrm{GL}_2(F_\infty)}$ is a holomorphic vector in the discrete series of weight $(I, 0)$ satisfying $f(\gamma x z u \exp(\theta)) = \varepsilon(u)\omega(z) f(x) \exp_F(2I\theta)$. See my Oxford book (2006) $\S$2.3.2–5.

§**11. Jacques-Langlands correspondence.** Let

$$\widehat{\Delta} = \{\alpha \in M_2(\widehat{O}) | \alpha_N \in \begin{pmatrix} O_N & O_N \\ NO_N & O_N^\times \end{pmatrix}, \det(x) \in F^\times_{\mathbb{A}(\infty)}\} = \bigsqcup_{\mathfrak{n}} T(\mathfrak{n})$$

with $T(\mathfrak{n})$ made of x whose determinant span the ideal $\mathfrak{n}\widehat{O}$. The character ε extends naturally to $\widehat{\Delta}$. Decomposing $T(\mathfrak{n}) = \sqcup_\alpha \alpha \widehat{\Gamma}_0(N)$, define $f|T(\mathfrak{n})(x) = \sum_\alpha \varepsilon(\alpha)^{-1} f(x\alpha)$, the Hecke operator acts on $S(\mathbb{C})$ and $S_B(A)$ for A inside $W, \mathbb{C}$ linearly. Often we write $T(n)$ for $T(\mathfrak{n})$ with an idele generator n of $\mathfrak{n}\widehat{O}$.

Theorem 1 (Jacquet–Langlands)**.** *We have a non-canonical isomorphism* $JL : S_B(\mathbb{C}) \cong S(\mathbb{C})$ *such that* $JL \circ T(\mathfrak{n}) = T(\mathfrak{n}) \circ JL$ *for all integral ideals* $\mathfrak{n}$*.* (See Oxford book Corollary 2.33).

Put $S(A) \subset S(\mathbb{C})$ made of f with A-integral q-expansion. Define the Hecke algebra $\mathbf{h}_P = W[T(\mathfrak{n})]_{\mathfrak{n}} \subset \mathrm{End}_W(S_B(W))$. Fixing the embedding $W \hookrightarrow \mathbb{C}$, $\mathbf{h}_P \otimes_W \mathbb{C}$ acts on $S(\mathbb{C})$ also. By Control theorem, the local ring $\mathbb{T}_P$ is a factor of $\mathbf{h}_P$, while $\mathbb{T}$ is a factor of $\mathbf{h} = W[[T(\mathfrak{n})]]_{\mathfrak{n}} \subset \prod_P \mathbf{h}_P$ for P running over all arithmetic points.