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On p-adic Hecke Algebras for GL_2

HARUZO HIDA

In the theory of automorphic functions, holomorphic ones are particularly rich in arithmetic applications, although they are merely a part of the whole space of C^∞-class modular forms. The analogy between the theory of C^∞-class modular forms and that of p-adic ones recently becomes a prevalent idea; thus, one may naturally expect the existence of some special class of p-adic modular forms which is as rich in arithmetic structure as the holomorphic counterpart. This paper is intended to present a candidate for such a class of p-adic modular forms in the case of GL_2 over $\mathbf{Q}$ and to explain the reason why we believe that it is the right one.

Our approach is representation theoretic. As in the theory of Jacquet and Langlands in the complex case, we consider the Hecke algebra $\mathscr{h}$ which is the subalgebra of the endomorphism algebra of the space of p-adic modular forms generated topologically by Hecke operators $T(n)$. We then construct a certain idempotent e as a p-adic limit of powers of $T(p)$ in this algebra $\mathscr{h}$, which gives the projection to the special subspace already mentioned. The Hecke algebra $\mathscr{h}^{\mathrm{ord}} = e\mathscr{h}$ for this special space plays, in our theory for GL_2, the role of the algebra $\Lambda = \mathbf{Z}_p[[X]]$ in the Iwasawa theory for the cyclotomic $\mathbf{Z}_p$-extensions, and, naturally, $\mathscr{h}^{\mathrm{ord}}$ has a canonical Λ-algebra structure. With each irreducible component of $\mathrm{Spec}(\mathscr{h}^{\mathrm{ord}})$, we can associate a Galois representation into GL_2 with coefficients in the function field of the irreducible component. The infinite algebraic extension corresponding to the kernel of this representation is the object in our theory replacing the cyclotomic $\mathbf{Z}_p$-extensions. On the other hand, by virtue of Shimura's theory of algebraicity for the special values of zeta functions of modular forms, we are able to construct several p-adic L-functions on the spectrum of each irreducible component of $\mathrm{Spec}(\mathscr{h}^{\mathrm{ord}})$. Thus, what is awaited for further research is the study of the direct arithmetic relation between the Galois representation and the p-adic L-function attached to each irreducible component of $\mathrm{Spec}(\mathscr{h}^{\mathrm{ord}})$.

1. We begin with the definition of Hecke algebras. Let N be a positive integer, and put $\Gamma_1(N) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{SL}_2(\mathbf{Z}) | c \equiv 0 \bmod N,\ a \equiv d \equiv 1 \bmod N\}$. For each positive integer k, let $S_k(\Gamma_1(N))$ denote the space of holomorphic cusp forms on the upper half plane $\mathfrak{H}$ of weight k with respect to $\Gamma_1(N)$. Each element f in $S_k(\Gamma_1(N))$ has the following type of Fourier expansion:

$$f(z) = \sum_{n=1}^{\infty} a(n,f)q^n \qquad (q = \exp(2\pi i z),\ z \in \mathfrak{H})$$

for complex constants $a(n,f)$. By means of this, we may embed $S_k(\Gamma_1(N))$ into the power series ring $\mathbf{C}[[q]]$. One may then give a rational structure on $S_k(\Gamma_1(N))$ through defining the A-rational subspace $S_k(\Gamma_1(N);\ A)$ for each subalgebra A of $\mathbf{C}$ by $S_k(\Gamma_1(N);\ A) = S_k(\Gamma_1(N)) \cap A[[q]]$. For each integer l prime to N, by choosing $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ with the congruence: $c \equiv 0 \bmod N$ and $d \equiv l \bmod N$, we can let l act on $S_k(\Gamma_1(N))$ via

$$f \mid \langle l \rangle = l^{k-2} f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k}.$$

Then the Hecke operator $T(n)$ is defined as an endomorphism of $S_k(\Gamma_1(N))$ by

$$a(m, f \mid T(n)) = \sum_{\substack{l \mid m,\, l \mid n \\ (l,N)=1}} l a\left(mn/l^2, f \mid \langle l \rangle\right) \tag{1.1}$$

(see [**6**, (3.5.12)]). Then $S_k(\Gamma_1(N);\ A)$ is stable under $T(n)$ and $\langle l \rangle$. The Hecke algebra $\mathscr{h}_k(\Gamma_1(N);\ \mathbf{Z})$ is by definition the subalgebra of $\mathrm{End}_{\mathbf{C}}(S_k(\Gamma_1(N)))$ generated over $\mathbf{Z}$ by $T(n)$ for all n. For general commutative algebra A, we simply put $\mathscr{h}_k(\Gamma_1(N);\ A) = \mathscr{h}_k(\Gamma_1(N);\ \mathbf{Z}) \otimes_{\mathbf{Z}} A$. We let $\overline{\mathbf{Q}}_p$ for each rational prime p denote an algebraic closure of the p-adic field $\mathbf{Q}_p$. Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. We fix for each p an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$. Thus any algebraic number $a \in \overline{\mathbf{Q}}$ can be considered uniquely as a p-adic number as well as a complex number. Let $|\ |_p$ (resp. $|\ |$) denote the absolute value of $\overline{\mathbf{Q}}_p$ (resp. $\mathbf{C}$). We put $S_k(\Gamma_1(N);\ \overline{\mathbf{Q}}_p) = S_k(\Gamma_1(N);\ \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \overline{\mathbf{Q}}_p$ which is naturally a subspace of $\overline{\mathbf{Q}}_p[[q]]$. For each subalgebra A of $\overline{\mathbf{Q}}_p$, put $S_k(\Gamma_1(N);\ A) = S_k(\Gamma_1(N);\ \overline{\mathbf{Q}}_p) \cap A[[q]]$.

Now we shall define a pairing

$$\langle\ ,\ \rangle\colon \mathscr{h}_k(\Gamma_1(N);\ A) \times S_k(\Gamma_1(N);\ A) \to A \quad \text{by } \langle h, f \rangle = a(1, f \mid h). \tag{1.2a}$$

Then we see from (1.1)

$$\langle T(n), f \rangle = a(n,f) \quad \textit{for each positive integer } n. \tag{1.2b}$$

If A is one of the rings $\mathbf{Z}$, $\overline{\mathbf{Q}}_p$, or $\mathbf{C}$, then the pairing (1.2a) induces

$$\begin{aligned} \mathrm{Hom}_A(S_k(\Gamma_1(N);\ A), A) &\cong \mathscr{h}_k(\Gamma_1(N);\ A), \\ \mathrm{Hom}_A(\mathscr{h}_k(\Gamma_1(N);\ A), A) &\cong S_k(\Gamma_1(N);\ A). \end{aligned} \tag{1.3}$$

To each common eigenform f of all Hecke operators $T(n)$ in $S_k(\Gamma_1(N))$, we associate a $\mathbf{C}$-algebra homomorphism λ: $\mathscr{h}_k(\Gamma_1(N);\ \mathbf{C}) \to \mathbf{C}$ by $f \mid h = \lambda(h) f$. Note that

$$\lambda(T(n))\langle T(1), f \rangle = \langle T(n), f \rangle = a(n,f).$$

Thus if f is nontrivial, then $\langle T(1), f\rangle \neq 0$. Dividing f by $\langle T(1), f\rangle$, we may assume that $\langle T(1), f\rangle = a(1, f) = 1$. Common eigenforms of $T(n)$ under this normalization will be called *normalized* eigenforms. We say that two normalized eigenforms f and g are associated if f and g belong to the same eigenvalues of $T(l)$ except for finitely many primes l. In the class of normalized eigenforms associated with a given f, there exists a unique one contained in $S_k(\Gamma_1(C))$ with the smallest C [5], which is called *primitive*. The level C of the primitive form associated with f is called the *conductor* of f and will be written as $C(f)$. Let A be a subring of $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$. We denote by B the field $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$ according as $A \subset \overline{\mathbf{Q}}_p$ or $A \subset \mathbf{C}$. If λ: $h_k(\Gamma_1(N);\ A) \to B$ is an A-algebra homomorphism, the restriction of λ to $h_k(\Gamma_1(N);\ \mathbf{Z})$ has values in $\overline{\mathbf{Q}} \cap A$ since $h_k(\Gamma_1(N);\ \mathbf{Z})$ is free of finite rank over $\mathbf{Z}$ by (1.3). Thus we can extend λ to a $\mathbf{C}$-algebra homomorphism $\lambda_{\mathbf{C}}$: $h_k(\Gamma_1(N);\ \mathbf{C}) \to \mathbf{C}$. By the duality (1.3), we can find $f_\lambda \in S_k(\Gamma_1(N))$ so that $\langle h, f\rangle = \lambda_{\mathbf{C}}(h)$. Especially, we know from (1.2b) that $f_\lambda(z) = \sum_{n=1}^{\infty} \lambda(T(n))q^n$, and then f_λ is a normalized eigenform. Therefore, for each subalgebra A of $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$, the correspondence $\lambda \mapsto f_\lambda$ induces a bijection

$$\mathrm{Hom}_{A\text{-alg}}\big(h_k(\Gamma_1(N);\ A),\ B\big) \cong \big\{\textit{normalized eigenforms in } S_k(\Gamma_1(N))\big\}. \quad (1.4)$$

Let λ: $h_k(\Gamma_1(N);\ A) \to B$ be an A-algebra homomorphism and f be the corresponding normalized eigenform. Then there exists a Dirichlet character ψ: $(\mathbf{Z}/N\mathbf{Z})^\times \to \overline{\mathbf{Q}}$ such that $f|\langle l\rangle = \psi(l)l^{k-2}f$. This character ψ is called the *character* of f or λ.

Now we fix a prime p and a finite extension $K/\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$. Let $\mathcal{O}$ be the p-adic integer ring of K. By (1.3), $h_k(\Gamma_1(N);\ \mathcal{O})$ is free of finite rank over $\mathcal{O}$ and is then a product of complete local rings R. Let 1_R denote the idempotent of R, and define an idempotent $e_N \in h_k(\Gamma_1(N);\ \mathcal{O})$ by the sum of 1_R over the local rings R on which the image of $T(p)$ is invertible. Then

$$h_k^{ord}(\Gamma_1(N);\ \mathcal{O}) = e_N h_k(\Gamma_1(N);\ \mathcal{O})$$

is the maximal direct summand of $h_k(\Gamma_1(N);\ \mathcal{O})$ on which the image of $T(p)$ is invertible. The idempotent e_N can be explicitly given as a p-adic limit: $e_N = \lim_{n\to\infty} T(p)^{p^n(p^m-1)} \in h_k(\Gamma_1(N);\ \mathcal{O})$ for a suitable positive integer m. If λ is an $\mathcal{O}$-algebra homomorphism of $h_k^{\mathrm{ord}}(\Gamma_1(N);\ \mathcal{O})$ into $\overline{\mathbf{Q}}_p$, we know that $|\lambda(T(p))|_p = 1$. Thus the correspondence (1.4) induces a bijection:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}\text{-alg}}\big(h_k^{\mathrm{ord}}(\Gamma_1(N);\ \mathcal{O}), \overline{\mathbf{Q}}_p\big) && (1.5)\\ &\qquad\cong \big\{\textit{normalized eigenforms in } S_k(\Gamma_1(N)) \textit{ with } |a(p, f)|_p = 1\big\}.\end{aligned}$$

2. Hereafter we suppose that $p \geqslant 5$ and N is prime to p. If $r \geqslant s \geqslant 1$, we have by (1.1) a commutative diagram for all n:

$$\begin{array}{ccc} S_k(\Gamma_1(Np^s);\ A) & \to & S_k(\Gamma_1(Np^r);\ A) \\ \downarrow T(n) & & \downarrow T(n) \\ S_k(\Gamma_1(Np^s);\ A) & \to & S_k(\Gamma_1(Np^r);\ A) \end{array} \quad (2.1)$$

where the horizontal arrows are the natural inclusion. Then the restriction of each operator in $h_k(\Gamma_1(Np^r); A)$ to the subspace $S_k(\Gamma_1(Np^s); A)$ is again contained in $h_k(\Gamma_1(Np^s); A)$; thus, we have a surjective A-algebra homomorphism:

$$h_k(\Gamma_1(Np^r); A) \to h_k(\Gamma_1(Np^s); A) \quad \textit{for each pair } r \geqslant s \geqslant 1. \tag{2.2a}$$

Since $T(p)$ is sent to $T(p)$ by (2.2a), the image of e_{Np^r} under (2.2a) coincides with e_{Np^s}, and (2.2a) induces

$$h_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathcal{O}) \to h_k^{\mathrm{ord}}(\Gamma_1(Np^s); \mathcal{O}). \tag{2.2b}$$

We shall take the limits:

$$h_k(Np^\infty; \mathcal{O}) = \varprojlim_r h_k(\Gamma_1(Np^r); \mathcal{O}), \qquad e = \varprojlim_r e_{Np^r} \in h_k(Np^\infty; \mathcal{O}),$$

$$h_k^{\mathrm{ord}}(Np^\infty; \mathcal{O}) = e h_k(Np^\infty; \mathcal{O}) = \varprojlim_r h_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathcal{O}),$$

$$S_k(Np^\infty; \mathcal{O}) = \bigcup_{r=1}^{\infty} S_k(\Gamma_1(Np^r); \mathcal{O}) \quad \text{in } \mathcal{O}[[q]].$$

We define a p-adic norm on $S_k(Np^\infty; \mathcal{O})$ by

$$|f|_p = \mathrm{Sup}_n |a(n, f)|_p.$$

Let $\bar{S}_k(Np^\infty; \mathcal{O})$ be the completion of $S_k(Np^\infty; \mathcal{O})$ under this norm. By continuity, $h_k(Np^\infty; \mathcal{O})$ naturally acts on $\bar{S}_k(Np^\infty; \mathcal{O})$. Put $\bar{S}_k^{\mathrm{ord}}(Np^\infty; \mathcal{O}) = \bar{S}_k(Np^\infty; \mathcal{O})|e$. Then the duality (1.3) extends to the following (topological and also algebraic) duality [**3**]:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{O}}(\bar{S}_k(Np^\infty; \mathcal{O}), \mathcal{O}) \cong h_k(Np^\infty; \mathcal{O}), \\ &\mathrm{Hom}_{\mathcal{O}}(\bar{S}_k^{\mathrm{ord}}(Np^\infty; \mathcal{O}), \mathcal{O}) \cong h_k^{\mathrm{ord}}(Np^\infty; \mathcal{O}). \end{aligned} \tag{2.3}$$

Therefore we can formulate results for the structure of the space of p-adic modular forms in terms of the Hecke algebras without refering to the space, which we shall do in the rest of the paper.

Define a semigroup $\mathcal{L}$ by $\mathcal{L} = \{l \in \mathbf{Z} | (l, Np) = 1\}$ and a compact group Z by $Z = \varprojlim_r (\mathbf{Z}/Np^r\mathbf{Z})^\times$. We consider $\mathcal{L}$ as a sub-semigroup of Z. We know from (1.1) that the homomorphism of semigroup: $\mathcal{L} \ni l \mapsto \langle l \rangle$ has values in $h_k(\Gamma_1(Np^r); \mathcal{O})$ for each $r \geqslant 1$ and is continuous under the topology induced from Z. By continuity, we extend this homomorphism to Z as a continuous character with values in $h_k(\Gamma_1(Np^r); \mathcal{O})$ and hence obtain a continuous character: $Z \to h_k(Np^\infty; \mathcal{O})$. We can naturally identify Z with $\Gamma \times (\mathbf{Z}/Np\mathbf{Z})^\times$, where $\Gamma = \Gamma_1$ and $\Gamma_r = \{x \in \mathbf{Z}_p^\times | x \equiv 1 \bmod p^r\mathbf{Z}_p\}$. We shall define continuous group algebras by $\mathcal{A} = \varprojlim_r \mathcal{O}[(\mathbf{Z}/Np^r\mathbf{Z})^\times]$ and $\Lambda = \Lambda_{\mathcal{O}} = \varprojlim_r \mathcal{O}[\Gamma/\Gamma_r]$. By the universality of continuous group algebras, we have canonical $\mathcal{O}$-algebra homomorphisms: $\mathcal{A} \to h_k(Np^\infty; \mathcal{O})$ and $\Lambda \to h_k(Np^\infty; \mathcal{O})$. The following fact is essentially due to Shimura.

THEOREM 2.1. *For each $k \geqslant 2$, we have canonical $\mathcal{A}$-algebra isomorphisms*

$$h_k(Np^\infty; \mathcal{O}) \cong h_2(Np^\infty; \mathcal{O}) \quad \textit{and} \quad h_k^{\mathrm{ord}}(Np^\infty; \mathcal{O}) \cong h_2^{\mathrm{ord}}(Np^\infty; \mathcal{O}),$$

which take $T(m)$ of weight k to $T(m)$ of weight 2 for all m.

A proof of this fact for $h_k^{\mathrm{ord}}(Np^\infty;\ \mathcal{O})$ is given in [2, Theorem 1.1]. By this theorem and (2.3), the space $\bar{S}_k(Np^\infty;\ \mathcal{O})$ is also independent of $k \geqslant 2$. We write $h(N;\ \mathcal{O})$ for the universal Hecke algebra defined by the theorem and $h^{\mathrm{ord}}(N;\ \mathcal{O})$ for $eh(N;\ \mathcal{O})$.

We write $[\gamma]$ for the image of $\gamma \in \Gamma$ under the natural embedding of Γ into Λ. When we consider $\gamma \in \Gamma$ as an element of $\mathbf{Z}_p$, hence, as an element in the coefficient ring $\mathcal{O}$ of Λ, we simply write it as $\gamma \in \Lambda$. We fix a topological generator u of Γ and define an element of Λ by $\omega_{n,r} = [u^{p^{r-1}}] - u^{np^{r-1}}$ $(n, r \in \mathbf{Z}, r \geqslant 1)$. Then $\Lambda/\omega_{n,r}\Lambda$ is the maximal quotient of Λ on which Γ_r acts via the character: $\Gamma_r \ni \gamma \mapsto \gamma^n \in \mathcal{O}$. For each $n \geqslant 0$ and $r \geqslant 1$, we have by definition a surjective Λ-algebra homomorphism $\rho_{n,r}$: $h(N; \mathcal{O}) \to h_{n+2}(\Gamma_1(Np^r);\ \mathcal{O})$ which sends $T(m)$ in $h(N; \mathcal{O})$ to $T(m)$ in $h_{n+2}(\Gamma_1(Np^r);\ \mathcal{O})$. The following result determines the structure of $h^{\mathrm{ord}}(N;\ \mathcal{O})$:

THEOREM 2.2. *The Λ-algebra $h^{\mathrm{ord}}(N;\ \mathcal{O})$ is free of finite rank over Λ. Moreover, for each pair of integers $n \geqslant 0$ and $r \geqslant 1$, $\rho_{n,r}$ induces a Λ-algebra isomorphism*

$$h^{\mathrm{ord}}(N;\ \mathcal{O}) \otimes_\Lambda \Lambda/\omega_{n,r}\Lambda \cong h_{n+2}^{\mathrm{ord}}(\Gamma_1(Np^r);\ \mathcal{O}),$$

which sends $T(m)$ to $T(m)$ for all m.

For the proof, see [1] and [2]. When it is unlikely to cause misunderstanding, we simply write h^{ord} for $h^{\mathrm{ord}}(N;\ \mathcal{O})$. By the universality of the continuous group ring, each continuous character ξ: $\Gamma \to \bar{\mathbf{Q}}_p$ can be extended to an $\mathcal{O}$-algebra homomorphism P_ξ: $\Lambda \to \bar{\mathbf{Q}}_p$. When $\xi(\gamma) = \gamma^n\varepsilon(\gamma)$ for a nonnegative integer n and a finite order character ε: $\Gamma \to \bar{\mathbf{Q}}$, we write $P_{n,\varepsilon}$ for P_ξ. Let $\mathscr{L}$ be the quotient field of Λ, and we fix an algebraic closure $\bar{\mathscr{L}}$ of $\mathscr{L}$. We consider $\bar{\mathbf{Q}}_p$ as a subfield of $\bar{\mathscr{L}}$. Let $\mathscr{K}$ be a finite extension of $\mathscr{L}$ in $\bar{\mathscr{L}}$. We denote by $\mathscr{I}$ the integral closure of Λ in $\mathscr{K}$. Then $\mathscr{I}$ is known to be free of finite rank over Λ. Put $\mathscr{X}(\mathscr{I}) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathscr{I}, \bar{\mathbf{Q}}_p)$, and let $\mathscr{X}_{\mathrm{alg}}(\mathscr{I})$ denote the subset of $\mathscr{X}(\mathscr{I})$ consisting of all $\mathcal{O}$-algebra homomorphisms $\mathscr{I} \to \bar{\mathbf{Q}}_p$ whose restriction to Λ is of the form $P_{n,\varepsilon}$ for some $0 \leqslant n \in \mathbf{Z}$ and some finite order character ε: $\Gamma \to \bar{\mathbf{Q}}$. When $\mathscr{I} = \Lambda$, $\mathscr{X}(\Lambda) \cong \{x \in \bar{\mathbf{Q}}_p \| x|_p < 1\}$ via $P \mapsto P([u]) - 1 \in \bar{\mathbf{Q}}_p$, and in general $\mathscr{X}(\mathscr{I})$ is a covering space of $\mathscr{X}(\Lambda)$. For each $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$, we define $n(P) \in \mathbf{Z}$ and ε_P: $\Gamma \to \bar{\mathbf{Q}}$ by $P|_\Lambda = P_{n(P),\varepsilon_P}$. The order of ε_P is written as $p^{r(P)-1}$. Any element F of $\mathscr{I}$ may be regarded as a p-adic analytic function F: $\mathscr{X}(\mathscr{I}) \to \bar{\mathbf{Q}}_p$ whose value at $P \in \mathscr{X}(\mathscr{I})$ is given by $P(F) \in \bar{\mathbf{Q}}_p$.

Now we consider $\lambda \in \mathrm{Hom}_{\Lambda\text{-alg}}(h^{\mathrm{ord}}, \bar{\mathscr{L}})$, the combination of λ with the canonical character: $Z \to h^{\mathrm{ord}}$ gives a continuous character of Z and induces a Dirichlet character ψ: $(\mathbf{Z}/Np\mathbf{Z})^\times \to \bar{\mathbf{Q}}$ since $Z = \Gamma \times (\mathbf{Z}/Np\mathbf{Z})^\times$. This character ψ is called the character of λ. Since h^{ord} is of finite rank over Λ, we can find a finite extension $\mathscr{K}$ such that $\lambda(h^{\mathrm{ord}}) \subset \mathscr{I}$. The integral closure of $\mathcal{O}$ in $\mathscr{I}$ is of finite rank over $\mathcal{O}$. When it coincides with $\mathcal{O}$, we say that λ is *defined* over $\mathcal{O}$. Changing K by its finite extension, if necessary, we may (and will) assume

$$\lambda \text{ is defined over } \mathcal{O}. \tag{2.4}$$

For each $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$, we consider the composition $\lambda_P = P \circ \lambda\colon \mathscr{h}^{\mathrm{ord}} \to \overline{\mathbf{Q}}_p$. Since $P(\omega_{n(P),r(P)}) = 0$, λ_P factors through $\mathscr{h}^{\mathrm{ord}}/\omega_{n,r}\mathscr{h}^{\mathrm{ord}} \cong \mathscr{h}^{\mathrm{ord}}_{n+2}(\Gamma_1(Np^r);\ \mathscr{O})$ for $n = n(P)$ and $r = r(P)$. Then by (1.5) there is a unique normalized eigenform f_P in $S_{n(P)+2}(\Gamma_1(Np^{r(P)}))$ with q-expansion: $f_P = \sum_{m=1}^{\infty} \lambda_P(T(m))q^m$. This form is called the *ordinary form belonging to* λ *at* $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$. The character of f_P (or λ_P) is given by $\varepsilon_P \psi \omega^{-n(P)}$, where ψ is the character of λ and ω is the Teichmüller character. More generally, we say that a normalized eigenform $f \in S_k(\Gamma_1(Np^r))$ is *ordinary* if $r \geqslant 1$ and $|a(p,f)|_p = 1$. This notion is intrinsic (i.e. independent of r) because of (2.1). When f_0 is a normalized eigenform in $S_k(\Gamma_1(N))$ ($k \geqslant 2$) with $|a(p,f_0)|_p = 1$, there is a unique ordinary form $f \in S_k(\Gamma_1(Np))$ such that $a(n,f) = a(n,f_0)$ for all n prime to p. This form f is called the ordinary form *associated* with f_0. By abusing the language, we say that f_0 belongs to λ if the associated ordinary form f belongs to λ. The following fact is obvious from Theorem 2.2:

(2.5) *Each ordinary form $f \in S_k(\Gamma_1(Np^r))$ belongs to some Λ-algebra homomorphism $\lambda\colon \mathscr{h}^{\mathrm{ord}}(N;\ \mathscr{O}) \to \bar{\mathscr{L}}$.*

Let J be another positive integer prime to p. We say that two Λ-algebra homomorphisms $\lambda\colon \mathscr{h}^{\mathrm{ord}}(N;\ \mathscr{O}) \to \bar{\mathscr{L}}$ and $\mu\colon \mathscr{h}^{\mathrm{ord}}(J;\ \mathscr{O}) \to \bar{\mathscr{L}}$ are *associated* if $\lambda(T(l)) = \mu(T(l))$ except for finitely many primes l.

THEOREM 2.3. *Let $\lambda\colon \mathscr{h}^{\mathrm{ord}}(N;\ \mathscr{O}) \to \bar{\mathscr{L}}$ be a Λ-algebra homomorphism. In the class of all Λ-algebra homomorphisms associated with λ, there is a unique λ_0: $\mathscr{h}^{\mathrm{ord}}(C;\ \mathscr{O}) \to \bar{\mathscr{L}}$ with the smallest level C.*

The homomorphism λ_0 satisfying the condition of Theorem 2.3 is called *primitive* and the level C of λ_0 will be called the *conductor* of λ.

THEOREM 2.4. *If an ordinary form $f \in S_k(\Gamma_1(Np^r);\ \mathscr{O})$ has conductor divisible by N and $k \geqslant 2$, then there is a unique $\lambda \in \mathrm{Hom}_{\Lambda\text{-alg}}(\mathscr{h}^{\mathrm{ord}}(N;\ \mathscr{O}),\ \bar{\mathscr{L}})$ defined over $\mathscr{O}$ to which f belongs. The homomorphism λ as above is primitive and has conductor N. Conversely, suppose that $\lambda \in \mathrm{Hom}_{\Lambda\text{-alg}}(\mathscr{h}^{\mathrm{ord}}(N;\ \mathscr{O}),\ \mathscr{K})$ is primitive and is of conductor N. Then the conductor of each ordinary form belonging to λ is divisible by N. Moreover, let ψ be the character of λ and ψ_p be its restriction to $(\mathbf{Z}/p\mathbf{Z})^{\times}$. Then the ordinary form f_P belonging to λ at $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$ is primitive and is of conductor $Np^{r(P)}$ unless $\varepsilon_P \psi_p \omega^{-n(P)}$ is trivial.*

For the proofs of Theorem 2.3 and Theorem 2.4, see [**3**].

3. By a result of Deligne, one can attach to $\xi \in \mathrm{Hom}_{\mathscr{O}\text{-alg}}(\mathscr{h}_k(\Gamma_1(Np^r);\ \mathscr{O}),\ \overline{\mathbf{Q}}_p)$ a simple representation $\pi(\xi)\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$, which is characterized by

(3.1a) *$\pi(\xi)$ is unramified outside Np.*

(3.1b) *Let σ_l be the Frobenius element of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *for each prime l outside Np. Then we have that* $\det(1 - \pi(\xi)(\sigma_l)X) = 1 - \xi(T(l))X + l\xi(\langle l \rangle)X^2$.

We shall attach to each Λ-algebra homomorphism $\lambda\colon \mathscr{h}^{\mathrm{ord}}(N;\ \mathscr{O}) \to \mathscr{K}$ a representation $\pi(\lambda)\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathscr{K})$ which interpolates p-adically the Deligne representation $\pi(\lambda_P)$ as in (3.1) for all $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$. A representation

π: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathscr{K})$ is said to be *continuous* if the following conditions are satisfied:

(3.2a) *there exists an $\mathscr{I}$-submodule L of $\mathscr{K}^2$ stable under π such that $L \otimes_{\mathscr{I}} \mathscr{K} = \mathscr{K}^2$ and L is of finite type over $\mathscr{I}$;*

(3.2b) *the representation π*: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}_{\mathscr{I}}(L)$ *is continuous under the $\mathfrak{m}$-adic topology for the maximal ideal $\mathfrak{m}$ of $\mathscr{I}$.*

THEOREM 3.1. *For each Λ-algebra homomorphism λ: $h^{\mathrm{ord}}(N;\ \mathcal{O}) \to \mathscr{K}$, there exists a unique representation $\pi(\lambda)$*: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathscr{K})$ *characterized by*

(3.3a) *$\pi(\lambda)$ is continuous and simple;*

(3.3b) *$\pi(\lambda)$ is unramified outside Np;*

(3.3c) *for the Frobenius element σ_l for each prime l outside Np, we have that* $\det(1 - \pi(\lambda)(\sigma_l)X) = 1 - \lambda(T(l))X + l\lambda(\langle l\rangle)X^2$.

For the proof, see [2]. For each prime ideal $\mathscr{P}$ of $\mathscr{I}$, let $k(\mathscr{P})$ be the quotient field of $\mathscr{I}/\mathscr{P}$. We say that a representation $\tilde{\pi}$: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(k(\mathscr{P}))$ is a residual representation modulo $\mathscr{P}$ of a continuous representation π: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathscr{K})$ if $\tilde{\pi}$ is semisimple and the characteristic polynomial of $\tilde{\pi}(\sigma)$ is the reduction of that of $\pi(\sigma)$ modulo $\mathscr{P}$ for every $\sigma \in$ Gal$(\overline{\mathbf{Q}}/\mathbf{Q})$. We write the residual representation $\tilde{\pi}$ modulo $\mathscr{P}$ as $\pi \bmod \mathscr{P}$. When π: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathscr{K})$ is continuous, we now see the existence of the residual representation $\pi \bmod P$ for each $P \in \mathscr{X}(\mathscr{I})$. Let L be the $\mathscr{I}$-lattice of $\mathscr{K}^2$ stable under π as in (3.2a). Since the localization $\mathscr{I}_P$ of $\mathscr{I}$ at P is a discrete valuation ring, $L_P = L \otimes_{\mathscr{I}} \mathscr{I}_P$ is free of rank 2 over $\mathscr{I}_P$. Therefore π induces a representation π: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathscr{I}_P)$. Then $\pi \bmod P$ is given by the semisimplification of $P \circ \pi$: Gal$(\overline{\mathbf{Q}}/\mathbf{Q}) \xrightarrow{\pi} \mathrm{GL}_2(\mathscr{I}_P) \xrightarrow{P} \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$.

By condition (3.3c), $\pi(\lambda) \bmod P$ is equivalent to $\pi(\lambda_P)$ for all $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$. This is why we think that the representation $\pi(\lambda)$ is a p-adic interpolation of the Deligne representations $\pi(\lambda_P)$. Since we may assume that $\pi(\lambda_P)$ has values in $\mathrm{GL}_2(\mathcal{O}')$ for a valuation ring $\mathcal{O}'$ finite over $\mathcal{O}$, $\pi(\lambda_P)$ has its residual representation modulo $\not{p}$ for the maximal ideal $\not{p}$ of $\mathcal{O}'$. The representation $\pi(\lambda_P) \bmod \not{p}$ can be identified with the residual representation of $\pi(\lambda)$ modulo the maximal ideal $\mathfrak{m}$ of $\mathscr{I}$. Thus

(3.4) *The residual representation $\pi(\lambda) \bmod \mathfrak{m}$ exists* [4].

It should be noted that Mazur and Wiles [4] have shown that the image of $\pi(\lambda)$ contains $\mathrm{SL}_2(\Lambda_{\mathbf{Z}_p})$ for λ: $h^{\mathrm{ord}}(1;\ \mathbf{Z}_p) \to \Lambda_{\mathbf{Z}_p}$ if the image of $\pi(\lambda) \bmod \mathfrak{m}$ contains $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$. (This condition is verified, for example, if the unique normalized eigenform $\Delta = q\prod_{n=1}^{\infty}(1 - q^n)^{24} \in S_{12}(\mathrm{SL}_2(\mathbf{Z}))$ belongs to λ and $p \neq 11, 23$, and 691.)

4. Finally, we shall refer to the result for p-adic L-functions. Let J be another positive integer prime to p. For two normalized eigenforms $f \in S_k(\Gamma_1(Np^r))$ and $g \in S_\kappa(\Gamma_1(Jp^s))$, we shall define algebraic numbers α_l, β_l, α_l', β_l' for each prime l

by

$$\sum_{n=1}^{\infty} a(n,f)n^{-s} = \prod_l \left[(1-\alpha_l l^{-s})(1-\beta_l l^{-s})\right]^{-1},$$
$$\sum_{n=1}^{\infty} a(n,g)n^{-s} = \prod_l \left[(1-\alpha'_l l^{-s})(1-\beta'_l l^{-s})\right]^{-1}.$$

Let ψ' be the character of f and ψ be the primitive character which induces ψ'. Then we define

$$\mathscr{D}(s,f) = \prod_l \left[(1-\bar{\psi}(l)\alpha_l^2 l^{-s})(1-\bar{\psi}(l)\alpha_l\beta_l l^{-s})(1-\bar{\psi}(l)\beta_l^2 l^{-s})\right]^{-1},$$
$$\mathscr{D}(s,f,g) = \prod_l \left[(1-\alpha_l\alpha'_l l^{-s})(1-\alpha_l\beta'_l l^{-s})(1-\beta_l\alpha'_l l^{-s})(1-\beta_l\beta'_l l^{-s})\right]^{-1}.$$

We denote by $\mathscr{D}_p(s,f,g)$ the Euler product obtained from $\mathscr{D}(s,f,g)$ by excluding its Euler p-factor. These L-functions can be continued to meromorphic functions of $s \in \mathbf{C}$, and their algebraicity at certain integer points (for details, see below) is shown by Shimura [7] for the latter series and by Sturm [**8**] for the former. If f is primitive, we can define a canonical transcendental factor $U_\infty(f) \in \mathbf{C}^\times$ [**2**, §10; **3**, §4] independent of p such that $\mathscr{D}(k,f)/U_\infty(f)$ and $\mathscr{D}(m,f,g)/\pi^{2m-\kappa-k}U_\infty(f)$ for $\kappa \leqslant m < k$ are algebraic numbers ($U_\infty(f)$ is independent of g). If Ω: $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{Q}}_p)$ (resp. Ω') denotes the Deligne representation as in (3.1) associated with f (resp. g) and $\tilde{\Omega}$ denotes the contragredient representation of Ω, then, up to finitely many Euler factors, $\mathscr{D}(s,f,g)$ is the zeta function of $\Omega \otimes \Omega'$: $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_4(\overline{\mathbf{Q}}_p)$ and $\mathscr{D}(s,f)$ corresponds to the unique three-dimensional subrepresentation of $\Omega \otimes \tilde{\Omega}$.

Let λ: $h^{\mathrm{ord}}(N;\, \mathcal{O}) \to \mathscr{I}$ be a primitive Λ-algebra homomorphism defined over $\mathcal{O}$. Let $\mathscr{R}$ be the unique local ring of $h^{\mathrm{ord}}(N;\, \mathcal{O})$ through which λ factors. We suppose

$$\mathrm{Hom}_\Lambda(\mathscr{R}, \Lambda) \cong \mathscr{R} \quad \textit{as } \mathscr{R}\textit{-modules}. \tag{4.1}$$

This condition can be verified when $\pi(\lambda) \bmod m$ is simple for the maximal ideal m of $\mathscr{I}$ and when the restriction ψ_p of the character ψ of λ to $(\mathbf{Z}/p\mathbf{Z})^\times$ is nontrivial and different from ω^{-1} [**4**, **9**]. For example, if the discriminant function $\Delta \in S_{12}(SL_2(\mathbf{Z}))$ belongs to λ, (4.1) is satisfied if $p \neq 11, 691$. By the multiplication in $\mathscr{I}$, we have a natural $\mathscr{I}$-algebra homomorphism: $\mathscr{I} \otimes_\Lambda \mathscr{I} \to \mathscr{I}$. Combining $\lambda \otimes \mathrm{id}$: $\mathscr{R} \otimes_\Lambda \mathscr{I} \to \mathscr{I} \otimes_\Lambda \mathscr{I}$ with this morphism, we can extend λ to an $\mathscr{I}$-algebra homomorphism of $\mathscr{R} \otimes_\Lambda \mathscr{I}$ onto $\mathscr{I}$, which we denote by $\hat{\lambda}$. We can then decompose $\mathscr{R} \otimes_\Lambda \mathscr{K} = \mathscr{K} \oplus \mathscr{A}$ as an algebra direct sum such that the projection of $\mathscr{R} \otimes_\Lambda \mathscr{I}$ to the first factor $\mathscr{K}$ coincides with $\hat{\lambda}$. Let $\mathscr{R}(\mathscr{A})$ be the projected image of $\mathscr{R} \otimes_\Lambda \mathscr{I}$ in $\mathscr{A}$. Then we have the diagonal map δ: $\mathscr{R} \otimes_\Lambda \mathscr{I} \to \mathscr{I} \oplus \mathscr{R}(\mathscr{A})$. Under the assumption (4.1), we can find $0 \neq H \in \mathscr{I}$ so that $\mathrm{Coker}(\delta) \cong \mathscr{I}/H\mathscr{I}$ *as $\mathscr{I}$-modules*. The function H: $\mathscr{X}(\mathscr{I}) \to \overline{\mathbf{Q}}_p$ is determined up to unit factors in $\mathscr{I}$. Let ψ be the character of λ, and write the p-part of ψ as ω^a for $0 \leqslant a < p-1$.

THEOREM 4.1. *Suppose* (4.1) *and that* $a \neq 0$ *or* $N = 1$. *Let* f_P *be the ordinary form belonging to* λ *at* $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$. *Then we can find* $U_p(f_P) \in \overline{\mathbf{Q}}_p$ *with* $|U_p(f_P)|_p = 1$ *such that*

$$H(P) = \mathscr{D}(n(P)+2, f_P)/U_\infty(f_P)U_p(f_P).$$

A part of this result is given in [2] and [3], and the proof in}i the general case will be given in a forthcoming paper.

Now let μ: $h^{\mathrm{ord}}(J;\ \mathscr{O}) \to \mathscr{I}$ be another primitive Λ-algebra homomorphism defined over $\mathscr{O}$. For each $Q \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$, we denote by g_Q the ordinary form belonging to μ at Q. Put, for each finite order character χ: $\mathbf{Z}_p^\times \to \overline{\mathbf{Q}}$ and complex conjugation ρ,

$$f|\chi = \sum_{n=1}^{\infty} \chi(n)a(n,f)q^n, \quad f^\rho = \sum_{n=1}^{\infty} a(n,f)^\rho q^n \qquad (f \in \mathbf{C}[[q]]).$$

If f is a modular form, $f|\chi$ and f^ρ are again modular forms. We write the restriction of the character of μ (resp. λ) to $(\mathbf{Z}/p\mathbf{Z})^\times$ (resp. $(\mathbf{Z}/N\mathbf{Z})^\times$) as ω^b for $0 \leqslant b < p-1$ (resp. ψ_0). Put, for each triple $(P,Q,R) \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})^2 \times \mathscr{X}_{\mathrm{alg}}(\Lambda)$,

$$k = n(P)+2, \qquad m = n(R), \qquad j = 2+2m+n(Q),$$
$$v = r(Q)+2r(R), \qquad r = r(P),$$

and

$$t(P,Q,R) = 2^{-k-j}(\sqrt{-1})^{j-k}\phi(Np)\phi(N/C)^{-1}C^{-1}N^{-1-k/2}J^{m+k/2} \times [N,J]^{1-m+(j-k)/2}p^{-r+(vj-rk)/2}\Gamma(j-m)\Gamma(m+1)\Gamma(k)^{-1},$$

where $[N, J]$ is the least common multiple of N and J, C is the conductor of ψ_0, ϕ is the Euler function, and $\Gamma(x)$ is the gamma function. For each primitive form $f \in S_k(\Gamma_1(M))$, we define a constant $W(f)$ with $|W(f)| = 1$ by the first Fourier coefficient of $M^{-k/2}f(-1/Mz)z^{-k}$. Let $\mathscr{I}\hat{\otimes}_{\mathscr{O}}\mathscr{I}\hat{\otimes}_{\mathscr{O}}\Lambda$ (resp. $\mathscr{I}\hat{\otimes}_{\mathscr{O}}\mathscr{I}$) be the p-adic completion of $\mathscr{I}\otimes_{\mathscr{O}}\mathscr{I}\otimes_{\mathscr{O}}\Lambda$ (resp. $\mathscr{I}\otimes_{\mathscr{O}}\mathscr{I}$). Any element $F \in \mathscr{I}\hat{\otimes}_{\mathscr{O}}\mathscr{I}\hat{\otimes}_{\mathscr{O}}\Lambda$ can be regarded as a p-adic analytic function F: $\mathscr{X}(\mathscr{I}) \times \mathscr{X}(\mathscr{I}) \times \mathscr{X}(\Lambda) \to \overline{\mathbf{Q}}_p$.

THEOREM 4.2. *Let the assumption and the notation be as in Theorem* 4.1 *for* λ. *Then for each integer* $0 \leqslant c < p-1$, *we have a unique p-adic L-function* $D \in \mathscr{I}\hat{\otimes}_{\mathscr{O}}\mathscr{I}\hat{\otimes}_{\mathscr{O}}\Lambda$ *such that for* $(P,Q,R) \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I}) \times \mathscr{X}_{\mathrm{alg}}(\mathscr{I}) \times \mathscr{X}_{\mathrm{alg}}(\Lambda)$,

$$D(P,Q,R) = t(P,Q,R)a(p,f_P)^{r-v}W(f_P)^{-1}W(g_Q|\varepsilon_R)\frac{\mathscr{D}_p\left(j-m, f_P, g_Q^\rho|\omega^{-c}\varepsilon_R^{-1}\right)}{\pi^{j-k}U_\infty(f_P)U_p(f_P)}$$

if $0 \leqslant n(R) < n(P) - n(Q)$, $\omega^{-c}\varepsilon_R^{-1} \neq \varepsilon_Q\omega^b$, $\varepsilon_R\omega^c \neq 1$, *and* $\omega^a\varepsilon_P \neq 1$.

The condition for characters ε_P, ε_Q, ε_R, ω^a, ω^b, and ω^c is not necessary for the evaluation of $D(P,Q,R)$, but without this assumption, a new type of Euler p-factor appears in the right-hand side of the formula and the result becomes a little more complicated. If one wants to interpolate the value $\mathscr{D}$ without the modification at the Euler p-factor, there appears a singularity for the p-adic

L-function obtained:

THEOREM 4.3. *Put* $t(P,Q) = t(P,Q,P_{0,\iota})$ *for* $(P,Q) \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})^2$, *where* ι *is the identity character of* Γ. *Let the assumption and the notation be as in Theorem* 4.1 *for* λ. *Let* $X, Y \in \mathscr{I}\hat{\otimes}_{\mathscr{O}}\mathscr{I}$ *be the functions on* $\mathscr{X}(\mathscr{I})^2$ *defined by* $X(P,Q) = u^{n(P)}\varepsilon_P(u) - 1$ *and* $Y(P,Q) = u^{n(Q)}\varepsilon_Q(u) - 1$ *on* $\mathscr{X}_{\mathrm{alg}}(\mathscr{I})^2$. *Then we have a unique p-adic L-function* Δ *on* $\mathscr{X}(\mathscr{I})^2$ *such that*

(i) $(X - Y)\Delta \in \mathscr{I}\hat{\otimes}_{\mathscr{O}}\mathscr{I}$,

(ii)

$$\Delta(P,Q) = t(P,Q)a(p,f_P)^{r(P)-r(Q)-2}W(f_P)^{-1}W(f_Q) \times \frac{\mathscr{D}\left(n(Q)+2, f_P, f_Q^\rho\right)}{\pi^{n(Q)-n(P)}U_\infty(f_P)U_p(f_P)}$$

if $n(P) > n(Q) \geqslant 0$ *and* $\varepsilon_P\omega^a \neq 1$, $\varepsilon_Q\omega^a \neq 1$,

(iii) $((X - Y)\Delta)(P,Q)|_{P=Q} = (1 + Y(P))(p-1)p^{-1}(\log(u))H(P)$,

where $H \in \mathscr{I}$ *is as in Theorem* 4.1 *and* $\log(u)$ *is the p-adic logarithm of the generator* $u \in \Gamma$.

For the proof, see [**3**, Theorem 10.1]. The above formula (iii) is the p-adic analogue of the well known residue formula of the complex case:

$$\mathrm{Res}_{s=k}\mathscr{D}(s,f,f^\rho) = \Gamma(k)^{-1}N^{-2}2^{2k-1}\pi^{k+1}\phi(N)(f,f)$$

for each normalized eigenform $f \in S_k(\Gamma_1(N))$, where

$$(f,f) = \int_{\Gamma_0(N)\backslash H} |f(z)|^2 y^{k-2}\,dx\,dy$$

(see [**7**, (2.5)]).

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HOKKAIDO UNIVERSITY, SAPPORO 060, JAPAN