NON-VANISHING OF INTEGRALS OF A MOD p MODULAR FORM.

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Abstract. The proof of [H04, Theorem 3.2], [H07, Theorem 4.2] and [EAI, Theorem 8.25] is based on the assertion claiming that the Zariski closure in the Hilbert modular Shimura variety of an infinite set of CM points stable under the action of a CM torus contains an irreducible component of positive dimension with a CM point in the starting infinite set. More than 7 years ago, Akshay Venkatesh pointed me out that this fact might not be true for a non-noetherian pro-variety like Shimura variety. I would like to present an argument proving this fact under an extra requirement on the starting infinite set of CM points. Thereby the assertion of [H04, Theorem 3.3], [H07, Theorem 4.3] and [EAI, Theorem 8.31] on non-vanishing modulo p of Hecke L-values is valid for "Zariski dense" characters in the sense of these articles. In some special cases, non-vanishing is claimed for "except finitely many characters" in these articles, which is still an open question.

Contents

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Recall from [H04] the base totally real field F with integer ring O and the CM quadratic extension $M_{/F}$ with integer ring R. We fix a prime $p > 2$ unramified in $M/\mathbb{Q}$ each of whose prime factors in O splits in M/F. We call such a CM field *p-ordinary*. Fix a prime ideal $\mathfrak{l}$ of O prime to p with residual characteristic ℓ. Let $R_n = O + \mathfrak{l}^n R$ (the order of conductor $\mathfrak{l}^n$) and put $Cl_n = \mathrm{Pic}(R_n)$. Since $O \subset R_n$, we have a natural map $Cl_F := \mathrm{Pic}(O) \to Cl_n$. We write $Cl_n^- := \mathrm{Coker}(Cl_F \to Cl_n)$. Let $Cl_\infty := \varprojlim_n Cl_n$ and $Cl_\infty^- := \varprojlim_n Cl_n^-$ under natural projections. The group of fractional R-ideals prime to $\mathfrak{l}$ is naturally embedded into Cl_∞ whose image in Cl_∞ (resp. Cl_∞^-) we write as $Cl^{alg} \subset Cl_\infty$ (resp, $C^{alg} \subset Cl_\infty^-$). Decompose $Cl_\infty^- = \Delta^- \times \Gamma$ for the maximal finite group Δ^- and $\mathbb{Z}_\ell$-free Γ. Since Cl_F is finite, Γ can be identified with the torsion-free part of Cl_∞, and we have a decomposition $Cl_\infty = \Gamma \times \Delta$ with Δ surjecting down to Δ^- under the projection $Cl_\infty \twoheadrightarrow Cl_\infty^-$. Write $d = \mathrm{rank}_{\mathbb{Z}_\ell}\Gamma$ and choose a basis $\gamma_1, \ldots, \gamma_d$ of Γ over $\mathbb{Z}_\ell$. Let $\mathbb{F}$ (resp. $\overline{\mathbb{Q}}_\ell$) be an algebraic closure of $\mathbb{F}_p$ (resp. $\mathbb{Q}_\ell$). We identify $\mu_{\ell^\infty}(\mathbb{F}) = \mu_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$ as an ℓ-divisible group, and write it just as μ_{ℓ^∞}.

For each projective fractional R_n-ideal $\mathcal{A}$, we defined in [H04, §2.1 and §3.1] a CM abelian variety $X(\mathcal{A})$ of ordinary CM type Σ and the associated CM point $x(\mathcal{A}) = x_\Sigma(\mathcal{A})$ on the Shimura variety Sh for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)$ (see Section 3), which only depends on the ideal $\mathcal{A}$ and a chosen p-ordinary CM type Σ. For an open compact subgroup $K = \mathrm{GL}_2(O_p) \times K^{(p)}$, the point $x(\mathcal{A})$ gives rise to a point $x_K(\mathcal{A}) = (x(\mathcal{A}) \bmod K^{(p)}) \in Sh_K = Sh^{(p)}/K^{(p)}$, which often we write $x(\mathcal{A})$. Choose suitably an irreducible component V of the Shimura variety of prime-to-p level defined over an algebraic closure $\mathbb{F} = \overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. We just fix a finite extension W of $W(\mathbb{F})$ inside $\mathbb{C}_p$ and put $\mathcal{W} = \mathcal{W}_p = W \cap \overline{\mathbb{Q}}$ for the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}_p$. We embed $\overline{\mathbb{Q}}$ into $\mathbb{C}$. We take a $U(\mathfrak{l})$-eigenform $g_{/\mathcal{W}}$ and put $f = \theta^\kappa g$ for the Ramanujan differential operator θ^κ given by $\prod_\sigma \left(q_\sigma \frac{d_\sigma}{dq_\sigma}\right)^{\kappa_\sigma}$ with $\kappa_\sigma \geq 0$ for the q-expansion variables $q_\sigma = \exp(2\pi i z_\sigma)$. We use the same symbol f also for $f_{/\mathbb{F}} := f \mod \mathfrak{m}_W$ defined over $\mathbb{F}$. Here g can have Γ_0-type level $\mathfrak{l}^n$ at $\mathfrak{l}$ for any finite $n > 0$. In [H04] and [H07], we only allowed Γ_0-type level $\mathfrak{l}$ for g just because we needed only an Eisenstein series of level $\mathfrak{l}$ for g. The argument works well for a $U(\mathfrak{l})$-eigenform g with any high power level $\mathfrak{l}^n$ of Γ_0-type.

Write Γ_n for the image of Γ in Cl_n^- (for small n, it can be just $\{1\}$). We fix a character $\psi : \Delta^- \to \mathbb{F}^\times$ to project the measure originally defined on Cl_n^- to Γ_n (see Lemma 4.2 in the text). To define the measure, we need to replace $f(x(\mathcal{A}))$ by $f([\mathcal{A}]) := \lambda^{-1}(\mathcal{A})f(x(\mathcal{A}))$, choosing a Hecke character of infinity type $k\Sigma + \kappa(1-c)$ and of conductor $\mathfrak{C}$ prime to $p\ell$ so that $f([\mathcal{A}])$ only depends on the class $[\mathcal{A}] \in Cl_n^-$ for all n (see §4.1 for more details of the choice of λ). This allows us to define a "measure" $d\varphi_f = d\varphi_{f,n}$ on the finite group Cl_n^- by $\int_{Cl_n^-} \phi d\varphi_{f,n} = \sum_{[\mathcal{A}] \in Cl_n^-} \phi([\mathcal{A}])f([\mathcal{A}])$. If $f|U(\mathfrak{l}) = af$ with $a \neq 0$, the measures $(\lambda(\mathfrak{l})N(\mathfrak{l})a^{-1})^n d\varphi_{f,n}$ patch into a unique measure $d\varphi_f$ on Cl_∞^-, but if $f|U(\mathfrak{l}) = 0$, this is just a collection of measures $\{d\varphi_{f,n}\}_n$. In the application to Hecke L-values, g is given by an Eisenstein series, hence $f|U(\mathfrak{l}) \neq 0$ always (i.e., the automorphic representation spanned by an Eisenstein series is never super-cuspidal at any finite places). However for a cusp form g which is highly ramified at $\mathfrak{l}$, $f|U(\mathfrak{l}) = 0$ can happen (even if we assume that λ is unramified at $\mathfrak{l}$).

Let $\mathbb{F}_\mathbf{q}$ be the field of rationality of $f_{/\mathbb{F}}$, ψ and λ modulo $\mathfrak{m}_W$, and define an integer $r > 0$ such that ℓ-Sylow subgroup of $\mathbb{F}_\mathbf{q}[\mu_\ell]^\times$ has order ℓ^r (i.e., $\mu_{\ell^\infty}(\mathbb{F}_\mathbf{q}[\mu_\ell]) = \mu_{\ell^r}(\mathbb{F}_\mathbf{q}[\mu_\ell])$ and $\ell^r \| (\mathbf{q}-1)$). Though the measure is defined in the earlier papers for f with non-zero eigenvalue for $U(\mathfrak{l})$, in this paper we define a measure on Cl_n^- for each finite n even for f with $f|U(\mathfrak{l}) = 0$, and the argument goes through even for f killed by $U(\mathfrak{l})$. The non-vanishing of the $U(\mathfrak{l})$-eigenvalue is necessary to patch the measure on Cl_n^- for each n to get a measure on Cl_∞^-, but this patching argument is not essential in the proof of non-vanishing results. Also if $f|U(\mathfrak{l}) = 0$, $\int_{Cl_n^-} \chi\psi d\varphi_{f,n} \neq 0$ can happen only for the minimal n for which the integral is well defined. To project the measure $d\varphi_{f,n}$ to Γ_n, we need to modify f into a modular form f_ψ and further to a function $f_\psi^{\mathcal{Q}} : \bigsqcup_n Cl_n^- \to \mathbb{F}$ which involve a transcendental operation depending on a choice of a finite subset $\mathcal{Q}$ of Cl_∞^-/C^{alg} (see (4.7)) so that $\int_{\Gamma_n} \chi d\varphi_{f_\psi^{\mathcal{Q}},n} = \int_{Cl_n} \chi\psi d\varphi_{f,n}$ for all n and all characters $\chi : \Gamma_n \to \mathbb{F}^\times$. Indeed, we embed $\bigsqcup_n Cl_n^-$ into the product $V^{\mathcal{Q}}$ of $\mathcal{Q}$-copies of V, choose an infinite subset Ξ of the disjoint union $\bigsqcup_n Cl_n^-$ and study the Zariski density in $V^{\mathcal{Q}}$ of the embedded image $\Xi \hookrightarrow V^{\mathcal{Q}}$. Roughly speaking, defining $\underline{n}$ by $m \in \underline{n} \Leftrightarrow P \in \Xi$ comes from Cl_m^- (so, $\Xi \subset \bigsqcup_{m \in \underline{n}} Cl_m^-$) and assuming that Ξ is stable under a natural action of $K^\times$ associated to the embedding $\bigsqcup_n Cl_n^- \hookrightarrow V^{\mathcal{Q}}$, we prove in Corollary 2.12 the desired Zariski density if Ξ contains an infinite arithmetic progression. This is the key new result in this paper which was not in the earlier works [H04], [H07] and [EAI, Chapter 8]. Indeed, Corollary 2.12 is used in [He25] to include the conjugate self dual characters for non-vanishing not treated in the above papers. Assuming $F = \mathbb{Q}$, by a method different from the one presented in this article, [BHKO] proves, not only for self dual cases, but the stronger non-vanishing, meaning "except for finitely many" (resurrecting some result claimed in [H07, Theorem 4.3]).

We regard the set of continuous characters $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ as a subset of $\mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$ by sending a character χ to $(\chi(\gamma_1), \dots, \chi(\gamma_d)) \in \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell) \subset \mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$. A subset $\mathcal{X}$ of $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ is said to be Zariski dense if $\mathcal{X}$ is Zariski dense in $\mathbb{G}_m^d$ over $\overline{\mathbb{Q}}_\ell$. This notion of density is independent of the choice of the basis $\{\gamma_j\}_j$. Write $\mathrm{cond}(\chi)$ for the conductor of χ which is a power of $\mathfrak{l}$.

Here is a new version of [H04, Theorem 3.3] and (a part of) [H07, Theorem 4.2]:

Theorem 0.1. *Suppose that there exists $\xi \in F \cap O_{\mathfrak{l}}$ in each class $v \in (O_{\mathfrak{l}}/\mathfrak{l}^j O_{\mathfrak{l}})^\times$ for a sufficiently large $j \geq r$ (for a specific $r > 0$ defined in (4.16)) only dependent on $\mathfrak{l}$ (not v) such that the q-expansion coefficient $a(\xi, f_\psi) \neq 0$ in $\mathbb{F}$ at an infinity cusp of V. Then the set of characters $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}(\mathbb{F}))$ such that $\int_{Cl_n^-} \chi\psi d\varphi_{f,n} \neq 0$ for n given by $\mathrm{cond}(\chi) = \mathfrak{l}^n$ is Zariski dense in $\mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$. If $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$, j can be taken to be equal to r.*

In [H04, Theorem 3.2], [H07, Theorem 4.2] and [EAI, Theorem 8.25], when $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$, we claimed a stronger version of Theorem 0.1 asserting non-vanishing of the integral for "except finitely many characters" χ. This stronger version is still an open question and is used to provide strong arithmetic applications in [Hs14a] and [O20]. Hence at this moment, the proof of these applications (including the Eisenstein divisibility towards the Iwasawa CM main conjecture in [Hs14a]) is still incomplete. Providing a proof of non-vanishing of the integral for "except finitely many characters χ" (at least in the rank 1 case) is an ongoing focus of current research (cf. [B20], [He25] and [BHKO]). Indeed, in [BHKO], the authors announced that the proof of the Eisenstein divisibility by Hsieh in [Hs14a] would be completed by a strategy related to their paper.

Taking an Eisenstein series as the starting modular form g, for any $v \in O/\mathfrak{l}^j$ and any $j \geq r$, the assumption of the theorem is satisfied except for a very rare case which satisfies conditions (M1–3) (see Theorem 0.2 below). For cusp forms, things are more complicated, and Hsieh [Hs14b] uses Galois representations of the given cusp form as its traces is basically q-expansion coefficients. Of course, one needs to assume that the root number is not -1 in addition to some extra assumptions (as the square of the integral is the central critical values by Waldspurger).

Geometrically irreducible components of the Shimura variety of the level group $\Gamma_0(\mathfrak{N})$ are indexed by polarization (strict) ideal classes of F. Then infinity cusps of a component V are indexed by equivalence classes of pairs $(\mathfrak{a}, \mathfrak{b})$ of ideals with $(\mathfrak{a}\mathfrak{b})^{-1}$ giving a polarization ideal of V (e.g., [PAF, §4.1.5]). The condition of the existence of ξ with $a(\xi, f_\psi) \neq 0$ does not depend on the choice of $\mathfrak{a}, \mathfrak{b}$. If $f|U(\mathfrak{l}) = 0$, $\int_{Cl_n} \chi\psi d\varphi_{f,n} \neq 0$ implies that $\mathfrak{l}$-conductor $\mathfrak{l}^\nu$ of χ is exactly $\mathfrak{l}^n$ (i.e., $\nu = n$), while this non-vanishing holds for all $n \geq \nu$ once it holds for $n = \nu$ if $f|U(\mathfrak{l}) = af$ with $a \neq 0$.

If $\pi : V' \to U$ is a finite étale irreducible cover for a dense open subset $U \subset V$ containing Ξ, Zariski density of Ξ in $V^{\mathcal{Q}}$ and density of a lift Ξ' in $U^{\mathcal{Q}}$ such that π induces an isomorphism $\Xi' \cong \Xi$ are equivalent (cf. Remark 3.1). This can be applied to the finite layer U of the Igusa tower Ig over V (or the Igusa tower twisted by an unramified character) which classifies Hilbert modular abelian varieties with μ-type level structure in addition to prime-to-p level structure. Here "twisting" means that we replace the standard Igusa level structure $\mu_{p^j} \otimes_{\mathbb{Z}} O \hookrightarrow A$ by $\mu \hookrightarrow A$ for a (multiplicative type) group scheme μ which is the Cartier dual of $(O/p^nO)(\chi)$ with the Galois action given by an unramified character χ of the base Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/W)$. Note that U is a partial component of the reduction modulo p of a level p-power Shimura variety (possibly twisted by χ) not the entire p-fiber. In this way, we can add finite p level to some extent.

Here are a more technical description and the reason why I take up this problem again. As will be explained in §1.1, more than 7 years ago, Akshay Venkatesh noticed a missing point from the proof of [H04, Proposition 2.7] (taken to be true in [H04] and [H07]): positive dimensional irreducible components of the Zariski closure of an infinite set Ξ of closed points in a non-noetherian variety may not contain any points in the starting set Ξ. For the proof of the above theorem, we need a Zariski density theorem of a thin infinite set Ξ of CM points in $V^{\mathcal{Q}}$. A key step is to show for the Zariski closure $\overline{\Xi}$ of Ξ to contain a positive dimensional irreducible component having at least one point of Ξ. This is because the density theorems Corollary 3.19 and Theorem 3.20 of [H10] we apply to show $\overline{\Xi} = V^{\mathcal{Q}}$ require (as its starting hypothesis) existence of at least one positive dimensional component with non-trivial intersection with Ξ. All the results of [H10] are valid and intact as the Zariski closure appearing in [H10] has at the onset the base point in the positive dimensional component. Unfortunately, under the setting of [H04] and [H07] and the present paper, the existence of a positive dimensional component with a point in Ξ is not evident a priori. In §2.3 and §2.4, if $\underline{n}$ contains an arithmetic progression and Ξ_m for every $m \in \underline{n}$ is sufficiently large stable under the action $\mathcal{A} \mapsto \alpha\mathcal{A}$ for $\alpha \in M^\times$ prime to $\mathfrak{l}p$, replacing $\underline{n}$ by a suitable sub-progression, we show that a semi-group generated by αs as above and a power of $U(\mathfrak{l})$ acts faithfully on Ξ and transitively on the set $\mathrm{Irr}_0(\overline{\Xi})$ of all 0-dimensional irreducible components of the Zariski closure $\overline{\Xi}$. To exhibit an absurdity under $\Xi \subset \mathrm{Irr}_0(\overline{\Xi})$, some points of $\mathrm{Irr}_0(\overline{\Xi})$ is shown to be projected onto a 0-dimensional irreducible components $P \in \Xi$ at a finite level (see Lemma 1.2 (3)). By the semi-group action, we show the orbit of P is an infinite set, while the set of irreducible components at finite level is a finite set (see Theorem 2.7 (1)). This is sufficient for a proof of Theorem 0.1 from the result of [H10].

When $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$, we obtain a slightly stronger result: Consider the sequence of vanishing integral:

$$(*) := \{0 < n \in \mathbb{Z} | \mathfrak{l}^n \text{ is the conductor of } \chi \text{ with vanishing integral}\}.$$

Then, under the condition in Theorem 0.1 on non-vanishing of q-expansion coefficients of f modulo p, this sequence contains no infinite arithmetic progressions if $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$ (see Corollary 5.3). This does not mean that the natural density of $(*)$ is zero [W72] (see Conjecture 5.4). To claim the density 0, we need a somewhat stronger input. Since the description of Ξ is technical, we postpone it to Section 2 of the main text. Here we just say that Ξ is essentially the set of points in $Sh^{(p)}$ corresponding classes in $\bigsqcup_n Cl_n^-$ which carries a character χ with vanishing integral.

Note here that V is a non-noetherian pro-variety of the form $V = \varprojlim_K V_K$ for noetherian schemes V_K with $K = \mathrm{Gal}(V/V_K)$, and hence the zero set of a modular form on V is infinite (of continuous cardinality) even if $\dim V = 1$ as its contain the entire fiber of the infinite étale covering $\pi_K : V \twoheadrightarrow V_K$ of the zeros of the modular form defined over the noetherian quotient V_K. Write $\overline{\Xi}_K$ for the Zariski closure of the image of Ξ in $V_K^{\mathcal{Q}}$. There is an example supplied by Venkatesh of a pro-curve in which any positive dimensional irreducible component of the Zariski closure of an infinite set Ξ is disjoint from Ξ (see §1.1). If $g_{\mathfrak{Q}}$ (appearing in the proof of [H04, Theorem 3.2] denoted by g_v in the text: see (5.2)) had a non-zero eigenvalue for $U(\mathfrak{l})$, the sequence like $(*)$ associated to $\{g_{\mathfrak{Q}}\}_{\mathfrak{Q}}$ would contain an infinite arithmetic progression (and thereby getting a contradiction). However it is easy to see $g_{\mathfrak{Q}}|U(\mathfrak{l}) = 0$; so, for the version of [H04, Theorem 3.2] and the part of [H07, Theorem 4.2] in the case where $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$, we are forced to assume that $\underline{n}$ contains an infinite arithmetic progression.

For the sake of the reader's convenience, we state the corrected version of [H07, Theorem 4.3] which is identical to the original one removing the stronger assertion in the case where $\mathrm{rank}\, \Gamma = 1$ (and its proof is also identical replacing [H07, Theorem 4.3] by the above Theorem 0.1). Let us recall some notation from [H07] (and refer undefined objects to [H07, §4.3]). We assume that W is sufficiently large containing all ℓ-power of roots of unity and the CM abelian variety $X(R)$ with $X(\mathbb{R})(\mathbb{C}) \cong \mathbb{C}^\Sigma / R^\Sigma$ for $R^\Sigma = \{(\nu(a))_{\nu\in\Sigma} \in \mathbb{C}^\Sigma | a \in R\}$. We write $\Omega_\infty = (\Omega_\nu)_{\nu\in\Sigma}$ for the Néron period defined in [H07, §2.6] with $\Omega_\infty^k = \prod_\nu \Omega_\nu^{k_\nu} \in \mathbb{C}^\times$ for $k = \sum_\nu k_\nu \nu \in \mathbb{Z}[\Sigma]$. Put $\Gamma_\Sigma(k) := \prod_\nu \Gamma(k_\nu)$ for the Gamma function $\Gamma(s)$. We fix a character $\nu : \Delta \to \mathcal{W}^\times$ for $\mathcal{W} = W \cap \overline{\mathbb{Q}}$.

Theorem 0.2. *Let $p > 2$ and ℓ be as above. Let λ be a Hecke character of M of conductor $\mathfrak{C}$ prime to $p\ell$ and of infinity type $k\Sigma + \kappa(1-c)$ with $0 < k \in \mathbb{Z}$ and $0 \le \kappa \in \mathbb{Z}[\Sigma]$ for a CM type Σ ordinary with respect to p. Suppose that $\mathfrak{C}$ is a product of split primes in M/F. Then $\frac{\pi^\kappa \Gamma_\Sigma(k\Sigma+\kappa) L^{(\mathfrak{l})}(0,\nu^{-1}\chi^{-1}\lambda)}{\Omega_\infty^{k\Sigma+2\kappa}} \in \mathcal{W}$ for all characters $\chi : Cl_\infty \to \mu_{\ell^\infty}(\mathcal{W})$ factoring through Γ, where $L^{(\mathfrak{l})}$ is the L-function with the Euler $\mathfrak{l}$-factor removed. Moreover, for Zariski densely populated character χ in $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$, we have*

$$\frac{\pi^\kappa \Gamma_\Sigma(k\Sigma+\kappa) L^{(\mathfrak{l})}(0,\nu^{-1}\chi^{-1}\lambda)}{\Omega_\infty^{k\Sigma+2\kappa}} \not\equiv 0 \mod \mathfrak{m}_{\mathcal{W}},$$

unless the following three conditions are satisfied by ν and λ simultaneously:

(M1) *M/F is unramified everywhere;*

(M2) *The strict ideal class (in F) of the polarization ideal $\mathfrak{c}$ of $X(R)$ (given in §3.1) is not a norm class of an ideal class of M ($\Leftrightarrow \left(\frac{M/F}{\mathfrak{c}}\right) = -1$);*

(M3) *The ideal character $\mathfrak{a} \mapsto (\lambda\nu^{-1}N(\mathfrak{a}) \mod \mathfrak{m}_W) \in \mathbb{F}^\times$ of F is equal to the character $\left(\frac{M/F}{\cdot}\right)$ of M/F.*

If (M1-3) *are satisfied, the L–value as above vanishes modulo $\mathfrak{m}$ for all anticyclotomic characters χ.*

Since the proof of the anticyclotomic main conjecture for the CM field M in [H06] is based on the stronger version of this theorem to good extent, at the end of this paper (Section 6), we will reprove the key ingredient in [H06] used to prove the main conjecture in [H06] by modified arguments, and the final result of [H06, Theorem] and [H09, Theorem] in the introduction of these papers are valid intact.

1. Irreducible components of Zariski closure

We study a general theory of Zariski closure in a pro-étale variety of an infinite set of close points. We start with a pathologic example.

1.1. **An example.** To motivate the reader to go through this article dealing with technical topics, we first discuss an example of an affine pro-scheme $V = V_{\infty/\mathbb{C}}$ étale over the affine line $V_0 = \mathrm{Spec}(\mathbb{C}[X])$ such that the Zariski closure of an infinite set $\Xi \subset V(\mathbb{C})$ does not have a single positive dimensional irreducible component containing a point of Ξ. The example was supplied by Akshay Venkatesh in 2018 December.

For a (finite dimensional) scheme $S_{/k}$ for an algebraically closed field k, we write $\mathrm{Irr}(S)$ for the set of all irreducible components of S and $\pi_0(S)$ for the set of all connected components of a scheme S. Put

$$\mathrm{Irr}_d(S) := \{I \in \mathrm{Irr}(S) | \dim I = d\}.$$

Thus $S = \bigcup_{Z \in \mathrm{Irr}(S)} Z$ and $\mathrm{Irr}(S) = \bigsqcup_{d=0}^{\dim S} \mathrm{Irr}_d(S)$. Set $\mathrm{Irr}_+(S) = \bigsqcup_{d>0}^{\dim S} \mathrm{Irr}_d(S)$. If $S = \mathrm{Spec}(A)$, we write $\mathrm{Irr}(A) = \mathrm{Irr}(\mathrm{Spec}(A))$ and $\pi_0(A) = \pi_0(\mathrm{Spec}(A))$. The set $\mathrm{Irr}(A)$ is in bijection onto the set of minimal prime ideals of A, and we identify the two sets.

Take $k = \mathbb{C}$. Let $V_n := V_0 \times \mathbb{Z}/2^n\mathbb{Z}$ and the projection $\mathbb{Z}/2^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2^n\mathbb{Z}$ for $m > n$ induces étale morphism $V_m \twoheadrightarrow V_n$. Let $P_j := (X - j) \subset \mathbb{C}[X]$ $(0 < j \in \mathbb{Z})$ and regard it as a closed point j of V_0. We define $V := \varprojlim_n V_n \cong V_0 \times \mathbb{Z}_2$. Write $(P_j, j)_n$ for the maximal ideal of V_n giving rise to the point $(j, 2^j \mod 2^j) \in V_n$. Therefore $(P_j, 2^j)_\infty$ for finite j is the maximal ideal of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$ non-trivial equal to P_j only at 2^j-component of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$, and the prime ideal $(P_j, 2^j)_n$ is $P_j \oplus \bigoplus_{i \not\equiv 2^j \mod (2^n)} \mathbb{C}[X]$. Let $\Xi = \{(j, 2^j)_\infty \in V | j = 1, 2, \dots\}$, and write

$$\Xi_n := \{(j, 2^j)_n = (j, 2^j)_n \in V_n | j = 1, 2, \dots\}$$

for the image of Ξ in V_n. Note that $V_n = \mathrm{Spec}(\bigoplus_{\mathbb{Z}/2^n\mathbb{Z}} \mathbb{C}[X])$ and $V = \mathrm{Spec}(\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X])$. We have $\bigcap_j (P_j, 2^j)_\infty = ((0), 0)_\infty \oplus \bigcap_{0<j\in\mathbb{Z}} (P_j, 2^j)_\infty$, where $((0), 0)_\infty$ is the prime ideal of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$ equal to (0) only at the 0-component of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$. Thus $\overline{\Xi} = V_0 \sqcup \bigsqcup_{0<j\in\mathbb{Z}} (P_j, 2^j)_\infty \subset V$, where V_0 is inserted as the 0-component. Thus only positive dimensional irreducible (and connected) component of the Zariski closure $\overline{\Xi}$ in V is V_0 which does not contain any points of Ξ.

> *If we have a transitive action of a semi-group inside* $\mathrm{Aut}(V)$ *on $\overline{\Xi}$, we expect to be able to avoid such a pathologic example.*

Though $\alpha : (v, z) \mapsto (v + 1, 2z)$ acts transitively on Ξ, α is not an automorphism of V. It is an automorphism of $V_0 \times \mathbb{Q}_2$ which is an indo-pro-variety not a pro-variety. In the above example, we have

$$\begin{aligned} &\mathrm{Irr}_1(\overline{\Xi}_n) = \{(V_0 \times 0) | 0 \in \mathbb{Z}/2^n\mathbb{Z}\},\ \mathrm{Irr}_0(\overline{\Xi}_n) = \{(j \times 2^j)_n | j = 1, \dots, n-1 (2^j \neq 0 \in \mathbb{Z}/2^n\mathbb{Z})\}, \\ &\mathrm{Irr}_1(\overline{\Xi}) = \{(V_0 \times 0) | 0 \in \mathbb{Z}_2\} \ \text{ and } \ \mathrm{Irr}_0(\overline{\Xi}) = \{(j, 2^j)_\infty | 0 < j \in \mathbb{Z}, 2^j \in \mathbb{Z}_2\}. \end{aligned} \tag{1.1}$$

The action of any positive power of α brings some points in $\mathrm{Irr}_0(\overline{\Xi}_n)$ into a component in $\mathrm{Irr}_1(\overline{\Xi}_n)$ (non-stability of $\mathrm{Irr}_0(\overline{\Xi}_n)$ under α coming from the fact that α is not an automorphism of V). Writing

$\pi_n : V \to V_n$, we can consider the reduced image $\pi_n(I) \subset V_n$ for $I \in \mathrm{Irr}(\overline{\Xi})$. Let $\pi_{n,*}(\mathrm{Irr}(\overline{\Xi})) = \{\pi_n(I) | I \in \mathrm{Irr}(\overline{\Xi})\}$ and $\pi_{n,*}(\mathrm{Irr}_j(\overline{\Xi})) = \{\pi_n(I) | I \in \mathrm{Irr}_j(\overline{\Xi})\}$ as sets. Then
(1.2)
$$\pi_{0,*} : \mathrm{Irr}_1(\overline{\Xi})) \cong \mathrm{Irr}_1(\overline{\Xi}_0)),\ \pi_{0,*}(\mathrm{Irr}_0(\overline{\Xi})) \supset \mathrm{Irr}_0(\overline{\Xi}_n) \ \text{ with infinite } \pi_{0,*}(\mathrm{Irr}_0(\overline{\Xi})) - \mathrm{Irr}_0(\overline{\Xi}_n) \text{ in } V_0 \times 0.$$

(ne) The image of $\{(j, 2^j)_\infty \in \Xi | j \geq n\}$ lies in the one dimensional $(V_0 \times 0) \in \mathrm{Irr}_1(\overline{\Xi}_n)$ and the 0-dimensional scheme $(j, 2^j)_\infty$ $(j \geq n)$ is not étale over V_n.

If we take a 2-unit $u \in \mathbb{Z}$ and consider $\Xi = \{(j, u^j)_\infty | j = 1, 2, \dots\} \subset V$, one can show that $\mathrm{Irr}(\overline{\Xi}_n) = \{V_0 \times u^j | u^j \mod 2^n\}$ and $\mathrm{Irr}(\overline{\Xi}) = \{V_0 \times x | x \in \langle u \rangle\}$ for the subgroup $\langle u \rangle \subset \mathbb{Z}_2^\times$ topologically generated by u. The action $[1] : (j, u^j) \mapsto (j+1, u^{j+1})$ extends to an automorphism $[1] : (v, z) \mapsto (v+1, uz)$. A similar morphism $\alpha(v, z) = (v+1, 2z)$ for non-unit 2 in place of u is not an automorphism of V.

Taking an infinite sequence of irreducible polynomials $X - a_j$ of $\mathbb{F}[X]$ with distinct $a_j \in \mathbb{F}$, we can make an example similar to (ne) also over $\mathbb{F}$ taking $V_0 := \mathrm{Spec}(\mathbb{F}[X])$ and $\Xi = \{(P_j := ((X - a_j), 2^j)\}_j$ with $V_n = V_0 \times \mathbb{Z}/2^n\mathbb{Z}$. Then $\varprojlim_n V_n = V_0 \times \mathbb{Z}_2$.

1.2. **Geometry of irreducible components.** We prepare some notation and geometric lemmas to prove that $\mathrm{Irr}_0(\overline{\Xi})$ in the introduction is a union of the pull back image of $\mathrm{Irr}_0(\overline{\Xi}_K)$ of finite level (e.g., Lemma 1.2). After the lemmas, in the following Section 2, we study the correspondence action whose orbit of any 0-dimensional irreducible component at finite level is infinite (this infinity is a key to obtain absurdity under assuming $\Xi \subset \mathrm{Irr}_0(\overline{\Xi})$).

Let $\pi : \mathcal{V}_{/\mathbb{F}} \to \mathcal{V}_{K/\mathbb{F}}$ be an affine étale Galois covering with $\mathcal{V} = \mathrm{Spec}_{\mathcal{O}_{\mathcal{V}_K}}(\mathcal{O}_{\mathcal{V}})$ (as a relative spectrum). Here $K = \mathrm{Gal}(\mathcal{V}/\mathcal{V}_K)$ and $\mathcal{V} = \varprojlim_{U \lhd K} \mathcal{V}_U$ for U running over open subgroups of K with $\mathcal{V}_U = \mathcal{V}/U$. In the following lemmas, assume that $\mathcal{V}_K$ is noetherian (so, $\mathcal{V}_U = \mathcal{V}/U$ is also noetherian for an open subgroup U of K. Let $\Xi \subset \mathcal{V}(\mathbb{F})$ be an *infinite* set of closed points with image Ξ_K in $\mathcal{V}_K(\mathbb{F})$.

Lemma 1.1. *Let X' (resp. X) be the Zariski closure of Ξ (resp. Ξ_K) in $\mathcal{V}$ (resp. $\mathcal{V}_K$). Then*
(1) *X' and X are reduced scheme, X'/X is finite if $\mathcal{V}/\mathcal{V}_K$ is finite.*
(2) *The projection $\pi_X : X' \to X$ is dominant inducing a surjection of $\mathbb{F}$-points: $X'(\mathbb{F}) \twoheadrightarrow X(\mathbb{F})$, and X' is unramified over X.*

As described in (ne), even if $\Xi \cong \Xi_K$, the map $\pi_X : X' \to X$ may not be étale.

Proof. Regard $P' \in \Xi$ (resp. $P \in \Xi_K$) as a sheaf of $\mathcal{O}_{\mathcal{V}}$-ideal (resp. $\mathcal{O}_{\mathcal{V}_K}$-ideal) defining the point P' (resp. P); so, for example $\mathcal{O}_{\mathcal{V}_K}/P \cong \mathbb{F}(P) = \mathbb{F}$ as a skyscraper sheaf supported by P. By definition, we have $X' = \mathrm{Spec}(\mathcal{O}_{\mathcal{V}}/\bigcap_{P' \in \Xi} P')$ and $X = \mathrm{Spec}(\mathcal{O}_{\mathcal{V}_K}/\bigcap_{P \in \Xi_K} P)$.

We prove the lemma first in the absolute affine case; so, we put $\mathcal{V}_K = \mathrm{Spec}(A)$, $\mathcal{V} = \mathrm{Spec}(A')$, $B = A/\bigcap_{P \in \Xi_K} P$ and $B' = A'/\bigcap_{P' \in \Xi} P'$. Since $B' \hookrightarrow \prod_{P' \in \Xi} A'/P$ with the right-hand-side reduced, B' is reduced. In the same way, B is reduced.

If A'/A is étale finite, we have $\Xi_K = \{P' \cap A | P' \in \Xi\}$; so, putting $\mathfrak{b}' := \bigcap_{P' \in \Xi} P'$ and $\mathfrak{b} := \bigcap_{P \in \Xi_K} P$, we have $\mathfrak{b}' \cap A = \mathfrak{b}$. Thus the induced map $B \xrightarrow{i} B'$ is injective. If A'/A is not finite, we can write $A = \bigcup_i A_i$ with A_i/A finite étale, we still get the injectivity. Therefore the projection $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is dominant. Pick a maximal ideal $\mathfrak{m} \in \mathrm{Spec}(B)(\mathbb{F})$. Then by the going-up theorem [CRT, Theorem 9.3 (i)], we have a prime ideal $\mathfrak{p} \in \mathrm{Spec}(B')$ with $\mathfrak{p}' \cap B = \mathfrak{m}$. Take a maximal ideal $\mathfrak{m}'$ containing $\mathfrak{p}'$, $\mathfrak{m}' \cap B \supset \mathfrak{m}$ is still a proper ideal as B'/B is integral; so, $\mathfrak{m}' \cap B = \mathfrak{m}$. Thus $B'/\mathfrak{m}'$ is a finite extension of $B/\mathfrak{m} = \mathbb{F}$ which is algebraically closed, we conclude $B/\mathfrak{m}' = \mathbb{F}$ and $\mathfrak{m}' \in \mathrm{Spec}(B')(\mathbb{F})$; so, $\mathrm{Spec}(B')(\mathbb{F}) \to \mathrm{Spec}(B)(\mathbb{F})$ is onto.

Pick $\mathfrak{m}' \in \mathrm{Spec}(B')(\mathbb{F})$ and regard it as a maximal ideal of A'. Since $\mathfrak{m}' \supset \mathfrak{b}'$, $\mathfrak{m} := \mathfrak{m}' \cap A \supset \mathfrak{b}$; so, $\mathfrak{m} \in \mathrm{Spec}(B)(\mathbb{F})$. We have the following commutative diagram of the completions at $\mathfrak{m}'$ and $\mathfrak{m}$:

$$\begin{tikzcd}
\widehat{A}_{\mathfrak{m}} \arrow[r, hook] \arrow[d, two heads] & \widehat{A}'_{\mathfrak{m}} \arrow[r, two heads] \arrow[d, two heads] & \widehat{A}'_{\mathfrak{m}'} \arrow[d, two heads] \\
\widehat{B}_{\mathfrak{m}} \arrow[r, "i_{\mathfrak{m}}"] & \widehat{B}'_{\mathfrak{m}} \arrow[r, "p_{\mathfrak{m}'}", two heads] & \widehat{B}'_{\mathfrak{m}'}.
\end{tikzcd}$$

Since the top row composite: $\widehat{A}_{\mathfrak{m}} \hookrightarrow \widehat{A}'_{\mathfrak{m}} \twoheadrightarrow \widehat{A}'_{\mathfrak{m}'}$ is an isomorphism (as $A \hookrightarrow A'$ is étale), $p_{\mathfrak{m}'} \circ i_{\mathfrak{m}}$ is onto. Therefore B'/B is an unramified extension and is finite if A'/A is finite. This proves (1) and (2) in the absolute affine case.

Now we treat the general relative affine case. We cover $\mathcal{V}_K = \bigcup_A \mathrm{Spec}(A)$ for affine open subscheme $\mathrm{Spec}(A)$, and write $A' = \pi_*\mathcal{O}_{\mathcal{V}}(\mathrm{Spec}(A))$. Then $\mathrm{Spec}(A')$ is an open subscheme of $\mathcal{V}$ covering $\mathrm{Spec}(A)$. Then we have $X' \cap \mathrm{Spec}(A') = X' \times_{\mathcal{V}_K} \mathrm{Spec}(A) = \mathrm{Spec}(B')$ and $X \cap \mathrm{Spec}(A) = X \times_{\mathcal{V}_K} \mathrm{Spec}(A) = \mathrm{Spec}(B)$ with $(A'/A, B'/B, \Xi\cap\mathrm{Spec}(A'), \Xi_K \cap \mathrm{Spec}(A))$ satisfying the assumption of Lemma 1.1. Since B (resp. B') depends on A, if needed, we write $B = B_A$ and $B' = B'_A$ to emphasize the dependence. By the above argument, B' and B are reduced algebra, and B' is an unramified extension of B, B'/B is finite if A'/A is finite, and the projection $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is dominant and the induced map: $\mathrm{Spec}(B')(\mathbb{F}) \to \mathrm{Spec}(B)(\mathbb{F})$ is surjective. Since $\mathrm{Spec}(B')$ is the pull-back to X' of $\mathrm{Spec}(B)$ and $X' = \bigcup_A \mathrm{Spec}(B'_A) = \bigcup_A \pi^{-1}(\mathrm{Spec}(B_A))$ and $X = \bigcup_A \mathrm{Spec}(B_A)$, the above proof in the affine case implies the assertion in the general case. □

Assume that $\Xi \cong \Xi_K$ by π. We have another commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\hookrightarrow} & \prod_{P\in\Xi_K} A/P \\ {\scriptstyle \pi_B^*}\big\downarrow & & \wr\big\downarrow \\ B' & \xrightarrow[\hookrightarrow]{} & \prod_{P\in\Xi} A'/P'. \end{array}$$

The right vertical map is an isomorphism as $\Xi \cong \Xi_K$. Thus π_B^* is injective; so, again we see that $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is dominant.

Lemma 1.2. *Let the notation and the assumption be as in* Lemma 1.1. *Recall that $\mathcal{V}_K$ is a noetherian scheme. Let $\pi_*(\mathrm{Irr}(X')) := \{\pi(Z')|Z' \in \mathrm{Irr}(X')\}$ for the set of the reduced image $\pi(Z') \subset X$. Then we have*

1. *The image $\pi_*(\mathrm{Irr}(X'))$ contains $\mathrm{Irr}(X)$,*
2. *For $Y \in \mathrm{Irr}(X)$, if $Y' \in \mathrm{Irr}(\pi^{-1}(Y))$ is contained in X', we have $Y' \in \mathrm{Irr}(X')$, where $\pi^{-1}(Y) = Y \times_{\mathcal{V}_K} \mathcal{V}$.*
3. *If $\Xi \cong \Xi_K$ under the projection $\mathcal{V} \xrightarrow{\pi} \mathcal{V}_K$, we have a unique section $\mathrm{Irr}_0(X) \to \mathrm{Irr}_0(X')$ of $\mathrm{Irr}_0(X') \twoheadrightarrow \mathrm{Im}(\mathrm{Irr}_0(X')) \subset X$ and $\mathrm{Irr}_0(X') \subset \Xi$. Moreover writing X'_U for the image of X' in $\mathcal{V}/U$ for an open subgroup U of K, $\mathrm{Irr}_0(X') = \varinjlim_U \mathrm{Irr}_0(X'_U)$ for U running over all open subgroups of K.*
4. *If $\dim Z = \dim X$ for $Z \in \mathrm{Irr}_{\dim X}(X)$, then Z is in the image of $\mathrm{Irr}_{\dim X'}(X')$ in X. In particular, $\mathrm{Irr}_{\dim(X)}(X') \neq \emptyset$.*

Proof. Again we may assume that $\mathcal{V}_K = \mathrm{Spec}(A)$, $\mathcal{V} = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$ as in the proof of Lemma 1.1. Pick $\mathfrak{p}_Y \in \mathrm{Irr}(B)$ giving $Y \in \mathrm{Irr}(\mathrm{Spec}(B))$. Since B'/B is integral, we find a prime $P' \in \mathrm{Spec}(B')$ such that $P' \cap B = \mathfrak{p}_Y$ by going-up theorem [CRT, Theorem 9.3 (i)]. For each $P' \in \mathrm{Spec}(B')$ with $P' \cap B = \mathfrak{p}_Y$ (i.e., $P' \in \pi^{-1}(Y) = \mathrm{Spec}(B'/\mathfrak{p}_Y B')$), take a minimal prime $\mathfrak{p}' \subset P'$ (i.e., $\mathfrak{p}' \in \mathrm{Irr}(B')$). Then $\mathfrak{p}' \cap B$ is a prime ideal of B and $\mathfrak{p}_Y \supset \mathfrak{p}' \cap B$; so, by minimality of $\mathfrak{p}_Y$, we have $\mathfrak{p}_Y = \mathfrak{p}' \cap B$. Thus $\mathfrak{p}_Y$ is in the image of $\mathrm{Irr}(B')$. This proves the assertion (1).

As $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is étale, $\pi^{-1}(Y)$ is étale over Y; so, equi-dimensional. Suppose that $Y' \subset X'$ for $Y' \in \mathrm{Irr}(\pi^{-1}(Y))$. Then we find $Z' \in \mathrm{Irr}(X')$ such that $Z' \supset Y'$; so, $\pi(Z') \subset X$. We are going to show $Z' = Y'$. We have $X \supset \pi(Z') \supset Y$. Since $\pi(Z')$ is irreducible, $\pi(Z')$ containing $Y \in \mathrm{Irr}(X)$ implies $\pi(Z') = Y$. Thus $Z' \twoheadrightarrow Y$ is a integral dominant; so, $\dim Z' = \dim Y' = \dim Y$. This shows $Z = Z' \in \mathrm{Irr}(X')$, as desired. Thus the assertion (2) follows.

To show the assertion (3) for Irr_0, we first assume that B'/B is finite. We regard $\Xi_K \subset \mathrm{Spec}(B)$. Pick $\mathfrak{m} \in \mathrm{Irr}_0(B)$. Then $B = B^{(\mathfrak{m})} \oplus B/\mathfrak{m}$ for a subring $B^{(\mathfrak{m})} \subset B$ as $\mathrm{Spec}(B/\mathfrak{m})$ is a connected component of $\mathrm{Spec}(B)$. Thus $\mathrm{Irr}_0(B) = \{Z \in \pi_0(\mathrm{Spec}(B))|\dim Z = 0\}$. Since $B' \supset B$, the above decomposition induces an algebra direct sum $B' = B'^{(\mathfrak{m})} \oplus B'/\mathfrak{m}B'$. Since B' is finite over B, $B'/\mathfrak{m}B'$ has dimension 0. By reducedness of B', the direct summand $B'/\mathfrak{m}B'$ of B' is a direct sum of fields. This means that π induces a surjection of the upper row of the following diagram:

$$\begin{tikzcd} \pi_0(\mathrm{Spec}(B'/\mathfrak{m}B')) \arrow[r, two heads, "\pi_*"'] \arrow[d, "\cap"'] & \pi_0(\mathrm{Spec}(B/\mathfrak{m})) = \{\mathfrak{m}\} \\ \mathrm{Irr}_0(B') \end{tikzcd}$$

for each $\mathfrak{m} \in \mathrm{Irr}_0(B) \subset \pi_0(B)$. Therefore $\pi_*(\mathrm{Irr}_0(B')) \supset \mathrm{Irr}_0(B)$. Pick $\mathfrak{m} \in \mathrm{Irr}_0(B)$. If $\mathfrak{m} \notin \Xi_K$, $\Xi_K \subset \mathrm{Spec}(B^{(\mathfrak{m})})$ as $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}) \sqcup \mathrm{Spec}(B^{(\mathfrak{m})})$. This implies $B = A/\bigcap_{P \in \Xi_K} P$ is equal to $B^{(\mathfrak{m})}$, a contradiction. Thus $\mathfrak{m} \in \Xi_K$, and $\mathrm{Irr}_0(B) \subset \Xi_K$. Since $\Xi \cong \Xi_K$, π_* has a unique section $\pi^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(B')$. If B'/B is not finite, we can write $B' = \bigcup_j B_j$ for B-subalgebras $B_j \subset B'$ finite over B. We may assume that the index set is totally ordered so that $B_{j'} \supset B_j$ if $j' > j$. Let $X'_U = \mathrm{Spec}(B_U)$ for an open subgroup U of K. Then B_U/B is finite unramified. Then applying the above argument to finite B_U/B, we find natural inclusion $\mathrm{Irr}_0(B_U) \subset \pi_{U',U,*}(\mathrm{Irr}_0(B_{U'}))$ for open subgroups $U' \subset U \subset K$ with a unique section $\pi^*_{U',U} : \mathrm{Irr}_0(B_U) \hookrightarrow \mathrm{Irr}_0(B_{U'})$. In particular, the injective limit of $\pi^*_{U',U}$ gives rise to the section $\pi^* : \mathrm{Irr}_0(B) \hookrightarrow \mathrm{Irr}_0(B')$ and $\mathrm{Irr}_0(B') = \varinjlim_U \mathrm{Irr}_0(B_U)$. This proves the assertion (3).

Now suppose that $\dim B/\mathfrak{p} = \dim B$ for $\mathfrak{p} \in \mathrm{Irr}(B)$. Such $\mathfrak{p}$ always exists as B is noetherian. Since B'/B is integral, $\dim B = \dim B'$. Then we take $\mathfrak{p}' \in \mathrm{Spec}(B')$ such that $\mathfrak{p}' \cap B = \mathfrak{p}$. Such a prime exists as already remarked. Then $B/\mathfrak{p} \hookrightarrow B'/\mathfrak{p}'$ and hence $\dim B'/\mathfrak{p}' = \dim B/\mathfrak{p} = \dim B$ as $B'/\mathfrak{p}'$ is integral over $B/\mathfrak{p}$. Since $\dim B' = \dim B$, we conclude $\mathfrak{p}' \in \mathrm{Irr}_{\dim B'}(B')$; so, $\mathrm{Irr}_{\dim B'}(B') \neq \emptyset$. This proves the assertion (4). □

Lemma 1.3. *Suppose that $\pi_*(Z') := \pi(Z') \notin \mathrm{Irr}(X)$ for $Z' \in \mathrm{Irr}(X')$. Then there exists $Z_0 \in \mathrm{Irr}(X)$ such that $Z_0 \supset \pi_*(Z')$.*

Proof. Again we may assume that $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$ as in the proof of Lemma 1.1. Write $Z' = \mathrm{Spec}(B'/\mathfrak{p}')$. By the assumption, $\mathfrak{p}' \cap B \notin \mathrm{Irr}(B)$; therefore, $\mathfrak{p}' \cap B \supsetneq \mathfrak{p}_0$ for a minimal prime ideal $\mathfrak{p}_0$ of B. By definition, $\mathfrak{p}_0 \in \mathrm{Irr}(B)$ and $\mathfrak{p}' \cap B \supset \mathfrak{p}_0$ means $\mathfrak{p}' \cap B \in \mathrm{Spec}(B/\mathfrak{p}_0)$. Thus $Z_0 = \mathrm{Spec}(B/\mathfrak{p}_0)$ does the job. □

Lemma 1.4. *If $\Xi_{K,0}$ is a subset of Ξ_K with finite $\Xi_K - \Xi_{K,0}$, then the Zariski closure X of Ξ_K in $\mathcal{V}_K$ and that X_0 of $\Xi_{K,0}$ share irreducible components of positive dimension (i.e., $\mathrm{Irr}_+(X) = \mathrm{Irr}_+(X_0)$), and $\mathrm{Irr}(X) - \mathrm{Irr}(X_0)$ is a finite subset of $\Xi_K - \Xi_{K,0}$.*

Proof. Again we may assume that $X = \mathrm{Spec}(B)$ as in the proof of Lemma 1.1. Write $\Xi_K - \Xi_{K,0} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_h\}$ for maximal ideals $\mathfrak{m}_i$ of A and put $\mathfrak{a} = \bigcap_i \mathfrak{m}_i$. Then for $\mathfrak{b}_0 = \bigcap_{P \in \Xi_{K,0}} P$ and $\mathfrak{b} = \bigcap_{P \in \Xi_K} P$, we have $\mathfrak{b} = \mathfrak{b}_0 \cap \mathfrak{a}$. For each i, either $\mathfrak{m}_i \supset \mathfrak{b}_0$ or $\mathfrak{m}_i + \mathfrak{b}_0 = A$ as $\mathfrak{m}_i$ is maximal. Thus we may assume that $\Xi_K - \Xi_{K,0} = \{\mathfrak{m}_i | \mathfrak{m}_i + \mathfrak{b}_0 = A\}$. Then $\mathfrak{a} + \mathfrak{b}_0 = A$ as $|\Xi_K - \Xi_{K,0}|$ is finite. Thus $A/\mathfrak{b} = A/\mathfrak{b}_0 \cap \mathfrak{a} = A/\mathfrak{b}_0 \oplus A/\mathfrak{a}$, and hence $X = X_0 \sqcup (\Xi_K - \Xi_{K,0})$ as desired. □

2. Corresponding action on subvarieties in the Hilbert modular variety

We give a definition and a detailed analysis of the correspondence toric action on the tower of subvarieties given by the Zariski closure of Ξ_K in the product of copies of the Hilbert modular Shimura variety. Assuming four axioms (∞), (T), (F) and (N), we prove a general result Theorem 2.7 in §2.3 assuring the existence of a component in $\mathrm{Irr}_+(\overline{\Xi})$ with a point in Ξ. We prove the axioms later in §2.4 in our original setting and obtain Corollary 2.12 implying Theorem 0.1.

Recall the CM quadratic extension $M_{/F}$ with its integer ring R from the introduction and their class groups $Cl_\infty = \varprojlim_n Cl_n$ and $Cl^-_\infty = \varprojlim_n Cl^-_n$, where Cl_n is the ring class group of $R_n = O + \mathfrak{l}^n R$ and $Cl^-_n = Cl_n/Cl_F$. Write $[\mathcal{A}]_n$ for the class of a proper R_n-ideal $\mathcal{A}$ in Cl_n. As in [H04, page 755], define a subgroup Cl^{alg} of Cl_∞ by

$$Cl^{alg} := \{[x]_\infty = \varprojlim_n [x\widehat{R}_n \cap M] \in Cl_\infty | x \in M^\times_{\mathbb{A}} \text{ with } x_{\mathfrak{l}\infty} = 1\} \subset Cl_\infty,$$

where $\widehat{R}_n = R_n \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ with $\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$. By multiplication, Cl^{alg} acts on Cl_∞ and Cl^-_∞.

Let $G := \mathrm{Res}_{O/\mathbb{Z}}\mathrm{GL}(2)$ and $Sh_{/\mathbb{Q}}$ be the Hilbert modular Shimura variety associated to G. Since $G(\mathbb{A}^{(\infty)})$ acts on Sh as automorphisms, we define the prime-to-p level Shimura variety $Sh^{(p)}$ by $Sh/G(\mathbb{Z}_p)$. The Shimura variety $Sh^{(p)}$ extend canonically to a smooth pro-scheme over $\mathcal{W}$ (e.g.

[PAF, Chapter 4]). Recall the irreducible component $V = V^{(p)}$ of the Shimura variety $Sh^{(p)}_{/\overline{\mathbb{Q}}}$ we fixed. By smoothness, $V_{/\mathbb{F}} := V \times_{\mathcal{W}} \mathbb{F}$ is an irreducible component of $Sh^{(p)}_{/\mathbb{F}}$.

Let $\mathcal{Q} \subset Cl_\infty$ be a finite subset independent modulo C^{alg}; i.e., $\delta C^{alg} \neq \delta' C^{alg}$ for any pair $(\delta, \delta') \in \mathcal{Q}^2$ with $\delta \neq \delta'$. Since Cl^{alg} naturally contains Cl_F, for the image $\mathcal{Q}^-$ in Cl^-_∞, we have $\mathcal{Q} \cong \mathcal{Q}^-$ and $\mathcal{Q}^-$ is still independent modulo Cl^{alg}. We identify the two sets $\mathcal{Q}$ and $\mathcal{Q}^-$. For a closed subgroup $K^{(p)} \subset G(\mathbb{A}^{(p\infty)})$, we put $K = G(\mathbb{Z}_p) \times K^{(p)}$ and write V_K for the image of V in $Sh^{(p)}_K = Sh/K$. We set $\mathcal{V}_{/B} := V^{\mathcal{Q}}_{/B}$ for $B = \overline{\mathbb{Q}}, \mathcal{W}, \mathbb{F}$ (the product of $\mathcal{Q}$ copies of V) and $\mathcal{V}_{K/B} := V^{\mathcal{Q}}_{K/\mathbb{F}}$. We can embed Cl_n into $\mathcal{V}$ by $[\mathcal{A}] \mapsto \mathbf{x}(\mathcal{A}) = \mathbf{x}([\mathcal{A}]) := (x([\mathcal{A}]\delta))_{\delta \in \mathcal{Q}} \in \mathcal{V}$, and write its image with C_n. Put $C^{(\infty)} = \bigsqcup_n C_n \subset \mathcal{V}$ as abelian variety sitting over $\mathbf{x}(\mathcal{A})$ is uniquely determined by $[\mathcal{A}]$. Though the modular form f is a function on V, we normalize it by multiplying a suitable Hecke character value later so that the normalized values at $x(\mathcal{A})$ and $x(\mathcal{A}')$ are identical if $[\mathcal{A}] = [\mathcal{A}']$ in Cl^-_∞. Because of this normalization, we may regard f as a function on $C^{(\infty)}$ modulo Cl_F. We fix an infinite subset Ξ of $C^{(\infty)}$. When it is necessary to indicate the level group K for which $x(\mathcal{A})$ resides in $\mathcal{V}_K$ (or V_K), we write $x_K(\mathcal{A})$ in place of $x(\mathcal{A})$. Here K can be a closed subgroup of $\mathrm{GL}_2(F^{(\infty)}_{\mathbb{A}})$. Actually we only deal with the tower raising $\mathfrak{l}$-power level; so, K can be a closed subgroup of $\mathrm{GL}_2(O_{\mathfrak{l}})$ which acts on $\mathcal{V}$ and V.

We fix a CM type Σ of M and write Σ_p for the set of p-adic places induced by the embedding in Σ by the identification $\mathbb{C} \cong \mathbb{C}_p$ we fixed.

Notation 2.1. *Hereafter, we simply write X (resp. X_K) for the Zariski closure of Ξ (resp. Ξ_K).*

We recall two assumptions (unr) and (ord) in [H04, §2.1] for p in addition to $\Xi \cong \Xi_K$ under the projection $\mathcal{V} \to \mathcal{V}_K$:

(ord) Σ *is p–ordinary: $\Sigma_p \cap \Sigma_p c = \emptyset$ for the generator c of* $\mathrm{Gal}(M/F)$.

Such a CM type Σ is called a *p–ordinary* CM type. The existence of a p–ordinary CM type is equivalent to the fact that all prime factors of p in F split into a product of two distinct primes in M. We suppose

(unr) *p is unramified in $F/\mathbb{Q}$.*

2.1. **Toric action.** The Zariski closure $X \subset \mathcal{V}$ (resp. $X_K \subset \mathcal{V}_K$) of Ξ (resp. Ξ_K) forms a tower $\{X \to X_K\}_K$ of varieties, and the tower induces a correspondence action on each noetherian layer X_K. If we have an appropriate action of a torus $\mathbf{T}$ in $G(\mathbb{A}^{(p\infty)})$ on Ξ, the correspondence action on $\varinjlim_K \mathrm{Irr}_0(X_K) = \mathrm{Irr}_0(X) \subset \Xi$ (by Lemma 1.2 (3)) coincides with the action of $\mathbf{T}$ (see (2.5)). The idea of the proof of Theorem 2.7 is to show

(1) $\mathrm{Irr}_0(X_K) \neq \emptyset$ for sufficiently small open K if $\mathrm{Irr}_0(X) \neq \emptyset$;
(2) If $\mathrm{Irr}_0(X_K) \neq \emptyset$, assuming the infinity (∞) of orbits in Ξ under the action of $\mathbf{T}$, $\mathrm{Irr}_0(X_K)$ has to be of infinite order, against the noetherian property of X_K.

We start with a list of conditions for proving the assertions (1)–(2) above under the correspondence action of $\mathbf{T}$. After this, we state five lemmas about the action under these conditions before starting with supplying the missing argument/fact (stated as Theorem 2.7).

If K is an open compact subgroup of $G(\mathbb{A}^{(\infty)})$, $\mathcal{V}_K$ is noetherian. On $\mathcal{V} = V^{\mathcal{Q}}$, $\mathrm{Aut}(V/\mathbb{F})$ diagonally acts. Let us denote by Ξ an infinite set of CM points in $\mathcal{V}$ for which we would like to prove density in $\mathcal{V}$. We suppose to have a semi-group $\mathbf{T} \subset \mathrm{Aut}(V/\mathbb{F})$ as in (T) below acting on Ξ under the diagonal action. The action of $\mathbf{T}$ is supposed to come from the action of elements in $G(\mathbb{A}^{(p\infty)})$ on $Sh^{(p)}$. Since it is a semi-group action, $\beta \in \mathbf{T}$ embeds Ξ into Ξ; so, $\beta(\Xi) \subset \Xi$ and $\beta^{-1}(\Xi) \supset \Xi$, where β^{-1} may not be in $\mathbf{T}$ but in $\mathrm{Aut}(\mathcal{V}/\mathbb{F})$.

Let $N := \{\varrho(u) | u \in O_{\mathfrak{l}}\}$ for $\varrho(u) := \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ and B be the normalizer of N in $\mathrm{GL}_2(O_{\mathfrak{l}})$ (i.e., B is the upper triangular Borel subgroup). We may regard N and B as group schemes over $O_{\mathfrak{l}}$; for example, $N(A) = \{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) | u \in A\}$ for an $O_{\mathfrak{l}}$-algebra A. Decompose $\widehat{O} := \varprojlim_{0<N\in\mathbb{Z}} O/NO = O_{\mathfrak{l}} \times O^{(\mathfrak{l})} = O_{p\mathfrak{l}} \times O^{(p\mathfrak{l})}$. The group $\mathrm{GL}_2(\widehat{O})$ is decomposed accordingly.

We consider the following conditions for K:

(K) K is closed of the form $K^{(p\mathfrak{l})} \times K_p \times K_{\mathfrak{l}}$ with $K^{(p\mathfrak{l})} \subset \mathrm{GL}_2(F^{(p\mathfrak{l}\infty)}_{\mathbb{A}})$, $\mathrm{GL}_2(O_p) \subset K_p \subset \mathrm{GL}_2(F_p)$ and $N \subset K_{\mathfrak{l}} \subset \widehat{\Gamma}_0(\mathfrak{l})$,

(I) $\pi : \mathcal{V} \twoheadrightarrow \mathcal{V}_K$ induces $\Xi \cong \Xi_K$,

where $F_{\mathbb{A}}^{(p\mathfrak{l}\infty)}$ is the adele ring of F away from $p\mathfrak{l}\infty$, $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p \subset F_{\mathbb{A}}$, and

$$\widehat{\Gamma}_0(\mathfrak{l}^\nu) = \{g \in \mathrm{GL}_2(O_{\mathfrak{l}}) | (g \bmod \mathfrak{l}^\nu) \in B(O/\mathfrak{l}^\nu)\}.$$

We put

$$\widehat{\Gamma}_1(\mathfrak{l}^\nu) = \{g \in \mathrm{GL}_2(O_{\mathfrak{l}}) | (\overline{g} \bmod \mathfrak{l}^\nu) \in N(O/\mathfrak{l}^\nu)\}$$

for the image $\overline{g} \in \mathrm{PGL}_2(O_{\mathfrak{l}})$ of $g \in \mathrm{GL}_2(O_{\mathfrak{l}})$. For general $g \in \mathrm{GL}_2(F_{\mathfrak{l}})$, we write $S^g := g^{-1}Sg$ for a subgroup S of $G(\mathbb{A}^{(\infty)})$.

In the application in [H04], we assumed K to be $\widehat{\Gamma}_0(\mathfrak{l}) \times \mathrm{GL}_2(\widehat{O}^{(\mathfrak{l})})$, and in [H07], we kept $K_{\mathfrak{l}} = \widehat{\Gamma}_0(\mathfrak{l})$ but allowed $K^{(p\mathfrak{l})}$ to be a Γ_0-type level group (cf. Definition 6.5). In these papers, the measure φ_f is defined using a level $\mathfrak{l}$ Eisenstein series f, but the measure is defined on Cl_n^- for any high power $\mathfrak{l}^n$. If we start with a $U(\mathfrak{l})$-eigenform f of level $\mathfrak{l}^n$, the same construction of the measure works well. In any case, for Theorem 0.1, we do not need to assume $K_{\mathfrak{l}} = \widehat{\Gamma}(\mathfrak{l})$ and we allow $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^n)$ for any $n > 0$.

Here is the first axiom. We assume

(T) $\quad \mathbf{T} = \mathcal{T} \times \alpha^{\mathbb{N}}$ *for* $\alpha^{\mathbb{N}} = \{\alpha^n | 0 \le n \in \mathbb{Z}\}$ *and a group* $\mathcal{T}$ *acts on* Ξ,

where $\alpha \in \mathrm{GL}_2(F_{\mathfrak{l}})$ is upper triangular and $\alpha N \alpha^{-1} \supsetneq N$. Here the semi-group $\mathbf{T} \subset \mathrm{Aut}(V/\mathbb{F})$ acts on Ξ under the diagonal action. The action of $\mathcal{T}$ is basically multiplication by elements in Cl^{alg} (coming from the non-split torus $M^\times \hookrightarrow G(\mathbb{A}^{(p\infty)})$) which permutes elements in Cl_n^- and is essential in the proof of [H10, Theorem 3.20] which shows that $X = \mathcal{V}$ once we know $\mathrm{Irr}(X) = \mathrm{Irr}_+(X)$.

In this article, the action of the semi-group $\alpha^{\mathbb{N}}$ plays a central role to prove $\mathrm{Irr}(X) = \mathrm{Irr}_+(X)$. The condition $\alpha N \alpha^{-1} \supsetneq N$ implies that $\alpha \in B \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right) B$ for some $m > 0$ with a uniformizer $\varpi_{\mathfrak{l}}$ of $O_{\mathfrak{l}}$, and if $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$ ($\nu > 0$), $S = S_K := K \cap K^\beta$ is normalized by K and a representative set of $S_K \backslash K$ can be chosen in N. Note that $N\alpha^{\mathbb{N}}N := \bigcup_{\beta \in \alpha^{\mathbb{N}}} N\beta N \subset \mathrm{GL}_2(F_{\mathfrak{l}})$ is a multiplicative semi-group.

Consider the following condition

(∞) $\quad$ *every* $\mathbf{T}$*-orbit in* Ξ *is infinite.*

This condition will be verified for our choice of Ξ in Proposition 2.11 for the above α well chosen. Since $\alpha \in B \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right) B$ ($m > 0$) does not have a fixed point in $\mathcal{V}$, if one orbit $\mathbf{T}(x)$ for $x \in \Xi$ is infinite, every orbit is indeed infinite.

For simplicity, we assume hereafter $K_{\mathfrak{l}} = \widehat{\Gamma}_0(\mathfrak{l}^\nu)$ or $\widehat{\Gamma}_1(\mathfrak{l}^\nu)$ with $\nu > 0$ and that K is open in $G(\mathbb{A})$ satisfying (K). Since α is supposed to preserve the irreducible component V of $Sh^{(p)}$, we may assume that $\mathfrak{l}^m = (\varpi)$ with $\varpi = \varphi\varphi^c$ for some $\varphi \in R$. Replacing m by a positive integer multiple of m, we may further assume

(2.1) $\quad$ for $a := \varpi_{\mathfrak{l}}^m/\varpi$, elements $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ in $G(\widehat{\mathbb{Z}})$ belong to K.

Indeed, by replacing m by mn and ϖ by ϖ^n, a is replaced by a^n which is sufficiently close to 1. Hereafter, for simplicity, we assume that $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ for $\varpi = \varphi\varphi^c$ and write β for a general element in $N\alpha^{\mathbb{N}}N$. We write $?(\Xi)_K$ ($? = \beta, \beta^{-1}$) for the Zariski closure in $\mathcal{V}_K$ of the image $?(\Xi)_K = ?(\Xi_K)$ of $?(\Xi)$ in $\mathcal{V}_K$.

As we recall from [H04] in §3.1, the left action of $g \in G(\mathbb{A}^{(p\infty)})$ on the point $x = (A, \eta) \in \mathcal{V}$ is given by $g(x) = \tau(g)^{-1}(x)$, where the right action $\tau(g)$ is by definition given by $\eta \mapsto \eta \circ g$ for the level structure η associated to the point $x \in Sh^{(p)}$. If $\beta \in \alpha^{\mathbb{N}}$, $K \supset N$ may not be normalized by β. Thus β acts on $\mathcal{V}_K$ as a correspondence.

Let us explain the correspondence action in some more details. Recall $S = S_K := K \cap \beta^{-1}K\beta = K \cap K^\beta$. By definition $S^{\beta^{-1}} = K^{\beta^{-1}} \cap K \subset K$ and $S^{\beta^{-1}}S \subset K$ (so, $S^{\beta^{-1}}$ satisfies the condition (K) while S is not). Then S is normalized by N if $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$ but $S \not\supset N$. We have $N\beta N = \bigsqcup_{u \in \mathcal{N}} N\beta\varrho(u)$ for a finite set $\mathcal{N} = \left\{\varrho(u) | u \bmod \mathfrak{l}^j\right\}$ for $0 < j$ given by $(\det(\beta)) = \mathfrak{l}^j$ (so, $m|j$), and $K\beta K = \bigsqcup_{u \in \mathcal{N}} K\beta\varrho(u)$. Then we have the correspondence $U(\beta) \subset \mathcal{V}_K \times \mathcal{V}_K$ (with respect to the tower $\{\mathcal{V} \twoheadrightarrow \mathcal{V}_K\}_K$) defined by the following commutative diagram

$$\begin{array}{ccc} \mathcal{V}_S & \xrightarrow[\sim]{v\mapsto\beta(v)=v\beta^{-1}} & \mathcal{V}_{S^{\beta^{-1}}} \\ {\scriptstyle p_{S,K}}\downarrow & & {\scriptstyle p_{S^{\beta^{-1}},K}}\downarrow \\ \mathcal{V}_K & \xrightarrow[U(\beta)]{} & \mathcal{V}_K, \end{array}$$

where $U(\beta)$ is identified with a subvariety given by the diagonal image of $\mathcal{V}_S$ under the product of the projections $p_{S,K} \times (p_{S^{\beta^{-1}},K} \circ \beta)$. It is easy to see $U(\beta^n) = U(\beta)^n$ under the correspondence action. The correspondence $U(\beta)$ brings a point $x \in \mathcal{V}_K$ to a finite set $U(\beta)(x) := (p_{S^{\beta^{-1}},K} \circ \beta)(p_{S,K}^{-1}(x))$. We assume, for K satisfying (K),

(N) *The action of* $\mathbf{T}$ *on* Ξ *extends to a correspondence action of the semi-group* $N\mathbf{T}N$ *on* Ξ_K.

If it is necessary to indicate the dependence of the level group K, we write Ξ_K for the image of Ξ in $\mathcal{V}_K$. We write $U(\beta^n)(\Xi_K) := \bigcup_{x\in\Xi_K} U(\beta^n)(x)$. The condition (N) means that $U(\beta)$ acts on Ξ (i.e., $U(\beta)(\Xi_N) \subset \Xi_N$).

Since $\alpha_{\mathfrak{l}} \in \mathrm{GL}_2(F_{\mathfrak{l}})$, by (2.1), the correspondence $U(\beta)$ for $\beta \in S\alpha^{\mathbb{N}}S$ only depends on the double coset $N\beta N$. We need the following finiteness condition (which will be verified in Lemma 2.10 and (2.9)):

(F) $$\Xi_N - U(\alpha^n)(\Xi_N) \text{ and } \alpha^{-n}(\Xi) - \Xi \text{ are finite for all } n > 0.$$

Since $\mathcal{T}$ is a group, (F) implies finiteness of $\Xi_N - U(\beta)(\Xi_N)$ and $\beta^{-1}(\Xi) - \Xi$ for all $\beta \in \mathbf{T}$. We actually use only the finiteness of $\beta^{-1}(\Xi) - \Xi$ in the proof of the key result (Theorem 2.7).

Let $X = X_\Xi$ (resp. $X_S = X_{\Xi,S}$) be the Zariski closure of Ξ in $\mathcal{V}$ (resp. of the image Ξ_S in $\mathcal{V}_S$) for a closed subgroup S satisfying (K). Since $U(\beta)(\Xi) \subset \Xi$, we find $X_\Xi \supset X_{U(\beta)(\Xi)} = \bigcup_{u\in\mathcal{N}} \beta\varrho(u)(X_\Xi)$. Thus we have a tower $\{X_S\}_S$ of reduced schemes with projections $p_{S',S} : X_{S'} \to X_S$ for $S' \subset S$ (which we write simply $p_{S'}$ if S is clear in the context). Therefore, we can think of the corresponding action of β on X_S with respect to the tower $\{X_S\}_S$.

If S is open compact, X_S is a reduced variety (i.e., reduced noetherian). The semi-group $N\mathbf{T}N$ acts on X sending $X = X_\Xi$ to $\beta(X) = X_{\beta(\Xi)}$ and also $U(\beta)(X) = X_{U(\beta)(\Xi)}$. For $\beta \in N\mathbf{T}N$ and an open compact subgroup $K \subset G(\mathbb{A}^{(\infty)})$ satisfying (K), taking an open compact subgroup S of K such that $SS^{\beta^{-1}} \subset K$, we have a diagram

$$\begin{array}{ccc} X_S & \xrightarrow{v\mapsto\beta(v)} & \beta(X)_{S^{\beta^{-1}}} \subset X_{S^{\beta^{-1}}} \\ {\scriptstyle p_{S,K}}\downarrow & & {\scriptstyle p_{S^{\beta^{-1}},K}}\downarrow \\ X_K & \xrightarrow[C(\beta)]{} & X_K, \end{array} \tag{2.2}$$

where $C(\beta)$ is a subvariety given by the diagonal image of X_S under $p_{S,K} \times p_{S^{\beta^{-1}},K} \circ \beta$. We regard $C(\beta)$ as a correspondence from X_K into X_K. This correspondence is specifically on X_K and its points and is possibly *different* from the operator $U(\beta)$ for the tower $\{\mathcal{V} \twoheadrightarrow \mathcal{V}_S\}_S$.

Lemma 2.2. *Assume that* $\mathcal{V}_S \to \mathcal{V}_K$ *for* $S = S_K := K \cap K^\beta$ *is étale. Let* $Y^S := p_{S,K}^{-1}(Z_K) = Z_K \times_{\mathcal{V}_K} \mathcal{V}_S$ *for* $Z_S \in \mathrm{Irr}(X_S)$ *and* $Z_K := p_{S,K}(Z_S)$, *and write* $Y^S = \bigcup_{Z\in\mathrm{Irr}(Y^S)} Z$. *If* $Z \neq Z'$ *for* $Z, Z' \in \mathrm{Irr}(Y^S)$, *we have* $Z \cap Z' = \emptyset$; *so,* $Y^S = \bigsqcup_{Z\in\mathrm{Irr}(Y^S)} Z$. *If* $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$ *and* $Z \in \mathrm{Irr}(Y^S)$, *then* K *normalizes* S *and for* $u \in K/S$, *either* $u(Z) = Z$ *or* $u(Z) \cap Z = \emptyset$.

Proof. Note that $Y^S := p_{S,K}^{-1}(Z_K)$ is étale finite over Z_K as $\mathcal{V}_S \to \mathcal{V}_K$ is étale. Thus Y^S is equi-dimensional with $\dim Z = \dim Y_S = \dim Z_S = \dim Z_K$ for $Z \in \mathrm{Irr}(Y^S)$. If $\emptyset \neq Z \cap Z' \subsetneq Z$ for $Z \neq Z'$ ($Z, Z' \in \mathrm{Irr}(Y^S)$), $Z \to Z_K$ and $Z' \to Z_K$ are dominant by the equi-dimensionality. Thus by the étale property of $Y^S \to Z_K$, $Z(\mathbb{F}) \to Z_K(\mathbb{F})$ and $Z'(\mathbb{F}) \to Z_K(\mathbb{F})$ are onto. For $x \in (Z \cap Z')(\mathbb{F})$, $|((Z \cup Z') \times_{Z_K} x_K)(\mathbb{F})| < \deg(Z'/Z_K) + \deg(Z/Z_K)$; so, $Z \cup Z'$ ramifies over $p_{S,K}(Z_S)$ since $\dim Z = \dim Z'$, which is impossible as $Z \cup Z' \hookrightarrow Y^S \to Z_K$ is unramified by Lemma 1.1 (2). This shows the first assertion.

Suppose that $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$ and $Z \in \mathrm{Irr}(Y^S)$. Since $SN = K$ and N normalizes $S = S_K = K \cap K^\beta$, K normalizes S. If $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$, $Y^S = \bigcup_{u\in K/S} u(Z)$; so, $u(Z)$ is still an irreducible component of Y^S, and K/S acts on $\mathrm{Irr}(Y^S)$. Thus the intersection is either empty or $u(Z) \cap Z = Z$. If $u(Z) \cap Z = Z$, we have $Z \subset u(Z)$. Since they are irreducible and have equal dimension, we conclude $Z = u(Z)$. $\square$

Lemma 2.3. *Suppose* (F). *Then for $\beta \in \mathbf{T}$ and K satisfying* (K), *we have* $\mathrm{Irr}_+(U(\beta)(X)_K) = \mathrm{Irr}_+(X_K)$ *and* $\mathrm{Irr}(X_K) - \mathrm{Irr}(U(\beta)(X)_K) \subset (\Xi_K - U(\beta)(\Xi_K))$. *Similarly we have* $\mathrm{Irr}_+(\beta^{-1}(X)_K) = \mathrm{Irr}_+(X_K)$ *and* $\mathrm{Irr}(\beta^{-1}(X)_K) - \mathrm{Irr}(X_K) \subset (\beta^{-1}(\Xi) - \Xi)$.

Proof. As remarked after (F), $\Xi - U(\beta)(\Xi)$ is finite for all $\beta \in \mathbf{T}$. Since $U(\beta)(\Xi) \subset \Xi$, we have a closed immersion $U(\beta)(X_K) \subset X_K$. Since $U(\beta)(X)_K$ is the Zariski closure of $U(\beta)(\Xi_K)$, the finiteness of $\Xi - U(\beta)(\Xi)$ implies $\mathrm{Irr}_+(X_K) = \mathrm{Irr}_+(U(\beta)(X)_K)$ and $\mathrm{Irr}(X_K) - \mathrm{Irr}(U(\beta)(X)_K) \subset (\Xi - U(\beta)(\Xi))$ by Lemma 1.4. The last assertion follows from finiteness of $\beta^{-1}(\Xi) - \Xi$ assumed in (F). $\square$

The semi-group element $\beta \in \mathbf{T}$ acts on $\pi_0(X)$ and $\mathrm{Irr}(X)$ in the sense that β sends $\pi_0(X)$ and $\mathrm{Irr}(X)$ isomorphically onto $\pi_0(\beta(X))$ and $\mathrm{Irr}(\beta(X))$, respectively. Therefore $\beta : x \mapsto \beta(x) = x\beta^{-1}$ induces an isomorphism $\beta_* : \mathrm{Irr}(X_S) \cong \mathrm{Irr}(\beta(X)_{U^{\beta^{-1}}})$. Let Z_S be an irreducible component of X_S and write $\beta(Z_S) \in \mathrm{Irr}(\beta(X)_{S^{\beta^{-1}}})$.

Lemma 2.4. *Suppose that $S \subset K$ is a closed subgroup for an open compact subgroup $K = G(\mathbb{Z}_p) \times K^{(p)}$ in $G(\mathbb{A}^{(\infty)})$. Take $Z_S \in \mathrm{Irr}(X_S)$ with $\dim Z_S = \dim X_S$ and write Z_K for the image of Z_S in X_K. Then $Z_K \in \mathrm{Irr}(X_K)$, and there exists $x \in \Xi$ such that its image x_K lies in an open subscheme of Z_K made of smooth points of Z_K.*

Proof. By Lemma 1.2, we have $Z_K \in \mathrm{Irr}(X_K)$. Thus we prove the existence of the point $x \in \Xi$ as in the lemma. If $Z_K = X_K$, nothing to prove. We suppose that $Z_K \neq X_K$. Since X_K is noetherian, the Zariski closure $Z_K^\perp$ of $X_K - Z_K$ is a proper closed subscheme of X_K; so, by Zariski density of Ξ_K in X_K, if $\Xi_K \subset Z_K^\perp$, we find $X_K = Z_K^\perp$, a contradiction. Therefore $(Z_K - Z_K^\perp) \cap \Xi_K \neq \emptyset$. For the Zariski closure Z'_K of $(Z_K - Z_K^\perp) \cap \Xi_K$ in Z_K, $Z'_K \cup Z_K^\perp$ contains $((Z_K - Z_K^\perp) \cap \Xi_K) \cup (Z_K^\perp \cap \Xi_K) = \Xi_K$ as $X_K = Z_K \cup Z_K^\perp$. Thus $Z'_K \cup Z_K^\perp = X_K = Z_K \cup Z_K^\perp$. Since $Z_K^\perp$ is a union of irreducible components of X_K different from Z_K, this implies $Z_K \subset Z'_K$, and $(Z_K - Z_K^\perp) \cap \Xi_K$ is Zariski dense in Z_K. We can thus pick x_K in the open subscheme $Z_K - Z_K^\perp$ in Z_K. Since the subscheme of smooth points of $Z_K - Z_K^\perp$ is non-empty and open in Z_K [CRT, Theorem 24.4], we may assume that x_K is a smooth point of $Z_K - Z_K^\perp$. $\square$

For each reduced Zariski closed subset $\mathcal{Y}$ of $\mathcal{V}_S$, we put $\Xi^{\mathcal{Y}} = \mathcal{Y} \cap \Xi_S$.

Lemma 2.5. *Suppose that K is an open compact subgroup as in* (K). *Let $Z_K \in \mathrm{Irr}(X_K)$. Then Ξ^{Z_K} is dense in Z_K.*

Proof. Since $\Xi_K \cap (Z_K - Z_K^\perp)$ is dense in Z_K as seen in the proof of Lemma 2.4, Ξ^{Z_K} containing $\Xi_K \cap (Z_K - Z_K^\perp)$ is dense in Z_K.

We can argue differently. For an irreducible component Z_K of X_K, $Z_K - Z_K^\perp$ is an open subset of X_K; so, any open subset $Y' \subset (Z_K - Z_K^\perp)$, $Y' \cap \Xi_K \neq \emptyset$. Thus $\Xi^{Z_K} = \Xi_K \cap Z_K$ is dense in Z_K. $\square$

Take $x \in \Xi$ and $\beta \in \mathbf{T}$ and fix an open compact subgroup K satisfying (K). Suppose $\mathcal{V} \to \mathcal{V}_K$ is étale. Let $S = S_K = K \cap K^\beta \subset K$ such that $SS^{\beta^{-1}} \subset K$. Take $Y_K \in \mathrm{Irr}_d(X_K)$ with $Y_K \ni x_K$. Let $Y^S := p_{S,K}^{-1}(Y_K)$ for the projection $p_{S,K} : \mathcal{V}_S \to \mathcal{V}_K$. By Lemma 2.2, $Y^S = \bigsqcup_{Z \in \mathrm{Irr}(Y^S)} Z$ (disjoint union). By $\Xi \cong \Xi_K$, $\Xi^{Y^S} \cong \Xi^{Y_K}$. We have a partition $\Xi^{Y^S} = \bigsqcup_{Z \in \mathrm{Irr}(Y^S)} \Xi^Z$ for $\Xi^Z = \Xi_S \cap Z$.

Suppose $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$. Assume that $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is étale. Since the diagram

$$\begin{array}{ccc} Y^S & \xrightarrow{\hookrightarrow} & \mathcal{V}_S \\ \big\downarrow & & {\scriptstyle p_{S,K}}\big\downarrow{\scriptstyle \text{étale}} \\ Y_K & \xrightarrow{\hookrightarrow} & \mathcal{V}_K \end{array}$$

is Cartesian, $Y^S \to Y_K$ is étale. Therefore, Y^S is equi-dimensional with $\dim Y^S = \dim Y_K$. By Lemma 1.2, $\mathrm{Irr}(Y^S) \cap \mathrm{Irr}(X_S) \neq \emptyset$; so, we can define a non-empty subscheme Y_S of Y^S by

$$Y_S := \bigcup_{Z \in \mathrm{Irr}(X_S) \cap \mathrm{Irr}(Y^S)} Z \overset{(*)}{=} \bigsqcup_{Z \in \mathrm{Irr}_d(X_S) \cap \mathrm{Irr}(Y^S)} Z \subset X_S, \tag{2.3}$$

which is equi-dimensional with dimension $d := \dim Y_K$, and the identity $(*)$ follows from Lemma 2.2 under étaleness of $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$. Thus $\mathrm{Irr}(X_S) \cap \mathrm{Irr}(Y^S) = \mathrm{Irr}_d(X_S) \cap \mathrm{Irr}(Y^S)$. Note that taking intersection $\mathrm{Irr}(Y^S) \cap \mathrm{Irr}(X_S) \neq \emptyset$ means that we can pick irreducible components of X_S which dominates Y_K (so, each member of $\mathrm{Irr}(Y^S) \cap \mathrm{Irr}(X_S) \neq \emptyset$ has dimension equal to $\dim Y_K$). By Lemma 2.2, Y^S is a disjoint union of Y_S and $\bigsqcup_{Z \in \mathrm{Irr}(Y^S) - \mathrm{Irr}(Y_S)} X$, and hence $Y_S \to Y_K$ is étale finite dominant.

Lemma 2.6. *Suppose that K is an open compact subgroup as in* (K) *and pick $Y_K \in \mathrm{Irr}_d(X_K)$ for $0 \le d \le \dim X$. Suppose $\mathcal{V}_S \to \mathcal{V}_K$ is étale. Then we have $Y^S = \bigsqcup_{Z \in \mathrm{Irr}(Y^S)} Z$ and $Y_S = \bigsqcup_{Z \in \mathrm{Irr}(Y^S), Z \subset X_S} Z$. The set Ξ^Z is either empty or Zariski dense in Z for $Z \in \mathrm{Irr}(Y^S)$, and for each $x \in \Xi^{Y_S}$, there is a unique irreducible component $Z \in \mathrm{Irr}(Y_S)$ with $x \in Z$.*

Proof. The first assertion is proven before the statement of the lemma. We prove the remaining assertion. If $Z \subset X_S$ for $Z \in \mathrm{Irr}(Y^S)$, it is an irreducible component of X_S by Lemma 1.2. Thus Ξ^Z is Zariski dense in Z by Lemma 2.5. In other words, if $Z \not\subset X_S$, Ξ^Z is an empty set, and for each $x \in \Xi^{Y_S}$, there is a unique irreducible component $Z \in \mathrm{Irr}(Y_S)$ with $x \in Z$ as Y_S is a disjoint union of Z. $\square$

2.2. **Modular correspondences acting on irreducible components of X_K.** Pick an irreducible component $Y_K \in \mathrm{Irr}_d(X_K)$ for $0 \le d \le \dim X_K$ with an open compact subgroup K satisfying (K).

2.2.1. *Definition of the correspondence.* Choosing $x \in \Xi$ so that $x_S \in Y_S$ for Y_S in (2.3), we have $\beta(x)_S := x_S\beta^{-1} \in \beta(Y_S) \subset X_{S^{\beta^{-1}}}$, and there is a unique irreducible component Z of Y_S containing x_S by Lemma 2.6. Since $Y_S \xrightarrow[\beta]{\sim} \beta(Y_S) \subset \beta(X)_{S^{\beta^{-1}}} \subset X_{S^{\beta^{-1}}}$, we have $\dim Y_K = \dim Y_S = \dim \beta(Y_S) = \dim \beta(Y_S)_K$ for the projection $\beta(Y_S)_K$ of $\beta(Y_S)$ in X_K.

For any pair of open compact subgroups (K, S) with $K \supset SS^{\beta^{-1}}$ (so, $S \subset K \cap \beta^{-1}K\beta$), we have a diagram similar to (2.2):

$$\begin{array}{ccccc} Y_S & \xrightarrow[\sim]{v \mapsto \beta(v)} & \beta(Y_S) & \xrightarrow{\subset} & X_{S^{\beta^{-1}}} \\ {\scriptstyle p_{S,K}}\big\downarrow{\scriptstyle \text{finite}} & & \big\downarrow & & {\scriptstyle p_{S^{\beta^{-1}},K}}\big\downarrow{\scriptstyle \text{finite}} \\ Y_K & \xrightarrow[C_S(\beta)]{} & \beta(Y_S)_K := p_{S^{\beta^{-1}},K}(\beta(Y_S)) & \xrightarrow[\subset]{} & X_K, \end{array}$$

for the correspondence $C_S(\beta)$ given by the reduced image $\mathrm{Im}(p_{S,K} \times p_{S^{\beta^{-1}},K} \circ \beta : Y_S \to \mathcal{V}_K \times \mathcal{V}_K)$ whose support is contained in $C(\beta)$ in (2.2). Note that

$$C_S(\beta) \text{ is independent of the choice of } S \tag{2.4}$$

as $p_{S,K} \times p_{S^{\beta^{-1}},K} \circ \beta = (p_{S_K,K} \times p_{S_K^{\beta},K} \circ \beta) \circ p_{S,S_K}$ for $S_K = K \cap \beta K \beta^{-1}$ (so, $C_S(\beta) = C_{S_K}(\beta)$). As mentioned below (2.2), the correspondence $C_S(\beta)$ is with respect to the tower $\{X_K\}_K$ and is possibly *different* from $U(\beta)$ with respect to the tower $\{\mathcal{V}_S\}_S$.

Hereafter we choose S to be S_K and still write it as S (so, the correspondence action of $C_S(\beta)$ on irreducible components we introduce in the proof of the following Theorem 2.7 only depends on β (and K)). Note that $\beta(Y_S)_K = \bigcup_u \beta u(Z)_K$ for some $u \in \mathcal{N} \cong K/S_K$, where $\beta u(Z)_K$ is the image under $p_{S^{\beta^{-1}},K}$ of $\beta u(Z)$ for a component $Z \in \mathrm{Irr}(Y_S)$ (cf. Lemma 2.2). Since $\beta : X_S \cong \beta(X)_{S^{\beta^{-1}}}$, $\mathrm{Irr}(\beta(Y_S)) \subset \mathrm{Irr}_d(\beta(X)_{S^{\beta^{-1}}})$ ($d = \dim Y_K$). By the above diagram with dominant $p_{S,K}$ and $p_{S^{\beta^{-1}},K}$, we again find $\dim \beta(Y_S)_K = \dim Y_K$ as $\beta(Y_S)_K \subset p_{S^{\beta^{-1}},K}(\beta(p_{S,K}^{-1}(Y_K))$.

2.3. **Positive dimensionality of irreducible components of X.** We now prove the following fact not described in [H04]:

Theorem 2.7. *Suppose* (unr) *and* (ord) *at the beginning of* Section 2 *for p. Let $\Xi \subset \mathcal{V}(\mathbb{F})$ be an infinite subset injecting into $\mathcal{V}_K$ for any open compact subgroup K satisfying* (K) *and* (I). *We assume that a semi-group $\mathbf{T} \subset \mathrm{Aut}(V/\mathbb{F})$ as in* (T) *embedded in* $\mathrm{Aut}(\mathcal{V}/\mathbb{F})$ *acts on Ξ, and assume* (F) *and* (N).

(1) *If the condition (∞) is satisfied, all irreducible components of X has positive dimension;*
(2) *If $\mathbf{T}$ acts on Ξ transitively, $\dim X > 0$, X is equi-dimensional, and the irreducible component containing a given $x \in \Xi$ is unique.*

Under (∞), we can replace Ξ by an infinite orbit $\mathbf{T}(x)$ and apply the result and conclude the Zariski closure of $\mathbf{T}(x)$ is equidimensional of positive dimension; so, one of them contains x.

Proof of (1).[1] We need to describe the correspondence action of β on $Y_K \in \mathrm{Irr}_d(X_K)$. First suppose that $d = \dim X$ as this is the easiest case. Then $\dim \beta(Y_S)_K = \dim X$, and hence $\beta(Z)_K \in \mathrm{Irr}_{\dim X}(X_K)$ for $Z \in \mathrm{Irr}(Y_S)$. In this way, $\beta \in \mathbf{T}$ acts on $Y_K \in \mathrm{Irr}_{\dim X}(X_K)$ as correspondences (i.e., Y_K is brought to a subset $\beta(Y_K) = \{\beta(Z)_K | Z \in \mathrm{Irr}(X_S) \cap \mathrm{Irr}(Y^S)\} \subset \mathrm{Irr}_{\dim X}(X_K)$ whose member has equal dimension). Since $\mathrm{Irr}(X_S) \cap \mathrm{Irr}(Y^S)$ is made of $u(Z)$ for $u \in N$, for a finite subset B of $N\beta N$, we have the correspondence action of $C(\beta)$ given by the image set $\beta(Y_K) := \bigcup_{\beta' \in B}\{\beta'(Y_K)\}$ under $C(\beta)$ on $\mathrm{Irr}_{\dim X}(X_K)$.

Though we only need the result for $d = 0$, we give an argument for the intermediate dimension $0 < d < \dim X$ now as this introduces necessary notation for the case $d = 0$. Pick $Y_K \in \mathrm{Irr}_d(X_K)$ and start with $Y_K \in \mathrm{Irr}_d(X_K)$. As above to define the action of $C(\beta)$ on $\mathrm{Irr}_d(X_K)$, we only need to give a good definition of the image set $\beta(Y_K)$ for a $\beta \in N\mathbf{T}N$. For simplicity, write $S' := S^{\beta^{-1}}$. Let us recall a general notation: For an irreducible component Y'_K of X_K, we define as before ${Y'}^{S'} := p^{-1}_{S',K}(Y'_K)$ and $Y'_{S'} := \bigsqcup_{Z' \in \mathrm{Irr}_+(X_{S'}) \cap \mathrm{Irr}_+({Y'}^{S'})} Z'$ (by (2.3)). Now recall the irreducible component $Z \in \mathrm{Irr}_+(Y_S)$ containing the base point $x_S \in \Xi_S$ chosen in §2.2.1 and we apply the above notation to the irreducible component Y'_K of X_K such that $\beta(Z) \subset Z'$ for an irreducible component Z' of $Y'_{S'}$ (so, $\beta(x_S) \in Z'$). To see the existence of an irreducible component Y'_K of X_K as above, we argue as follows. Since $\beta(Z)$ is an irreducible closed variety of $X_{S'}$, $p_{S',K}(\beta(Z))$ is an irreducible closed variety of X_K. Then there exists an irreducible component Y'_K containing $p_{S',K}(\beta(Z))$ of X_K by Lemma 1.2 (1). Therefore $\beta(Z) \subset Y'_{S'}$ which is contained in $Z' \in \mathrm{Irr}(Y'_{S'})$. So $\dim Z' = \dim Y'_K \geq d$ by Lemma 2.2. Replacing (β, Y_K, S, K) by $(\beta^{-1}, Y'_K, S', K)$, we apply the above argument. Note that $\beta^{-1}(Z') \subset \beta^{-1}(X_{S'})$; so, $\beta^{-1}(Z')_K \subset \beta^{-1}(X)_K$. By the choice of Y'_K, Lemma 2.6 tells us that Z is determined by the two conditions (i) $\beta^{-1}(X)_K \supset \beta^{-1}(Z')_K \supset Y_K$ and (ii) $x_S \in Z$. Since $\mathrm{Irr}_+(\beta^{-1}(X)_K) = \mathrm{Irr}_+(X_K)$ by Lemma 2.3 and $\beta^{-1}(Z')_K$ is irreducible, we conclude from $\beta^{-1}(Z')_K \supset Y_K$ that $\beta^{-1}(Z')_K = Y_K$ (as Y_K is an irreducible component of $\mathrm{Irr}_+(\beta^{-1}(X)_K) = \mathrm{Irr}_+(X_K)$); in particular, $\dim Z' = \dim Y_K = d$. So, $Y'_K = \beta(Z)_K$ and that $\beta(Z)_K$ is an element in $\mathrm{Irr}_d(X_K)$ (Lemma 2.3). Therefore, again $\beta \in \mathbf{T}$ acts on $\mathrm{Irr}_d(X_K)$ as correspondences (i.e., Y_K is brought to a subset $\beta(Y_K) = \{\beta(Z)_K | Z \in \mathrm{Irr}(X_S) \cap \mathrm{Irr}(Y^S)\} \subset \mathrm{Irr}_d(X_K)$ whose member has equal positive dimension).

Now suppose $d = 0$. Since the correspondence action preserves $\mathrm{Irr}_+(X_K)$, it also preserves the complement $\mathrm{Irr}_0(X_K)$. The following argument to see the correspondence action is really an action sending a point to a point also gives an alternative proof of the stability of $\mathrm{Irr}_0(X_K)$ under the action of $\mathbf{T}$. We proceed similarly to the case where $0 < d < \dim X$ using the same notation. Then $x_K = Y_K \in \mathrm{Irr}_0(X_K)$ falls in the image Ξ_K in $\mathcal{V}_K$ of Ξ by Lemma 1.2 (3). By (I), the projection $\pi : \mathcal{V} \twoheadrightarrow \mathcal{V}_K$ induces $\Xi \cong \Xi_K$; so, $p^{-1}_{S,K}(x_K)$ is a finite set of points above x_K and $\{x' \in p^{-1}_{S,K}(x_K) | x' \in X_S\}$ is a singleton by Lemma 1.2 (2–3). Thus $p^{-1}_{S,K}(Y_K) \cap X_S = p^{-1}_{S,K}(x_K) \cap X_S$ is a singleton. Therefore $Y_S = \{Z := x_S\}$ is a singleton. Take an irreducible component Y'_K of X_K such that $\beta(Z) \subset Z'$ for an irreducible component Z' of $Y'_{S'}$ (so, $\beta(x_S) \in Z'$). Such a Y'_K exists by Lemma 1.2 (1). So $\dim Z' = \dim Y'_K \geq 0$. We want to prove $\dim Y'_K = 0$. Since $\mathrm{Irr}_+(\beta^{-1}(X)_S) = \mathrm{Irr}_+(X_S)$ by Lemma 2.3 and (F), if $\dim Z' > 0$, we have $\dim \beta^{-1}(Z') > 0$ and $\beta^{-1}(Z')$ is an irreducible component of X_S. Since $\beta^{-1}(Z') \supset Z = x_S$ by construction and the two are irreducible components of X_S, we find that $\beta^{-1}(Z') = Z = x_S$, a contradiction against $\dim Z' > 0$. Hence $\dim Z' = 0$ and $Z' = \beta(Z) = \beta(x_S)$. This implies that β brings $\mathrm{Irr}_0(X_K)$ into $\mathrm{Irr}_0(X_K)$. It is now clear that this is really an action (not a correspondence action) of $\mathbf{T}$ on $\mathrm{Irr}_0(X_K)$, and

(2.5) the action is compatible with the action of $\mathbf{T}$ on Ξ

as $\mathrm{Irr}_0(X_K) \subset \Xi_K \cong \Xi$.

In particular, $\mathrm{Irr}_0(X_K)$ contains $\mathbf{T}(x_K)$ for each $x_K \in \mathrm{Irr}_0(X_K) \subset \Xi_K$. Then by (∞), $\mathrm{Irr}_0(X_K)$ is infinite, a contradiction as X_K is a noetherian scheme. Therefore $\mathrm{Irr}_0(X_K) \cap \mathbf{T}(x_K) = \emptyset$ for every open compact subgroup K of $G(\mathbb{A}^{(p)})$ satisfying (K) and $x_K \in \mathrm{Irr}_0(X_K)$. This implies $\mathrm{Irr}_0(X_K) =$

[1]In the proof, we use the existence of $\beta^{-1} \in \mathrm{Aut}(V)$ essentially, while $\alpha : (v, z) \mapsto (v + 1, 2z)$ in §1.1 cannot be extended to an automorphism of V there.

$\emptyset$ for every open compact subgroup K of $G(\mathbb{A}^{(p)})$ satisfying (K), and therefore $\mathrm{Irr}_0(X) = \emptyset$ by Lemma 1.2 (3). This shows that all irreducible components of X have positive dimension.

Proof of (2). We have proven positive dimensionality of irreducible components of X. We need to prove equi-dimensionality of X and the uniqueness of the component containing $x \in \Xi$ under equi-dimensionality. Since the smooth locus X_K^{sm} of X_K is open dense in X_K by [CRT, Theorem 24.4], $\Xi_K^{sm} := \Xi \cap X_K^{sm}$ is still dense in X_K. Since $\mathrm{Irr}(X_K) = \pi_0(X_K^{sm})$, for each $x \in \Xi_K^{sm}$, the irreducible component of X_K^{sm} containing x_K is unique. Since $\mathbf{T}$ acts on $\mathrm{Irr}(X_K) = \pi_0(X_K^{sm})$ as correspondence, for any $Z, Z' \in \mathrm{Irr}(X_K)$, we find $x \in \Xi \cap Z^{sm}$ and $y \in \Xi \cap Z'^{sm}$.

Interchanging Z and Z' if necessary, by (T) and transitivity of the action, we can choose $\beta \in \mathbf{T}$ with $\beta(x) = y$. Then $\beta(Z) \in \mathrm{Irr}(X_K)$ and $y \in \beta(Z) \cap Z'$. Thus $y \in Z'^{sing} = Z' - Z^{sm}$ (i.e., $y \in \beta(Z) \cap Z'$ with Z' different from any irreducible components of $\beta(Z)$) or $\beta(Z) \supset Z'$ or $\beta(Z) \subset Z'$. The case: $y \in Z'^{sing} = Z' - Z^{sm}$ does not occur as we have chosen $y \in Z'^{sm}$. Since the correspondence action of $\mathbf{T}$ preserves $\mathrm{Irr}_d(X_K)$ for any given $d > 0$, the remaining cases $\beta(Z) \supset Z'$ or $\beta(Z) \subset Z'$ imply $\dim Z = \dim \beta(Z) = \dim Z'$ and $Z' \in \mathrm{Irr}(\beta(Z))$. Choosing one of Z and Z' to have maximal dimension $\dim X$, the other has to have maximal dimension; so, $\mathrm{Irr}(X_K) = \mathrm{Irr}_{\dim X}(X_K)$; so, X_K is equidimensional. This implies X is equidimensional.

By the first fundamental sequence of differentials and unramifiedness of X_S/X_K in Lemma 1.1, the projection induces a surjection:

$$\Omega_{X_K/\mathbb{F}} \otimes_{\mathcal{O}_{X_K}} \mathbb{F}(x_K) \twoheadrightarrow \Omega_{X_S/\mathbb{F}} \otimes_{\mathcal{O}_{X_S}} \mathbb{F}(x_S)$$

for $S \subset K$. By the proof of the equi-dimensionality, for $\dim \mathcal{O}_{X_S,x_S} = \dim \mathcal{O}_{X_K,x_K}$ for any point $x_S \in X_S$ with projection x_K in X_K. Thus

$$\dim_{\mathbb{F}} \Omega_{X_K/\mathbb{F}} \otimes_{\mathcal{O}_{X_K}} \mathbb{F}(x_K) \geq \dim_{\mathbb{F}} \Omega_{X_S/\mathbb{F}} \otimes_{\mathcal{O}_{X_S}} \mathbb{F}(x_S) \geq \dim \mathcal{O}_{X_S,x_S} = \dim \mathcal{O}_{X_K,x_K}.$$

Here "$\dim_{\mathbb{F}}$" indicates dimension of an $\mathbb{F}$-vector space, and $\dim R$ for a ring R means the Krull dimension of the ring R. Thus the singular locus

$$X_S^{sing} := \{x_S \in X_S | \dim_{\mathbb{F}} \Omega_{X_S/\mathbb{F}} \otimes_{\mathcal{O}_{X_S}} \mathbb{F}(x_S) > \dim \mathcal{O}_{X_S,x_S}\}$$

of X_S is sent to X_K^{sing}, where $\mathbb{F}(x)$ is the residue field of x. Thus $X^{sing} = \varprojlim_S X_S^{sing}$, and hence $\dim X^{sing} < \dim X = \dim Y$ for any $Y \in \mathrm{Irr}(X)$. Plainly $\mathbf{T}$ preserves X^{sing}. If $x \in \Xi \cap X^{sing}$, then $\Xi = \mathbf{T}(x) \subset X^{sing}$; so, $X = X^{sing}$, a contradiction. Thus $\Xi \cap X^{sing} = \emptyset$. Since $X^{sm} := X - X^{sing}$ is a dense open subscheme of X, $\pi_0(X^{sm}) \cong \mathrm{Irr}(X^{sm}) \cong \mathrm{Irr}(X)$ with $X^{sm} = \bigsqcup_{Y \in \mathrm{Irr}(X)} Y^{sm}$. Thus for each given $x \in \Xi \subset X^{sm}$, $Y^{sm} \in \mathrm{Irr}(X^{sm})$ containing x is unique. □

Since X_K has positive dimension for an open compact level K (as $|\Xi_K|$ is infinity; cf. Lemma 1.2 (4)), by the above proposition, all components of X have positive dimension. Taking $x \in \Xi$ and an irreducible component of X containing x, we get

Corollary 2.8. *Let the notation and the assumption be as in* Theorem 2.7 (1). *Then X contains an irreducible component X_0 of positive dimension with a point $x \in \Xi$. Moreover for each element ξ of the stabilizer of x in $\mathbf{T}$, we have $\xi(X_0) = X_0$.*

Proof. We need to prove the last assertion: $\xi(X_0) = X_0$. Since $\xi \in \mathbf{T}$, $\xi(X_0)$ is another irreducible component of X containing x. Taking a level group K sufficiently small, $\xi(X_0) \cup X_0 \to X_K$ is unramified. Since $\xi(X_0) \cap X_0 \ni x$, unramifiedness and positive dimensionality of X_0 tells us that $\xi(X_0) = X_0$.

There is another argument. Replacing Ξ by the orbit $\Xi' := \mathbf{T}(x)$, we may assume that $\mathbf{T}$ acts transitively on Ξ. Then X_0 and $\xi(X_0)$ are irreducible components of the Zariski closure X' of Ξ'. Then we can apply Theorem 2.7 (2) to X' and Ξ'. Since there is only one irreducible component containing $x = \xi(x)$, we have $X_0 = \xi(X_0)$. □

Remark 2.9. Note that the stabilizer of $x \in \Xi_n$ is given by $\mathcal{T}_x := (M^\times \cap R_{n,\mathfrak{l}}^\times \cap R_p^\times)/(F^\times \cap O_{\mathfrak{l}}^\times \cap O_p^\times)$ embedded into $\mathrm{GL}_2(F_{\mathbb{A}}^{(\mathfrak{l}p\infty)}) \subset \mathrm{GL}_2(F_{\mathbb{A}}^{(p\infty)})$ which after p-adic completion contains a p-adically open subgroup. We can take $\mathcal{T}$ in (T) to be this group or the bigger group $(M^\times \cap R_{\mathfrak{l}}^\times \cap R_p^\times)/(F^\times \cap O_{\mathfrak{l}}^\times \cap O_p^\times)$, and under this choice, we can apply Corollary 2.8 to $\xi \in \mathcal{T}_x$. The stability of X in Corollary 2.8 is a requirement of [H10, Corollary 3.19, Theorem 3.20], and the choice $\mathcal{T}_x \subset \mathcal{T}$ is sufficient for this purpose. Since the central elements in $F^\times$ acts on V trivially, we take $\mathcal{T}_x$ as above rather than $M^\times \cap R_{n,\mathfrak{l}}^\times \cap R_p^\times$.

2.4. **Verification of (F) and (N) for infinite arithmetic progression.** We briefly describe the choice of $\Xi \subset \mathcal{V}$. For the details of the definition of CM point $x(\mathcal{A})$, see Section 3.

Let

$$\mathcal{T} := (M^\times \cap R^\times_{(\mathfrak{l})} \cap R^\times_{(p)})/(F^\times \cap O^\times_{(\mathfrak{l})} \cap O^\times_{(p)}), \tag{2.6}$$

but for our convenience, we often shrink $\mathcal{T}$ slightly to a subgroup of finite index to define $\mathbf{T}$ as remarked in Remark 2.9. We write $R^\times_{(\mathfrak{a})}$.

For each $\xi \in R^\times_{(p)}$ (in [H04, page 755] the symbol "α" is used for the letter "ξ" here), we have $x(\mathcal{A}) := (X(\mathcal{A}), \overline{\Lambda}(\mathcal{A}), \eta^{(p)}(\mathcal{A})) \xrightarrow[\xi]{\sim} x(\xi\mathcal{A})$ as in the middle of [H04, page 755], and as seen in [H04, page 756], $\rho_R(\xi^{(\mathfrak{l})})(x(\mathcal{B})) = x([\xi^{(\mathfrak{l})}]\mathcal{B})$ for the class $[\xi^{(\mathfrak{l})}] = [(\xi)]$ of the ideal (ξ) in $Cl_\infty = \varprojlim_n Cl_n$. Recall $C_n = \{\mathbf{x}(\mathcal{A}) := (x([\mathcal{A}]\delta))_{\delta\in\mathcal{Q}} | \mathcal{A} \in Cl_n\}$ and $C^{(\infty)} = \bigsqcup_n C_n \subset \mathcal{V}$. Thus $\xi \in \mathcal{T}$ acts on $C^{(\infty)}$ by $[\mathcal{A}] \mapsto [(\xi)][\mathcal{A}]$.

Let $\varpi_\mathfrak{l}$ be a prime element of $O_\mathfrak{l}$. As specified in [H04, §2.1 and §3.1], for each proper fractional ideal $\mathcal{A}$ of R_n, we have a specific CM point $x(\mathcal{A}) \in Sh^{(p)}(\mathbb{F})$. In our application, Ξ is made of the set of points of the form $x(\mathcal{A})$. Note that (see (3.3))

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{l}^m \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{u}{\varpi_\mathfrak{l}^m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{l}^m \end{pmatrix}.$$

By [H04, (3.2)] (see (3.2) in the text), writing $\alpha_m := \begin{pmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{l}^m \end{pmatrix}$ and $\varrho(u) := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, we have

$$\alpha_m\varrho(u)(x_N(R_n)) = \varrho(\frac{u}{\varpi_\mathfrak{l}^m})\alpha_m(x_N(R_n)) = x_N(\mathcal{A}) \quad (0 < m \in \mathbb{Z}) \tag{2.7}$$

for $\mathcal{A}$ given by $x(\mathcal{A}) = x(R_n)/\mathcal{C}_u$ for a suitable subgroup $\mathcal{C}_u \subset X(R_n)$ with $\mathcal{C}_u \cong O/\mathfrak{l}^m$ depending on u, x_N indicates the image of x in $\mathcal{V}_N$, and $\mathcal{A} = R_{n+m}$ if $u = 0$. By (2.1), in (2.7), we can replace $\varpi_\mathfrak{l}^m$ by $\varpi = \varphi\varphi^c$ and x_N by x_K, and the identity is valid on $\mathcal{V}_K$ (in place of $\mathcal{V}_N$). Any $\mathcal{A}$ in $\mathrm{Ker}(Cl_{n+m} \to Cl_n)$ with $n > 0$ can be written as in (2.7).

Set $\Xi^n_j = \{\mathbf{x}(\mathcal{A}) \in \mathcal{V} | \mathcal{A} \in \mathrm{Ker}(Cl_n \to Cl_j)\}$ for each $n > j \geq 0$ with a given j. Since

$$\Xi^n_j = \{\mathbf{x}([\mathcal{A}]\delta) | \mathcal{A} = \xi R_n \text{ with } \xi \in R^\times_{(p\mathfrak{l})} \cap (1 + \mathfrak{l}^j R_\mathfrak{l})\},$$

defining $\mathcal{T}_j \subset \mathcal{T}$ (for $\mathcal{T}$ in (2.6)) by

$$\mathcal{T}_j := \{\xi \in (M^\times \cap R^\times_{(\mathfrak{l})} \cap R^\times_{(p)}) | (\xi \bmod \mathfrak{l}^j) \in (R_{(p\mathfrak{l})}/\mathfrak{l}^j)^\times\}/O^\times_{(p\mathfrak{l})}, \tag{2.8}$$

the group $\mathcal{T}_j$ acts transitively on Ξ^n_j for every n with $n > j \geq 0$. Here $R_{(p\mathfrak{l})}$ and $O_{(p\mathfrak{l})}$ are the localization at $p\mathfrak{l}$ of R and O not the completion, and note $(\mathcal{T} : \mathcal{T}_j) < \infty$.

Lemma 2.10. *Assume that $\mathfrak{l}^m$ is generated by an element of $N_{M/F}(R)$ and write $\mathfrak{l}^m = (\varpi)$ with $\varpi \in N_{M/F}(R)$. Define $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$ and let $\mathbf{T} = \mathbf{T}_{j,m} = \mathcal{T}_j \times \bigcup_{k\geq 0} N\alpha^k N$ as a semi-group. If $\underline{n}$ is an infinite arithmetic progression of difference m, for $\Xi = \Xi_{\underline{n}} = \Xi_{\underline{n},j} := \bigsqcup_{i\geq 0} \Xi^{n_0+im}_j$, we have $\Xi \supset U(\beta)(\Xi)$ for $\beta \in \mathbf{T}_{j,m}$ (which implies that the condition* (N) *is satisfied), and $\Xi - U(\beta)(\Xi)$ is finite.*

Proof. By (2.7), we have $U(\alpha)(C_n) = C_{n+m}$ and $U(\alpha)(\Xi^n_j) = x(\Xi^{n+m}_j)$. Thus the semi-group $\bigcup_{k\geq 0} N\alpha^k N$ acts on $\Xi_{\underline{n}}$ for $\underline{n} = \{n_0 + im | i = 0, 1, 2 \dots\}$ for $U(\alpha^k)$ $(0 \leq k \in \mathbb{Z})$ sending $\Xi^{n_0+im}_j$ into $\Xi^{n_0+(i+k)m}_j$. Any element of $\Xi^{n_0+(i+k)m}_j$ is an image of an element of $\Xi^{n_0+im}_j$ under the action of $\beta \in N\alpha^k N$. Then we have

$$\Xi_{\underline{n}} - U(\alpha^k)(\Xi_{\underline{n}}) = \bigsqcup_{i\geq 0} \Xi^{n_0+im}_j - \bigsqcup_{i'\geq 0} \Xi^{n_0+(k+i')m}_j = \bigsqcup_{i=0}^{k-1} \Xi^{n_0+im}_j$$

which is finite. Since $\mathcal{T}_j$ is a group acting transitively on $\Xi^{n_0+im}_j$, this implies $\Xi - U(\beta)(\Xi)$ is finite for all $\beta \in N\mathbf{T}N$ and hence we get (N) and (F) for $U(\beta)$. $\square$

The point $x(\mathcal{A})$ is given by identifying $\widehat{\mathcal{A}}^{(p)} = \mathcal{A} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^{(p)}$ with the prime-to-p Tate module of the corresponding CM abelian variety $X(\mathcal{A})$; so, strictly speaking, it is more precise to write $x(\widehat{\mathcal{A}}^{(p)})$ (or $x(\widehat{\mathcal{A}})$) in place of $x(\mathcal{A})$. Under this notation, $\alpha(x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n,\mathfrak{l}})) = x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n+m,\mathfrak{l}})$ and $\alpha^{-1}(x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n,\mathfrak{l}})) = x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n-m,\mathfrak{l}})$ as long as $n > m$. See §5.3 for what happens when $n \leq m$.

It appears that the map α^{-1} is non-injective, but this comes from the fact that $K \supset N$ (satisfying (K)) but S_K is not; in other words, $p_{S,K}$ is not injective but shrinking K to K' at $\mathfrak{l}$ so that $K' \cap \mathcal{N} = \{1\}$ (with $K/N \cong K'/(K' \cap N)$), the fiber of $p_{S_K,K}$ will be separated modulo $S_K \cap K'$ (but the fiber of $p_{S_{K'},K'}$ is non-trivial again). Thus $\alpha^{-1}(C_n) = C_{n-m}$ as long as $n > m$. This shows (F) for α^{-i} and $\Xi_{\underline{n},j}$:

$$\alpha^{-i}(\Xi) - \Xi \textit{ is finite for all } i > 0 \tag{2.9}$$

as long as $\underline{n}$ is an infinite arithmetic progression of difference m, $\mathfrak{l}^m$ is generated by an element of $N_{M/F}(R)$ and $\alpha \operatorname{diag}[1, \varpi_{\mathfrak{l}}]^{-m} \in K$. Here we write $\operatorname{diag}[a, b]$ for the diagonal matrix with diagonal entries a, b from top to bottom. Therefore in this case the condition (F) is valid.

If necessary, we also write sometime $x(\widehat{\mathcal{A}})$ ($\widehat{\mathcal{A}} = \mathcal{A} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$) for $x(\mathcal{A})$ assuming $\mathcal{A}_p = R_p$. The group $\mathcal{T}$ acts on $x(\mathcal{A})$ as follows: For $\xi \in \mathcal{T}$,

$$x(\mathcal{A}) = x(\widehat{\mathcal{A}}) \mapsto x(\xi^{(\mathfrak{l})}\widehat{\mathcal{A}}) = x((\xi)\mathcal{A}),$$

where $\xi^{(\mathfrak{l})} \in M_{\mathbb{A}}^{\times}$ is the finite idele with $\mathfrak{l}$-component equal to 1 and every component at finite place outside $\mathfrak{l}$ is equal to ξ.

In the idele class group $\mathcal{I} := M_{\mathbb{A}}^{\times}/M^{\times}M_{\infty}^{\times}$, ξ is trivial but $\xi^{(\mathfrak{l})}$ is not trivial; so, the action of ξ on Cl_n is non-trivial for a sufficiently large n. Regard $Cl_n = \operatorname{Pic}(R_n)$ as a quotient of $\mathcal{I}$, and write $(\xi) = (\xi)_n$ for the image of $\xi^{(\mathfrak{l})}$ in Cl_n. Since $\operatorname{Ker}(Cl_n \to Cl_0)$ is spanned by $(\xi)_n$ with ξ running in $\mathcal{T}$, $\mathcal{T}$ acts transitively on $\operatorname{Ker}(Cl_n \to Cl_0)$. More generally, noting that $\mathcal{T}_r \subset \mathcal{T}$ is the stabilizer of $x(R_r)$ in Cl_r, $\mathcal{T}_r$ acts transitively on $\operatorname{Ker}(Cl_n \to Cl_r)$ for all $n \geq r$. From $\widehat{\mathcal{A}}$ with $\mathcal{A}_{\mathfrak{l}} = R_{n,\mathfrak{l}}$, we can create $\widehat{\mathcal{A}}_i := \widehat{\mathcal{A}}^{(\mathfrak{l})} \times R_{i,\mathfrak{l}}$. Then even if $\mathcal{A} = \xi R_n$ with $\xi_{\mathfrak{l}} \in R_{n,\mathfrak{l}}$ (i.e., $\mathcal{A}$ is trivial in Cl_n), for $i > n$ with $\xi_{\mathfrak{l}} \notin R_{i,\mathfrak{l}}$, $\widehat{\mathcal{A}}_i$ is non-trivial in Cl_i. In this way, the group $\mathcal{T} := R_{(p\mathfrak{l})}^{\times}/O_{(p\mathfrak{l})}^{\times}$ acts on C^{alg} as in [H04, page 755].

Let $\underline{n} = \{0 < n_0 < n_1 < n_2 < \cdots < n_i < \cdots\}$ be an infinite sequence of integers such that $\mathfrak{l}^{n_i}$ is generated by an elements in $N_{M/F}(R)$. If m is an exponent such that $\mathfrak{l}^m$ is generated by an elements in $N_{M/F}(R)$, then any infinite arithmetic progression $\underline{n} = \{n_i = n_0 + im | 0 \leq i \in \mathbb{Z}\}$ for an initial value $0 < n_0$ satisfies this condition. Recall $\Xi_j^{n_i} = \{(x([\mathcal{A}]\delta))_{\delta \in \mathcal{Q}} \in \mathcal{V} | [\mathcal{A}] \in \operatorname{Ker}(Cl_{n_i} \to Cl_j)\}$ for $0 < j \leq n_0$ as in [H04, Proposition 2.7]. Define $\Xi = \Xi_{\underline{n},j} = \bigsqcup_i \Xi_j^{n_i} \subset \mathcal{V}$. Since Cl_{n_i} and Cl_j is stable under the action of $\mathcal{T}_j$ and the projection $Cl_{n_i} \to Cl_j$ is compatible with the action of $\mathcal{T}_j$, Ξ_{n_i} is stable under $\mathcal{T}_j$, and hence $\Xi_{\underline{n},j}$ is also stable under $\mathcal{T}_j$. Thus we get

Theorem 2.11. *Choose $0 < m \in \mathbb{Z}$ so that $\mathfrak{l}^m$ is principal generated by $\varpi = \varphi\varphi^c$ with $\varphi \in R$ and define α as in* Lemma 2.10. *Suppose $\alpha \operatorname{diag}[1, \varpi_{\mathfrak{l}}]^{-m} \in K$. If $\underline{n}$ is an infinite arithmetic progression* (*with initial value n_0 and difference m*)*, the semi-group $\mathbf{T}_{j,m}$ generated by the group $\mathcal{T}_j$ in* (2.8) *and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ acts transitively on $\Xi_{\underline{n},j}$ and satisfies* (T), (N) *and* (F) (*for $\mathcal{T} = \mathcal{T}_j$*).

Theorem 2.7 combined with this result, Corollary 2.8 and [H10, Corollary 3.19, Theorem 3.20] gives

Corollary 2.12. *If $\underline{n}$ contains an arithmetic progression, then $\Xi_{\underline{n},j}$ for any $j \geq r$ is Zariski dense in $V^{\mathcal{Q}}$.*

2.5. **Characteristic 0 version.** We consider $Sh^{(p)}_{/\mathcal{W}}$ and its geometric irreducible component $V_{/\mathcal{W}}$ and define $\mathcal{V} = V^{\mathcal{Q}}_{/\mathcal{W}}$ in the same manner as above. Consider $\mathcal{V}_{/\mathbb{F}} = \mathcal{V}_{/\mathcal{W}} \otimes_{\mathcal{W}} \mathbb{F}$. Note that $V_{/\mathcal{W}}$ is smooth over $\mathcal{W}$ (see [PAF, Theorem 7.1]).

Lemma 2.13. *Let A be a smooth $\mathcal{W}$-domain and Ξ be a countable set of $\mathcal{W}$-points of* $\operatorname{Spec}(A)$ *and as a subscheme of* $\operatorname{Spec}(A)$*, Ξ is étale over $\mathcal{W}$. Write $\overline{X} := X \otimes_{\mathcal{W}} \mathbb{F}$ for $X = A, A_i, \Xi$ as a subscheme of* $\operatorname{Spec}(\overline{A})$*. Then if $\overline{\Xi}$ is Zariski dense in* $\operatorname{Spec}(\overline{A})$*, then the schematic closure of Ξ in* $\operatorname{Spec}(A)$ *is equal to* $\operatorname{Spec}(A)$ *and $\Xi_\eta = \Xi \times_{\mathcal{W}} \eta$ for the generic point $\eta \in \operatorname{Spec}(A)$ is Zariski dense in* $\operatorname{Spec}(A) \times_{\mathcal{W}} \eta$.

Order $\Xi = \{P_1, P_2, \ldots\}$ with $\Xi_n := \{P_1, \ldots, P_n\}$. Write $\widehat{X} := \varprojlim_n X/\mathfrak{m}_{\mathcal{W}}^n X$ for $X = A, A_i, \Xi_i, P_i$ (the formal completion along the special fiber).

Proof. Since A is smooth over $\mathcal{W}$, $\widehat{A}$ (resp. $\overline{A}$) is smooth over W (resp. $\mathbb{F}$); in particular, $\overline{A}$ is a domain. Since Ξ_n is étale over $\mathcal{W}$; so, is $\widehat{\Xi}_n$ over W. Thus $\Xi_n \cong \widehat{\Xi}_n \cong \overline{\Xi}_n$ as point sets; hence $\Xi \cong \overline{\Xi}$ as sets.

We have $A/\bigcap_{j=1}^n P_j \hookrightarrow \prod_j A/P_j = \prod_{j=1}^n \mathcal{W}$. Thus $A/\bigcap_{j=1}^n P_j$ is $\mathcal{W}$-flat. In the same manner, $\widehat{A}/\bigcap_{j=1}^n \widehat{P}_j$ is W-flat. We have a short exact sequence $(\bigcap_{j=1}^n \widehat{P}_j)\otimes_W \mathbb{F} \hookrightarrow \overline{A} \twoheadrightarrow (\widehat{A}/\bigcap_{j=1}^n \widehat{P}_j)\otimes_W \mathbb{F}$. For an $\widehat{A}$-ideal $\mathfrak{a}$ with W-flat quotient $\widehat{A}/\mathfrak{a}$, we have an exact sequence $\overline{\mathfrak{a}} = \mathfrak{a}\otimes_W \mathbb{F} \hookrightarrow \overline{A} \twoheadrightarrow (A/\mathfrak{a})\otimes_W \mathbb{F}$. We identify $\overline{\mathfrak{a}} = \mathfrak{a}\otimes_W \mathbb{F}$ as an ideal of $\overline{A}$ and $\overline{A}/\overline{\mathfrak{a}}$ with $(A/\mathfrak{a})\otimes_W \mathbb{F}$. Take another ideal $\mathfrak{b}$ with W-flat $\widehat{A}/\mathfrak{b}$. Then $\widehat{A}/(\mathfrak{a}\cap\mathfrak{b}) \hookrightarrow \widehat{A}/\mathfrak{b}\oplus\widehat{A}/\mathfrak{b}$ implies $\widehat{A}/(\mathfrak{a}\cap\mathfrak{b})$ is W-flat. From the short exact sequence: $\widehat{A}/(\mathfrak{a}\cap\mathfrak{b}) \hookrightarrow \widehat{A}/\mathfrak{a}\oplus\widehat{A}/\mathfrak{b} \twoheadrightarrow \widehat{A}/(\mathfrak{a}+\mathfrak{b})$ for the two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we obtain a three term exact sequence $\overline{A}/\overline{\mathfrak{a}\cap\mathfrak{b}} = (\widehat{A}/\mathfrak{a}\cap\mathfrak{b})\otimes_W \mathbb{F} \xrightarrow{\overline{i}} \overline{A}/\overline{\mathfrak{a}}\oplus\overline{A}/\overline{\mathfrak{b}} \twoheadrightarrow \overline{A}/(\overline{\mathfrak{a}}+\overline{\mathfrak{b}})$. Thus $\mathrm{Im}(\overline{i}) \cong \overline{A}/(\overline{\mathfrak{a}}\cap\overline{\mathfrak{b}})$ and $\mathrm{Coker}(\overline{i}) \cong \overline{A}/(\overline{\mathfrak{a}}+\overline{\mathfrak{b}})$ which implies $\overline{\mathfrak{a}\cap\mathfrak{b}} \subset \overline{\mathfrak{a}}\cap\overline{\mathfrak{b}}$. By induction on n, we thus have $\overline{\bigcap_{j=1}^n P_j} \subset \bigcap_{j=1}^n \overline{P}_j$ and hence $(\bigcap_{P\in\Xi}\widehat{P})\otimes_W \mathbb{F} = \overline{\bigcap_{P\in\Xi} P} \subset \bigcap_{P\in\Xi}\overline{P}_j$, whose right-hand-side is (0) by Zariski-density. For $\mathbf{P} := \bigcap_{P\in\Xi}\widehat{P}$, we have $\widehat{\mathbf{P}}\otimes_W \mathbb{F} = \mathbf{P}\otimes_W \mathbb{F}$. Therefore we conclude $\widehat{\mathbf{P}}\otimes_W \mathbb{F} = (0)$. By Nakayama's lemma for adically complete modules over a complete ring (e.g., [CAG, Exercise 7.2]), we conclude $\mathbf{P} = (0)$. Since $\bigcap_{P\in\Xi} P \subset \bigcap_{P\in\Xi}\widehat{P} \subset \mathbf{P} = (0)$, we conclude $\bigcap_{P\in\Xi} P = 0$. Thus Ξ is schematically dense in $\mathrm{Spec}(A)$. Since $K = \mathrm{Frac}(\mathcal{W})$ is flat over $\mathcal{W}$, we have $\bigcap_{P\in\Xi}(P\otimes_{\mathcal{W}} K) = (\bigcap_{P\in\Xi} P)\otimes_{\mathcal{W}} K = 0$; so, $\Xi\otimes_{\mathcal{W}}\eta$ is Zariski dense in $\mathrm{Spec}(A)\times_{\mathcal{W}}\eta = \mathrm{Spec}(A\otimes_{\mathcal{W}} K)$. □

The definition of $\Xi \subset \mathcal{V}$ in Theorem 0.1 works well over $\mathcal{W}$; so, we take a geometrically irreducible component V of $Sh^{(p)}_{/\mathcal{W}}$ with $x(R_n) \in V(\mathcal{W})$ for sufficiently large n and define $\mathcal{V} = V^{\mathcal{Q}}$ and $\Xi \subset \mathcal{V}$ as in Theorem 0.1.

Proposition 2.14. *Assume $\Xi\otimes_{\mathcal{W}}\mathbb{F}$ is Zariski dense in $\mathcal{V}\otimes_{\mathcal{W}}\mathbb{F}$. Then $\Xi\otimes\eta$ is Zariski dense in the generic fiber $\mathcal{V}\otimes_{\mathcal{W}}\eta$.*

Proof. Since $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is affine, covering $\mathcal{V}_K$ by open affine schemes $\mathrm{Spec}(A_{K,i})$ and pulling them back to $\mathrm{Spec}(A_{S,i}) \subset \mathcal{V}_S$ for open subgroups $S \subset K$, we apply Lemma 2.13 to $\mathrm{Spec}(A_i) \subset \mathcal{V}$ for $A_i = \lim_S A_{S,i}$ assuming Zariski density of Ξ in the special fiber $\mathcal{V}\otimes_{\mathcal{W}}\mathbb{F}$ and conclude Zariski density in the generic fiber. □

3. Geometric modular forms and CM points

The Hilbert modular Shimura variety $Sh^{(p)}$ is the moduli (up to prime-to-p O-linear isogeny) of triples $(X, \overline{\Lambda}, \eta)$ for an abelian variety X of dimension $d = [F:\mathbb{Q}]$ with multiplication by O, an O-linear polarization class $\overline{\Lambda}$ up to multiplication by $(O_{(p)+})^\times$ (see [H04, §2.2]) and an O-linear level structure $\eta : V^{(p)}(X) = \mathcal{T}(X)\otimes_{\widehat{\mathbb{Z}}}\mathbb{A}^{(p)} \cong (F^{(p)}_{\mathbb{A}})^2$ for the Tate module $\mathcal{T}(X)$ of X. For the Hilbert modular Shimura variety $Sh^{(p)}$, we use the definition and notation introduced in [H04, Section 2]. See also [HMI, Section 4.3] for a more detailed description of the Shimura variety and modular forms. Geometric modular forms can be defined as global sections of weight κ Hodge bundles over the Shimura variety, or equivalently a functorial rule assigning a value to classified abelian varieties with extra structure. Out of the assigned value at CM points, we create a distribution interpolating L-values in the next section Section 4.

3.1. **CM points** $x(\mathcal{A})$**.** We recall the definition of the CM points $x(\mathcal{A})$ from [H04]. Let $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)$ (so, $G(A) = \mathrm{GL}_2(A\otimes_{\mathbb{Q}} F)$). We write the left action: $G(\mathbb{A}^{(p\infty)})\times Sh^{(p)} \to Sh^{(p)}$ simply as $(g,x)\mapsto g(x) := \tau(g)^{-1}(x)$. Here the action of $\tau(g)$ is a right action induced by $\eta\mapsto\eta\circ g$ for the level structure . For each point $x = (X,\overline{\Lambda},\eta)\in Sh$, we can associate a lattice $\widehat{L} = \eta^{-1}(\mathcal{T}(X)) \subset (F^{(\infty)}_{\mathbb{A}})^2$. Then the level structure η is determined by the choice of a base $w = (w_1, w_2)$ of $\widehat{L}$ over $\widehat{O}$. In view of the base w, the inverted action $x\mapsto g(x)$ is matrix multiplication: ${}^t w \mapsto g^t w$, because $(\eta\circ g^{-1})^{-1}(\mathcal{T}(X)) = g\eta^{-1}(\mathcal{T}(X)) = g\widehat{L}$.

For each O-lattice $\mathcal{A}$, we recall a description of a CM point $x(\mathcal{A}) = (X(\mathcal{A}),\Lambda(\mathcal{A}),\eta(\mathcal{A}))\in Sh^{(p)}$ from [H04, §2.1], where $X(\mathcal{A})_{/\mathcal{W}}$ is an abelian scheme of CM-type (M,Σ) with $H^1(X(\mathcal{A})(\mathbb{C}),\mathbb{Z}) = \mathcal{A}$ in the sense $X(\mathcal{A})(\mathbb{C}) = \mathbb{C}^\Sigma/\mathcal{A}^\Sigma$ for $\mathcal{A}^\Sigma = \{(a^{\sigma_1},\dots,a^{\sigma_d})\in\mathbb{C}^\Sigma | a\in\mathcal{A}\}$ writing $\Sigma = \{\sigma_1,\dots,\sigma_d\}$. For the order $R_{\mathcal{A}} := \{\alpha\in M|\alpha\mathcal{A}\subset\mathcal{A}\}$ and an ideal $\mathfrak{a}$ of $R_{\mathcal{A}}$, we write, as a finite flat group scheme over $\mathcal{W}$,

$$X(\mathcal{A})[\mathfrak{a}] := \{x\in X(\mathcal{A})|\mathfrak{a}x = 0\} = \bigcap_{\alpha\in\mathfrak{a}}\mathrm{Ker}(\alpha : X(\mathcal{A})\to X(\mathcal{A})).$$

It is well known that there is an O-ideal $\mathfrak{f}(\mathcal{A})$ such that $R_{\mathcal{A}} = O + \mathfrak{f}(\mathcal{A})R$. The ideal $\mathfrak{f}(\mathcal{A})$ is called the conductor of $R_{\mathcal{A}}$ (and $\mathcal{A}$).

Recall the order $R_m = O + \mathfrak{l}^m R \subset M$ with conductor $\mathfrak{l}^m$, the groups $Cl^-_m = \mathrm{Coker}(\mathrm{Pic}(O) \to \mathrm{Pic}(R_m))$ and $Cl^-_\infty = \varprojlim_m Cl^-_m$. By class field theory, Cl^-_m gives the Galois group of the maximal anticyclotomic class field in the ring class field of conductor $\mathfrak{l}^m$ over M. The ideal $\mathfrak{l}_m = \mathfrak{l} + \mathfrak{l}^m R = \mathfrak{l}R_{m-1}$ is a prime ideal of R_m but is not proper (it is a proper ideal of R_{m-1}). Since $X(R_m)[\mathfrak{l}_m] \cong R_m/\mathfrak{l}_m = O/\mathfrak{l}$ and $\mathfrak{l}_m R_{m-1} \subset R_m$, we find that $X(R_m)[\mathfrak{l}_m] = R_{m-1}/R_m$ and $X(R_m)/X(R_m)[\mathfrak{l}_m] \cong X(R_{m-1})$. We pick a subgroup $C \subset X(R_m)[\mathfrak{l}]$ isomorphic to $O/\mathfrak{l}$ but different from $X(R_m)[\mathfrak{l}_m]$. We look into the quotient $X(R_m)/C$. Take a lattice $\mathfrak{A}$ so that $X(R_m)/C = X(\mathfrak{A}) \Leftrightarrow \mathfrak{A}/R_m = C$. Since C is an O–submodule, $\mathfrak{A}$ is an O–lattice of M. Since $\mathfrak{l}C = 0$, we find $\mathfrak{l}R_m\mathfrak{A} \subset \mathfrak{A}$. Thus $\mathfrak{A}$ is R_{m+1}–ideal, because $R_{m+1} = O + \mathfrak{l}R_m$. Since C is not an R_m–submodule, the ideal $\mathfrak{A}$ is not R_m–ideal; so, it is a proper R_{m+1}–ideal. Since C generates over R_m all $\mathfrak{l}$–torsion points of $X(R_m)$, we find $R_m\mathfrak{A} = \mathfrak{l}^{-1}R_m$. In this way, we have created ℓ proper R_{m+1}–ideals $\mathfrak{A}$ with $\mathfrak{A}R_m = \mathfrak{l}^{-1}R_m$.

We choose a base $w_0 = (w_1, w_2)$ of $\widehat{R}$ over $\widehat{O}$ in [H04, §2.1] with a specific choice at p and $\mathfrak{l}$: at p, for the choice of the ordinary p-adic CM-type $\Sigma = \{\mathfrak{P}|p\}$, writing $R_\Sigma = \prod_{\mathfrak{P}\in\Sigma} R_{\mathfrak{P}}$ and $R_{\Sigma^c} = \prod_{\mathfrak{P}\in\Sigma} R_{\mathfrak{P}^c}$ for complex conjugation c, $R_p = R_{\Sigma^c} \oplus R_\Sigma$ and $w_{0,p} = ((1,0),(0,1))$.

For an R_m-proper ideal $\mathcal{A}$ prime to $p\mathfrak{l}$, we choose a level structure $\eta(\mathcal{A})$ of $X(\mathcal{A})$ with $\eta(\mathcal{A})(\widehat{O}^2) = \widehat{\mathcal{A}}$ in the following way. We are going to specify the base w_0 of $\widehat{R}$ now at $\mathfrak{l}$. So the base of $\widehat{R}^{(\mathfrak{l})}_m$ to be the prime-to-p part $w^{(\mathfrak{l})}_0$ as above, because $\widehat{R}^{(\mathfrak{l})}_m = \widehat{R}^{(\mathfrak{l})}$. To specify the base $w_{0,\mathfrak{l}}$ of $R_\mathfrak{l}$, we take $d \in O_\mathfrak{l}$ so that $R_\mathfrak{l} = O_\mathfrak{l}[\sqrt{d}] \subset M_\mathfrak{l}$. We assume that d is a $\mathfrak{l}$–adic unit if $\mathfrak{l}$ is unramified in M/F and d generates $\mathfrak{l}O_\mathfrak{l}$ if $\mathfrak{l}$ ramifies in M/F. Then we choose $w_{0,\mathfrak{l}} = (1, \sqrt{d})$ and put $w_0 := (w_{0,\mathfrak{l}}, w^{(\mathfrak{l})}_0)$. We use the base $w := w_0 \cdot g$ of $\widehat{R}_m$ for a suitable $g = g(\mathcal{A}) \in \mathrm{GL}_2(F_\mathfrak{l})$ to define the level structure for an R_m-proper ideal $\mathcal{A}$.

First we choose a representative set $\{\mathfrak{A}_j\}$ of ideal classes of M (prime to $p\mathfrak{l}$). Then we can write $\widehat{\mathfrak{A}}_j = a_j\widehat{R}$ for an idele a_j with $a_j = a_j^{(\mathfrak{l}p\infty)}$ and choose $\alpha \in M$ so that $\mathcal{A}R = \alpha\mathfrak{A}_j$ for a fractional R-ideal $\mathcal{A}$. Here for an idele $a \in F^\times_{\mathbb{A}}$ (resp. an adele $a \in F_{\mathbb{A}}$) and an integral ideal $\mathfrak{a}$, $a^{(\mathfrak{a}\infty)}$ indicates $a_v = 1$ (resp. $a_v = 0$) for each place v appearing in $\mathfrak{a}$ or ∞. We define the level structure $\eta(\mathcal{A})$ by $(F^{(\infty)}_{\mathbb{A}})^2 \ni (a,b) \mapsto a\alpha a_j w_1 + b\alpha a_j w_2 \in M^{(\infty)}_{\mathbb{A}} = V(X(\mathcal{A}))$.

Here are the choices of g. When $m = 0$, $g = 1$ (the identity matrix). When $m > 0$, we first suppose that $\mathfrak{l}^m = (\varphi\varphi^c)$ for $\varphi \in M$. Our choice of g is $\mathrm{diag}[1, \varphi\varphi^c]\varrho(u)$ for a suitable $u \in O_\mathfrak{l}$ so that the $\mathfrak{l}$-component of wg gives the base of $\widehat{R}_m$. The element g is equal to $\alpha_m\varrho(u)$ modulo $\widehat{\Gamma}_0(\mathfrak{l}^m)$.

Suppose that $\mathfrak{l}^m$ is not generated by a norm from M. We choose $g = \alpha_m\varrho(u)$ with $u \in O_\mathfrak{l}$ so that $w(\mathcal{A}) = \alpha a_j w \cdot g$ gives a base over $\widehat{O}$ of $\widehat{\mathcal{A}}$, and define $\eta(\mathcal{A})$ by using $w(\mathcal{A})$. For a general $\mathcal{A}$ (not necessarily prime to $p\mathfrak{l}$), taking $\mathcal{A}_0$ in the same class of $\mathcal{A}$ prime to $p\mathfrak{l}$ such that $\mathcal{A} = \beta\mathcal{A}_0$ for $\beta \in M^\times$, we replace the basis w by βw and define $\eta(\mathcal{A})$. There is an ambiguity of the choice of α, β and φ up to units in R, but this does not cause any trouble as what really matter is the embedding $\rho_{\mathcal{A}} : M^\times_{\mathbb{A}} \to G(\mathbb{A}^{(\infty)})$ given by $\alpha\eta(\mathcal{A}) = \eta(\mathcal{A}) \cdot \rho_{\mathcal{A}}(\alpha)$, which is independent of the choice of α, β and φ.

By our choice, we have $\rho_{\mathcal{A}} = \rho_R$ on $M^{(\mathfrak{l})\times}_{\mathbb{A}}$, and

$$\det(g(\mathcal{A})) \in F^\times_+ \text{ if } \mathfrak{l}^m \text{ is generated by a norm from } M. \tag{3.1}$$

Regarding Σ as a set of p-adic places (i.e., field embeddings of M into $\overline{\mathbb{Q}}_p$) and composing with $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ we fixed, we may regard Σ as a set of complex embeddings. We write $\Sigma(\mathcal{A}) := \{(\sigma(a))_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma | a \in \mathcal{A}\}$ as a lattice in $\mathbb{C}^\Sigma := \prod_{\sigma\in\Sigma}\mathbb{C}$.

We choose a totally imaginary $\delta \in M$ with $\mathrm{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$. Then the alternating form $(a,b) \mapsto (c(a)b - ac(b))/2\delta$ gives an identity $R \wedge_O R = \mathfrak{c}^*$ for a fractional ideal $\mathfrak{c}$ of F. Here $\mathfrak{c}^* = \{x \in F | \mathrm{Tr}_{F/\mathbb{Q}}(x\mathfrak{c}) \subset \mathbb{Z}\} = \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ for the different $\mathfrak{d}$ of $F/\mathbb{Q}$. Identifying $M \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^\Sigma$ by $m \otimes r \mapsto (\sigma(m)r)_{\sigma\in\Sigma}$, we find that $(a, \sqrt{-1}a) = \frac{\sqrt{-1}}{\delta}a\overline{a} \gg 0$ for $a \in M^\times$. Here the symbol "$\gg$" means total positivity. Thus $\mathrm{Tr}_{F/\mathbb{Q}} \circ (\cdot,\cdot)$ gives a Riemann form for the lattice $\Sigma(\mathcal{A})$, and therefore, a projective embedding of $\mathbb{C}^\Sigma/\Sigma(R)$ onto a projective abelian variety $X(\mathcal{A})_{/\mathbb{C}}$. The complex abelian scheme $X(\mathcal{A})$ extends to an abelian scheme over $\mathcal{W}$ (unique up to isomorphisms). In this way, we get a $\mathfrak{c}$–polarization $\Lambda(\mathcal{A}) : X(\mathcal{A})(\mathbb{C}) \otimes \mathfrak{c} \cong {}^tX(\mathcal{A})(\mathbb{C})$ for the dual abelian scheme ${}^tX(\mathcal{A}) = \mathrm{Pic}^0_{X(\mathcal{A})/\mathcal{W}}$.

For an order $\mathcal{R}$ of conductor $\mathfrak{f}$ and a proper $\mathcal{R}$-ideal $\mathcal{A}$,

$$\mathcal{R}\wedge\mathcal{R}=\mathfrak{f}(O\wedge R)+\mathfrak{f}^2(R\wedge R)=(\mathfrak{f}^{-1}\mathfrak{c})^* \text{ and } \mathcal{A}\wedge\mathcal{A}=(N_{M/F}(\mathcal{A})^{-1}\mathfrak{f}^{-1}\mathfrak{c})^*,$$

where the exterior product is taken over O. Hereafter we fix δ so that $\mathfrak{c}$ is prime to $p\mathfrak{f}(\mathcal{A})\mathfrak{d}$, and write $\mathfrak{c}(\mathcal{A})$ for $N_{M/F}(\mathcal{A})^{-1}\mathfrak{f}(\mathcal{A})^{-1}\mathfrak{c}$ (so, $\mathfrak{c}=\mathfrak{c}(R)$). We can always choose such a δ, since in this paper we only treat $\mathcal{A}$ with $\mathfrak{l}$–power conductor.

Since an isogeny defined over the field of fractions $\mathrm{Frac}(\mathcal{W})$ of $\mathcal{W}$ between abelian schemes over $\mathcal{W}$ extends to the entire abelian scheme (e.g. [GME] Lemma 4.1.16), we have a well defined $\mathfrak{c}(\mathcal{A})$–polarization $\Lambda(\mathcal{A}):X(\mathcal{A})\otimes\mathfrak{c}(\mathcal{A})\cong{}^tX(\mathcal{A})$. Replacing $X(\mathcal{A})$ by an isomorphic $X(\alpha\mathcal{A})$ for $\alpha\in M$, we may assume that $\mathcal{A}_p=R_p$. Then

$$X(\mathcal{A})[\mathfrak{p}_F]=X(\mathcal{A})[\mathfrak{p}]\oplus X(\mathcal{A})[\mathfrak{p}^c]$$

for $\mathfrak{p}_F=\mathfrak{p}\cap F$ is isomorphic by $\Lambda(\mathcal{A})$ to its Cartier dual. Since the Rosati-involution $a\mapsto a^*=\Lambda(\mathcal{A})\circ{}^ta\circ\Lambda(\mathcal{A})^{-1}$ is the complex conjugation c, $X(\mathcal{A})[\mathfrak{p}]_{/\mathcal{W}}$ is multiplicative (étale locally) if and only if $X(\mathcal{A})[\mathfrak{p}^c]$ is étale over $\mathcal{W}$.

Since the base of $R_{n,\mathfrak{l}}$ is given by $\alpha_n{}^t(1,\sqrt{d})$ for $\alpha_n=\left(\begin{smallmatrix}1&0\\0&\varpi_\mathfrak{l}^n\end{smallmatrix}\right)$ with a prime element $\varpi_\mathfrak{l}$ of $O_\mathfrak{l}$, we find that $\alpha_n(x(R))=x(R_n)$ and $\alpha_1(x(R_{n-1}))=x(R_n)$. Moreover, for a suitable $u\in O$

$$\varpi_\mathfrak{l}(x(\mathcal{A}))=\begin{pmatrix}1&\frac{u}{\varpi_\mathfrak{l}}\\0&1\end{pmatrix}(x(R_{n+1})) \text{ if } \mathcal{A}=R_n/C \text{ for } O/\mathfrak{l}\cong C\neq X(R_n)[\mathfrak{l}_n], \tag{3.2}$$

because the base of $\varpi_\mathfrak{l}\mathcal{A}_\mathfrak{l}$ is given by $\begin{pmatrix}1+\varpi_\mathfrak{l}^n u\sqrt{d}\\ \varpi_\mathfrak{l}^{n+1}\sqrt{d}\end{pmatrix}=\begin{pmatrix}1&\frac{u}{\varpi_\mathfrak{l}}\\0&1\end{pmatrix}\alpha_{n+1}\begin{pmatrix}1\\ \sqrt{d}\end{pmatrix}$. Here the action of $\varpi_\mathfrak{l}: x(\mathcal{A})\mapsto\varpi_\mathfrak{l}(x(\mathcal{A}))$ may bring $x(\mathcal{A})$ on a geometrically irreducible component of $Sh^{(p)}$ to a different one.

Now we consider $x(\mathcal{A})$ in V_K for an open subgroup $K\subset G(\mathbb{A}^{(\infty)})$ containing $\widehat{O}^\times$. By repeating (3.2), if $x(\mathcal{A})=x(R_n)/C$ for $C\cong O/\mathfrak{l}^m$ with $C\cap X(R_n)[\mathfrak{l}_n]=\{0\}$, then $\mathcal{A}$ is a proper R_{n+m}–ideal. If further $\mathfrak{l}^m$ is generated by an element $\varpi\in F$, we get $x(\mathcal{A})=x(\varpi\mathcal{A})=\varpi_\mathfrak{l}^m(x(\mathcal{A}))$ in V_K (because $\varpi/\varpi_\mathfrak{l}^m\in K$) and

$$x(\mathcal{A})=\begin{pmatrix}1&\frac{u}{\varpi_\mathfrak{l}^m}\\0&1\end{pmatrix}(x(R_{n+m}))=\left(\begin{smallmatrix}1&\frac{u}{\varpi}\\0&1\end{smallmatrix}\right)(x(R_{n+m})) \text{ for a suitable } u\in O. \tag{3.3}$$

The set $\{x(\mathcal{A})|[\mathcal{A}R_n]=[\mathfrak{A}]\}$ for $\mathcal{A}\in Cl_{n+m}$ running through ideal classes $\mathcal{A}$ projecting down to a given ideal class $[\mathfrak{A}]\in Cl_n$ is in bijection with $O/\mathfrak{l}^m$ by associating u to $\mathcal{A}$ in (3.3) (see [H04, Proposition 4.2]).

3.2. **Geometric modular forms.** Let k be a weight of $T=\mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$, that is, $k:T(A)=(A\otimes_\mathbb{Z}O)^\times\to A^\times$ is a homomorphism given by $(a\otimes\xi)^k=\prod(a\xi^\sigma)^{k_\sigma}$ for integers k_σ indexed by field embeddings $\sigma:F\hookrightarrow\overline{\mathbb{Q}}$. Let B be a base ring, which is a $\mathcal{W}$–algebra. We consider quadruples $(X,\overline{\Lambda},\eta^{(p)},\omega)_{/A}$ for a B–algebra A with a differential ω generating $H^0(X,\Omega_{X/A})$ over $A\otimes_\mathbb{Z}O$. We impose the following condition:

$$\eta^{(p)}(\widehat{L}_\mathfrak{c}^{(p)})=\mathcal{T}(X)\otimes_\mathbb{Z}\widehat{\mathbb{Z}}^{(p)} \text{ for } L_\mathfrak{c}=O\oplus\mathfrak{c}^* \text{ with a fixed } \mathfrak{c}. \tag{3.4}$$

Under this condition, as seen in [H04, §2.3] and [HMI, §4.3.1], the classification up to prime-to–p isogenies of the quadruples is equivalent to the classification up to isomorphisms. A modular form f (integral over B) of weight k is a functorial rule of assigning a value $f(X,\overline{\Lambda},\eta^{(p)},\omega)\in A$ to (the A–isomorphism class of) each quadruple $(X,\overline{\Lambda},\eta^{(p)},\omega)_{/A}$ (called a *test object*) defined over a B–algebra A. Here Λ is a $\mathfrak{c}$–polarization which (combined with $\eta^{(p)}$) induces $L_\mathfrak{c}\wedge L_\mathfrak{c}\cong\mathfrak{c}^*$ given by $((a\oplus b),(a'\oplus b'))\mapsto ab'-a'b$. The Tate test object at the cusp $(\mathfrak{a},\mathfrak{b})$ for two fractional ideals with $\mathfrak{a}^*\mathfrak{b}=\mathfrak{c}^*$ is an example of such test objects. The Tate semi-AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is defined over $\mathbb{Z}[[q^\xi]]_{\xi\in(\mathfrak{a}\mathfrak{b})_+}$ and is given by the algebraization of the formal quotient $(\widehat{\mathbb{G}}_m\otimes\mathfrak{a}^*)/q^\mathfrak{b}$ (see [HMI, §4.2.5] for details of this construction). The rule f is supposed to satisfy the following three axioms:

(G1) For a B–algebra homomorphism $\phi:A\to A'$, we have

$$f((X,\overline{\Lambda},\eta^{(p)},\omega)\times_{A,\phi}A')=\phi(f(X,\overline{\Lambda},\eta^{(p)},\omega)).$$

(G2) f is finite at all cusps, that is, the q–expansion of f at every Tate test object does not have a pole at $q=0$.

(G3) $f(X,\overline{\Lambda},\eta^{(p)},\xi\omega)=\xi^{-k}f(X,\overline{\Lambda},\eta^{(p)},\omega)$ for $\xi\in T(A)$.

We write $G_k(\mathfrak{c};B)$ for the space of all modular forms f satisfying (G1-3) for B–algebras A. Since we do not place any level p-structure, the modular forms we consider has level prime to p (i.e., the level subgroup at p is $\mathrm{GL}(\widehat{L}_{\mathfrak{c},p}) = \mathrm{GL}_2(O_p)$). We put

$$G_k(B) = \bigoplus_{\mathfrak{c}} G_k(\mathfrak{c};B), \tag{3.5}$$

where $\mathfrak{c}$ prime to ℓp runs over a representative set of strict ideal classes of F.

An element $g \in G(\mathbb{A}^{(p\infty)})$ fixing $\widehat{L}_{\mathfrak{c}}^{(p)}$ acts on $f \in G_k(\mathfrak{c};B)$ by

$$f|g(X,\overline{\Lambda},\eta^{(p)},\omega) = f(X,\overline{\Lambda},\eta^{(p)}\circ g,\omega).$$

For a closed subgroup $K^{(p)} \subset K_{\mathfrak{c}}^{(p)} = \mathrm{GL}(\widehat{L}_{\mathfrak{c}}^{(p)}) \cap G_1(\mathbb{A}^{(p\infty)})$, setting $K = K^{(p)} \times \mathrm{GL}_2(O_p) \subset \mathrm{GL}_2(\mathbb{A}^{(\infty)})$, we write $G_k(\mathfrak{c};K;B)$ for the space of all $K^{(p)}$–invariant modular forms; thus,

$$G_k(\mathfrak{c};K;B) = H^0(K^{(p)}, G_k(\mathfrak{c};B)).$$

Since elements of $G_k(\mathfrak{c};B)$ is of level prime to p, the notation $G_k(\mathfrak{c};K;B)$ is legitimate. Take an O–ideal $\mathfrak{N}$ prime to $p\mathfrak{c}$. Then the $\mathfrak{N}$–component of $K_{\mathfrak{c}}$ is $\mathrm{GL}_2(O_{\mathfrak{N}})$. Let

$$\Gamma_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(O_{\mathfrak{N}}) \middle| c \in \mathfrak{N}O_{\mathfrak{N}} \right\} \quad \text{and} \quad \Gamma_1(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N}) \middle| a \equiv d \equiv 1 \mod \mathfrak{N}O_{\mathfrak{N}} \right\}.$$

Assume that $\mathfrak{N}$ is prime to $p\mathfrak{l}$ and define for an open subgroup $K_{\mathfrak{l}\mathfrak{N}} \subset \mathrm{SL}_2(O_{\mathfrak{l}\mathfrak{N}})$

$$G_k(K_{\mathfrak{l}\mathfrak{N}};B) = \bigoplus_{\mathfrak{c}} G_k(\mathfrak{c};K_{\mathfrak{l}\mathfrak{N}} \times K_{\mathfrak{c}}^{(\mathfrak{l}\mathfrak{N})};B).$$

A W–algebra B is called a p–adic algebra if $B = \varprojlim_n B/p^nB$. We write η_{ord} for the pair of level structures $(\eta_p^{ord} : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \hookrightarrow X[p^\infty], \eta^{(p)})$. A p–adic modular form f over a p–adic W–algebra B is a functorial rule of assigning a value in A to triples $(X,\overline{\Lambda},\eta_{ord})_{/A}$ with $\mathfrak{c}$–polarization class $\overline{\Lambda}$ satisfying an obvious version of (G1-2) for p–adic B–algebras A (not just B–algebras). In general, we do not impose (G3) on p–adic modular forms. See [HMI, §4.2.8] for more details about p-adic modular forms. We write $V(\mathfrak{c};B)$ for the space of p–adic modular forms defined over B. We again define

$$V(B) = \bigoplus_{\mathfrak{c}} V(\mathfrak{c};B) \quad \text{and} \quad V(K_{\mathfrak{l}\mathfrak{N}};B) = \bigoplus_{\mathfrak{c}} V(\mathfrak{c};K_{\mathfrak{l}\mathfrak{N}} \times K_{\mathfrak{c}}^{(p\mathfrak{l}\mathfrak{N})};B), \tag{3.6}$$

where $V(\mathfrak{c};K^{(p)};B) = H^0(K^{(p)}, V(\mathfrak{c};B))$ with $g \in K^{(p)}$ acting through $\eta^{(p)}$. For $f \in V(B)$, we write $f_{\mathfrak{c}} \in V(\mathfrak{c};B)$ for the $\mathfrak{c}$–component of f, and we say that f is of *level* $\mathfrak{N}$ if f in either in $G_k(K_{\mathfrak{l}\mathfrak{N}};B)$ or in $V(K_{\mathfrak{l}\mathfrak{N}};B)$ for $K_{\mathfrak{l}\mathfrak{N}} \subset \mathrm{GL}_2(O_{\mathfrak{l}\mathfrak{N}})$ with $K_{\mathfrak{N}} = \Gamma_0(\mathfrak{N})$ or $\Gamma_1(\mathfrak{N})$.

Since η_p^{ord} induces the identification $\widehat{\eta}_p^{ord} : \widehat{\mathbb{G}}_m \otimes O^* \cong \widehat{X}$ for the formal completion of X along the origin, by pushing forward the differential $\frac{dt}{t}$, we can associate $(X,\overline{\Lambda},\eta^{(p)},\widehat{\eta}_{p,*}^{ord}\frac{dt}{t})$ to a quadruple $(X,\overline{\Lambda},\eta_p^{ord},\eta^{(p)})$. In this way, any modular form f satisfying (G1-3) can be regarded as a p–adic modular form by

$$f(X,\overline{\Lambda},\eta_{ord}) = f(X,\overline{\Lambda},\eta^{(p)},\widehat{\eta}_{p,*}^{ord}\frac{dt}{t}). \tag{3.7}$$

By the q–expansion principle (cf. [HMI, Corollary 4.16] or [PAF, Corollary 4.23]), we have a canonical embedding of $G_k(B)$ into $V(B)$ which keeps the q–expansion. A p–adic modular form associated to a modular form in $G_k(B)$ satisfies the following replacement of (G3):

(g3) $f(X,\overline{\Lambda},\xi\cdot\eta_p^{ord},\eta^{(p)}) = \xi^{-k} f(X,\overline{\Lambda},\eta_p^{ord},\eta^{(p)})$ for $\xi \in O_p^\times$.

Although we do not impose the condition (G3) on p–adic modular forms f, we limit ourselves to the study of forms satisfying the following condition (G3′) in order to define the modified value $f([\mathcal{A}])$ later at CM points $x(\mathcal{A})$ truly independent of the choice of $\mathcal{A}$ in its proper ideal class. Here abusing our notation, $x(\mathcal{A})$ is the quadruple $(X(\mathcal{A}),\Lambda(\mathcal{A}),\eta_{ord}(\mathcal{A}),\omega(\mathcal{A}))_{/\mathcal{W}}$ introduced in [H04, §2.1]). We consider the torus $T_M = \mathrm{Res}_{R/\mathbb{Z}}\mathbb{G}_m$ and identify its character group $X^*(T_M)$ with the module $\mathbb{Z}[\Sigma \sqcup \Sigma c]$ of formal linear combinations of embeddings of M into $\overline{\mathbb{Q}}$. By the identity: $(X(\xi\mathcal{A}),\Lambda(\xi\mathcal{A}),\eta_{ord}(\xi\mathcal{A}) = \xi\eta_{ord}(\mathcal{A}))_{/\mathcal{W}} \cong (X(\mathcal{A}),\xi\xi^c\Lambda(\mathcal{A}),\eta_{ord}(\mathcal{A})\circ\rho_{\mathcal{A}}(\xi))_{/\mathcal{W}}$, we may assume that for $k,\kappa \in \mathbb{Z}[\Sigma]$,

(G3′) $f(x(\xi\mathcal{A})) = f(\rho_R(\xi^{(\mathfrak{l})})(x(\mathcal{A}))) = \xi^{-k-\kappa(1-c)} f(x(\mathcal{A}))$ for $\xi \in T_M(\mathbb{Z}_{(\ell)})$.

It is known that for the p–adic differential operator d_σ of Dwork-Katz ([K78] 2.5-6) corresponding to $\frac{1}{2\pi i}\frac{\partial}{\partial z_\sigma}$ for $\sigma \in \Sigma$, $\theta^\kappa f$ ($\theta^\kappa = \prod_\sigma d_\sigma^{\kappa_\sigma}$) satisfies (G3$'$) if $f \in G_k(B)$.

Remark 3.1. The recipe (3.1) of regarding a geometric modular form as a p-adic one works well even if we add an ordinary level p-structure $\phi_p : \mu_{p^n} \otimes_{\mathbb{Z}} O \hookrightarrow X$ to the quadruples $(X, \overline{\Lambda}, \eta^{(p)}, \omega)$ defining $G_k(\mathfrak{c}; B)$. By doing this, we can allow geometric modular forms of level $\Gamma_1(p^n)$ at p in the picture. However this is redundant as our result is valid for any $f \in V(\mathfrak{c}; \overline{\mathbb{F}}_p)$ which covers any p-integral classical modular forms level $\Gamma_1(p^n)$ at p.

3.3. **Hecke operators.** Suppose that the $\mathfrak{l}$–component $K_\mathfrak{l}$ of the level subgroup is equal to $\Gamma_0(\mathfrak{l}^\nu)$ ($\nu \geq 0$). Let $e_1 = {}^t(1,0), e_2 = {}^t(0,1)$ be the standard basis of $F^2 \otimes \mathbb{A}^{(p\infty)}$. Then, under (3.4), for each triple $(X, \overline{\Lambda}, \eta_{ord})_{/A}$ with $\eta_{ord} = \eta_p^{ord} \times \eta^{(p)}$,

$$C = \eta_\mathfrak{l}(\mathfrak{l}^{-\nu} O_\mathfrak{l} e_1 + O_\mathfrak{l} e_2)/\eta_\mathfrak{l}(O_\mathfrak{l}^2)$$

gives rise to an A–rational cyclic subgroup of X of order $\mathfrak{l}^\nu$, that is, a finite group subscheme defined over A of $X_{/A}$ isomorphic to $O/\mathfrak{l}^\nu$ étale locally. Since $\Gamma_0(\mathfrak{l}^\nu)$ fixes $(\mathfrak{l}^{-\nu} O_\mathfrak{l} e_1 + O_\mathfrak{l} e_2)/O_\mathfrak{l}^2$, the level $\Gamma_0(\mathfrak{l}^\nu)$ moduli problem is equivalent to the classification of quadruples $(X, \overline{\Lambda}, C, \overline{\eta}_{ord}^{(\mathfrak{l})})_{/A}$ for a subgroup C of order $\mathfrak{l}^\nu$ in X, where $\eta_{ord}^{(\mathfrak{l})}$ is the (p–ordinary) level structure outside $\mathfrak{l}$. Thus we may define for $f \in G_k(\Gamma_0(\mathfrak{l}^\nu \mathfrak{N}); B)$ the value of f at $(X, \overline{\Lambda}, C, \eta^{(p\mathfrak{l})}, \omega)$ by $f(X, \overline{\Lambda}, C, \eta^{(p\mathfrak{l})}, \omega) := f(X, \overline{\Lambda}, \eta^{(p)}, \omega)$. When f is a p–adic modular form, we replace the ingredient ω by the ordinary level structure η_p^{ord} in order to define the value $f(X, \overline{\Lambda}, C, \eta^{(p\mathfrak{l})}, \eta_p^{ord})$.

We shall define Hecke operators $T(1, \mathfrak{l}^n)$ and $U(\mathfrak{l}^n)$ over (p–adic) modular forms of level K (with $K_\mathfrak{l} = \Gamma_0(\mathfrak{l}^\nu)$). The operator $U(\mathfrak{l}^n)$ is defined when $\nu > 0$, and $T(1, \mathfrak{l}^n)$ is defined when $\nu = 0$. Since $\mathfrak{l}$ is prime to p (and B is a $\mathcal{W}$–algebra), any cyclic subgroup C' of X of order $\mathfrak{l}^n$ is isomorphic to $O/\mathfrak{l}^n$ étale locally. We make the quotient $\pi : X \twoheadrightarrow X/C'$, and Λ, η_p^{ord} and ω induce canonically a polarization $\pi_*\Lambda$, a canonical level structure $\pi_*\eta_p^{ord} = \pi \circ \eta_p^{ord}$, $\pi_*\eta^{(p\mathfrak{l})} = \pi \circ \eta^{(p\mathfrak{l})}$ and a differential $(\pi^*)^{-1}\omega$ on X/C'. If $C' \cap C = \{0\}$ for the $\Gamma_0(\mathfrak{l}^\nu)$–structure C (in this case, we call that C' and C are disjoint), $\pi(C) = C + C'/C'$ gives rise to the level $\Gamma_0(\mathfrak{l}^\nu)$–structure on X/C'. We write $\underline{X/C'}$ for the new test object of the same level as the test object $\underline{X} = (X, \overline{\Lambda}, C, \eta_{ord}^{(\mathfrak{l})}, \omega)$ we started with. When f is p–adic, we suppose not to have ω in $\underline{X}$, and when f is classical, we ignore the ingredient η_p^{ord} in $\underline{X}$. Then we define (for $\nu > 0$)

$$f|U(\mathfrak{l}^n)(\underline{X}) = \frac{1}{N(\mathfrak{l}^n)} \sum_{C'} f(\underline{X/C'}), \tag{3.8}$$

where C' runs over all étale cyclic subgroups of order $\mathfrak{l}^n$ disjoint from C. The newly defined $f|U(\mathfrak{l}^n)$ is a modular form of the same level as f and $U(\mathfrak{l}^n) = U(\mathfrak{l})^n$. Since the polarization ideal class of X/C' is given by $\mathfrak{c}\mathfrak{l}^n$ for the polarization ideal class $\mathfrak{c}$ of X, the operators $U(\mathfrak{l}^n)$ permute the components $f_\mathfrak{c}$.

We recall some other isogeny actions on modular forms. For fractional ideals $\mathfrak{z}$ in F, we can think of the association $X \mapsto X \otimes_O \mathfrak{z}$ for each AVRM X. This operation will be made explicit in terms of the lattice $L = \pi_1(X)$ in $Lie(X)$. There are a natural polarization and a level structure on $X \otimes \mathfrak{z}$ induced by those of X. Writing $(X, \Lambda, \eta) \otimes \mathfrak{z}$ for the triple made out of (X, Λ, η) after tensoring $\mathfrak{z}$, we define $f|\langle \mathfrak{z} \rangle(X, \Lambda, \eta) = f((X, \Lambda, \eta) \otimes \mathfrak{z})$ (see [PAF, §4.1.9] for more details of this definition, though $\langle \mathfrak{z} \rangle$ here is $\langle \mathfrak{z}^{-1} \rangle$ in [PAF, §4.1.9]). For $X(\mathcal{A})$, we have $\langle \mathfrak{z} \rangle(\underline{X}(\mathcal{A})) = \underline{X}(\mathfrak{z}\mathcal{A})$.

(3.9) *The effect of* $\langle \mathfrak{z} \rangle$ *on the Fourier expansion at* $(\mathfrak{a}, \mathfrak{b})$ *is given by that at* $(\mathfrak{z}\mathfrak{a}, \mathfrak{z}^{-1}\mathfrak{b})$

(e.g., by [PAF, §4.2.9], noting $\langle \mathfrak{z} \rangle$ here is $\langle \mathfrak{z}^{-1} \rangle$ in [PAF]).

Let $\mathfrak{q}$ be a prime ideal of F outside $p\mathfrak{l}$. For a test object $(X, \overline{\Lambda}, C, \eta_{ord}^{(\mathfrak{q})}, \omega)$ of level $\Gamma_0(\mathfrak{q})$, we can construct canonically its image under $\mathfrak{q}$–isogeny:

$$[\mathfrak{q}](X, \overline{\Lambda}, C, \eta_{ord}^{(p\mathfrak{q})}, \omega) = (X', \overline{\Lambda}, \pi_*\eta_{ord}^{(p\mathfrak{q})}, \overline{\eta}_\mathfrak{q}, (\pi^*)^{-1}\omega)$$

for the projection $\pi : X \twoheadrightarrow X' = X/C$, where $\overline{\eta}_\mathfrak{q} = \eta_\mathfrak{q} \cdot \mathrm{GL}_2(O_\mathfrak{q})$ for any level $\mathfrak{q}$–structure $\eta_\mathfrak{q}$ identifying $\mathcal{T}_\mathfrak{q}(X')$ with $O_\mathfrak{q}^2$. Then Then we have a linear operator $[\mathfrak{q}] : V(\Gamma_1(\mathfrak{l}^\nu \mathfrak{N}); B) \to V(\Gamma_0(\mathfrak{q}\mathfrak{l}^\nu \mathfrak{N}); B)$ given by $f|[\mathfrak{q}](\underline{X}) = f([\mathfrak{q}](\underline{X}))$. See [H04, (4.14)] for the description of this operator in terms of the lattice of X.

If $\mathfrak{q}$ splits into $\mathfrak{Q}\overline{\mathfrak{Q}}$ in M/F, choosing $\eta_{\mathfrak{q}}$ induced by

$$X(\mathcal{A})[\mathfrak{q}^\infty] \cong M_{\mathfrak{Q}}/R_{\mathfrak{Q}} \times M_{\overline{\mathfrak{Q}}}/R_{\overline{\mathfrak{Q}}} \cong F_{\mathfrak{q}}/O_{\mathfrak{q}} \times F_{\mathfrak{q}}/O_{\mathfrak{q}},$$

we always have a canonical level $\mathfrak{q}$–structure on $X(\mathcal{A})$ dependent on the choice of the factor $\mathfrak{Q}$. Then $[\mathfrak{q}](X(\mathcal{A})) = X(\mathcal{A}[\mathfrak{Q}]^{-1})$ for $[\mathfrak{Q}] \in Cl_\infty$. When $\mathfrak{q}$ ramifies in M/F as $\mathfrak{q} = \mathfrak{Q}^2$, $X(\mathcal{A})$ has a subgroup $C = X(\mathcal{A})[\mathfrak{Q}_n]$ isomorphic to $O/\mathfrak{q}$ for $\mathfrak{Q}_n = \mathfrak{Q} \cap R_n$; so, we can still define $[\mathfrak{q}](X(\mathcal{A})) = X(\mathcal{A}\mathfrak{Q}_n^{-1}) = X(\mathcal{A}[\mathfrak{Q}]^{-1})$.

The effect on the q–expansion of the operator $[\mathfrak{q}]$ can be computed similarly to $\langle \mathfrak{z} \rangle$ (e.g. [DR80] 5.8; see also [PAF, §4.2.9]), and the q–expansion of $f|[\mathfrak{q}]$ at the cusp $(\mathfrak{a}, \mathfrak{b})$ is given by the q–expansion of f at the cusp $(\mathfrak{q}\mathfrak{a}, \mathfrak{b})$.

These operators $[\mathfrak{q}]$ and $\langle \mathfrak{z} \rangle$ change polarization ideals (as we will see later in [H04, §4.2]); so, they permute components $f_{\mathfrak{c}}$. By the q–expansion principle, $f \mapsto f|[\mathfrak{q}]$ and $f \mapsto f|\langle \mathfrak{z} \rangle$ are injective.

4. Distribution attached to $U(\mathfrak{l})$-eigenform

We recall notation and construction of a measure $d\varphi_{f,n}$ on Cl_n^- for a mod p modular form $f_{/\mathbb{F}}$ such that $f|U(\mathfrak{l}) = af$. If $0 \neq a \in \mathbb{F}$, we can patch together into a measure $d\varphi_f$ on Cl_∞^-. If $a = 0$, this is just a collection of infinitely many measures $\{d\varphi_{f,n}\}_n$ (see Remark 4.1).

4.1. **Anti-cyclotomic measure.** Choose a $U(\mathfrak{l})$-eigenform $f \in V(\Gamma_1(\mathfrak{l}^\nu\mathfrak{N}); A)$ with a central character for a p-adic ring A in which ℓ is invertible. We suppose that $f|U(\mathfrak{l}) = (a/\lambda(\mathfrak{l})N(\mathfrak{l}))f$ for either a unit $a \in A$ or $a = 0$. This f is an element of $V(\Gamma_1(\mathfrak{l}^\nu\mathfrak{N}); A)$ defined over the non-connected Hilbert modular Shimura variety whose geometrically connected components are indexed by the strict ray class group $Cl_F^+(\mathfrak{l}^\nu\mathfrak{N})$ of F. Our geometrically irreducible component V carries $x(\mathcal{A})$ for $\mathcal{A} \in Cl^{alg} \cap K_0$ for $K_0 := \mathrm{Ker}(Cl_\infty \twoheadrightarrow Cl_0)$. Anyway $f(x(\mathcal{A}))$ is well defined for all $\mathcal{A} \in Cl^{alg}$ possibly $x(\mathcal{A})$ sitting in another geometrically connected component.

Choose a Hecke character λ of M such that

- (f1) λ has infinity type $k + \kappa(1-c)$ of conductor $\mathfrak{C}$ prime to $p\ell$ which is a product of split primes over F ($k, \kappa \in \mathbb{Z}[\Sigma]$),
- (f2) Decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c$ for integral ideals $\mathfrak{F}$ and $\mathfrak{F}_c$ such that $\mathfrak{F} + \mathfrak{F}_c = R$, $\mathfrak{F} \subset \mathfrak{F}_c^c$, the Neben character of f as in [H07, (S1-3)] is given by $(\lambda_{\mathfrak{F}_c}, \lambda_{\mathfrak{F}}, (\lambda|_{F_{\mathbb{A}}^\times})| \cdot |_{F_{\mathbb{A}}}^2)$.

The existence of the character satisfying (f2) implies $k_\sigma = k_\tau$ for any two embeddings $\sigma, \tau \in \Sigma$; so, hereafter, often we identify k with the integer k_σ and write $k\Sigma$ in place of k (i.e., Σ is identified $\sum_{\sigma\in\Sigma} \sigma \in \mathbb{Z}[\Sigma]$). It might appear strange to have the absolute value character $|\cdot|_{F_{\mathbb{A}}}^2$ in the description of the central character $(\lambda|_{F_{\mathbb{A}}^\times})| \cdot |_{F_{\mathbb{A}}}^2$ of f, but when we extend a geometric modular form to an automorphic form on $G(\mathbb{A})$, we multiply the factor $|\det(g)|_{\mathbb{A}}$ as the adelic Fourier expansion has the factor $|\det(g)|_{\mathbb{A}}$ in front of the Fourier expansion sum in [HMI, (2.3.15)]; so, the central action on a geometric modular form and the adelic one has this discrepancy. See [HMI, §2.3.2, §4.3.7] for more details on the relation of geometric Hilbert modular forms and adelic ones. Then by (f1) and (G3′), $f([\mathcal{A}]) = \lambda(\mathcal{A})^{-1} f(x(\mathcal{A}))$ for $\mathcal{A}$ prime to p depends only on the class of $\mathcal{A}$ in $Cl_n^- = Cl_n/Cl_F$.

For the p–adic avatar $\widehat{\lambda}(x) = \lambda(xR)x_p^{k\Sigma+\kappa(1-c)}$, we also have $f([\mathcal{A}]) = \widehat{\lambda}(\mathcal{A})^{-1} f(x(\mathcal{A}))$. This new definition is valid even for $\mathcal{A}$ with non-trivial common factor with p. Then often we regard f as a function of $C^{(\infty)} = \bigsqcup_n C_n$ (embedded into $Sh^{(p)}_{/\mathcal{W}}$ or $Ig_{/\mathbb{F}}$ by $\mathcal{A} \mapsto x(\mathcal{A})$).

Writing $X(\mathcal{A})/C = X(\mathfrak{A})$ for $C \neq X(\mathcal{A})[\mathfrak{l}_n]$ for R_n-proper ideal $\mathcal{A}$ prime to $\mathfrak{l}$, $\mathfrak{A}$ is a proper R_{n+1}-ideal such that $\mathfrak{l}(R_n\mathfrak{A}) = \mathcal{A}$. Since there are $N(\mathfrak{l})$ proper R_{n+1}-ideal such that $\mathfrak{l}(R_n\mathfrak{A}) = \mathcal{A}$ if $n > 0$, we have

$$(a/\lambda N(\mathfrak{l}))\lambda(\mathcal{A})f([\mathcal{A}]) = (a/\lambda N(\mathfrak{l}))f(x(\mathcal{A})) = f|U(\mathfrak{l})(x(\mathcal{A})) = N(\mathfrak{l})^{-1} \sum_{\mathfrak{A}:\mathfrak{l}(R_n\mathfrak{A})=\mathcal{A}} f(x(\mathfrak{A}))$$

$$= N(\mathfrak{l})^{-1}\lambda(\mathfrak{A}) \sum_{\mathfrak{A}:\mathfrak{l}(R_n\mathfrak{A})=\mathcal{A}} f([\mathfrak{A}]) \overset{\lambda(\mathfrak{A})=\lambda(\mathfrak{l})^{-1}\lambda(\mathcal{A})}{=} \lambda N(\mathfrak{l})^{-1}\lambda(\mathcal{A}) \sum_{\mathfrak{A}:\mathfrak{l}(R_n\mathfrak{A})=\mathcal{A}} f([\mathfrak{A}]) \quad \text{if } n > 0.$$

Since $f([\mathfrak{A}])$ only depends on the class of Cl_{n+1}^-, this implies

(1) $a \cdot f([\mathcal{A}]_n) = \sum_{[\mathcal{B}]_{n+1}:Cl_{n+1}^- \ni [\mathcal{B}]_{n+1} \mapsto [\mathcal{A}]_n} f([\mathcal{B}]_{n+1})$,

(2) $f|U(\mathfrak{l})([\mathcal{A}]_n) = \lambda N(\mathfrak{l})^{-1} \sum_{[\mathcal{B}]_{n+1} \in Cl_{n+1}^-:[\mathcal{B}]_{n+1} \mapsto [\mathcal{A}]_n} f([\mathcal{B}]_{n+1})$.

We can rewrite the above relation (1) as

$$a \cdot f([\mathcal{A}]_n) = \sum_{[\mathcal{B}]_{n+1} \in Cl^-_{n+1} : [\mathcal{B}]_{n+1} \mapsto [\mathcal{A}]_n} f([\mathcal{B}]_{n+1}) \ \text{ if } n > 0. \tag{4.1}$$

More generally as seen in [H04, (3.8)], we get, for integers $n > m \geq 1$,

$$\sum_{[\mathcal{B}] \in Cl^-_n,\ [\mathcal{B}]_n \mapsto [\mathcal{A}]_m \in Cl^-_m} f([\mathcal{B}]_n) = a^{n-m} f([\mathcal{A}]_m), \tag{4.2}$$

where $\mathcal{A}$ runs over all elements in Cl^-_n which project down to $\mathfrak{l}^{m-n}\mathfrak{A} \in Cl^-_m$. The second relation (2) can be written

(C1) $f|U(\mathfrak{l})([\mathcal{A}]_n) = \lambda(\mathfrak{l})^{-1} f([\mathcal{B}]_{n+1})$ if $f([\mathcal{A}]_{n+1}) = f([\mathcal{B}]_{n+1})$ for any $[\mathcal{B}]_{n+1}$ with $[R_n\mathcal{B}]_n = [\mathcal{A}]_n$,
(C2) $f|U(\mathfrak{l}^m)([\mathcal{A}]_n) = \lambda N(\mathfrak{l})^{-m} \sum_{[\mathcal{B}]_{n+m} \in Cl^-_{n+m} : [\mathcal{B}]_{n+m} \mapsto [\mathcal{A}]_n} f([\mathcal{B}]_{n+m})$.

For each function $\phi : Cl^-_\infty \to A$ factoring through Cl^-_n, assuming $a \in A^\times$, we define

$$\int_{Cl^-_\infty} \phi d\varphi_f = a^{-n} \sum_{\mathcal{A} \in Cl^-_n} \phi(\mathcal{A}^{-1}) f([\mathcal{A}]). \tag{4.3}$$

Then for $n > m \geq 1$, assuming $a \in A^\times$, we find

$$\begin{aligned} a^{-n} \sum_{\mathcal{A} \in Cl^-_n} \phi(\mathcal{A}^{-1}) f([\mathcal{A}]) &= a^{-m} \sum_{\mathfrak{A} \in Cl^-_m} \phi(\mathfrak{A}^{-1}) a^{m-n} \sum_{\mathcal{A} \in Cl^-_n,\ \mathcal{A} \mapsto \mathfrak{A}} f([\mathcal{A}]) \\ &\overset{(4.2)}{=} a^{-m} \sum_{\mathfrak{A} \in Cl^-_m} \phi(\mathfrak{A}^{-1}) a^{m-n} f|U((\mathfrak{l})^{n-m})([\mathfrak{A}]) = \int_{Cl^-_\infty} \phi(x) d\varphi_f(x). \end{aligned}$$

Thus φ_f gives an A–valued distribution on Cl^-_∞ well defined independently of the choice of m for which ϕ factors through Cl^-_m, because $U(\mathfrak{l}^m) = U(\mathfrak{l})^m$.

Remark 4.1. The assumption that $a \in A^\times$ is not essential. If $a = 0$, we just define for each finite n and a function $\phi : Cl^-_n \to A$

$$\int_{Cl^-_n} \phi d\varphi_f = \sum_{\mathcal{A} \in Cl^-_n} \phi(\mathcal{A}^{-1}) f([\mathcal{A}]) \tag{4.4}$$

without dividing by a. Though we lose the distribution relation (4.4) above, we have well defined value $\int_{Cl^-_n} \phi d\varphi_f$ dependent on n. Changing ∞ by n, all the formulas independent of the distribution relation holds even when $a = 0$. So hereafter we allow the case where $a = 0$, and as a convention, we use n in place of ∞. If $a \in A^\times$, we can replace n by ∞ since $\int_{Cl^-_\infty} = \int_{Cl^-_n}$ as long as the integral factors through Cl^-_n. If $a = 0$, by (4.1), $\int_{Cl^-_n} \phi d\varphi_f \neq 0$ happens for a unique $n > 0$. This n is a minimal n for which ϕ factors through Cl^-_n. To write formulas uniform, we define $\mathbf{a} = 1$ if $a = 0$ and $\mathbf{a} = a$ if $a \neq 0$ in $\mathbb{F}$.

Classical modular forms can be defined over the integer ring of a number field; so, we assume that f is defined over a discrete valuation ring $\mathcal{V}$ (of residual characteristic p) in a number field E. We assume that E is the smallest field containing M' for the reflex (M', Σ') of (M, Σ) and the values $\lambda(\mathfrak{A})$ for all M-fractional ideals $\mathfrak{A}$. We write $\mathcal{P}|p$ for the prime ideal of the p–integral closure $\overline{\mathcal{V}}$ of $\mathcal{V}$ in $\overline{\mathbb{Q}}$ corresponding to $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. More generally, if $f = \theta^\kappa g$ for a classical modular form g integral over $\mathcal{V}$, the value $f([\mathcal{A}])$ is algebraic, abelian over M' and $\mathcal{P}$–integral over $\mathcal{V}$ by a result of Shimura and Katz (see [EAI, §8.1.1] and [K78]).

Let $f = \theta^\kappa g$ for $g \in G_k(\Gamma_0(\mathfrak{l}); \mathcal{V})$. Suppose $f|U(\mathfrak{l}) = (a/\lambda(\mathfrak{l})N(\mathfrak{l}))f$ for a giving a unit of $\overline{\mathcal{V}}/\mathcal{P}$. For the moment, let φ be the measure associated to f with values in $A = \overline{\mathcal{V}}$. We have a well defined measure $\varphi \mod \mathcal{P}$. Let E_f be the field of rationality of $x(\mathcal{A})$ for all $[\mathcal{A}] \in Cl^{alg}$ over $E[\mu_{\ell^\infty}]$. Then E_f/E is an abelian extension unramified outside ℓ, and we have the Frobenius element $\sigma_\mathfrak{b} \in \mathrm{Gal}(E_f/E)$ (that is, the image of $\mathfrak{b}$ under the Artin reciprocity map) for each ideal $\mathfrak{b}$ of E prime to ℓ. By Shimura's reciprocity law ([ACM] 26.8), writing (M', Σ') for the reflex CM type of (M, Σ), we find for $\sigma = \sigma_\mathfrak{b}$, $x(\mathcal{A})^\sigma = x(N(\mathfrak{b})^{-\Sigma'}\mathcal{A})$ for the norm $N : E \to M'$. As for $\eta_p^{ord}(\mathcal{A})$, we find $\sigma \circ \eta_p^{ord}(\mathcal{A}) = u\eta_p^{ord}$ for $u \in R^\times_{\Sigma_p}$. Since $\mathcal{A}_p \cong R_p$, we have $X(R)[p^\infty] \cong X(\mathcal{A})[p^\infty]$ as a Galois module. Thus we conclude $u = \psi_E(\mathfrak{b})$ for the Hecke character ψ_E of $E^\times_\mathbb{A}/E^\times$ giving rise to the zeta function of $X(R)$. From this, we see $f([\mathcal{A}])^\sigma = f([N(\mathfrak{b})^{-\Sigma'}\mathcal{A}])$ for any ideal $\mathfrak{b}$, since $\psi_E(\mathfrak{b}) \in M$ generates the ideal $N(\mathfrak{b})^{\Sigma'} \subset M$ ([ACM] Sections 13 and 19) and hence $\psi_E(\mathfrak{b})^{k\Sigma+\kappa(1-c)} = \lambda(N(\mathfrak{b})^{\Sigma'})$. We then have

$$\sigma \cdot \left(\int_{Cl_n^-} \phi(x) d\varphi_f(x)\right) = \int_{Cl_n^-} \sigma \circ \phi(N(\mathfrak{b})^{\Sigma'}x) d\varphi_f(x), \tag{4.5}$$

where $N(\mathfrak{b})$ is the norm of $\mathfrak{b}$ over M'. Writing $\mathbb{F}_q$ for $q := p^{r_0}$ for the residue field of $E \cap \mathcal{P}$, any modular form defined over $\mathbb{F}_q$ is a reduction modulo $\mathcal{P}$ of a classical modular form defined over $\mathcal{V}$ of sufficiently high weight. Since $\xi^\Sigma \in M'$ for $\xi \in M$ as the reflex of Σ' is a sub-CM-type of Σ, we have $\mathbb{F}_q \subset \mathbb{F}_{\mathfrak{q}}$. Thus the above identity is valid for $\sigma = \Phi^s$ ($s \in \mathbb{Z}$) for the Frobenius element $\Phi \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. In this case, $N(\mathfrak{b})$ is a power of a prime ideal $\mathfrak{p}|p$ in M'.

We now assume that $A = \mathbb{F} = \overline{\mathcal{V}}/\mathcal{P}$ and regard the measure φ_f as having values in $\mathbb{F}$. Then (4.5) shows that if ϕ is a character χ of Cl_n^- with arbitrary $n > 0$, for $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$,

$$\int_{Cl_n^-} \chi(x) d\varphi_f(x) = 0 \iff \int_{Cl_n^-} \sigma \circ \chi(x) d\varphi_f(x) = 0. \tag{4.6}$$

Let $\mathbb{F}_q[\mu_\ell]$ be the finite subfield of $\mathbb{F}$ generated by all ℓ–th roots of unity over $\mathbb{F}_q$; so, it is the field of rationality of λ, f and μ_ℓ over the residue field of $M' \cap \mathcal{P}$.

4.2. **Measure projected to Γ and Γ_n.** Recall Γ_n which is the image of Γ in Cl_n^-. Since each fractional R–ideal $\mathfrak{A}$ prime to $\mathfrak{l}$ defines a class $[\mathfrak{A}]$ in Cl_∞^-, we can embed the ideal group of fractional ideals prime to $\mathfrak{l}$ into Cl_∞^-. We write C^{alg} for its image. Thus the projection of $[\mathfrak{Q}]$ in Cl_n^- is $[\mathfrak{Q}]_n$ as specified for the integral ideal $\mathfrak{Q}$ above. Then $\Delta^{alg} = \Delta^- \cap C^{alg}$ is generated by prime ideals of M ramified over F. We choose a complete representative set for Δ^{alg} made of product of prime ideals in M ramified over F prime to $p\mathfrak{l}$. We may choose this set as $\{\mathfrak{R}^{-1}|\mathfrak{r} \in \mathcal{R}\}$, where $\mathcal{R}$ is made of square-free product of primes outside $\mathfrak{l}$ in F non-principal ramifying in M/F, and $\mathfrak{R}$ is a unique ideal in M with $\mathfrak{R}^2 = \mathfrak{r}$. Note that $\{\mathfrak{R}|\mathfrak{r} \in \mathcal{R}\}$ is a complete representative set for 2–torsion elements in the quotient Cl_0^-.

In [H04] and [H07], we used Cl_n in place of Cl_n^-; so, we had to choose a complete representative set $\mathcal{S}$ of the image $\overline{Cl}_F$ of Cl_F in Cl_n, which is not necessary. Indeed, since $f([\mathcal{A}]) = f([\mathfrak{s}\mathcal{A}])$ for an O-ideal $\mathfrak{s}$ by our choice of λ, we have $\overline{h}f([\mathcal{A}]) = \sum_{\mathfrak{s}\in\mathcal{S}} f([\mathfrak{s}\mathcal{A}])$ for $\overline{h} := |\overline{Cl}_F|$, and if we make our choice of λ, this implies the triviality of the measure if $p|\overline{h}$. To avoid this, we do not sum over $\mathcal{S}$. We fix a character $\psi : \Delta^- \to \mathbb{F}^\times$, and define

$$f_\psi = \sum_{\mathfrak{r}\in\mathcal{R}} \lambda\psi^{-1}(\mathfrak{R}) f|[\mathfrak{r}]. \tag{4.7}$$

In [H04] and [H07], f_ψ is defined by

$$\sum_{\mathfrak{r}\in\mathcal{R}} \lambda\psi^{-1}(\mathfrak{R}) \left(\sum_{\mathfrak{s}\in\mathcal{S}} \psi\lambda^{-1}(\mathfrak{s}) f|\langle\mathfrak{s}\rangle\right) |[\mathfrak{r}],$$

and we do not follow this definition.

Choose a complete representative set $\mathcal{Q}$ for $Cl_\infty^-/\Gamma\Delta^{alg}$ made of primes $\mathfrak{Q}$ of M split over F outside $p\mathfrak{l}$ except for the trivial element R representing $1 \in Cl_\infty^-/\Gamma\Delta^{alg}$. Thus $\mathfrak{q} := N_{M/F}(\mathfrak{Q})$ is a prime ideal of O if $\mathfrak{Q} \neq R$ (and $\mathfrak{q} = O$ if $\mathfrak{Q} = R$). We choose $\eta_n^{(p)}$ out of the base (w_1, w_2) of $\widehat{R}_n$ so that at $\mathfrak{q} = \mathfrak{Q} \cap F$, $w_{1,\mathfrak{q}} = (1, 0) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$ and $w_{2,\mathfrak{q}} = (0, 1) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$. Since all operators $\langle\mathfrak{s}\rangle$, $[\mathfrak{q}]$ and $[\mathfrak{r}]$ commute with $U(\mathfrak{l})$, $f_\psi|[\mathfrak{q}]$ is still an eigenform of $U(\mathfrak{l})$ with the same eigenvalue as f. Thus in particular, we have a measure $\varphi_{f_\psi|[\mathfrak{q}]}$. We then define another measure φ_f^ψ on Γ by

$$\int_{\Gamma_n} \phi d\varphi_f^\psi = \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) \int_{\Gamma_n} \phi|\mathfrak{Q} d\varphi_{f_\psi|[\mathfrak{q}]}, \tag{4.8}$$

where $\phi|\mathfrak{Q}(y) = \phi(y[\mathfrak{Q}]_\Gamma)$ for the projection $[\mathfrak{Q}]_\Gamma$ in Γ of the class $[\mathfrak{Q}] \in Cl_\infty^-$. As already remarked, $\phi \mapsto \phi|\mathfrak{Q}$ is a transcendental action unless $\mathfrak{Q} = R$. If $\mathfrak{Q} = R$, $\phi|\mathfrak{Q} = \phi$ and $f|[\mathfrak{q}] = f$.

Lemma 4.2. *If $\chi : \Gamma_n \to \mathbb{F}^\times$ and $\psi : \Delta^- \to \mathbb{F}^\times$ are characters, we have*
$$\int_{\Gamma_n} \chi d\varphi_f^\psi = \int_{Cl_n^-} \chi\psi d\varphi_f.$$
Here recall the image Γ_n of Γ in Cl_n^-.

Proof. For a proper R_n–ideal $\mathcal{A}$, by the above definition of these operators,
$$f|[\mathfrak{r}]|[\mathfrak{q}]([\mathcal{A}]) = \lambda(\mathcal{A})^{-1} f(x(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathcal{A})).$$
Since $\chi = \psi$ on Δ^-, we have
$$\begin{aligned}\int_{\Gamma_n} \chi d\varphi_f^\psi &= \sum_{\mathfrak{Q}\in\mathcal{Q}}\sum_{\mathfrak{r}\in\mathcal{R}} (\lambda\chi^{-1}\psi^{-1})(\mathfrak{Q}\mathfrak{R}) \sum_{\mathcal{A}\in\Gamma_n} \chi(\mathcal{A}) f|[\mathfrak{r}]|[\mathfrak{q}]([\mathcal{A}]) \\ &= \sum_{\mathcal{A},\mathfrak{Q},\mathfrak{r}} \chi\psi(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathcal{A}) f([\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathcal{A}]) = \int_{Cl_n^-} \chi\psi d\varphi_f,\end{aligned}$$
because $Cl_n^- = \bigsqcup_{\mathfrak{Q}\mathfrak{R}} [\mathfrak{Q}^{-1}\mathfrak{R}^{-1}]\Gamma_n$. □

We write $\mathbb{F}_{\mathbf{q}} := \mathbb{F}_q[\psi] \subset \mathbb{F}$ for the field of rationality of ψ over $\mathbb{F}_q$. Then $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{\mathbf{q}})$ preserves $d\varphi_f^\psi$. Then (4.6) shows that if χ is a character of Γ, for $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{\mathbf{q}})$,
$$\int_{\Gamma_n} \chi(x) d\varphi_f^\psi(x) = 0 \iff \int_{\Gamma_n} \sigma\circ\chi(x) d\varphi_f^\psi(x) = 0. \tag{4.9}$$

4.3. **Trace relation.** For any finite extensions $\kappa/\kappa'/\mathbb{F}_{\mathbf{q}}[\mu_\ell]$, we consider the trace map: $\mathrm{Tr}_{\kappa/\kappa'}(\xi) = \sum_{\sigma\in\mathrm{Gal}(\kappa/\kappa')} \sigma(\xi)$ for $\xi \in \kappa$. Recall the image Γ_n of Γ in Cl_n. Define
$$f_\psi^{\mathcal{Q}}([\mathcal{A}]) = \sum_{\mathfrak{Q}\in\mathcal{Q}} \psi(\mathfrak{Q})^{-1} f_\psi([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma) \text{ for the projection } [\mathfrak{Q}]_\Gamma \in \Gamma \text{ of } [\mathfrak{Q}]. \tag{4.10}$$
Let $\chi : \Gamma_n \to \mathbb{F}^\times$ be a character. Suppose that $\mathrm{Im}(\chi) \cap \mathbb{F}_{\mathbf{q}}[\mu_\ell]^\times$ has order ℓ^r and that χ has order ℓ^ν. Note that $1 \le r \in \mathbb{Z}$. Fix $j \ge r$, and write
$$\Phi = \Phi_n := \Gamma_n \cap \chi^{-1}(\mathbb{F}_{\mathbf{q}}[\mu_{\ell^j}]^\times) \tag{4.11}$$
and $[\mathcal{A}_y] = [\mathcal{A}_y]_n$ for the image of $y \in \Gamma$ in Γ_n. By (3.3), we have an isomorphism of O-modules:
$$O/\mathfrak{l}^j \cong \Phi_n \text{ by } u \mapsto \varrho(u/\varpi^j) x(R_{n+j}). \tag{4.12}$$
Note that $[R_{n-1}\mathcal{A}_y]_n = [\mathcal{A}_y]_{n-1}$ for all n. Recall $\mathbf{a} \in \mathbb{F}^\times$ defined in Remark 4.1. If $\nu \ge j$, for
$$d = [\mathbb{F}_{\mathbf{q}}[\chi] : \mathbb{F}_{\mathbf{q}}[\mu_{\ell^j}]] = [\mathrm{Im}(\chi) : \mathrm{Im}(\chi) \cap \mathbb{F}_{\mathbf{q}}[\mu_\ell]^\times] = \ell^{\nu-j},$$
$$\begin{aligned}&\int_{\Gamma_n} \mathrm{Tr}_{\mathbb{F}_{\mathbf{q}}[\chi]/\mathbb{F}_{\mathbf{q}}[\chi^{(\ell)},\mu_\ell]} \circ \chi(y^{-1}x) d\varphi_f^\psi(x) \\ &= \frac{d}{\mathbf{a}^n} \sum_{\mathcal{A}\in\Gamma_n:\mathcal{A}y^{-1}\in\Phi_n} \chi(y^{-1}\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}]) = \frac{d}{\mathbf{a}^n} \sum_{[\mathcal{A}]\in\Phi_n} \chi(\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}][\mathcal{A}_y]),\end{aligned} \tag{4.13}$$
because for an ℓ–power root of unity and a finite extension $\kappa/\mathbb{F}_{\mathbf{q}}[\mu_{\ell^j}]$, $\zeta \in \mu_{\ell^\nu} - \mu_{\ell^j}$,
$$\mathrm{Tr}_{\kappa[\mu_{\ell^\nu}]/\kappa}(\zeta^s) = \begin{cases} \ell^{\nu-j}\zeta^s & \text{if } \zeta^s \in \kappa \text{ and } \kappa \cap \mu_{\ell^\infty} = \mu_{\ell^j} \\ 0 & \text{otherwise.}\end{cases} \tag{4.14}$$
Thus by (4.9), we have
$$\sum_{[\mathcal{A}]\in\Phi_n} \chi(\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}][\mathcal{A}_y]) = 0 \text{ if } \int_{Cl_n^-} \chi\psi d\varphi_f = 0. \tag{4.15}$$

Let $\mathcal{F}(\Phi_n[\mathcal{A}_y], \mathbb{F})$ be the space of functions $\phi : \Phi_n[\mathcal{A}_y] \to \mathbb{F}$. Consider the linear form $\ell_\chi : \mathcal{F}(\Phi_n[\mathcal{A}_y], \mathbb{F}) \to \mathbb{F}$ given by $\ell_\chi(\phi) = \sum_{[\mathcal{A}]\in\Phi_n} \chi([\mathcal{A}])\phi([\mathcal{A}][\mathcal{A}_y])$. Since the orthogonal complement of the space spanned by $\{\ell_{\chi^\sigma}\}_{\sigma\in\mathrm{Gal}(\mathbb{Q}[\mu_{\ell^e}]/\mathbb{Q})}$ in $\mathcal{F}(\Phi_n, \mathbb{F})$ under the pairing
$$\langle \phi, \phi' \rangle = \sum_{[\mathcal{A}]\in\Phi_n} \phi([\mathcal{A}][\mathcal{A}_y])\phi'([\mathcal{A}])$$

is spanned by characters of order $\leq \ell^{\varepsilon-1}$. If $\varepsilon = 1$, the orthogonal complement is made of constant functions on Φ_n. Thus assuming that the integral (4.13) vanishes with $\Phi_n \cong \mu_\ell$ and that $\mathrm{Gal}(\mathbb{F}_{\mathbf{q}}[\mu_\ell]/\mathbb{F}_{\mathbf{q}}) = \mathrm{Gal}(\mathbb{Q}[\mu_\ell]/\mathbb{Q})$, $[\mathcal{A}] \mapsto f_\psi^{\mathcal{Q}}([\mathcal{A}][\mathcal{A}_y])$ is a constant function of $[\mathcal{A}]$ whose value is $f([\mathcal{A}_y])$, for $\alpha_1 = \mathrm{diag}[1, \varpi_{\mathfrak{l}}]$, we have

$$\ell f_\psi^{\mathcal{Q}}([\mathcal{A}_y]_n) = \sum_{[\mathcal{B}]\in\Phi_n y} f_\psi^{\mathcal{Q}}([\mathcal{B}]_n) = \sum_{u \mod \mathfrak{l}} f_\psi^{\mathcal{Q}}|(\varrho(u/\varpi_{\mathfrak{l}})\alpha_1)([\mathcal{A}_y]_{n-1}) = a f_\psi^{\mathcal{Q}}([\mathcal{A}_y]_{n-1}).$$

This is easy to see if we choose $\mathcal{A}_y$ prime to $\mathfrak{l}$ (i.e., $\mathcal{A}_{y,\mathfrak{l}} = R_{n,\mathfrak{l}}$). Hereafter exclusively the latter r for the integer defined by

$$\mu_{\ell^\infty} \cap \mathbb{F}_{\mathbf{q}}[\mu_\ell]^\times = \mu_{\ell^r}. \tag{4.16}$$

Thus $r \geq 1$.

5. Proof of Theorem 0.1

Write $\widehat{\mathbb{G}}_{m/\mathbb{Z}_\ell}$ for the formal completion over $\mathbb{Z}_\ell$ at the origin of $\mathbb{G}_m(\mathbb{F}_\ell)$. Let $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ embed into $\mathbb{G}^d_{m/\mathbb{Z}_\ell}$ and $\widehat{\mathbb{G}}^d_{m/\mathbb{Z}_\ell}$ by choosing a basis $(\gamma_1, \ldots, \gamma_d)$ of Γ over $\mathbb{Z}_\ell$ and sending $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ to $(\chi(\gamma_1), \ldots, \chi(\gamma_d))$. A subset S of $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ therefore has its Zariski closure $\overline{S}$ (resp. $\widehat{S}$) in $\mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell)$ (resp. $\widehat{\mathbb{G}}^d_m(\overline{\mathbb{Q}}_\ell)$). Since $\mathrm{Aut}(\widehat{\mathbb{G}}^d_m) = \mathrm{GL}_d(\mathbb{Z}_\ell)$, the isomorphism class of $\widehat{S}$ is independent of the choice of the basis. As we will see later for our choice of S that $\dim \widehat{S} = \dim \overline{S}$, and hence $\overline{S}$ being a proper Zariski closed set is independent of the choice of basis.

Fix a character $\psi : \Delta \to \mathbb{F}^\times$. Let

$$\mathcal{X} = \mathcal{X}_\psi := \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) | \int_{Cl_n^-} \chi\psi d\varphi_f \neq 0 \ \text{ for some } n\}.$$

If $a = 0$, as seen in Remark 4.1, $\int_{Cl_n^-} \chi\psi d\varphi_f \neq 0$ for one value n; in other words, for n given by $\mathrm{cond}(\chi) = \mathfrak{l}^n$, the integral is not defined over $Cl_{n'}$ for $n' < n$ and the integral vanishes for $n' > n$. On the contrary, if $a \neq 0$, the vanishing (and non-vanishing) of the integral is independent of n as long as it is well defined.

Assume the following condition:

$$\textit{The Zariski closure } \overline{\mathcal{X}}_\psi \textit{ in } \mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell) \textit{ of the set } \mathcal{X}_\psi \textit{ has dimension} < d, \tag{5.1}$$

and we are going to deduce absurdity.

5.1. **Proof.** We prepare a lemma. Let $\mathbb{C}_\ell$ be the ℓ-adic completion of $\overline{\mathbb{Q}}_\ell$. Let $\mathbf{W}$ be a discrete valuation ring finite over the Witt vector ring $W(\overline{\mathbb{F}}_\ell)$ inside $\mathbb{C}_\ell$ for an algebraic closure $\overline{\mathbb{F}}_\ell$ of $\mathbb{F}_\ell$, and write $\mathcal{K}$ for its quotient field. For a formal subscheme X of $\widehat{\mathbb{G}}_{m/\mathbf{W}}$, we write $X(\mathbb{C}_\ell) := X(\overline{\mathbf{W}})$ for the integral closure $\overline{\mathbf{W}}$ of $\mathbf{W}$ in $\mathbb{C}_\ell$. The map $t \mapsto t^{\overline{z}_n}$ is an automorphism of μ_{ℓ^n} for $\overline{z}_n \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. Take a sequence of $z_n \in \mathbb{Z}$ lifting $\overline{z}_n$ and assuming $\{z_n\}$ converges to $z \in \mathbb{Z}_\ell^\times$. Then $\zeta \mapsto t^{\overline{z}_n}$ gives rise to an automorphism $z \in \mathbb{Z}_\ell^\times$ of μ_{ℓ^∞}. In this way, ℓ-adic unit z acts on $\mu^d_{\ell^\infty}$. If $z \in \mathbb{Q} \cap \mathbb{Z}_\ell^\times$ prime to ℓ, this automorphism of $\mu^d_{\ell^\infty}$ extends to an isogeny $t \mapsto t^z$ of $\mathbb{G}^d_m$. If we identify $\mu_{\ell^\infty} = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, $t \mapsto t^z$ turns into a multiplication $\tau \mapsto z\tau$ by z on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. In the following lemma, we take $z = p^m$ for $m \in \mathbb{Z}$ and a prime $p \neq \ell$.

Lemma 5.1. *Let p and ℓ be distinct primes and $r > 0$ be an integer. Let $\mathcal{X}$ be a subset of $\mu^d_{\ell^\infty}$ and $\overline{\mathcal{X}}$ be the Zariski closure of $\mathcal{X}$ in $\mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell)$ for $d \geq 1$. Suppose that $\overline{\mathcal{X}}$ is a subscheme stable under $t \mapsto t^{p^{rn}}$ for all $n \in \mathbb{Z}$ and a fixed $r > 0$ (this means $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$). Assume $\dim \overline{\mathcal{X}} < d$. Identify $\mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$ with $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^d$ as ℓ-divisible groups. Then, for a given d-tuple $(a_1, \ldots, a_d)$ of non-negative integers, we can find a sufficiently large p^r-power $P = p^j$ ($j = rj'$, $0 < j' \in \mathbb{Z}$) and a positive integer N such that there exists a sequence of subsets $\{\Upsilon_n\}_{n=N}^\infty$ outside $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell)$ given by*

$$\Upsilon_n = \left\{ \left(\frac{P^{k_1}e_1}{\ell^{n+a_1}} + \cdots + \frac{P^{k_d}e_d}{\ell^{n+a_d}}\right) \mod \mathbb{Z}_\ell^d \middle| (k_i) \in \mathbb{Z}^d \right\}$$

if we choose a basis $\{e_i\}$ of $\mathbb{Z}_\ell^d$ suitably.

Proof. We choose the p^r-power P so that $P \equiv 1 \mod \ell$. Let $\Gamma_P = P^{\mathbb{Z}_\ell} \subset \mathbb{Z}_\ell^\times$, which is an open subgroup of $1+\ell\mathbb{Z}_\ell$. Let $\overline{\mathcal{X}}[\ell^\infty] := \overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$. Since $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$, we have $\overline{\mathcal{X}}[\ell^\infty]^{p^r} \subset \overline{\mathcal{X}}[\ell^\infty]$; so, the Zariski closure of $\overline{\mathcal{X}}[\ell^\infty]$ is stable under $t \mapsto t^{p^r}$. We may replace $\overline{\mathcal{X}}$ by the Zariski closure of $\overline{\mathcal{X}}[\ell^\infty]$ as the lemma only concerns about $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$, and after the replacement, the stability $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$ is intact. If $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$ is a finite set, the assertion plainly follows; so, we may assume that $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$ is infinite. Since $\overline{\mathcal{X}}$ is noetherian, we may also assume that $\dim \overline{\mathcal{X}} > 0$ as otherwise, $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell)$ is finite.

The variety $\overline{\mathcal{X}}$ is defined over a finite extension $\mathcal{K}$ of $\mathrm{Frac}(W(\overline{\mathbb{F}}_\ell))$. Take $\mathbf{W}$ to be the ℓ-adic integer ring of $\mathcal{K}$ with maximal ideal $\mathfrak{m}_{\mathbf{W}}$. Let $\overline{\mathcal{X}}_{/\mathbf{W}}$ be the schematic closure of $\overline{\mathcal{X}}_{/\mathcal{K}}$ in $\mathbb{G}^d_{m/\mathbf{W}}$. Writing $\mathbb{G}^d_{m/\mathbf{W}} = \mathrm{Spec}(A)$ for $A = \mathbf{W}[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$ and $\overline{\mathcal{X}} = \mathrm{Spec}((A \otimes_{\mathbf{W}} \mathcal{K})/\mathfrak{a})$ with an ideal $\mathfrak{a}$ of $A \otimes_{\mathbf{W}} \mathcal{K}$, $\overline{\mathcal{X}}_{/\mathbf{W}} = \mathrm{Spec}(A/\mathfrak{A})$ for $\mathfrak{A} := \mathfrak{a} \cap A$. Thus $\overline{\mathcal{X}}_{/\mathbf{W}}$ is flat over $\mathbf{W}$. Let

$$\mathfrak{m} = \mathfrak{m}_{\mathbf{W}} + (t_1 - 1, \dots, t_d - 1) \subset \mathbf{W}[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$$

and $\widehat{\mathfrak{A}}$ be the $\mathfrak{m}$-adic closure of $\mathfrak{A}$ in $\widehat{A} = \mathbf{W}[[T_1, \dots, T_d]] = \varprojlim_n A/\mathfrak{m}^n$ with $T_i = t_i - 1$.

We write $\widehat{M}$ for $\mathfrak{m}$-adic completion of an A-module M and $\mathrm{gr}(M) = \bigoplus_{j=0}^\infty \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ (the graded module over $\mathrm{gr}(A)$). Note that $\widehat{\mathbb{G}}^d_{m/\mathbf{W}} = \mathrm{Spf}(\widehat{A})$ is the formal completion of $\mathbb{G}^d_{m/\mathbf{W}}$ along the identity of $\widehat{\mathbb{G}}_{m/\overline{\mathbb{F}}_\ell}$.

Define $\widehat{\mathcal{X}} := \mathrm{Spf}(X)$ for $X := \mathbf{W}[[T_1, \dots, T_d]]/\widehat{\mathfrak{A}}$, which is a formal subscheme of $\widehat{\mathbb{G}}^d_{m/\mathbf{W}}$ over $\mathbf{W}$. Since $X = A/\mathfrak{A} \otimes_A \mathbf{W}[[T_1, \dots, T_n]] = \widehat{A/\mathfrak{A}}$, $\widehat{\mathcal{X}}$ is a flat over $\mathbf{W}$. Since $\dim(X) = \dim \mathrm{gr}(X) = \dim \mathrm{gr}(A/\mathfrak{A}) = \dim(A/\mathfrak{A})$ (e.g., [CRT, Theorem 15.7]), we find $\dim \widehat{\mathcal{X}} = \dim \overline{\mathcal{X}} < d$.

Since $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell) \neq \emptyset$ and $\mu^d_{\ell^\infty} \times_{\mathbf{W}} \overline{\mathbb{F}}_\ell$ has only one geometric point, we see $\widehat{\mathcal{X}}(\mathbf{W}[\mu_{\ell^\infty}]) \supset \overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$. Since $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$, we still have $\widehat{\mathcal{X}}^{p^r} \subset \widehat{\mathcal{X}}$ inside $\widehat{\mathbb{G}}^d_{m/\mathbf{W}}$. Thus $\widehat{\mathcal{X}}$ is stable under an open subgroup U of $1+\ell\mathbb{Z}_\ell$. Here an element $s \in 1+\ell\mathbb{Z}_\ell$ acts on $\widehat{\mathbb{G}}^d_m$ by $t \mapsto t^s$.

Since $\widehat{\mathcal{X}}$ is noetherian, it has finitely many geometrically irreducible components, and U permutes them. Thus, replacing U by its open subgroup, we may assume that U fix each geometrically irreducible component. By extending scalars, we may assume that each geometrically irreducible component is defined over $\mathbf{W}$. Then by Lemma 5.5 below, $\widehat{\mathcal{X}} = \bigcup_{\zeta \in Z, i} \zeta \mathcal{T}_{\zeta,i}$, where $\mathcal{T}_{\zeta,i} \subsetneq \widehat{\mathbb{G}}^d_{m/\mathbf{W}}$ is a formal subtorus of $\widehat{\mathbb{G}}^d_{m/\mathbf{W}}$ and Z is a finite subset of $\mu^d_{\ell^\infty}(\mathbf{W}[\mu_{\ell^\infty}])$.

We first assume that $Z = \{1\}$. By this assumption, $\widehat{\mathcal{X}}$ is a union of subtori $\{\mathcal{T}_i\}_{i \in J}$ with $|J| < \infty$ and $\dim \mathcal{T}_i < d$. Thus we have its ℓ-adic Tate module $T\mathcal{T}_i = \varprojlim_n \mathcal{T}[\ell^n] \subset T := T\mu^d_{\ell^\infty}$. Put $T\widehat{\mathcal{X}} := \bigcup_i T\mathcal{T}_i$. We identify $\mu^d_{\ell^n} = \ell^{-n}T/T \cong T/\ell^n T$; so, $\mu^d_{\ell^\infty} = \mathbb{Q}_\ell T/T$. In particular, $\overline{\mathcal{X}}[\ell^n] = \overline{\mathcal{X}} \cap \mu^d_{\ell^n}$ is the image of $\bigcup_i \mathbb{Q}_\ell T\mathcal{T}_i$ in $\mu^d_{\ell^n} = \ell^{-n}T/T$. Then we can choose a base $\{e_1, \dots, e_n\}$ of T over $\mathbb{Z}_\ell$ outside $T\widehat{\mathcal{X}}$ so that $\mathbb{Z}_\ell e \cap T\widehat{\mathcal{X}} = \{0\}$ for $e = e_1 + e_2 + \dots + e_d$. Then the ℓ–adic distance from the $\mathbb{Q}_\ell$-span $\mathbb{Q}_\ell T\widehat{\mathcal{X}} = \bigcup_i \mathbb{Q}_\ell T\mathcal{T}_i$ to the point $\frac{e}{\ell^n}$ is larger than or equal to $c\ell^n$ for a positive constant c independent of n. Thus we can find a sufficiently large power P of p^r (ℓ–adically very close to 1) so that $U_n = \Gamma_P \frac{e_1}{\ell^{a_1+n}} + \dots + \Gamma_P \frac{e_d}{\ell^{a_d+n}}$ for $\Gamma_P = P^{\mathbb{Z}_\ell}$ gives rise to an open neighborhood of $\frac{e}{\ell^n}$ disjoint from $\mathbb{Q}_\ell T\widehat{\mathcal{X}}$. Then the image Υ_n of U_n in $\mu^d_{\ell^\infty}$ is disjoint from $\widehat{\mathcal{X}}[\ell^\infty]$ and hence from $\overline{\mathcal{X}}$ for all $n \geq 1$.

When $Z \neq \{1\}$, we consider the subgroup $\langle Z \rangle$ of $\mu^d_{\ell^\infty}$ generated by Z. The group $\langle Z \rangle$ is finite. Consider the projection $\pi : \widehat{G}^d_m \to \widehat{G}^d_m/\langle Z \rangle$. The image of $\pi(\widehat{\mathcal{X}})$ under π is a union of formal subtori and hence stable under scalar multiplication by elements in $\mathbb{Z}_\ell$. Using the result proven under the condition $Z = \{1\}$ applied to $\pi(\widehat{\mathcal{X}})$, we write Υ'_n for the sets constructed for $\pi(\mu^d_{\ell^\infty}) = \mu_{\ell^\infty}/\langle Z \rangle$. Then we find that for $n > N$ any Γ^d_P–orbit of an element in the pull-back image $\Upsilon_n := \pi^{-1}(\Upsilon'_n)$ gives a desired set $\Upsilon_n \subset \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$. This finishes the proof. □

Choose a $\mathbb{Z}_\ell$-basis $\gamma_1, \dots, \gamma_d$ for $d = \mathrm{rank}_{\mathbb{Z}_\ell} \Gamma$. Then identify $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ with $\mu^d_{\ell^\infty}$ by $\chi \mapsto (\chi(\gamma_1), \dots, \chi(\gamma_d)) \in \mu^d_{\ell^\infty} \subset \mathbb{G}^d_m$. Here is a more accurate version of Theorem 0.1.

Theorem 5.2. *Suppose that for a given class $v \in (O_{\mathfrak{l}}/\mathfrak{l}^j)^\times$ with a sufficiently large $j \geq r > 0$ for r as in* Theorem 0.1 *and a cusp $(\mathfrak{a}, \mathfrak{b})$ with $\mathfrak{a}$ and $\mathfrak{b}$ coprime to $\mathfrak{l}$, there exists $\xi \in \mathfrak{ab} \cap -v$ such that $a(\xi, f_\psi) \neq 0$ in $\mathbb{F}$. Then the set of characters $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ with $v(\chi) = v$ and non-vanishing*

$\int_{Cl_n} \psi\chi d\varphi_f \neq 0$ *for* $n > 0$ *given by* $\mathrm{cond}(\chi) = \mathfrak{l}^n$ *is Zariski dense in* $\mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$. *Here* $v(\chi) \in (O_\mathfrak{l}/\mathfrak{l}^j)^\times$ *is defined in the following proof. If* $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$, *we can take* $j = r$ *for* r *as in* (4.16).

Though the minimal possible r depends on $\mathfrak{l}$, the assumption in the theorem is in appearance weaker than

(h) There exists a strict ideal class $\mathfrak{c}$ of F such that $\mathfrak{c}(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s})$ is in $\mathfrak{c}$ for some $(\mathfrak{Q},\mathfrak{R},\mathfrak{s}) \in \mathcal{Q} \times \mathcal{S} \times \mathcal{R}$ and for any given integer $j \geq r > 0$, the $N(\mathfrak{l})^j$ modular forms $f_{\psi,\mathfrak{c}}|\left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right)$ for $u \in \mathfrak{l}^{-j}/O$ are linearly independent,

which is assumed in [H04, Theorem 3.3].

Proof. Let

$$\mathcal{X} = \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) \hookrightarrow \mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell} | \int_\Gamma \chi d\varphi_f^\psi \neq 0 \text{ and } v(\chi) = v\}$$

and $\widehat{\mathcal{X}}$ (resp. $\overline{\mathcal{X}}$) be the formal Zariski (resp. Zariski) closure of $\mathcal{X}$ in $\widehat{\mathbb{G}}^d_{m/\mathbf{W}}$ (resp. $\mathbb{G}^d_m$). Note that $\overline{\mathcal{X}}^P \subset \overline{\mathcal{X}}$ and $\widehat{\mathcal{X}}^P \subset \widehat{\mathcal{X}}$. Suppose $\overline{\mathcal{X}}$ is a proper Zariski closed subset of $\mathbb{G}^d_m$ and get a contradiction.

First suppose $d = 1$. Then $\overline{\mathcal{X}}$ is a finite set. Take $j := r$ in (4.11) and (4.12). So there exists $B > 0$ such that if the conductor of a character χ is $\mathfrak{l}^n$ with $n > B$, by (4.15), identifying $\Phi_n = O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^r$ as in (4.12) by $x(\mathcal{A}_u) = \varrho(u/\ell^r)x(R_{n+r}) \leftrightarrow O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^r$, χ induces a character of $O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^r$ which is of the form $u \mapsto \zeta_r^{v(\chi)u}$ for $v(\chi) \in (O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^r)^\times$. Then we have

$$\int_\Gamma \chi d\varphi_f^{\mathcal{Q}} = \sum_{[\mathcal{A}]\in\Phi_n} \chi(\mathcal{A}) f_\psi^{\mathcal{Q}}|\varrho(u/\varpi_\mathfrak{l}^r)([\mathcal{A}\mathcal{A}_y]) = \sum_{u \mod \mathfrak{l}^r} \zeta_r^{vu} f_\psi^{\mathcal{Q}}|\varrho(u/\ell^r)([\mathcal{A}\mathcal{A}_y]) = 0.$$

Here $[\mathcal{A}_y]$ is any element in Γ_n. Let $\Xi_n = \{x([\mathcal{A}])|[\mathcal{A}] \in \Gamma_n\}$ and $\Xi := \bigsqcup_{n>B} \Xi_n \cap V$. Then Ξ is associated to an infinite arithmetic progression of difference m (for minimal exponent $\mathfrak{l}^m$ is generated by $N_{M/F}(R)$).

Since $\chi|_{\Phi_n} : O/\mathfrak{l}^\nu \to \mathbb{F}^\times$ is an arbitrary character of order ℓ^r, we may fix a character $\chi_v(u) = \zeta_r^{vu}$ for $v \in (O/\mathfrak{l}^\nu)^\times$ independent of $n > B$ as an additive character of $O/\mathfrak{l}^\nu$. Writing

$$g_v := \sum_{u\in O/\mathfrak{l}^\nu} \chi_v(u) f_\psi|\varrho(u/\varpi_\mathfrak{l}^\nu), \tag{5.2}$$

we find $\sum_{\mathfrak{Q}\in\mathcal{Q}} \psi^{-1}(\mathfrak{Q}) g_v|[\mathfrak{q}]([\mathcal{A}][\mathfrak{Q}]_\Gamma) = 0$ for all $[\mathcal{A}] \in \Xi$. By Corollary 2.12, Ξ is Zariski dense in $\mathcal{V} = V^{\mathcal{Q}}$, and hence we conclude $g_v|[\mathfrak{q}] = 0$. Since the q-expansion of a modular form $h|[\mathfrak{q}]$ at $(\mathfrak{a},\mathfrak{b})$ is given by the q-expansion of h at $(\mathfrak{q}\mathfrak{a},\mathfrak{b})$; so, by q-expansion principle, $g_v|[\mathfrak{q}] = 0 \Leftrightarrow g_v = 0$ (e.g., [H10, (5.10)]). Note $a(\xi, g_v) = N(\mathfrak{l})^r a(\xi, f_\psi)$ as long as $\xi \equiv -v \mod \mathfrak{l}^\nu$. Since v is arbitrary, we can choose v so that ξ as in the theorem satisfies $\xi \equiv -v \mod \mathfrak{l}^\nu$; so, $g_v \neq 0$, a contradiction.

We now assume $d \geq 2$. Take a base $\gamma_1, \dots, \gamma_d$ of Γ of Γ over $\mathbb{Z}_\ell$, which gives rise to an identification $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) = \mu^d_{\ell^\infty}$ by $\chi \mapsto (\chi(\gamma_1), \dots, \chi(\gamma_d))$. Regard $\mu^d_{\ell^\infty} \subset \mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$ and apply Lemma 5.1 to $\overline{\mathcal{X}} \subset \mathbb{G}^d_m$. Thus we have the base $e_1, \dots, e_d$ as in Lemma 5.1 of the Tate module $T\mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) = \varprojlim_n \mathrm{Hom}(\Gamma, \mu_{\ell^n})$. We rewrite the corresponding basis of Γ as $\gamma_1, \dots, \gamma_d$; so, the $\mathbb{Z}_\ell$-module $\gamma_i^{\mathbb{Z}_\ell}$ is sent isomorphically onto $\mathbb{Z}_\ell e_i$ for each i. Recall $Cl^-_\infty = \varprojlim_n Cl^-_n$ and $Cl_\infty = \Gamma \times \Delta$ for a finite group Δ. Pick the smallest integer $0 < a \in \mathbb{Z}$ so that $\mathrm{Ker}(Cl_\infty \to Cl_a) \subset \Gamma$. Choose $a_1, \cdots, a_d$ so that $\prod_i \gamma_i^{\ell^{a_i+n}\mathbb{Z}_\ell} = \mathrm{Ker}(Cl_\infty \to Cl_{a+n})$ for $n \geq 0$. Let $P = p^j$ with $j \geq r$ as in Lemma 5.1.

Suppose $\mathfrak{l}^m$ is principal generated by $\varpi = \varphi\varphi^c$ for $\varphi \in R$. Then $\Upsilon = \bigcup_{i\geq N} \Upsilon_i$ is disjoint from $\overline{\mathcal{X}}$ by Lemma 5.1 for some positive integer N. Put $\Xi_{a+im} = \{x(\mathcal{A})|\mathcal{A} \in \mathrm{Ker}(Cl_{a+im} \to Cl_a)\}$, replacing m by a positive multiple so that $m \geq N - a$. Define an infinite arithmetic progression $\underline{n} := \{a + im|i = 1, 2, \dots\}$. Then $\mathbf{T}_{a,m}$ acts transitively on Ξ, and by Theorem 2.7 and the proof of [H04, Proposition 2.8], Ξ embedded in $V^{\mathcal{Q}}$ by $\mathcal{A} \mapsto \mathbf{x}([\mathcal{A}]) := (x([\mathcal{A}][\mathfrak{Q}]_\Gamma))_{\mathfrak{Q}\in\mathcal{Q}}$ is Zariski dense.

For each $\chi \in \Upsilon$,

$$\sum_{\mathfrak{Q}} \psi(\mathfrak{Q})^{-1} \sum_{\mathcal{A}\in y\widetilde{\chi}^{-1}(\mu^d_{\ell^j})} \chi(\mathcal{A}) f_{\psi,\mathfrak{c}}([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma) = 0$$

holds by (4.15) (see also [H04, page 770]) for $\mathcal{A}$ with $\mathbf{x}([\mathcal{A}]) \in \Xi_{\underline{n}}$.

Identify again $\Phi_n = O_{\mathfrak{l}}/(\mathfrak{l}O_{\mathfrak{l}})^j$ and define $v(\chi) \in (O_{\mathfrak{l}}/(\mathfrak{l}O_{\mathfrak{l}})^j)^\times$ by $\chi(u) = \zeta_j^{\mathrm{Tr}(vu)}$ for all $u \in O_{\mathfrak{l}}/(\mathfrak{l}O_{\mathfrak{l}})^j = \Phi_n$. Let $g_v := \sum_{u \in O/\mathfrak{l}^j} \chi_v(u) f_\psi | \varrho(u/\varpi_{\mathfrak{l}}^j)$ for $\chi_v(u) = \zeta_j^{\mathrm{Tr}(vu)}$ for $\mathrm{Tr} := \mathrm{Tr}_{O_{\mathfrak{l}}/\mathbb{Z}_\ell}$. Then

$$\sum_{\mathfrak{Q}} \psi(\mathfrak{Q})^{-1} \sum_{\mathcal{A} \in \Phi_n} g_v|[\mathfrak{q}]([\mathcal{A}][\mathfrak{Q}]_\Gamma) = 0 \quad \text{for } \mathbf{x}([\mathcal{A}]) \in \Xi. \tag{5.3}$$

By Zariski density of Ξ in V^v, we conclude $g_v|[\mathfrak{q}] = 0$. Since $[\mathfrak{q}] \in \mathrm{Aut}(Sh^{(p)})$, we conclude $g_v = 0$.

For a chosen class $v \in (O/\mathfrak{l}^j)$, we find ξ such that $\xi \in -v$ and $a(\xi, f_\psi) \neq 0 \Leftrightarrow a(\xi, g_v) \neq 0$, and from this we conclude contradiction against $a(\xi, f_\psi) \neq 0$. $\square$

Here is an obvious corollary of the proof of Theorem 5.2:

Corollary 5.3. *Let the notation be as in* Theorem 5.2. *Suppose $d = 1$ and $a(\xi, f_\psi) \neq 0$ for some $\xi \in -v$ for a given $v \in (O/\mathfrak{l}^\nu)^\times$. For a character $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}(\mathbb{F}))$, define $n(\chi)$ by* $\mathrm{Ker}(\chi) = \mathrm{Ker}(\Gamma \to Cl_{n(\chi)})$. *Define a subset of $\mathbb{Z}$ by*

$$\underline{n}_v := \{n(\chi) | v(\chi) = v \quad \text{and} \quad \int_{Cl^-_{n(\chi)}} \chi\psi d\varphi_\Gamma = 0\}.$$

Then $\underline{n}_v$ cannot contain any infinite arithmetic progression.

We can interpret heuristically the above corollary into a natural density 0 result. Let

$$\underline{n} = \{0 < n_0 < n_1 < n_2 < \cdots < n_i < \cdots\}$$

be an infinite sequence of integers. We define the density of $\underline{n}$ by

$$D(\underline{n}) := \lim_{|x| \to \infty} \frac{|\{j | n_j \le |x|\}|}{|x|}.$$

Consider the function $\phi = \phi_{\underline{n}} : j \mapsto n_j$ defined on the set of natural numbers $\mathbb{N} := \{n \in \mathbb{Z} | n > 0\}$. We study $D(\underline{n})$ in terms of ϕ. Suppose

(E) $\underline{n}$ does not contain any arithmetic progression.

Let $\Delta\phi(x) = \phi(x+1) - \phi(x)$. Suppose that $\Delta\phi(x)$ is bounded by an integer $B > 0$. Then the map $\mathbb{Z} \ni x \mapsto \phi(x) \mod B$ has a fiber F over $a \in [0, B) \cap \mathbb{Z}$ with infinitely many element by the pigeon hole principle. Arrange the set $F' := \{m | a + mB \in F\}$ in increasing order, if F' contains an additive subgroup, then $\underline{n}$ contains an arithmetic progression, a contradiction to (E). Thus $\Delta\phi(x)$ is unbounded. Therefore $\lim_{x \to \infty} \phi(x)/Bx = \infty$ for all $B > 0$, and we have $|\Delta\phi(x)| \le B$ for $x \gg 0$. This implies $D(\underline{n}) = 0$ if $\underline{n}$ does not contains arithmetic progression. As mentioned in the introduction, there exist a sequence of positive lower density without containing an arithmetic progression [W72]. However we are inclined to believe at least

Conjecture 5.4. *Let the notation and the assumption be as in* Corollary 5.3. *Then $D(\underline{n}_v) = 0$.*

5.2. **A rigidity lemma.** If a formal completion at a point v of a scheme $V_{/\mathbf{W}}$ has a canonical structure of a formal torus, a subvariety passing through v is interesting in the sense that such a variety is often quite limited (and such a result is called a rigidity result). If V is a good integral model of a Shimura variety, such subvariety is close to a Shimura subvariety (as Chai calls it a Tate linear subvariety of a Shimura variety). The study of the linearity was an essential tool in the study of density of CM points in [H10] which we used. Here we study formal subschemes of $\widehat{G} := \widehat{\mathbb{G}}^n_{m/\mathbf{W}}$ stable under the action of $t \mapsto t^z$ for all z in an open subgroup U of $\mathbb{Z}_\ell^\times$. Intuitively, one might expect that such a formal scheme is a union of a coset of formal subtori. We can prove this if the formal subscheme has large quantity of torsion points (cf. Lemma 5.1). There is a general result due to Chai of this type (see [EAI, Theorem 10.6]). Here we recall with a proof an easy case of such rigidity results used in the proof of Lemma 5.1 from [H14, Lemma 4.1]:

Lemma 5.5. *Let $\mathcal{K}$ be a finite extension of* $\mathrm{Frac}(W(\overline{\mathbb{F}}_\ell))$ *and $\mathbf{W}$ be the integral closure of $W(\overline{\mathbb{F}}_\ell)$ in $\mathcal{K}$. Let $T = \mathrm{Spf}(\mathcal{T})$ be a closed formal subscheme of $\widehat{G} = \widehat{\mathbb{G}}^n_{m/\mathbf{W}}$ flat geometrically irreducible over $\mathbf{W}$ (i.e., $\mathcal{T} \cap \overline{\mathbb{Q}}_\ell = \mathbf{W}$). Suppose there exists an open subgroup U of $\mathbb{Z}_\ell^\times$ such that T is stable under the action $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$ for all $u \in U$. If T contains a Zariski dense subset $\Omega \subset T(\mathbb{C}_\ell) \cap \mu^n_{\ell^\infty}(\mathbb{C}_\ell)$, then we have $\omega \in \Omega$ and a formal subtorus $\mathbf{T}$ such that $T = \mathbf{T}\omega$.*

A similar assertion is not valid for a formal group $\widehat{\mathbb{G}}^2_{m/K} = \mathrm{Spec}(K[[T, T']])$ over a characteristic 0 field K. Writing $t = 1+T$ and $t' = 1+T'$ for multiplicative variables t, t', the formal subscheme Z defined by $t^{\log(t')} = 1$ is not a formal torus, but it is stable under $(t, t') \mapsto (t^m, t'^m)$ for any $m \in \mathbb{Z}$. See [C02, Remark 6.6.1 (iv)] for an optimal expected form of the assertion similar to the above lemma.

Proof. Let T_s be the singular locus of the associated scheme $T^{sh} = \mathrm{Spec}(\mathcal{T})$ over $\mathbf{W}$, and put $T^\circ = T^{sh} \setminus T_s$. The scheme T_s is a closed formal subscheme of T with $\dim T_s < \dim T$ as $\mathcal{T}$ is excellent [CRT, §32]. To see this, we note, by the structure theorem of complete noetherian ring, that $\mathcal{T}$ is finite over a power series ring $\mathbf{W}[[X_1, \dots, X_d]] \subset \mathcal{T}$ for $d = \dim_{\mathbf{W}} T$ (cf. [CRT, §29]). The sheaf of continuous differentials $\Omega_{\mathcal{T}/\mathrm{Spf}(\mathbf{W}[[X_1,\dots,X_d]])}$ is a torsion $\mathcal{T}$-module, and T_s is the support of the formal sheaf of $\Omega_{\mathcal{T}/\mathrm{Spf}(\mathbf{W}[[X_1,\dots,X_d]])}$ (which is a closed formal subscheme of T). The regular locus T° of T is open dense in the generic fiber $T^{sh}_{/\mathcal{K}} := T^{sh} \times_{\mathbf{W}} \mathcal{K}$ of T^{sh}. Then $\Omega^\circ := T^\circ \cap \Omega$ is Zariski dense in $T^{sh}_{/\mathcal{K}}$.

In this proof, by extending scalars, we always assume that $\mathbf{W}$ is sufficiently large so that for $\zeta \in \Omega$ we focus on, we have $\zeta \in \widehat{G}(\mathbf{W})$ and that we have a plenty of elements of infinite order in $T(\mathbf{W})$ and in $T^\circ(\mathcal{K}) \cap T(\mathbf{W})$, which we simply write as $T^\circ(\mathbf{W}) := T^\circ(\mathcal{K}) \cap T(\mathbf{W})$.

Note that the stabilizer U_ζ of $\zeta \in \Omega$ in U is an open subgroup of U. Indeed, if the order of ζ is equal to ℓ^a, then $U_\zeta = U \cap (1 + \ell^a\mathbb{Z}_\ell)$. Thus making a variable change $t \mapsto t\zeta^{-1}$ (which commutes with the action of U_ζ), we may assume that the identity $\mathbf{1}$ of $\widehat{G}$ is in Ω°.

Let $\widehat{G}^{an}$, T_{an} and T^s_{an} be the rigid analytic spaces associated to T and T^s (in Berthelot's sense in [J95, §7]). We put $T^\circ_{an} = T_{an} \setminus T^s_{an}$, which is an open rigid analytic subspace of T_{an}. Then we apply the logarithm $\log : \widehat{G}^{an}(\mathbb{C}_\ell) \to \mathbb{C}_\ell^n = Lie(\widehat{G}^{an}_{/\mathbb{C}_\ell})$ sending $(t_i)_i \in \widehat{G}^{an}(\mathbb{C}_\ell)$ (the ℓ-adic open unit ball centered at $\mathbf{1} = (1, 1, \dots, 1)$) to $(\log_\ell(t_i))_i \in \mathbb{C}_\ell^n$ for the ℓ-adic Iwasawa logarithm map $\log_\ell : \mathbb{C}_\ell^\times \to \mathbb{C}_\ell$. Then for each smooth point $x \in T^\circ(\mathbf{W})$, taking a small analytic open neighborhood V_x of x (isomorphic to an open ball in $\mathbf{W}^d$ for $d = \dim_{\mathbf{W}} T$) in $T^\circ(\mathbf{W})$, we may assume that $V_x = G_x \cap T^\circ(\mathbf{W})$ for an n-dimensional open ball G_x in $\widehat{G}(\mathbf{W})$ centered at $x \in \widehat{G}(\mathbf{W})$. Since $\Omega^\circ \neq \emptyset$, $\log(T^\circ(\mathbf{W}))$ contains the origin $0 \in \mathbb{C}_\ell^n$. Take $\zeta \in \Omega^\circ$. Write T_ζ for the Tangent space at ζ of T. Then $T_\zeta \cong \mathbf{W}^d$ for $d = \dim_{\mathbf{W}} T$. The space $T_\zeta \otimes_{\mathbf{W}} \mathbb{C}_\ell$ is canonically isomorphic to the tangent space T_0 of $\log(V_\zeta)$ at 0.

If $\dim_{\mathbf{W}} T = 1$, there exists an infinite order element $t_1 \in T(\mathbf{W})$. We may (and will) assume that $U = (1 + \ell^m\mathbb{Z}_\ell)$ for $0 < m \in \mathbb{Z}$. Then T is the (formal) Zariski closure $\overline{t_1^U}$ of

$$t_1^U = \{t_1^{1+\ell^m z} | z \in \mathbb{Z}_\ell\} = t_1\{t_1^{\ell^m z} | z \in \mathbb{Z}_\ell\},$$

which is a coset of a formal subgroup Z. The group Z is the Zariski closure of $\{t_1^{\ell^m z} | z \in \mathbb{Z}_\ell\}$; in other words, regarding t_1^u as a $\mathbf{W}$-algebra homomorphism $t_1^u : \mathcal{T} \to \mathbb{C}_\ell$, we have $t_1 Z = \mathrm{Spf}(\mathcal{Z})$ for $\mathcal{Z} = \mathcal{T}/\bigcap_{u \in U} \mathrm{Ker}(t_1^u)$. Since t_1^U is an infinite set, we have $\dim_{\mathbf{W}} Z > 0$. From geometric irreducibility and $\dim_{\mathbf{W}} T = 1$, we conclude $T = t_1 Z$ and $Z \cong \widehat{\mathbb{G}}_m$. Since T contains roots of unity $\zeta \in \Omega \subset \mu^n_{\ell^\infty}(\mathbf{W})$, we confirm that $T = \zeta Z$ for $\zeta \in \Omega \cap \mu^n_{\ell^{m'}}$ for $m' \gg 0$. Replacing t_1 by $t_1^{\ell^m}$ for m as above if necessary, we have the translation $\mathbb{Z}_\ell \ni s \mapsto \zeta t_1^s \in Z$ of one parameter subgroup $\mathbb{Z}_\ell \ni s \mapsto t_1^s$. Thus we have $\log(t_1) = \frac{dt_1^s}{ds}|_{s=0} \in T_\zeta$, which is sent by "$\log : \widehat{G} \to \mathbb{C}_\ell^n$" to $\log(t_1) \in T_0$. This implies that $\log(t_1) \in T_0$ and hence $\log(t_1) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ (under the identification of the tangent space at any $x \in \widehat{G}$ with $Lie(\widehat{G})$). Therefore T_ζ's over $\zeta \in \Omega^\circ$ can be identified canonically. This is natural as Z is a formal torus, and the tangent bundle on Z is constant, giving $Lie(Z)$.

Suppose that $d = \dim_{\mathbf{W}} T > 1$. Consider the Zariski closure Y of t^U for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^\circ$). Since U permutes finitely many geometrically irreducible components, each component of Y is stable under an open subgroup of U. Therefore $Y = \bigcup \zeta' \mathcal{T}_{\zeta'}$ is a union of formal subtori $\mathcal{T}_{\zeta'}$ of dimension ≤ 1, where ζ' runs over a finite set inside $\mu^n_{\ell^\infty}(\mathbb{C}_\ell) \cap T(\mathbb{C}_\ell)$. Since $\dim_{\mathbf{W}} Y = 1$, we can pick $\mathcal{T}_{\zeta'}$ of dimension 1 which we denote simply by $\mathcal{T}$. Then $\mathcal{T}$ contains t^u for some $u \in U$. Applying the argument in the case of $\dim_{\mathbf{W}} T = 1$ to $\mathcal{T}$, we find $u\log(t) = \log(t^u) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ and $t \in V_\zeta$. Summarizing our argument, we have found

(t) The Zariski closure of t^U in T for an element $t \in V_\zeta$ of infinite order contains a coset $\xi\mathcal{T}$ of one dimensional subtorus $\mathcal{T}$, $\xi^{\ell^{m'}} = 1$ and $t^{\ell^{m'}} \in \mathcal{T}$ for some $m' > 0$;

(D) Under the notation as above, we have $\log(t) \in T_\zeta$ for all $\zeta \in \Omega^\circ$.

Moreover, the image $\overline{V}_\zeta$ of V_ζ in $\widehat{G}/\mathcal{T}$ is isomorphic to $(d-1)$-dimensional open ball. If $d > 1$, therefore, we can find $\overline{t}' \in \overline{V}_\zeta$ of infinite order. Pulling back $\overline{t}'$ to $t' \in V_\zeta$, we find $\log(t), \log(t') \in T_\zeta$, and $\log(t)$ and $\log(t')$ are linearly independent in T_ζ. Inductively arguing this way, we find infinite order elements $t_1, \dots, t_d$ in V_ζ such that $\{\log(t_i)\}_{i=1,\dots,d}$ spans over the quotient field K of $\mathbf{W}$ the tangent space $T_{\zeta/K} = T_\zeta \otimes_{\mathbf{W}} \mathcal{K} \hookrightarrow T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{\mathbf{1}/\mathcal{K}} \subset T_0$ with $T_{\zeta/\mathcal{K}} \subset T_0$. Thus the tangent bundle over $T^\circ_{/\mathcal{K}}$ is constant as it is constant over the Zariski dense subset Ω°. Therefore T° is close to an open dense subscheme of a coset of a formal subgroup. We pin-down this fact.

Take $t_i \in V_\zeta$ as above $(i = 1, 2, \dots, d)$ which give rise to a basis $\{\partial_i = \log(t_i)\}_i$ of the tangent space of $T_{\zeta/K} = T_{\mathbf{1}/K}$. Note that $t_i^u \in T$ and $u\partial_i = \log(t_i^u) = u\log(t_i) \in T_{\mathbf{1}/K}$ for $u \in U$. The embedding $\log : V_\zeta \hookrightarrow T_{\mathbf{1}} \subset Lie(\widehat{G}_{/\mathbf{W}})$ is surjective onto a open neighborhood of $0 \in T_{\mathbf{1}}$ (by extending scalars if necessary). For $t \in V_\zeta$, if we choose t closer to ζ, $\log(t)$ getting closer to 0. Thus by replacing $t_1, \dots, t_d$ inside V_ζ to elements in V_ζ closer to ζ, we may assume that $\log(t_i) \pm \log(t_i)$ for all $i \neq j$ is in $\log(V_\zeta)$.

So, for each pair $i \neq j$, we can find $t_{i\pm j} \in V_\zeta$ such that $\log(t_i t_j^{\pm 1}) = \log(t_i) \pm \log(t_j) = \log(t_{i\pm j})$. The element $\log(t_{i\pm j})$ is uniquely determined in $\log(\widehat{G}_{an}(\mathbb{C}_\ell)) \cong \widehat{G}_{an}(\mathbb{C}_\ell)/\mu^n_{\ell^\infty}(\mathbb{C}_\ell)$. Thus we conclude $\zeta'_{i\pm j} t_i t_j^{\pm 1} = t_{i\pm j}$ for some $\zeta'_{i\pm j} \in \mu^n_{\ell^N}$ for sufficiently large N. Replacing T by its image under the ℓ-power isogeny $\widehat{G} \ni t \mapsto t^{\ell^N} \in \widehat{G}$ and t_i by $t_i^{\ell^N}$, we may assume that $t_i t_j^{\pm 1} = t_{i\pm j}$ all in T. Since $t_i^U \subset T$, by (t), for a sufficiently large $m' \in \mathbb{Z}$, we find a one dimensional subtorus $\widehat{H}_i$ containing $t_i^{\ell^{m'}}$ such that $\zeta_i \widehat{H}_i \subset T$ with some $\zeta_i \in \mu^n_{\ell^{m'}}$ for all i. Thus again replacing T by the image of the ℓ-power isogeny $\widehat{G} \ni t \mapsto t^{\ell^{m'}} \in \widehat{G}$, we may assume that the subgroup $\widehat{H}$ (Zariski) topologically generated by $t_1, \dots, t_d$ is contained in T. Since $\{\log(t_i)\}_i$ is linearly independent, we conclude $\dim_{\mathbf{W}} \widehat{H} \geq d = \dim_{\mathbf{W}} T$, and hence T must be the formal subgroup $\widehat{H}$ of $\widehat{G}$. Since T is geometrically irreducible, $\widehat{H} = T$ is a formal subtorus. Pulling it back by the ℓ-power isogenies we have used, we conclude $T = \zeta\widehat{H}$ for the original T and $\zeta \in \mu^n_{\ell^{m'N}}(\mathbf{W})$. Since Ω is Zariski dense in T, we may assume that $\zeta \in \Omega$. This finishes the proof. $\square$

5.3. **Semi-group action.** An explicit description of the action of α_m in §2.4 is a key to prove the density corollary Corollary 2.12. Though we do not need it, we add here an explicit determination of the action of α_m^{-1} (and α_m in the case not treated yet) on the point $x(\mathcal{A})$ defined in [H04, §2.1].

More generally, we consider a pair $(L, \eta : \widehat{O}^2 \cong \widehat{L})$ of an O-lattice L of M and an $\widehat{O}$-linear isomorphism $\eta : (F_{\mathbb{A}}^{(p\infty)})^2 \cong \widehat{L} \otimes_O F_{\mathbb{A}}^{(p\infty)}$ with $\eta((\widehat{O}^{(p)})^2) = \widehat{L}^{(p)}$. We suppose that $L_p = R_p$. We define $Lg = \mathrm{Im}(\eta \circ g(\widehat{O}^2)) \cap M$ and $(L, \eta)g := (Lg, \eta \circ g)$. The pair gives rise to a point $x(L) \in Sh^{(p)}_{/W}$.

Choose a prime element $\varpi_\mathfrak{l}$ of $O_\mathfrak{l}$ and if $\mathfrak{l}$ ramifies in R, we suppose that $R_\mathfrak{l} = O_\mathfrak{l} + \sqrt{\varpi_\mathfrak{l}} O_\mathfrak{l}$. Recall $R_n = O + \mathfrak{l}^n R$. If ℓ is odd or ℓ dose not split in R, we write $R_\mathfrak{l} = O_\mathfrak{l} + \delta O_\mathfrak{l}$ so that $\delta = \sqrt{\varpi_\mathfrak{l}}$ if $\mathfrak{l}$ ramifies in R and $\delta = \sqrt{d}$ for $d \in O_\mathfrak{l}^\times$ if $\mathfrak{l}$ is unramified ($d = \delta^2$ is square if $\mathfrak{l} = \mathfrak{L}\mathfrak{L}^c$ splits and $d = (\delta, -\delta) \in R_\mathfrak{L} \times R_{\mathfrak{L}^c} = R_\mathfrak{l}$). If $\ell = 2$ and ℓ splits in R, we define $R'_\mathfrak{l} = \{x \in R_\mathfrak{l} | x \equiv x^c \mod 2\}$ and we start with this order, which has basis 1 and $(1, -1) \in O_\mathfrak{l} \times O_\mathfrak{l} = R_\mathfrak{l}$. We note in this case $R_1 = (R'_\mathfrak{l} \cap F) \cap \bigcap_{\mathfrak{q} \neq \mathfrak{l}} (R_\mathfrak{q} \cap F)$ in F for primes $\mathfrak{q}$, and we put $\delta = (1, -1) \in R_{1,\mathfrak{L}}$ (so, we start with non-maximal order R_1). Then we put $\alpha_\mathfrak{l} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{l} \end{smallmatrix}\right) \in \mathrm{GL}_2(O_\mathfrak{l})$. We often regard $\alpha_\mathfrak{l} \in G(\mathbb{A})$ so that its component at a prime $\mathfrak{q} \neq \mathfrak{l}$ is equal to 1. We simply write R_n for the pair (R_n, η_n) with $\eta_n(a, b) = a + \varpi_\mathfrak{l}^n b$ at $\mathfrak{l}$ and outside $\mathfrak{l}$, we choose the level structure in §3.1 and define the level structure η_n accordingly.

Then we put $\alpha_\mathfrak{l}^{\pm 1}(x(R_n)) = x(R_n \alpha_\mathfrak{l}^{\pm 1})$ under the action defined above. This action depends only on local component at $\mathfrak{l}$. As seen in (3.2) and (3.3), we have

$$\alpha_\mathfrak{l}(x(R_n)) = x(R_{n+1}) \text{ and } \alpha_\mathfrak{l}^{-1}(x(R_n)) = x(R_{n-1}) \text{ if } n > 0. \tag{5.4}$$

Since

$$\alpha_\mathfrak{l}^{-n} \left[\begin{smallmatrix} 1 \\ \delta \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ \varpi_\mathfrak{l}^{-n}\delta \end{smallmatrix}\right] = \varpi_\mathfrak{l}^{-n}\delta \left[\begin{smallmatrix} \varpi_\mathfrak{l}^n \delta^{-1} \\ 1 \end{smallmatrix}\right]$$

at $\mathfrak{l}$, we need to change the original level structure η_n to η'_n given by $\eta'_n(a, b) = \varpi_\mathfrak{l}^{-n}\delta^{-1}(a\varpi_\mathfrak{l}^n\delta^{-1} + b)$ at $\mathfrak{l}$ and outside $\mathfrak{l}$, the choice is the same as η_n. The lattice will change as follows

(un) $R_\mathfrak{l} \mapsto \mathfrak{l}^{-n} R_{n,\mathfrak{l}}$ with $R_0 = R$ if $\mathfrak{l}$ remains prime or $\mathfrak{l}$ is odd and split in R;

(ram) $R_\mathfrak{l} \mapsto \mathfrak{l}^{-n}\mathfrak{L} R_{n,\mathfrak{l}}$ with $R_0 = R$ if $\mathfrak{l} = \mathfrak{L}^2$ in R;

(sp2) $R_{0,\mathfrak{l}} \mapsto \mathfrak{l}^{-n} R_{n,\mathfrak{l}}$ with $R_0 = R_1$ if $\mathfrak{l}|2$ and $\mathfrak{l}$ splits in R.

Denote $x'(\mathcal{A}) = (\mathcal{A}, \eta'_n)$ with $\mathcal{A}_\mathfrak{l}$ equal to the ideal as in (un), (ram) and (sp2). Since

$$Cl_n = \frac{\{\text{fractional projective } R_n\text{-ideals}\}}{\{\text{principal } R_n\text{-ideals}\}},$$

we may allow R_n-ideals not prime to $\mathfrak{l}$. For an R_n-fractional ideal $\mathcal{A}$ prime to $\mathfrak{l}$, we denote $\mathcal{A}_n$ (resp. $\mathcal{A}'_n$) by the R_n-fractional ideal $\mathcal{A}_n$ (resp. $\mathcal{A}'_n$) with $\mathcal{A}_{n,\mathfrak{l}} = R_{n,\mathfrak{l}}$ (resp. $\mathcal{A}'_{n,\mathfrak{l}}$ given as in (un), (ram) and (sp2)), and outside $\mathfrak{l}$, it is equal to the given $\mathcal{A}$. We have the following effect of $\alpha_\mathfrak{l}^m$ on the points $x(\mathcal{A}_n)$ and $x'(\mathcal{A}'_n)$;

(+) $\alpha_\mathfrak{l}^m(x(\mathcal{A}_n)) = x(\mathcal{A}_{n+m})$ and $\alpha_\mathfrak{l}^{-m}(x'(\mathcal{A}'_n)) = x'(\mathcal{A}'_{n+m})$ if $n > 0$ and $m \geq 0$;

(0) $\alpha_\mathfrak{l}^m(x'(\mathcal{A}'_n)) = x(\mathcal{A}_{m-n})$ and $\alpha_\mathfrak{l}^{-m}(x(\mathcal{A}_n)) = x'(\mathcal{A}'_{m-n})$ if $m \geq n$;

(−) $\alpha_\mathfrak{l}^m(x'(\mathcal{A}'_n)) = x'(\mathcal{A}'_{n-m})$ and $\alpha_\mathfrak{l}^{-m}(x(\mathcal{A}_n)) = x(\mathcal{A}_{n-m})$ if $n > m$.

6. Key steps in the proof of anticyclotomic main conjecture in [H06]

In this section, we continue to assume (unr), (ord) and $p > 3$. Take W sufficiently large, and fix an anti-cyclotomic finite prime-to-p order character $\psi : M_\mathbb{A}^\times/M^\times \to W^\times$ with prime-to-p conductor $\mathfrak{C}(\psi)$. Here we say ψ is anti-cyclotomic if $\psi(x^c) = \psi^{-1}(x)$ for all $x \in M_\mathbb{A}^\times$. Define an integral F-ideal $\mathfrak{i} = \mathfrak{i}(\psi)$ by the product of all unramified inert primes in $\mathfrak{C}(\psi)$. Let

$$h_\mathfrak{i}(M/F) := (h(M)/h(F)) \prod_{\mathfrak{l}|\mathfrak{i}} (|O/\mathfrak{l}| + 1) \tag{6.1}$$

for the class numbers $h(M)$ and $h(F)$. For a Hecke character φ of M of type A_0, regarding it as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ by class field theory, we write $\varphi_c(\sigma) = \varphi(c\sigma c)$ for complex conjugation c and $\varphi^- = \varphi/\varphi_c$. It is a consequence of class field theory (see [H06, (7.18)] and [HMI, Lemma 5.31]) that we can always find another finite order Hecke character $\varphi : M_\mathbb{A}^\times/M^\times M_\infty^\times \to \mathbb{C}^\times$ such that its anti-cyclotomic projection φ^- is the starting anti-cyclotomic character ψ. Replacing φ by the Teichmüller lift of (φ mod $\mathfrak{m}_W$), we assume that φ also has order prime to p. A key step towards the proof of the anticyclotomic main conjecture is the following divisibility in the introduction of [H06] and [H09, (A)]:

$$h_\mathfrak{i}(M/F) L_p^-(\psi) \big| H(\varphi) \text{ in } W[[\Gamma_M]]. \tag{L}$$

We will recall the detailed meaning of the notation in (L) in the following subsection §6.2. Briefly, here $h(M)$ (resp. $h(F)$) is the class number of M (resp. F), and Γ_M is the Galois group over M of the composite of all $\mathbb{Z}_p$-extensions of M and Γ_M^- (resp. Γ_M^+) is its anti-cyclotomic (resp. cyclotomic) projection. The power series $L_p^-(\psi)$ in $W[[\Gamma_M^-]]$ is the anti-cyclotomic Katz p-adic L-function with branch character ψ, $H(\varphi)$ is the congruence power series associated to the ordinary p-adic analytic family $\boldsymbol{\theta}(\varphi)$ of modular form containing the theta series of φ. The power series $H(\varphi)$ is determined up to units in $W[[\Gamma_M]]$ (and the identity hereafter is always up to units in the Iwasawa algebras containing the two sides).

In [H06], the divisibility (L) is attributed to [H07, Corollary 5.6], whose proof relies on the stronger version [H07, Theorem 4.3] of Theorem 0.2 (actually the density 0 expectation in Conjecture 5.4 is sufficient). In [H06], this corollary was quoted as Corollary 5.5 in [H06, page 468], but it became Corollary 5.6 after publication of [H07] one year after the publication of [H06]. This stronger version is still an open question as remarked below Theorem 0.1. In any case, as Corollary 6.15, we give a different proof of (L) (without assuming any conjecture).

Here is an outline of the new proof. The new proof goes as follows. Following the technique of [HT93], the formula below was proven in [H07, Theorem 5.5] under some assumptions on ϕ which we recall later:

$$\frac{\mathcal{R}}{H(\varphi)} = \frac{\mathcal{L}_p(\varphi^{-1}\phi)\mathcal{L}_p(\varphi^{-1}\phi_c)}{h_\mathfrak{i}(M/F) L_p^-(\psi)}, \tag{RK0}$$

Here φ is as in (L) for which we want to prove (L) and ϕ is a well chosen character of conductor divisible by a high $\mathfrak{l}$-power. The numerator $\mathcal{R} \in W[[\Gamma_M \times \Gamma_M]]$ interpolating the Rankin product of the two CM families $\boldsymbol{\theta}(\phi)$ and $\boldsymbol{\theta}(\varphi)$. The numerator of the right-hand-side of (RK0) is the product of the two Katz p-adic L-functions $\mathcal{L}_p(\varphi^{-1}\phi)$ and $\mathcal{L}_p(\varphi^{-1}\phi_c)$ (with some Euler factors removed) with branch characters $\varphi^{-1}\phi$ and $\varphi^{-1}\phi_c$, respectively. The argument relies on the vanishing of the

μ-invariant of Katz p-adic L-functions [H11]. Indeed, in [HT93, Theorem I], (L) is proven under the vanishing of the μ-invariant of the numerator of the right-hand-side of (RK0) which we realize by (RK0) and a new formula (RK1) in [H09] below. In [H11], the vanishing of the μ-invariant of primitive Katz p-adic L-functions is proven under some restrictive assumptions (including (A1) below) 18 years later. Therefore we analyze carefully the missing Euler factors and arrange ϕ for them to have vanishing μ-invariant; thereby, the divisibility (L) follows. We note also that by the general formula (RK1), we removed the split conductor assumption (A1) in [H09].

We need an improvement of (RK0) in [H09] to remove the split conductor assumption (A1). We choose a CM quadratic extension M_1/F disjoint from M (or more precisely, if $M = M_1$, the formula we obtain is just (RK0); so, we assume for a while, $M \neq M_1$). For a Hecke character ξ of $X = M, M_1$ and the composite $K = MM_1$, write $\widehat{\xi} := \xi \circ N_{K/X}$ as a Hecke character of K. Adjusting the notation to the formula (RK0), in [H09, Theorem 3.5] (where slightly different notation was used), under some assumptions on ϕ (which we recall later), the following formula generalizing (RK0) is proven:

$$\text{(RK1)} \qquad \frac{\mathcal{R}}{H(\varphi)} = \frac{\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\phi})}{h_{\mathfrak{i}}(M/F)L_p^-(\psi)},$$

where ϕ is a branch Hecke character of M_1 such that $\widehat{\varphi}^{-1}\widehat{\phi}$ has split conductor in K (over its maximal real subfield), $\mathcal{R} \in W[[\Gamma_M \times \Gamma_{M_1}]]$ and $\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\phi})$ is the Katz p-adic L-function $\mathcal{K}$ in $W[[\Gamma_K]]$ with some Euler factors removed (of branch character $\widehat{\varphi}^{-1}\widehat{\phi}$) projected to $W[[\Gamma_M \times \Gamma_{M_1}]]$ by $N_{K/M}^{-1} \times N_{K/M_1}$.

We will choose ϕ well in (RK0) and (RK1) so that the imprimitive Katz p-adic L-functions appearing in the numerator also have vanishing μ-invariant; so, if p^μ exactly divides $h_{\mathfrak{i}}(M/F)L_p^-(\psi)$, p^μ has to divide the denominator of the left-hand-side $H(\varphi)$. Combining with [HT93, Theorem I], this completes the proof of (L).

In [H07], using the stronger version of Theorems 0.1 and 0.2, under (A1), we find ϕ in (RK1) so that the numerator of the right-hand-side is a unit in $W[[\Gamma_M \times \Gamma_{M_1}]]$. Whether the actual Theorems 0.1 and 0.2 are sufficient for this argument is not clear.

6.1. **Summary of assumptions, definitions and known facts.** Before going into a detailed description of the argument we sketched, we recall the assumptions and the notations we made in [H06]. In addition to (unr) and (ord), the assumptions we made in [H06] are:

- (A1) The character ψ has prime-to-p conductor made of split primes over F;
- (A2) The local character $\psi_{\mathfrak{P}}$ is non-trivial for all $\mathfrak{P} \in \Sigma_p$;
- (A3) The restriction ψ_* of ψ to $\mathrm{Gal}(\overline{F}/M[\sqrt{p^*}])$ for $p^* = (-1)^{(p-1)/2}p$ is non-trivial.

The assumption (A1) is later removed in [H09] and (A3) is used to prove the main conjecture from (L), and the argument of [H09] and the part of [H06] involving (A3) are nothing to do with weakening Theorem 0.1 (resp. Theorem 0.2) of the stronger assertion [H07, Theorem 4.2] (resp. [H07, Theorem 4.3]); so, we prove (L) for an odd prime p under (A2) (the argument in the case where non-split conductor will be exposed in §6.3.

Since $\psi_c = \psi^{-1}$, we find $\psi^- = \psi^2$ and $\psi_F^2 = \mathbf{1}$ for $\psi_F := \psi|_{F_{\mathbb{A}}^\times}$. Indeed, $\psi(N_{M/F}(x)) = \psi\psi_c(x) = 1$, and hence ψ_F factors through $F_{\mathbb{A}}^\times/F^\times N_{M/F}(M_{\mathbb{A}}^\times) \cong \mathrm{Gal}(M/F)$. Thus ψ_F is either trivial or $\chi_M := \left(\frac{M/F}{}\right)$. Write $\mathfrak{C}(?)$ for the conductor of a Hecke character ? of M.

Lemma 6.1. *We can find a finite order Hecke character φ of M such that $\varphi^- = \psi$ satisfying*

1. *the order of φ is prime to p;*
2. *for any given finite set S of primes outside $\mathfrak{C}(\psi)$, $\mathfrak{C}(\varphi)$ is outside S.*

Proof. As is well known (e.g., [HMI, Lemma 5.31]), we can always find a finite order Hecke character φ of M such that $\psi = \varphi^-$. Taking a high p-power p^m so that $\psi^{p^m-1} = 1$ and φ^{p^m} has order prime to p, replacing φ by φ^{p^m}, we assume that φ has order prime to p with $\varphi^- = \psi$.

Take a prime factor $\mathfrak{Q}|\mathfrak{C}(\varphi)$ outside $\mathfrak{C}(\psi)$ and put $\mathfrak{q} := \mathfrak{Q} \cap F$. Then $\varphi_{\mathfrak{q}} := \varphi|_{R_{\mathfrak{q}}^\times}$ satisfies $\varphi_{\mathfrak{q}} \circ c = \varphi_{\mathfrak{q}}$.

Consider the norm map: $M_{\mathfrak{q}}^\times \xrightarrow{N_{K/F}} F_{\mathfrak{q}}^\times$. Since $N_{M/F}(x) = x^{1+c}$, we have $H^1(M/F, M_{\mathfrak{q}}^\times) = \mathrm{Ker}(N_{M/F})/\mathrm{Im}(1-c) = \{1\}$ by Hilbert's theorem 90. Thus $\varphi_{\mathfrak{q}}$ is trivial on $\mathrm{Im}(1-c) = \mathrm{Ker}(N_{M/F})$; so, $\varphi_{\mathfrak{q}}$ factors through $N_{M/F}$, and we have a finite order character $\Phi_{\mathfrak{q}} : F_{\mathfrak{q}}^\times \to \mathbb{C}^\times$ such that

$\varphi_{\mathfrak{q}} = \Phi_{\mathfrak{q}} \circ N_{M/F}$. By a theorem of Chevalley [C51, Théorème 1], we can extend $\Phi_{\mathfrak{q}}$ to a finite order global Hecke character Φ of F whose prime-to-$\mathfrak{q}$ conductor is outside a given positive integer. Then $\phi := \varphi(\Phi^{-1} \circ N_{M/F})$ satisfies $\phi^- = \psi$ and $\mathfrak{C}(\phi)$ is prime to $\mathfrak{q}$. In this way, we can avoid any finite set of primes outside $\mathfrak{C}(\psi)$ in the conductor of ϕ with $\phi^- = \psi$. Then as above, we can modify ϕ further to have order prime to p. □

Write $\mathfrak{C}(\xi)$ for the conductor of a Hecke character ξ of M and decompose $\mathfrak{C}(\xi) = \mathfrak{F}(\xi)\mathfrak{F}_c(\xi)\mathfrak{I}(\xi)\mathfrak{R}(\xi)$ so that $\mathfrak{F}(\xi)$ is a product of primes of M split over F with $\mathfrak{F} + \mathfrak{F}_c = R$ and $\mathfrak{F}_c(\xi)^c \subset \mathfrak{F}(\xi)$, $\mathfrak{I}(\xi)$ is a product of primes of M inert unramified over F in M and $\mathfrak{R}(\xi)$ is a product of ramified primes of M over F. For the restriction of $?(\xi)$ to F, we use the lower case gothic character (e.g., $\mathfrak{c}(\xi) = \mathfrak{C}(\xi) \cap F$ and $\mathfrak{i} = \mathfrak{i}(\psi) = \mathfrak{I}(\psi) \cap F$). Then we define $Z(\mathfrak{C}) := \varprojlim_n Cl_M(\mathfrak{C}p^n)$. We have a canonical split exact sequence: $\Delta_{\mathfrak{C}} \hookrightarrow Z(\mathfrak{C}) \twoheadrightarrow \Gamma_M$ for the maximal finite group $\Delta_{\mathfrak{C}}$ and the torsion-free Γ_M. Over the maximal subgroup $\Delta_{\mathfrak{C}}^{(p)}$ of $\Delta_{\mathfrak{C}}$ with order prime to p, the above exact sequence is split canonically; so, we write the projection of $z \in Z(\mathfrak{C})$ to Γ_M (resp. $\Delta_{\mathfrak{C}}^{(p)}$) as $z \mapsto z_M$ (resp. $z \mapsto z^{(p)}$). We identify $Z(\mathfrak{C})$ with an appropriate quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ by class field theory. Let $\mathfrak{N}$ (resp. $\mathfrak{N}'$) be the prime-to–p part of the "minimal level" of φ (resp. the Artin conductor of $\mathrm{Ind}_M^F \varphi$). The Artin conductor of $\mathrm{Ind}_M^F \varphi$ is given by $\mathfrak{N}' = N_{M/F}(\mathfrak{C}(\varphi))d(M/F)$ for the relative discriminant $d(M/F)$ of M/F, which is equal to the conductor of the automorphic induction $\pi(\varphi)$ of φ. We write $\mathfrak{i}(\varphi^-) = \prod_{\mathfrak{l}} \mathfrak{l}^{e(\mathfrak{l})}$ and define

$$\mathfrak{N}_{sc} = \mathfrak{N}^{(\mathfrak{i}(\varphi^-))} \prod_{\mathfrak{l}|\mathfrak{i}(\varphi^-)} \mathfrak{l}^{-\lceil e(\mathfrak{l})/2 \rceil}, \tag{6.2}$$

where $\lceil \alpha \rceil$ is the integer such that $\alpha \le \lceil \alpha \rceil \le \alpha + \frac{1}{2}$ and $\mathfrak{N}^{(\mathfrak{i}(\varphi^-))}$ is the prime-to-$\mathfrak{i}(\varphi^-)$ part of $\mathfrak{N} = d(M/F)\mathfrak{f}(\varphi^-)N_{M/F}(\mathfrak{I}(\varphi^-)\mathfrak{R}(\varphi^-))$ for $\mathfrak{f}(\varphi^-) = \mathfrak{F}(\varphi^-) \cap F$. Note that $\mathfrak{N}_{sc} \supset \mathfrak{N} \supset \mathfrak{N}'$.

Out of the character $\varphi : \Delta_{\mathfrak{C}}^{(p)} \to W^\times$, we have the universal character $\widetilde{\varphi} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \twoheadrightarrow Z(\mathfrak{C}) \to W[[\Gamma_M]]^\times$ deforming φ given by sending $z \in Z(\mathfrak{C})$ to $\varphi(z^{(p)})z_M \in W[[\Gamma_M]]$. Hereafter, for simplicity, we write Λ for $W[[\Gamma_M]]$. The induced representation $\mathrm{Ind}_M^F \widetilde{\varphi}$ is modular nearly ordinary at p, and hence, for a suitably chosen ϵ dependent on φ, by the universality of the nearly p–ordinary Hecke algebra $\mathbf{h} := \mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ defined in [H09, §2.4] (see below for a description of the Neben character set ϵ), we have a unique algebra homomorphism $\lambda = \lambda_\varphi : \mathbf{h} \to \Lambda$ such that $\mathrm{Tr}(\mathrm{Ind}_M^F \widetilde{\varphi}) \cong \lambda \circ \mathrm{Tr}(\rho_{\mathbf{h}})$ for the 2-dimensional universal nearly ordinary modular Galois representation $\rho_{\mathbf{h}}$ with trace in $\mathbf{h}$. For $P \in \mathrm{Spf}(\Lambda)(\mathbb{C}_p)$, $\widetilde{\varphi}_P := (\widetilde{\varphi} \bmod P)$ can be regarded as a continuous character of $M^\times_{\mathbb{A}^{(\infty)}}/M^\times$, and if $\widetilde{\varphi}_P$ restricted to an open subgroup U of $R_p^\times$ coincides over U with an algebraic character of $\mathbb{G}_m(R_p) = R_p^\times$, we call P an arithmetic point. If P is arithmetic, we have a complex character $\varphi_P : M_{\mathbb{A}}^\times/M^\times \to \mathbb{C}^\times$ whose p-adic avatar is given by $\widetilde{\varphi}_P$; so, φ_P is the archimedean avatar of $\widetilde{\varphi}_P$.

Let Γ_F^+ be the maximal torsion-free quotient of $Cl_F(\mathfrak{N}p^\infty)$ and write $\mathbf{\Gamma}$ for the product of Γ_F^+ with the maximal torsion-free subgroup Γ_F^- of $O_p^\times$ (unique under (unr)). We have a natural isomorphism $\Gamma_F^+ \cong \Gamma_M^+$ induced by the restriction map $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Decomposing $R_p = O_p \oplus O_p$ so that $R_{\Sigma_p} \cong O_p$ gives the left factor, each $\gamma \in \Gamma_F^-$ can be embedded to $(\gamma^{-1}, \gamma) \in \Gamma_M^- \subset R_p^\times/R^\times \hookrightarrow Cl_M(p^\infty)$. In this way, $\mathbf{\Gamma}$ is embedded in Γ_M, and Λ is an algebra over $W[[\mathbf{\Gamma}]]$. The Hecke operator coming from each diagonal toric elements $\mathrm{diag}[\gamma^{-1}, \gamma]$ with $\gamma \in \Gamma_F^-$ and the central action gives rise to a natural group homomorphism $\mathbf{\Gamma} \hookrightarrow \mathbf{h}^\times$ and induces an algebra structure $W[[\mathbf{\Gamma}]] \overset{\iota}{\hookrightarrow} \mathbf{h}$. This makes λ_φ a $W[[\mathbf{\Gamma}]]$-algebra homomorphism.

The automorphic induction $\pi(\varphi_P)$ of φ_P for each arithmetic point $P \in \mathrm{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$ has corresponding local Galois representation of the form $\pi_{1,\mathfrak{q}} \oplus \pi_{2,\mathfrak{q}}$ for almost all places $\mathfrak{q}$ and at supercuspidal non-split places $\mathfrak{v}$ it is irreducible isomorphic to absolutely irreducible $\mathrm{Ind}_{M_{\mathfrak{v}}}^{F_{\mathfrak{v}}} \pi_{0,\mathfrak{v}}$, and the automorphic representation π attached to each arithmetic point of $\mathrm{Spf}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ satisfies the same property everywhere at finite places.

Definition 6.2. *We write $S_{sc} = S_{sc}(\varphi)$ for the supercuspidal primes $\mathfrak{v}$ which is denoted by S in* [H09, §3.1]. *Often we regard $\pi_{j,\mathfrak{q}}$ (resp. $\pi_{0,\mathfrak{v}}$) as a quasi-character of $F_{\mathfrak{q}}^\times$ (resp. $M_{\mathfrak{q}}^\times$) by local class field theory.*

Each arithmetic point $P \in \mathrm{Spf}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ has weight $\kappa = \kappa(P) = (\kappa_1 = \kappa_1(P), \kappa_2 = \kappa_2(P))$ with $\kappa_j \in \mathbb{Z}[I]$ and $\kappa_1 - \kappa_2 \ge I$ in the sense of [PAF, §4.3.1] where the role of κ_1 and κ_2 are reversed (i.e., κ gives the Hodge weight of the attached motive). A weight $k=\sum_\sigma k_\sigma\sigma\in\mathbb{Z}[I]$ is regarded often as a character of $O_p^\times$ by $O_p^\times\ni u\mapsto\prod_{\sigma\in I}\sigma(u)^{k_\sigma}$. Identifying $O_{\mathfrak{p}}^\times$ for $\mathfrak{p}|p$ with the inertia subgroup $I_{\mathfrak{p}}$ of the Galois group of the maximal abelian extension of $F_{\mathfrak{p}}$, the Galois representation $\rho_P:=(\rho_{\mathbf{h}} \bmod P)$ has upper triangular form on $I_{\mathfrak{p}}$ with Hodge–Tate weight $\kappa(P)$ [HMI, §2.3.8]; in particular, the infinity type of φ_P is

$$\infty(\varphi_P)=-\sum_{\sigma\in\Sigma}(\kappa_1(P)_{\sigma|_F}\sigma+\kappa_2(P)_{\sigma|_F}\sigma c). \tag{6.3}$$

The algebra $\mathbf{h}$ is finite flat over $W[[\mathbf{\Gamma}]]$ and is reduced [PAF, §4.2.11], and the residue ring $\mathbf{h}\otimes W[[\mathbf{\Gamma}]]/(P\cap W[[\mathbf{\Gamma}]])$ is isomorphic to the classical nearly ordinary weight κ Hecke algebra [PAF, Corollary 4.31]. In particular, $\lambda_P:=(\lambda_\varphi \bmod P)$ gives the Hecke eigenvalue $\lambda_P(T(\mathfrak{q}))$ of a Hecke eigenform $\theta(\varphi_P)$ in $\pi(\varphi_P)$. The eigenvalue $\lambda_P(T(\mathfrak{q}))$ is equal to $\mathrm{Tr}(\mathrm{Ind}_M^F\widetilde{\varphi}_P(\mathrm{Frob}_{\mathfrak{q}}))$ for primes $\mathfrak{q}$ outside $\mathfrak{N}'p$. The complex modular form $\theta(\varphi_P)$ is the minimal form in the sense of [H09, (L1–3) in §3.1] generating the automorphic induction $\pi(\varphi_P)$ for the archimedean avatar φ_P of $\widetilde{\varphi}_P$. We recall the definition of a minimal form later in Definition 6.5.

By (A2) and ψ having order prime to p, λ_φ is onto [H86, Corollary 4.2]; so, we identify $\mathrm{Im}(\lambda_\varphi)=\Lambda$ (this is the only place we use (A2) in this paper).

Definition 6.3. *Since* $\mathbf{h}$ *is finite flat over* $W[[\mathbf{\Gamma}]]$ *and is reduced* [PAF, §4.2.11], *we can write uniquely* $\mathrm{Spf}(\mathbf{h})=\mathrm{Spf}(\mathrm{Im}(\lambda_\varphi))\cup\mathrm{Spf}(\mathbf{h}^\perp)$ *for a unique complementary reduced closed subscheme* $\mathrm{Spf}(\mathbf{h}^\perp)\subset\mathrm{Spf}(\mathbf{h})$ [HT93, (6.9)]. *Then it is known that* $C_0=\mathrm{Im}(\lambda_\varphi)\otimes_{\mathbf{h}}\mathbf{h}^\perp$ *(i.e.,* $\mathrm{Spf}(\mathrm{Im}(\lambda_\varphi))\times_{\mathrm{Spf}(\mathbf{h})}\mathrm{Spf}(\mathbf{h}^\perp)=\mathrm{Spf}(C_0)$*) is isomorphic to* $\Lambda/(H(\varphi))$ *as* $\mathbf{h}$*-modules for a non-zero element* $H(\varphi)\in\Lambda$ [HT93, (6.9)], *which is called the congruence power series of* λ_φ*.*

We can replace $\mathbf{h}$ in the above definition of $H(\varphi)$ by the local ring $\mathbb{T}$ of $\mathbf{h}$ through which λ_φ factors through, since $\mathrm{Spf}(\mathbb{T})$ is the connected component of $\mathrm{Spf}(\mathbf{h})$ containing $\mathrm{Spf}(\mathrm{Im}(\lambda_\varphi))$.

Remark 6.4. By definition, $H(\varphi)\notin\Lambda^\times\Leftrightarrow\lambda_\varphi$ is not an injection $\Leftrightarrow\mathbf{h}\neq\Lambda\Leftrightarrow\mathbf{h}^\perp\neq 0$. The congruence power series $H(\varphi)$ only depends up to units on the isomorphism class of $\mathbf{h}$.

An automorphic representation $\pi=\pi_P$ associated to an arithmetic point of $P\in\mathrm{Spf}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ is factored as follows: $\pi=\pi^{(\infty)}\otimes\pi_\infty$ for representations $\pi^{(\infty)}$ of $G(\mathbb{A}^{(\infty)})$ and π_∞ of $G(\mathbb{R})$, and we further decompose

$$\pi^{(\infty)}=\pi^{(S_{sc})}\otimes\pi_{S_{sc}},\ \pi^{(S_{sc})}=\bigotimes_{\mathfrak{q}\notin S_{sc}}\pi(\pi_{1,\mathfrak{q}},\pi_{2,\mathfrak{q}}),\pi_{S_{sc}}=\bigotimes_{\mathfrak{v}\in S_{sc}}\pi(\pi_{0,\mathfrak{q}}) \tag{6.4}$$

for the principal series representation $\pi(\pi_{1,\mathfrak{q}},\pi_{2,\mathfrak{q}})$ of $GL_2(F_{\mathfrak{q}})$ induced from the upper triangular Borel subgroup from two characters $\pi_{1,\mathfrak{q}},\pi_{2,\mathfrak{q}}:F_{\mathfrak{q}}^\times\to\overline{\mathbb{Q}}^\times$ with central character $\omega_{\pi,\mathfrak{q}}=\pi_{1,\mathfrak{q}}\pi_{2,\mathfrak{q}}$ and local automorphic induction $\pi(\pi_{0,\mathfrak{v}})$. We always use the character $\mathfrak{v}$ for primes in S_{sc}.

$$\text{We put } \pi_+:=\prod_{\mathfrak{q}}\omega_{\pi,\mathfrak{q}}/|\omega_{\pi,\mathfrak{q}}|\ \text{(the unitarization of } \omega_\pi\text{)}, \tag{NB0}$$

which is a finite order Hecke character. We regard here the pair $(\pi_{1,\mathfrak{q}},\pi_{2,\mathfrak{q}})$ as a character of the Borel subgroup by $\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\mapsto\pi_{1,\mathfrak{q}}(a)\pi_{2,\mathfrak{q}}(d)$. We define a Γ_0-type level subgroup by

$$U_0(\mathfrak{a}):=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(\widehat{O})|c\in\mathfrak{a}\widehat{O}\}.$$

So its $\mathfrak{l}$-component is $\widehat{\Gamma}_0(\mathfrak{l}^n)$ in §2.1 if $\mathfrak{l}^n$ is the $\mathfrak{l}$-factor of $\mathfrak{a}$.

The infinity component π_∞ is described by a weight $\kappa=(\kappa_1,\kappa_2)\in\mathbb{Z}[I_F]$ of the diagonal torus $T=\mathbb{G}_m\times\mathbb{G}_m\subset\mathrm{GL}(2)_{/F}$ in the following way: We impose $\kappa_1+\kappa_2=[\kappa]I$ for $I=\sum_{\nu\in I_F}\nu$ with an integer $[\kappa]$. Then the central character ω_π has the following form $\omega_\pi(z)=z^{(1-[\kappa])I}=\prod_\nu z_\nu^{1-[\kappa]}$ for totally positive $z\in F_\infty^\times$. The weight of a holomorphic vector under $\mathrm{SO}_2(F_\infty)=(S^1)^I$ is given by $(S^1)^I\ni(\exp(\theta_\nu\sqrt{-1}))_\nu\mapsto\exp(-\sqrt{-1}\sum_\nu k_\nu\theta_\nu)$ for $k=\kappa_1-\kappa_2+I$.

Definition 6.5. *We call the set of characters* $\underline{\pi}:=(\pi_{1,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times},\pi_{2,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times},\pi_+)_{\mathfrak{q}}\cup(1,\pi_{+,\mathfrak{v}}|_{O_{\mathfrak{v}}^\times},\pi_{+,\mathfrak{v}})_{\mathfrak{v}}$ *as minimal Neben characters of* π*. We write often* π_j *for the character of* $O^{(S_{sc})\times}:=\prod_{\mathfrak{q}\notin S_{sc}}O_{\mathfrak{q}}^\times$ *with values in* $W^\times$ *The set* $\underline{\pi}$ *gives rise to a character of the level group* $U_0(\mathfrak{c}(\pi))$ *for*

$$\mathfrak{c}(\pi)=\prod_{\mathfrak{q}\notin S_{sc}}(\mathfrak{C}(\pi_{1,\mathfrak{q}})\cap\mathfrak{C}(\pi_{2,\mathfrak{q}}))\prod_{\mathfrak{v}\in S_{sc}}\mathfrak{C}(\pi_{0,\mathfrak{v}})d(M_{\mathfrak{v}}/F_{\mathfrak{v}})$$

for the local relative discriminant $d(M_{\mathfrak{v}}/F_{\mathfrak{v}})$ [H09, (L1–3) in §3.1]. *We call $\mathfrak{c}(\pi)$ the minimal level of π. A nonzero eigen vector under $U_0(\mathfrak{c}(\pi))$ in $\pi^{(S_{sc})}$ with this Neben character has minimal level among character twists. The character π_j may not be a Hecke character (i.e., may not be trivial over $F^\times \subset (F_{\mathbb{A}}^{(S_{sc}\infty)})^\times$), but π_+ is a Hecke character. For a given order of π_j, there is a unique nonzero minimal vector $v^{(\infty)}$ in $\pi^{(\infty)}$ up to scalars* [PAF, §5.1.1]. *Filling in the holomorphic lowest weight vector v_∞ at ∞ and a new vector at S_{sc}, $v = v^{(\infty)} \otimes v_\infty$ is called a minimal vector with Neben-type $\underline{\pi}$, which is unique up to scalars. There is a more classical choice of Neben type characters: $\underline{\pi}^{new} := (1, \pi_+|_{\widehat{O}^\times}, \pi_+)$, which is our choice at supercuspidal paces $\mathfrak{v}$. The vector in $\pi^{(\infty)}$ with this Neben character under $U_0(\mathfrak{c}(\pi))$ with lowest holomorphic vector at ∞ is the new vector (which is known to be unique up to scalars by the theory of new forms). Note that the local automorphic conductor $C(\pi_{\mathfrak{q}})$ of $\pi_{\mathfrak{q}}$ at $\mathfrak{q} \notin S_{sc}$ is given by $\mathfrak{C}(\pi_{1,\mathfrak{q}})\mathfrak{C}(\pi_{2,\mathfrak{q}})$ and $C(\pi_{\mathfrak{v}}) = \mathfrak{C}(\pi_{0,\mathfrak{v}})d(M_{\mathfrak{v}}/F_{\mathfrak{v}})$ for $\mathfrak{v} \in S_{sc}$; so, $C(\pi_{\mathfrak{q}}) \subset \mathfrak{C}(\pi_{1,\mathfrak{q}}) \cap \mathfrak{C}(\pi_{2,\mathfrak{q}})$ and the minimal vector has level $\mathfrak{C}(\pi_{1,\mathfrak{q}}) \cap \mathfrak{C}(\pi_{2,\mathfrak{q}})$ at $\mathfrak{q} \notin S_{sc}$.*

Remark 6.6. We used in [H09, §3.2] the minimal form in two automorphic representations π, π' to make an almost primitive Rankin product (the Rankin product with less missing Euler factor) under the classical convolution method in [H91] adapted to p-adic interpolation. However, we needed to assume that $\pi_1 = \pi'_1$ to make Rankin convolution of minimal forms in two everywhere principal automorphic representations π, π' [H09, (3.6)]. We can use the Neben-type $\pi_{\mathfrak{q}}^{new} = (1, \pi_{+,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times}, \pi_{+,\mathfrak{q}})$ and ${\pi'}_{\mathfrak{q}}^{new}$ at some places $\mathfrak{q}$ in the level with $\pi_{1,\mathfrak{q}} \neq \pi'_{1,\mathfrak{q}}$ to avoid this assumption as was done in [H91]. However by doing this, we miss some Euler factors at $\mathfrak{q}$ unless the Euler factor of the primitive Rankin product is trivial (and this is a reason for writing [H09] to have less missing factors). In this sense, as long as the Euler factor is trivial at $\mathfrak{q}$, the choice of Neben types $\underline{\pi}_{\mathfrak{q}}^{new}$ and $\underline{\pi}_{\mathfrak{q}} = (\pi_{1,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times}, \pi_{2,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times}, \pi_{+,\mathfrak{q}})$, does not matter to have the exact Petersson inner product formula [H09, (3.5)] by the adjoint L-value which is the key for the divisibility in [H09, Theorem 3.1]. By definition, the prime-to-p part $\underline{\pi}^{?,(p)}$ of $\underline{\pi}$ and the Teichmüller lift of Neben characters $\pi_{j,\mathfrak{p}}^?$ of $(\pi_{j,\mathfrak{p}}^?|_{O_{\mathfrak{p}}^\times} \bmod \mathfrak{m}_W)$ (? = new, nothing) for $\mathfrak{p}|p$ are independent of the arithmetic point of $\mathrm{Spf}(\mathbf{h})$ giving rise to π and depends only on φ.

We now regard the character $\kappa = (\kappa_1, \kappa_2)$ as a character of $T(O_p) = O_p^\times \times O_p^\times$ by $T(O_p) \ni \mathrm{diag}(x, y) \mapsto x^{\kappa_1}y^{\kappa_2}$. The Hecke algebra $\mathbf{h}$ is naturally an algebra over $W[[T(O_p \times O/\mathfrak{N}') \times Cl_F(\mathfrak{N}'p^\infty)]]$ and $\underline{\pi}$ induces a character

$$T(O_p \times O/\mathfrak{N}') \times Cl_F(\mathfrak{N}'p^\infty) \ni (\mathrm{diag}(x, y), z) \mapsto x_p^{\kappa_1}y_p^{\kappa_2}\pi_1(x_{\mathfrak{N}'})\pi_2(y_{\mathfrak{N}'})\pi_+(z) \in W^\times$$

which gives rise to an arithmetic point $P : W[[T(O_p \times O/\mathfrak{N}') \times Cl_F(\mathfrak{N}'p^\infty)]] \to W$. The quotient $\mathbf{h} \otimes_{W[[T(O_p\times O/\mathfrak{N}')\times Cl_F(\mathfrak{N}'p^\infty)]],P} W$ is the Hecke algebra $\mathbf{h}_\kappa^{n.ord}(\mathfrak{c}(\pi), \underline{\pi}; W)$ of nearly ordinary Hilbert modular forms on $U_0(\mathfrak{c}(\pi))$ of weight κ of Neben character $\underline{\pi}$. Here $\mathfrak{c}(\pi) \supset \mathfrak{N}$ and $\mathfrak{c}(\pi)/\mathfrak{N}$ is a product of prime factors of p. In particular, the algebra homomorphism $\lambda_\pi : \mathbf{h} \to W$ sending $T(\mathfrak{l})$ to the Hecke eigenvalue of $T(\mathfrak{l})$ of the minimal vector in π factors through $\mathbf{h}_\kappa(\mathfrak{c}(\pi), \underline{\pi}; W)$. All this follows from the theory exposed in [PAF, §4.2.12].

Definition 6.7. *Branch Neben characters are a set*

$$\epsilon = (\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}} : O_{\mathfrak{q}}^\times \to \mathcal{W}^\times, \epsilon_{+,\mathfrak{q}} : F_{\mathfrak{q}}^\times \to \mathcal{W}^\times)_{\mathfrak{q}\notin S_{sc}} \cup (1, \epsilon_{+,\mathfrak{v}} : O_{\mathfrak{v}}^\times \to \mathcal{W}^\times, \epsilon_{+,\mathfrak{v}})_{\mathfrak{v}} : F_{\mathfrak{v}}^\times \to \mathcal{W}^\times)_{\mathfrak{v}}$$

of finite order characters with order prime to p indexed by primes $\mathfrak{q}$ of F such that $\epsilon_{+,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times} = \epsilon_{1,\mathfrak{q}}\epsilon_{2,\mathfrak{q}}$, assuming that all these characters $\epsilon_{+,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times}, \epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}}$ are trivial over $O_{\mathfrak{q}}^\times$ for almost all primes $\mathfrak{q}$ and $\epsilon_+ = \prod_{\mathfrak{q}} \epsilon_{+,\mathfrak{q}}$ is induced by a Hecke character of F.

Here we say "branch Neben characters" because the p-adic analytic family is made of a specific vector (minimal or new at finite places) in automorphic representations π (indexed by $\mathrm{Spf}(\mathbf{h})$) which is an eigen vector under the action of a specific level group. The Neben-character set $\epsilon^{(p)}$ outside p and the Teichmüller part at p is independent of the members π, and $\underline{\pi}$ is congruent to ϵ modulo $\mathfrak{m}_W$ for all members of the family indexed by $\mathrm{Spf}(\mathbf{h})$.

To describe the local component at a prime $\mathfrak{q}$ of the branch Neben character set ϵ we choose for $\pi(\varphi_P)$, we use local class field theory and identify local characters of $R_{\mathfrak{Q}}^\times$ with the corresponding characters of the inertia group $I_{\mathfrak{Q}} \subset \mathrm{Gal}(\overline{M}_{\mathfrak{Q}}/M_{\mathfrak{Q}})$ (resp. $I_{\mathfrak{q}} \subset \mathrm{Gal}(\overline{M}_{\mathfrak{Q}}/F_{\mathfrak{q}})$ for $\mathfrak{q} = \mathfrak{Q} \cap O$) for each prime $\mathfrak{Q}$ of R. We define the *minimal level* $\mathfrak{N}$ of φ already mentioned by $\mathfrak{c}(\pi(\varphi))$ for automorphic induction $\pi(\varphi)$ of φ.

As a general rule, for $(\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}}, \epsilon_{+,\mathfrak{q}})$, we always assume

$$\mathfrak{C}(\epsilon_{1,\mathfrak{q}}) \supseteq \mathfrak{C}(\epsilon_{2,\mathfrak{q}}), \quad \text{and } \epsilon_+ = \varphi|_{F_{\mathbb{A}}^\times}\chi_M \text{ for } \chi_M = \left(\frac{M/F}{}\right). \tag{6.5}$$

Note that $\epsilon_+ = \omega_{\pi(\varphi)}/|\omega_{\pi(\varphi)}|$ (the unitarization of the central character of the automorphic induction $\pi(\varphi)$ of φ). Even under (6.5), there could be several choices.

First assuming $\mathrm{Ind}_M^F \varphi$ restricted to $\mathrm{Gal}(\overline{F}_\mathfrak{q}/F_\mathfrak{q})$ is reducible (so, $\pi(\varphi)_\mathfrak{q}$ is principal), we make explicit the "minimal" Neben character set $\epsilon^{min}_{\varphi,\mathfrak{q}} = (\epsilon^{min}_{1,\mathfrak{q}}, \epsilon^{min}_{2,\mathfrak{q}}, \epsilon_{+,\mathfrak{q}})$ as follows: for $\mathfrak{q}$ outside $\mathfrak{N}'p$, $\epsilon^{min}_{1,\mathfrak{q}} = \epsilon^{min}_{2,\mathfrak{q}} = 1$, and for primes $\mathfrak{q}|\mathfrak{N}'p$ of F,

$$\text{(NB1)} \qquad \begin{aligned} \epsilon^{min}_{1,\mathfrak{q}} &= \begin{cases} \varphi|_{I_\mathfrak{Q}} & \text{if } \mathfrak{q} = \mathfrak{Q}\overline{\mathfrak{Q}}, \\ \text{an extension of } \varphi|_{I_\mathfrak{Q}} \text{ to } I_\mathfrak{q} & \text{if } \mathfrak{q} = \mathfrak{Q}^2 \text{ or } \mathfrak{Q}, \end{cases} \\ \epsilon^{min}_{2,\mathfrak{q}} &= \begin{cases} \varphi|_{I_{\overline{\mathfrak{Q}}}} & \text{if } \mathfrak{q} = \mathfrak{Q}\overline{\mathfrak{Q}}, \\ \epsilon^{min}_{1,\mathfrak{q}}\chi_M|_{I_\mathfrak{q}} & \text{if } \mathfrak{q} = \mathfrak{Q}^2 \text{ or } \mathfrak{Q}, \end{cases} \end{aligned}$$

where $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$ are distinct primes in M. Here if $\mathfrak{q}|\mathfrak{C}p$ splits in M, we have chosen the order of decomposition $\mathfrak{q} = \mathfrak{Q}\overline{\mathfrak{Q}}$ so that $\mathfrak{Q} \in \Sigma_p$ and $\mathfrak{Q}|\mathfrak{F}(\varphi)$. If $\mathfrak{q}$ is outside $\mathfrak{N}p$, we have $\epsilon_{1,\mathfrak{q}} = \epsilon_{2,\mathfrak{q}}$.

Without assuming reducibility of $\mathrm{Ind}_M^F \varphi$ over $\mathrm{Gal}(\overline{F}_\mathfrak{q}/F_\mathfrak{q})$, we define branch characters $\epsilon^{new}_{\varphi,\mathfrak{q}} = (\epsilon^{new}_{1,\mathfrak{q}}, \epsilon^{new}_{2,\mathfrak{q}}, \epsilon_{+,\mathfrak{q}})$ of "new" type for any prime $\mathfrak{q}$:

$$\text{(NB2)} \qquad \epsilon^{new}_{1,\mathfrak{q}} = 1 \quad \text{and } \epsilon^{new}_{2,\mathfrak{q}} = \epsilon_+|_{O_\mathfrak{q}^\times}.$$

Plainly, for $\mathfrak{q}$ outside $\mathfrak{N}'$, $\epsilon^{min}_{\varphi,\mathfrak{q}} = \epsilon^{new}_{\varphi,\mathfrak{q}}$.

For the global branch characters, we often take mixed product of $\epsilon^{min}_{\varphi,\mathfrak{q}}$ and $\epsilon^{new}_{\varphi,\mathfrak{q}}$ and just write it as ϵ_φ with specific choices given each time (with $\epsilon^{new}_{\varphi,\mathfrak{v}}$ at least for places where $\mathrm{Ind}_{K_\mathfrak{v}}^{F_\mathfrak{v}} \varphi_\mathfrak{v}$ is irreducible). If we choose the *new* type everywhere simultaneously, we write ϵ^{new}_φ. If we choose minimal type everywhere possible, we write ϵ^{min}_φ (so, for $\mathfrak{v} \in S_{sc}$, we are forced to choose $\epsilon^{new}_\mathfrak{v}$ even if we write ϵ^{min}_φ). As exhibited in [H09, §3.1–2], to compute complex L-functions (either adjoint or Rankin-type) as primitive as possible via a classical method, we need to choose the minimal type wherever places minimal type allowable. Thus to define $H(\varphi)$, we use ϵ^{min}_φ.

We write $\epsilon^- := \epsilon_1^{-1}\epsilon_2$ which factors through $(O/\mathfrak{N})^\times$, and $U_0(\mathfrak{N}) \ni u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1(\det(u))\epsilon^-(u)$ with $\epsilon^-(u) = \epsilon_1(a)^{-1}\epsilon_2(d)$ is a well defined character of $U_0(\mathfrak{N}) \supset U_0(\mathfrak{N}')$, where

$$U_0(\mathfrak{a}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{O}) | c \in \mathfrak{a}\widehat{O}\}.$$

Remark 6.8. By tensoring a Hecke character θ of finite order, we can bring each Hilbert modular form f to $f \otimes \theta$ given by $(f \otimes \theta)(x) = f(x)\theta(\det(x))$, which induces an isomorphism $\mathbf{h}(\mathfrak{N}, \epsilon^{min}_\varphi; W) \cong \mathbf{h}(\mathfrak{N}, \epsilon^{min}_{\varphi\theta}; W)$, where $\epsilon^{min}_{\varphi\theta,\mathfrak{q}} = (\epsilon_{1,\mathfrak{q}}\theta_\mathfrak{q}|_{O_\mathfrak{q}^\times}, \epsilon_{2,\mathfrak{q}}\theta_\mathfrak{q}|_{O_\mathfrak{q}^\times}, \epsilon_{+,\mathfrak{q}}\theta_\mathfrak{q}^2)$ for $\mathfrak{q} \notin S_{sc}$ writing $\epsilon^{min}_\varphi = (\epsilon_1, \epsilon_2, \epsilon_+)$ (while ϵ^- is intact under this operation). This basically shows, by Remark 6.4, $H(\varphi)$ can be chosen not to have any variable in $W[[\Gamma_F^+]]$ in $W[[\mathbf{\Gamma}]]$. This fact will not be used in this paper.

Choice of φ: For a fixed ψ, we have a freedom to choose φ with $\varphi^- = \psi$. Since φ and $\varphi(\xi \circ N_{M/F})$ have equal $\varphi^- = \psi$ for finite order Hecke character ξ, we choose an optimal φ for our purpose in the following way: Recall $\mathfrak{C} = \mathfrak{C}(\varphi)$ and $\mathfrak{c} = \mathfrak{C} \cap F$ and the finite set $S = S(\varphi)$ of prime factors $\mathfrak{c}d(M/F)$ (or equivalently, prime factors of $\mathfrak{N}'$). Since $S_{sc} \subset S$ for $S_{sc} = S_{sc}(\varphi)$, we can decompose $S = S_{sc} \sqcup S_{ab}$ and $S_{ab} = S_i \sqcup S_r \sqcup S_f$ for $S_f := \{\mathfrak{q} \in S_{ab} | \mathfrak{q} \supset \mathfrak{f}_c(\varphi)\}$ for $\mathfrak{f}_c(\varphi) = \mathfrak{F}_c(\varphi) \cap F$, $S_i := \{\mathfrak{q} \in S_{ab} | \mathfrak{q} \supset \mathfrak{i}(\varphi)\}$ and $S_r := \{\mathfrak{q} \in S_{ab} | \mathfrak{q} \supset d(M/F)\}$. Write $\mathfrak{c}_?$ for $S_?$-component of $\mathfrak{c}(\varphi)$ for $? = i, r, f, ab$. Locally at $\mathfrak{q} \in S_{ab} \cup \{\mathfrak{p}|p\}$, $\mathrm{Ind}_M^F \varphi|_{D_\mathfrak{q}} \cong \varphi'_\mathfrak{q} \oplus \varphi''_\mathfrak{q}$ for two characters $\varphi'_\mathfrak{q}$ and $\varphi''_\mathfrak{q}$ of $D_\mathfrak{q}$. Regarding $\varphi'_\mathfrak{q}$ as a character of $O_\mathfrak{q}^\times$ which is the inertia group of the maximal abelian extension of $F_\mathfrak{q}$, we choose $\varphi'_\mathfrak{q} = \varphi|_{R_\mathfrak{Q}^\times}$ (as a character of $O_\mathfrak{q}^\times$ identifying $R_\mathfrak{Q} = O_\mathfrak{q}$) if $\mathfrak{q} = \mathfrak{Q}\overline{\mathfrak{Q}} \in S_f \cup \{\mathfrak{p}|p\}$ with $\mathfrak{Q}|\mathfrak{F}_c$ or $\mathfrak{Q} \in \Sigma_p c$, and otherwise, the choice is just a factor of $\mathrm{Ind}_M^F \varphi|_{D_\mathfrak{q}}$ for $\mathfrak{q} \in S_i \cup S_r$. Let $\varphi^{ab} := \prod_{\mathfrak{q} \in S_{ab} \cup \{\mathfrak{p}|p\}} \varphi'_\mathfrak{q}$, as a character of $O_{S_{ab}p}^\times \to W^\times$ for $O_{S_{ab}p} = \prod_{\mathfrak{q} \in S_{ab} \cup \{\mathfrak{p}|p\}} O_\mathfrak{q}$ (extending scalars W if necessary). By [C51, Théorème 1], we can find a character ξ_φ of $\widehat{O}^\times$ with conductor outside $S_{ab} \cup \{\mathfrak{p}|p\}$ such that $\xi_\varphi\varphi^{ab}$ extends to a finite order Hecke character $\Phi = \Phi_\varphi$ of $F^\times \mathbb{F}_\mathbb{A}^\times$. Replacing Φ_φ by the Teichmüller lift of $(\Phi_\varphi \mod \mathfrak{m}_W)$, we may assume that Φ_φ has order prime to p and $\Phi_\mathfrak{q} = \varphi'_\mathfrak{q}$ for all $\mathfrak{q} \in S_{ab} \cup \{\mathfrak{p}|p\}$. Replacing the starting $\varphi_0 := \varphi$ by $\varphi_0(\Phi \circ N_{M/F})^{-1}$ (i.e., $\pi(\varphi)$ by

$\pi(\varphi)\otimes\Phi^{-1}$ and writing φ for the new choice $\varphi(\Phi\circ N_{M/F})^{-1}$, we achieve $\epsilon^{min}_{\varphi,j,\mathfrak{q}} = \epsilon^{new}_{\varphi,j,\mathfrak{q}}$ $(j = 1, 2)$ for all $\mathfrak{q} \in S_{ab} \cup \{\mathfrak{p}|p\}$. At $\mathfrak{q} \in S_{\xi_\varphi} := \{\mathfrak{q}|\mathfrak{C}(\xi_\varphi)\}$, the local component $\pi_{\mathfrak{q}}(\varphi) = \pi_{\mathfrak{q}}(\varphi_0) \otimes \Phi_{\mathfrak{q}}^{-1}$ with spherical $\pi_{\mathfrak{q}}(\varphi_0)$ (or equivalently, $\mathrm{Ind}_M^F \varphi|_{I_{\mathfrak{q}}} = \xi_{\mathfrak{q}}^{-1} \oplus \xi_{\mathfrak{q}}^{-1}$), and we choose $\epsilon_{\varphi,\mathfrak{q}} := \epsilon^{new}_{\varphi,\mathfrak{q}} = (1, \Phi_{\mathfrak{q}}^{-2}, \omega_{\pi(\varphi_{\mathfrak{q}})})$. Thus $\epsilon_\varphi = \epsilon^{new}_\varphi$. This shows

$$S_{\xi_\varphi} \text{ is the set of places } \mathfrak{q} \text{ where } \epsilon_{\varphi,\mathfrak{q}} = \epsilon^{new}_{\varphi,\mathfrak{q}} \neq \epsilon^{min}_{\varphi,\mathfrak{q}}. \tag{6.6}$$

We choose a quadratic CM-extension $M_{1/F}$ with integer ring R_1 inside $\overline{\mathbb{Q}}$. The field M_1 can be equal to M. Write K for the composite of M and M_1 inside $\overline{\mathbb{Q}}$ if $M \neq M_1$, and $K = M \otimes_F M \cong M \oplus M$ if $M_1 = M$. When $M_1 = M$, we embed M (resp. $M_1 = M$) diagonally (resp. by $x \mapsto (x \oplus x^c)$) into K. Write F_1 for the maximal totally real subfield of K if K is a field, and otherwise, we put $F_1 = F \oplus F \subset K$. Then in K, there are three semi-simple quadratic extensions M, F_1 and M_1 of F:

$$\begin{array}{ccccc} & & M & & \\ & \nearrow & & \searrow & \\ F & \hookrightarrow & F_1 & \hookrightarrow & K. \\ & \searrow & & \nearrow & \\ & & M_1 & & \end{array} \tag{6.7}$$

When K is a field, we impose the following conditions:

- (K1) M_1/F is a p-ordinary CM field.
- (K2) For primes $\mathfrak{p}|p$ in F, the decomposition group of $\mathfrak{p}$ in $\mathrm{Gal}(K/F)$ is trivial (i.e., for any subextension of K/F, prime factors of p are all split).

Thus $F_{1,\mathfrak{P}} = F_{\mathfrak{P}}$ for all primes $\mathfrak{P}|p$ in K. We may also impose

- (K3) For places v either in S or archimedean, $M \otimes_F F_v \cong M_1 \otimes_F F_v$ (so, all primes in S splits in K/F_1).
- (K4) For every prime $\mathfrak{l} \notin S$ ramifying in M/F, $\mathfrak{l}$ is unramified in M_1/F.
- (K5) For the integer ring R_1 of M_1, $R_1^\times = O^\times$.

Since we impose how places decompose in F_1/F only at the finite set $S \cup \{\mathfrak{p}|p\} \cup \{\mathfrak{q}|\mathfrak{c} = \mathfrak{C} \cap F\} \cup \{\mathfrak{l}|d(M/F)\} \cup \{v|\infty\}$ of places of F, there will be infinitely many choices of (M_1, F_1, K) satisfying (K1–5). The field M_1/F and hence K/F_1 are a p-ordinary CM field by (K1–3) in which all primes in S and over p split. Since the finite group $R_1^\times/O^\times$ is represented by roots of unity, by adding ramification to M_1/F, we may assume (K4).

We now choose carefully a Hecke character ϕ of M_1 and the corresponding branch Neben character set ε so that we can prove vanishing of the μ-invariant of the Rankin product $\lambda_\varphi * \lambda_\phi$. We fix an isomorphism $\iota : M_{\mathfrak{c}p\infty} = \prod_{\mathfrak{l}|\mathfrak{c}p\infty} M_v \cong \prod_{\mathfrak{l}|\mathfrak{c}p\infty} M_{1,\mathfrak{l}} = M_{1,\mathfrak{c}p\infty}$ as in (K3), which induces an identification of their integer ring $R_{\mathfrak{c}p} = R_{1,\mathfrak{c}p}$ and transfers the chosen CM types Σ and Σ_p of M to Σ_1 and $\Sigma_{1,p}$, respectively. If $M \neq M_1$, the CM type Σ_1 depends on the choice of ι, but we fix it once and for all. If $M = M_1$, the top inclusion $M \hookrightarrow M \otimes_F M_1$ in (6.7) is given by $\xi \mapsto \xi \otimes 1$ and the bottom inclusion by $\xi \mapsto 1 \otimes \xi$, where we identify $M \otimes_F M_1$ with $M \oplus M$ by $\xi \otimes \eta \mapsto (\xi\eta \oplus \xi\eta^c)$. Therefore $\Sigma = \Sigma_1$.

Lemma 6.9. *If K is a field, assume* (K1–4). *Suppose that $\epsilon = \epsilon_\varphi = \epsilon^{new}_\varphi$ is as in* (6.6). *Pick a prime ℓ outside $S \cup \{\mathfrak{p}|p\}$ such that all prime factors $\mathfrak{l}$ of ℓ splits in M_1/F (so, $\mathfrak{l}$ splits in K/F_1). Fix an ℓ-adic CM type Σ_ℓ of M_1 so that $\{\mathfrak{l}|\ell\} = \Sigma_\ell \sqcup \Sigma_\ell c$. Then there exists a Hecke character ϕ of whatever high conductor $\prod_{\mathfrak{L}\in\Sigma_\ell c} \mathfrak{L}^n$ of order prime to p. Identify the ℓ-component $R^\times_{1,\ell}$ with $R^\times_{1,\Sigma_\ell} \times R^\times_{1,\Sigma_\ell c} = O_\ell^\times \times O_\ell^\times = T(O_\ell)$ in this order, and set $\varepsilon = \epsilon^{new}_\phi$. Then $\varepsilon = \epsilon^{new}_\phi = \epsilon^{min}_\phi$ and we have $\varepsilon_1 = \epsilon_1$ over $\widehat{O}^\times$ and an algebra homomorphism $\lambda_\phi : \mathbf{h}^{n.ord}(\mathfrak{N}_1, \varepsilon; W) \to \Lambda'$ for $\Lambda' := W[[\Gamma_{M_1}]]$ parameterizing automorphic inductions $\pi(\phi_Q)$ for arithmetic points $Q \in \mathrm{Spf}(\Lambda')(\overline{\mathbb{Q}}_p)$. Moreover $\mathfrak{N}_1' = \mathfrak{N}_1 = d(M_1/F) \prod_{\mathfrak{L}\in\Sigma_\ell c} (\mathfrak{L} \cap F)^n$.*

Proof. The class group $CL_{M_1}(\prod_{\mathfrak{L}\in\Sigma_\ell c} \mathfrak{l}^\infty)$ has a quotient isomorphic to $\mathbb{Z}_\ell$, which comes from $R^\times_{1,\Sigma_\ell c} \ni x \mapsto x^{\Sigma_\ell c} \in \mathbb{Z}_\ell^\times$ regarding $\Sigma_\ell c$ as an archimedean CM type. Then for a non-trivial character ϕ of $CL_{M_1}(\prod_{\mathfrak{L}\in\Sigma_\ell c} \mathfrak{L}^n)$ with high n and ℓ-power order, $\phi|_{R^\times_{1,\Sigma_\ell}} = 1$. Thus $\epsilon^{min}_\eta = \epsilon^{new}_\phi$ has the desired property. The last assertion is clear from the construction of ϕ, and the choice $\varepsilon = \epsilon^{new}_\phi = \epsilon^{min}_\phi$. $\square$

Lemma 6.10. *For a given p-ordinary CM type Σ of M, define a subset of embedding $\Sigma^K \subset \mathrm{Hom}_{field}(K, \overline{\mathbb{Q}}_p)$ for $K = M \otimes_F M_1$ as follows:*

$$\Sigma^K := \{\sigma \otimes \sigma_1 | \sigma \in \Sigma, \sigma_1 \in I_{M_1}, \sigma|_F = \sigma_1|_F\},$$

where $(\sigma \otimes \sigma_1)(x \otimes y) = \sigma(x)\sigma_1(y)$ for $x \in M$ and $y \in M_1$. Then Σ^K is a p-ordinary CM type of K and $\Sigma^K = \mathrm{Ind}_M^K \Sigma$ if K is a field.

If ξ is a Hecke character of M with $\xi(x_p) = x_p^{k\Sigma+\kappa(1-c)}$, we find $\widehat{\xi}(x) = \xi \circ N_{K/M}(x_p) = x_p^{-\mathrm{Inf}_M^K(k\Sigma+\kappa(1-c))} = x_p^{-k\Sigma^K - \mathrm{Inf}_M^K \kappa(1-c)}$.

Proof. We have $|\Sigma^K| = 2[F : \mathbb{Q}] = [F_1 : \mathbb{Q}]$. When $M = M_1$, we embed M into $M \otimes M = K$ diagonally, and M_1 by $M_1 \ni x \mapsto (x \oplus c(x)) \in M \oplus M$. Allowing K to be a field, let $\langle c_M \rangle = \mathrm{Gal}(K/M_1)$ and $\langle c_{M_1} \rangle = \mathrm{Gal}(K/M)$ with $c = c_M c_{M_1}$ which is the generator of $\mathrm{Gal}(K/F_1)$. We call c complex conjugation acting on K. By definition, every element in Σ^K (resp. $\Sigma^K c$) restricts to M to an element in Σ (resp. Σc_M). Since Σ is a CM type of M, we find $\Sigma^K \cap \Sigma^K c = \emptyset$. Counting number of elements, we find $\Sigma^K \sqcup \Sigma^K c = \mathrm{Hom}_{\mathrm{field}}(K, \overline{\mathbb{Q}}_p)$ as desired.

If $\sigma, \tau \in \Sigma^K$ with σ and τc inducing the same p-adic place of K, its restriction $\sigma|_M$ and $\tau c|_M = \tau|_M c_M$ to M also induces the same place of M, which is against the p-ordinarity of Σ as $\mathrm{Res}_K^M \mathrm{Inf}_M^K \Sigma = 2\Sigma$. Thus Σ^K is p-ordinary. $\square$

Choose a character $\phi : M_{\mathbb{A}}^\times / M^\times \to W^\times$ as in Lemma 6.9 of conductor $\mathfrak{C}_1 = \prod_{\mathfrak{L} \in \Sigma_\ell c} \mathfrak{l}^n$ and of order prime to p. In the same manner as above, we define the groups $(\Gamma_{M_1}, \Gamma_{M_1}^-, \Gamma_{M_1}^+, \Delta_{\mathfrak{C}_1})$ for M_1 in place of $(\Gamma_M, \Gamma_M^-, \Gamma_M^+, \Delta_{\mathfrak{C}})$ for M and the Hecke algebra $\mathbf{h}' := \mathbf{h}^{n.ord}(\mathfrak{N}_1, \varepsilon, W)$. Note $\mathfrak{F}_c(\phi) = \mathfrak{C}_1$ with $\mathfrak{F}(\varphi) = \mathfrak{I}(\varphi) = \mathfrak{R}(\varphi) = 1$ replacing $\Delta_{\mathfrak{C}}$ by $\Delta_{\mathfrak{C}_1}$, M by M_1 and φ by ϕ and $\varepsilon_1 = \epsilon_1$ for $\epsilon = \epsilon_\varphi^{new}$. In addition, we have $\mathfrak{N}_1 = \mathfrak{N}_1'$. Write $\mathbf{h}' := \mathbf{h}^{n.ord}(\mathfrak{N}_1, \varepsilon; W)$ (where $\varepsilon = \epsilon_\phi^{new} = \epsilon_\phi^{min}$ as in Lemma 6.9).

We recall $\Lambda' = W[[\Gamma_{M_1}]]$ and define the character $\widetilde{\phi} : Z(\mathfrak{C}_1) \to \Lambda'^\times$ for ϕ in the same way as the construction of $\widetilde{\varphi}$. For an arithmetic point $Q \in \mathrm{Spec}(\Lambda')$, $\widetilde{\phi}_Q$ has complex avatar ϕ_Q with infinity type $\phi_Q(x_\infty)x_\infty^{-m}$ for $m \in \mathbb{Z}[I_K]$. Then $m + mc = kI_K$ for an integer k. Since $(m_\sigma, m_{\sigma c})$ is the Hodge type at σ associated to the motive of ϕ_P, to have $s = 0$ as a critical evaluation point for $L(s, \phi_Q)$, we must have

(c1) $m_\sigma \neq m_{\sigma c}$,

(c2) $\Sigma_1' := \{\sigma \in I_K | m_\sigma > 0 \geq m_{\sigma c}\}$ is a CM type (since $m + mc = kI_K$ for $k \in \mathbb{Z}$).

Then for $m' := m - k\Sigma_1'$, $m' + m'c = 0$ and hence we can write $m' = \kappa(1 - c)$ for $\kappa \in \mathbb{Z}[\Sigma_1']$; so, $m = k\Sigma_1' + \kappa(1 - c)$. By (c1–2), $\kappa_\sigma \geq 0$ for all $\sigma \in \Sigma_1'$. The region of m satisfying $k > 0$ is called the right critical region (and functional equation brings it to the left defined by $k \leq 1$ and $\kappa_\sigma \geq 1 - k$ [HT93, Theorem II]). We assume that $\Sigma_1' = \Sigma_1$ (otherwise, we change ι to attain this if possible (which is always possible if p totally split in $K/\mathbb{Q}$). This choice of Σ_1 therefore determines the nearly ordinary character $\delta_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ specified in the nearly ordinary deformation problem for $\mathrm{Ind}_{M_1}^F \phi \bmod \mathfrak{m}_W$ (see below $(\mathfrak{p}_\kappa)$ in §6.5 and [HMI, (D4), §1.3.1]). Hereafter we always assume $k > 0$ and $\kappa_\sigma \geq 0$ for all $\sigma \in \Sigma_1$.

Lemma 6.11. *Write $\widehat{\varphi}_P = \varphi_P \circ N_{K/M}$ and $\widehat{\phi}_Q = \phi_Q \circ N_{K/M_1}$ for arithmetic points $P \otimes Q \in \mathrm{Spf}(\Lambda \widehat{\otimes} \Lambda')(\mathbb{C}_p)$. Then we have Zariski densely populated critical $P \otimes Q \in \mathrm{Spf}(\Lambda \widehat{\otimes} \Lambda')$ such that $\widehat{\varphi}_P^{-1} \widehat{\phi}_Q$ is critical at* 0.

Proof. Write the infinity type of $\widehat{\varphi}_P$ (resp. $\widehat{\phi}_P$) as $-\mathrm{Inf}_M^K(k\Sigma + \kappa(1-c))$ (resp. $-\mathrm{Inf}_{M_1}^K(\ell\Sigma_1 + \mu(1-c))$) with $k, \ell \in \mathbb{Z}$, $\kappa_P \in \mathbb{Z}[\Sigma]$ and $\mu_Q \in \mathbb{Z}[\Sigma_1]$. Then the Hodge type of the CM motive attached to $\varphi_P^{-1} \widehat{\phi}_Q$ at $\sigma \in \Sigma^K$ is given by

$$\begin{cases} (\kappa_\sigma - \mu_\sigma, -k - \kappa_\sigma + \ell + \mu_\sigma) & \text{if } \sigma \in \Sigma^K \cap \Sigma_1^K, \\ (\ell + \kappa_\sigma - \mu_{\sigma c}, -k - \kappa_\sigma + \mu_{\sigma c}) & \text{if } \sigma \in \Sigma^K - \Sigma_1^K, \\ (-\kappa_{\sigma c} + \mu_\sigma, -k + \kappa_{\sigma c} - \mu_\sigma) & \text{if } \sigma \in \Sigma_1^K - \Sigma^K, \\ (-\kappa_{\sigma c} + \mu_{\sigma c}, -k + \ell + \kappa_{\sigma c} - \mu_{\sigma c}) & \text{if } \sigma \notin \Sigma_1^K \cup \Sigma^K, \end{cases} \tag{6.8}$$

where $\Sigma_1^K = \mathrm{Inf}_{M_1}^K \Sigma_1$. To compare κ and μ, we identify $\mathbb{Z}[\Sigma^K] = \mathbb{Z}[I_{F_1}] = \mathbb{Z}[\Sigma_1^K]$ and regard κ and μ as an element of $\mathbb{Z}[I_{F_1}]$. Then (6.8) tells us if $k - \ell > \max_{\sigma \in I_{F_1}} |\mu_\sigma - \kappa_\sigma|$, $s = 0$ is critical for

$L(s,\varphi^{-1}\phi_Q)$. Thus we have Zariski densely populated critical $P\otimes Q\in \mathrm{Spf}(\Lambda\widehat{\otimes}\Lambda')$. In fact, allowing twists φ_P (resp. ϕ_Q) by finite order characters of Γ_M (resp. Γ_{M_1}), only one choice of Hodge type critical at 0 is sufficient for Zariski density of arithmetic $P\otimes Q$. □

In the automorphic induction $\pi(\phi_Q)$ relative to M_1/F, writing ϕ_Q for the archimedean avatar of $\widetilde{\phi}_P$ with automorphic induction $\pi(\phi_Q)$, we have a unique normalized Hecke eigenform $g(\widetilde{\phi}_Q)$ corresponding $\otimes v_{\mathfrak{q}}\in\pi(\phi_Q)^{(\infty)}$ such that $v_{\mathfrak{q}}$ is a spherical vector for $\mathfrak{q}$ outside $\mathfrak{N}_1'p$, $v_{\mathfrak{q}}$ is either a new vector or a minimal vector accordingly to the choice of new or minimal Neben character set $\varepsilon_{\mathfrak{q}}$ and a minimal vector at $\mathfrak{p}|p$ specified by $\varepsilon_{\mathfrak{p}}^{min}$. Thus we get an algebra homomorphism $\lambda_\phi:\mathbf{h}'\to\Lambda'$ which gives rise to the family of minimal modular forms $g(\widetilde{\phi}_Q)$ for each arithmetic points $Q\in\mathrm{Spec}(\Lambda')$.

The construction of the p-adic Rankin product of the Galois representation $\mathrm{Ind}_M^F\widetilde{\varphi}_P\otimes\mathrm{Ind}_{M_1}^F\widetilde{\phi}_Q^{-1}$ is carried out in [H91, Theorem 5.1] when $\varepsilon_{\mathfrak{q}}=\epsilon_{\phi,\mathfrak{q}}^{new}$ for all $\mathfrak{q}|\mathfrak{N}_1'$ and $\epsilon_{\mathfrak{q}}=\epsilon_{\varphi,\mathfrak{q}}^{new}$ for all $\mathfrak{q}|\mathfrak{N}'$ and in [H09, Theorem 3.3] when $\varepsilon_{\mathfrak{q}}=\epsilon_{\phi,\mathfrak{q}}^{min}$ for all $\mathfrak{q}|\mathfrak{N}_1'$ and $\epsilon_{\mathfrak{q}}=\epsilon_{\varphi,\mathfrak{q}}^{min}$ for all $\mathfrak{q}|\mathfrak{N}'$. In our case, $\epsilon=\epsilon_\varphi^{new}$ and $\varepsilon=\epsilon_\phi^{new}=\epsilon_\phi^{min}$, the computation of [H91] applies. However, by (6.6), $\epsilon_{\mathfrak{q}}\neq\epsilon_{\mathfrak{q}}^{min}$ possibly only at primes in S_{ab} and $\mathrm{Ind}_{M_1}^F\phi$ is unramified there by (K3), the primitive Rankin product has trivial Euler factor; so, the result of [H91] and [H09] match and produces Rankin-product primitive at all $\mathfrak{q}$ outside S_{sc}. At $\mathfrak{v}\in S_{nc}$, $\pi(\phi_Q)$ is unramified; so, no extra Euler factor appears by [H96, (7.4.1)] and [EMI, Proposition 9.1.1]. Therefore the result of [H91] and [H09] produces primitive Rankin product without missing factors for our choice of φ and ϕ.

We put $\mathcal{R}:=\mathcal{D}\cdot H(\varphi)$ for $\mathcal{D}=\lambda_\varphi*\lambda_\phi$. By the definition of $H(\varphi)$, $\mathcal{R}$ is an element of $\Lambda\widehat{\otimes}_W\Lambda'=W[[\Gamma_M\times\Gamma_{M_1}]]$ tautologically; so, we have $\mathcal{D}=\frac{\mathcal{R}}{H(\psi)}$. The modified norm map $N_{K/M}\times N_{K/M_1}^{-1}$ induces the homomorphism $\mathcal{N}:\Gamma_K\to\Gamma_M\times\Gamma_{M_1}$. Taking the Katz p-adic L-function $\mathcal{K}\in W[[\Gamma_K]]$ of K with branch character $\widehat{\varphi}^{-1}\widehat{\phi}$ ($\widehat{\varphi}:=\varphi\circ N_{K/M},\widehat{\phi}:=\phi\circ N_{K/M_1}$), we define $\mathcal{L}_p\in W[[\Gamma_M\times\Gamma_{M_1}]]=\Lambda\widehat{\otimes}_W\Lambda'$ by push forward via $\mathcal{N}:\Gamma_K\ni\gamma\mapsto(N_{K/M}(\gamma)^{-1},N_{K/M_1}(\gamma))\in\Gamma_M\times\Gamma_{M_1}$ of $\mathcal{K}$. Thus $\mathcal{L}_P\in\Lambda\widehat{\otimes}_W\Lambda'$ satisfies

$$\mathcal{L}_p(P,Q)=\mathcal{L}_p(P\otimes Q):=\mathcal{K}(\widehat{\varphi}_P^{-1}\widehat{\phi}_Q)=\int_{\Gamma_K}\widehat{\varphi}_P^{-1}\widehat{\phi}_Q d\mathcal{K}\tag{6.9}$$

for points $(P,Q)=P\otimes Q$ with $P\in\mathrm{Spf}(\Lambda)(\mathbb{C}_p)$ and $Q\in\mathrm{Spf}(\Lambda')(\mathbb{C}_p)$, where $\widehat{\phi}_Q=\widetilde{\phi}_Q|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ and $\widehat{\varphi}_P=\widetilde{\varphi}_Q|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ and $d\mathcal{K}$ is the measure on Γ_K associated to $\mathcal{K}$. In particular, if $F_1=F\oplus F$ and $K=M\oplus M$, we have $\Lambda\widehat{\otimes}_W\Lambda'=W[[\Gamma_M\times\Gamma_M]]$ and for the (possibly imprimitive) Katz p-adic L-function L_1 (resp. L_2) with branch character $\varphi\phi^{-1}$ (resp. $\varphi\phi_c^{-1}$)

$$\mathcal{L}_p(P,Q)=L_1(\widetilde{\varphi}_P^{-1}\widetilde{\phi}_Q)L_2(\widetilde{\varphi}_P^{-1}\widetilde{\phi}_Q\circ c).\tag{6.10}$$

Here in [HT93, Theorem 8.1] where we first dealt with this type of problem, we get an extra finite Euler factor Ψ_j at primes $\mathfrak{l}|(\mathfrak{N}'+\mathfrak{N}_1')$ which is a factor of a high power of $d(M/F)$ (as $\mathfrak{C}(\phi)$ is prime to $\mathfrak{C}(\varphi)$), and we define $L_j:=\Psi_j\mathcal{K}_j$ for the corresponding primitive Katz p-adic L-function $\mathcal{K}_j$. The conditions (K1–3) is automatic if $M=M_1$, but (K4) is impossible if S does not contain all ramifying primes in M/F and Ψ_j is a product of Euler factors at such primes (in [HT93], $\mathcal{K}_j$ is denoted by L_j). In [H09, Theorem 3.5] (and [HT93, Theorem 8.1]), we showed the equality of the values of the two sides of (RK1) at $P\otimes Q$ for $P\otimes Q$ critical at 0 as in Lemma 6.11.

Lemma 6.12. *If $M\neq M_1$, for p odd, we have a canonically split exact sequence*

$$1\to\Gamma_K\xrightarrow{\mathcal{N}}\Gamma_M\times\Gamma_{M_1}\xrightarrow{\mathcal{N}_F}\Gamma_F^+\to 1$$

for $\mathcal{N}_F$ sending $(\gamma,\gamma_1)\in\Gamma_M\times\Gamma_{M_1}$ to $\gamma_+\cdot\gamma_{1+}\in\Gamma_F^+$ for the projection $\gamma_+\in\Gamma_M^+=\Gamma_F^+$ (resp. $\gamma_{1+}\in\Gamma_{M_1}^+=\Gamma_F^+$) of γ (resp. γ_1). Indeed, $\Gamma_K^-=\Gamma_M^-\times\Gamma_{M_1}^-$ with $\Gamma_K^+=\Gamma_F^+$ by the restriction maps. Similarly, for $X=M,M_1,K$, $\Gamma_X=\Gamma_X^-\times\Gamma_X^+$ with $\Gamma_X^+=\Gamma_F^+$ by the restriction maps.

Proof. Writing X_∞^-/X for the composite of all anticyclotomic $\mathbb{Z}_p$-extensions of X. Then $\Gamma_X=\mathrm{Gal}(X_\infty^-/X)$ for $X=M,M_1$. From the proof of Lemma 6.10, recall c_M (resp. c_{M_1}) generating $\mathrm{Gal}(K/M_1)$ (resp. $\mathrm{Gal}(K/M)$). We write $\Gamma_K^{\tau=\pm1}$ for the τ-eigenspace of Γ_K^- with eigenvalue ±1 for $\tau=c_M,c_{M_1}$ and $c=c_Mc_{M_1}$. Then $\Gamma_K^-=\Gamma_K^{c_M=-1}\times\Gamma_K^{c_{M_1}=-1}$, as no non-trivial fixed point of $c=c_Mc_{M_1}$ in Γ_K^-. By restriction map, we have $\Gamma_K^-\twoheadrightarrow\Gamma_X^-$ for $X=M,M_1$, and $\Gamma_K^-\twoheadrightarrow\Gamma_M^-$ (resp.

$\Gamma_K^- \twoheadrightarrow \Gamma_{M_1}^-$) factors through $\Gamma_K^{c_M=-1}$ (resp. $\Gamma_K^{c_{M_1}=-1}$). As $\mathbb{Z}_p$-modules, we have $\operatorname{rank}\Gamma_X^- = \frac{1}{2}[X:\mathbb{Q}]$; so, $\operatorname{rank}\Gamma_K^- = \operatorname{rank}\Gamma_M^- + \operatorname{rank}\Gamma_{M_1}^-$ with $\operatorname{rank}\Gamma_M^- = \operatorname{rank}\Gamma_{M_1}^- = [F:\mathbb{Q}]$. By comparing ranks, we find $\Gamma_K^{c_M=-1} = \Gamma_M^-$ and $\Gamma_K^{c_{M_1}=-1} = \Gamma_{M_1}^-$. Thus $\Gamma_K = \Gamma_M^- \times \Gamma_{M_1}^- \times \Gamma_F^+$, and this shows the desired result. □

Since points $P \otimes Q$ critical at 0 are Zariski dense by Lemma 6.11, we conclude the identity. We restate [H09, Theorem 3.5] and [HT93, Theorem 8.1] for our choice of ϕ and φ with φ^- equals a given ψ:

Theorem 6.13. *Let φ (resp. ϕ) be a character of $\Delta_{\mathfrak{C}}^{(p)}$ (resp. $\Delta_{\mathfrak{C}_1}^{(p)}$) with values in $W^\times$ as in* Lemma 6.9. *According to* Lemma 6.9, *we choose the branch Neben characters ϵ for φ and ε for ϕ. Then we have, up to a unit in $\Lambda\widehat{\otimes}_W\Lambda'$, for $\mathcal{L}_p$ in (6.9),*

$$\frac{\mathcal{R}}{H(\varphi)} = \frac{\mathcal{L}_p}{h_{\mathrm{i}}(M/F)L_p^-(\psi)},$$

where $\psi = \varphi^-$. Here $L_p^-(\psi) \in W[[\Gamma_M^-]] \subset \Lambda \subset \Lambda\widehat{\otimes}_W\Lambda'$, $H(\varphi) \in \Lambda$, while $\mathcal{L}_p \in W[[\mathcal{N}(\Gamma_K)]] \subset \Lambda\widehat{\otimes}_W\Lambda'$ for $\mathcal{N}$ in Lemma 6.12, *though $H(\varphi)$ can be chosen in $W[[\Gamma_M^-]] \subset \Lambda$ as explained basically in* Remark 6.8

6.2. **Proof of (L).** We recall the argument (with a bit different formulation) in [HT93, page 257] just above Theorem 8.2 proving

Theorem 6.14. *Let the notation be as in* Theorem 6.13. *Then the GCD of L_1L_2 and $L_p^-(\psi)$ of* (6.10) *in the unique factorization domain $\Lambda\widehat{\otimes}_W\Lambda'$ is at worst a power of $(\varpi) = \varpi(\Lambda\widehat{\otimes}_W\Lambda')$ in $\Lambda\widehat{\otimes}_W\Lambda'$, where ϖ is a prime element of W. In particular, $h_{\mathrm{i}}(M/F)L_p^-(\psi)$ is a factor of $H(\varphi)$ in the unique factorization domain $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Proof. Take $M_1 = M$. We prove that GCD of L_1 and $L_p^-(\psi)$ is at most a power of ϖ. The same argument works well for L_2. Write $S = \mathrm{Spf}(\Lambda)$. In our case, $\Lambda = \Lambda'$; so, we write often Λ_L with $S_L = \mathrm{Spf}(\Lambda_L)$ (resp. Λ_R with $S_R = \mathrm{Spf}(\Lambda_R)$) for the left (resp. right) factor of $\Lambda\widehat{\otimes}_W\Lambda$. Then $\mathrm{Spf}(\Lambda_L\widehat{\otimes}_W\Lambda_R) = S_L \times_W S_R \cong S \times_W S$. The idea of the proof is that the zero locus of $L_p^-(\psi)$ in $(S_L \times S_R)(\mathbb{C}_p)$ is concentrated on $S_L(\mathbb{C}_p)$ as $L_p^-(\psi) \in \Lambda_L$, while $L_1(P,Q)$ considered as a function of Q of the factor $S_R(\mathbb{C}_p)$ cannot have zero concentrated on the factor $S_L(\mathbb{C}_p)$. To make this rigorous, for $P \in S_L(W)$ regarding P as an algebra homomorphism $P : \Lambda \twoheadrightarrow W$ (i.e., $\Lambda \supset \Gamma_M \ni \gamma \mapsto \varphi_P^{-1}(\gamma) \in W^\times$), we consider an automorphism $\iota_P : \Lambda' \cong \Lambda'$ induced by the inclusion $\Lambda' \cong 1 \otimes \Lambda' \subset \Lambda\widehat{\otimes}\Lambda'$ composed with the twisted projection $\Lambda\widehat{\otimes}\Lambda' \twoheadrightarrow W \otimes_{\Lambda\widehat{\otimes}\Lambda', P\otimes\mathrm{id}_{\Lambda'}} \Lambda' = \Lambda'$.

We now fix $P \in S_L(\mathbb{C}_p)$ and consider the p-adic formal function $S_R(\mathbb{C}_p) \ni Q \to \mathcal{L}_p(P,Q) \in \mathbb{C}_p$ as an element $L_P \in \Lambda'$. Recall $\mathcal{K}_1$ which is the Katz p-adic L-function with branch character $\varphi^{-1}\phi$. Note that $L_P := \iota_{P,*}\Psi_1\mathcal{K}_1$ (i.e., $\int_{\Gamma_M} \xi dL_P = \int_{\Gamma_M}(\xi \circ \iota_P)d(\Psi_1\mathcal{K}_1)$), and hence we get the identity $L_P = 0 \Leftrightarrow L_{P'} = 0$ for $P, P' \in S_R(\mathbb{C}_p)$.

By definition, $L_p^-(\psi) \in \Lambda$. If $L_p^-(\psi)$ is a power of ϖ times a unit, there is nothing to prove. So we assume to have a height 1 prime $\wp = (\pi)$ of Λ prime to (ϖ) such that $L_p^-(\psi) \in \wp$. Then as an element of $\Lambda\widehat{\otimes}\Lambda'$, $\wp\otimes\Lambda' = \pi(\Lambda\widehat{\otimes}\Lambda')$ is the height 1 prime containing $L_p^-(\psi)$. Writing $X = \mathrm{Spf}(\Lambda/\wp)$, we find $\mathrm{Spf}(\Lambda\widehat{\otimes}\Lambda'/\wp\otimes\Lambda') = X \times S'$.

Towards absurdity, we assume $\wp | L_1$. Since the Katz p-adic L-function $\mathcal{K}_1$ has an evaluation point P_0 with critical convergent Hecke L-value and $\Psi_j(P_0) \neq 0$, we have $L_1 \neq 0$ and $\Psi_1 \neq 0$. For $P \in S_R(\mathbb{C}_p)$ in X, we find $L_P = 0$ and hence $L_{P'} = 0$ for all $P' \in S_R(\mathbb{C}_p)$, which implies $\Psi_1\mathcal{K}_1 = L_1 = 0$ and $\mathcal{K}_1 = 0$, a contradiction. Thus L_1 and $L_p^-(\psi)$ has only common factor supported by the characteristic p-fiber of $S_L \times S_R$, as desired.

Alternatively, the identity $L_P := \iota_{P,*}\Psi_1\mathcal{K}_1$ implies the identity $\mu(L_P) = \mu(\Psi_1\mathcal{K}_1)$ of the μ-invariant. Taking $P \in X$, we find $\mu(L_P) = \infty$ which implies $\mathcal{K}_1 = 0$ again, a contradiction. The second argument is employed in [HT93, page 257].

The last assertion on divisibility follows from the identity of Theorem 6.13 as the denominator and the numerator of the right-hand-side of the identity are prime to each other in the unique factorization domain $\Lambda\widehat{\otimes}\Lambda' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $H(\varphi), L_p^-(\psi) \in \Lambda$. □

Corollary 6.15. *Assume $p > 2$ and that ψ is an anticyclotomic character of M. Then we have $h_{\mathrm{i}}(M/F)L_p^-(\psi) | H(\varphi)$ in Λ.*

Proof. We choose $M_1 \neq M$ satisfying (K1–5); so, K is a field and after base-change $\widehat{\varphi}^{-1}\widehat{\phi}$ has split conductor relative to K/F_1 (i.e., satisfying (A1) for K/F_1). for a well chosen (M,φ) (just above (6.6)) with $\varphi^- = \psi$ having conductor prime to $d(M/F)d(M_1/F)$, we choose ϕ as in Lemma 6.9. Then the branch character $\widehat{\varphi}^{-1}\widehat{\phi}$ has conductor made of split primes in K/F_1 by (K3). Then by [H11, Theorem I], the μ-invariant of the $\widehat{\varphi}^{-1}\widehat{\phi}$ of $\mathcal{K}$ is trivial, and hence $\mathcal{L}_p$ in (6.9) has trivial μ-invariant. Then by Theorem 6.13, for the μ-invariant, we have $\mu(h_{\mathfrak{i}}(M/F)L_p^-(\psi)) \leq \mu(H(\varphi))$, which completes the proof of $h_{\mathfrak{i}}(M/F)L_p^-(\psi)|H(\varphi)$ by Theorem 6.14. □

Remark 6.16. As is clear from the above proof of Corollary 6.15, for the divisibility of characteristic 0 factor, we used $M_1 = M$. For the characteristic p-factor, we need to choose a field $M_1 \neq M$ as in (K1–5) since $\widehat{\varphi}^{-1}\widehat{\phi}$ has split conductor relative to K/F_1 (i.e., (A1) for K/F_1) to apply the vanishing of μ in [H11] proven under (A1) for K/F_1. If $M = M_1$ and $\mathfrak{i}(\varphi)\mathfrak{r}(\varphi) \neq 1$, we have $\mathfrak{i}(\varphi^{-1}\phi))\mathfrak{r}(\varphi^{-1}\phi) = \mathfrak{i}(\varphi^{-1}\phi))\mathfrak{r}(\varphi^{-1}\phi_c) = \mathfrak{i}(\varphi)\mathfrak{r}(\varphi) \neq 1$, and we cannot apply the result in [H11].

6.3. **How the condition (A1) is removed for the main conjcture.** We proved the divisibility (L) without assuming (A1). However the proof of the main conjecture in [H06] requires (A1). Here is how to bring the argument to work without (A1) (or rather, we bring by quadratic base-change to the case where (A1) is satisfied). This is described in [H09], and we recall here the argument.

Recall $h_{\mathfrak{i}}(M/F) = (h(M)/h(F))\prod_{\mathfrak{l}|\mathfrak{i}}(|O/\mathfrak{l}|+1)$. Up to a p-adic unit, $h_{\mathfrak{i}}(M/F)$ is the relative class number of the ray class group modulo $\mathfrak{i}$, that is the ratio of the order of the ray class group $Cl_M(\mathfrak{i})$ of M and that of $Cl_F(\mathfrak{i})$ of F.

The idea of the proof is to reduce the conjecture to the case under (A1) treated in [H06] by quadratic base change to a well chosen totally real quadratic extension F'/F eliminating the assumption (A1). The three of the main ingredients of the proof is the $R = \mathbb{T}$ theorem proven by Fujiwara without (A1), the congruence power series $H(\varphi) \in W[[\Gamma_M]]$ of the CM-component of the universal nearly ordinary Hecke algebra $\mathbf{h} = \mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ we defined out of φ and the characteristic power series $\mathcal{F}_M(\psi)$ of the ψ-branch of the anticyclotomic Iwasawa module $X[\psi] = X_\Sigma[\psi]$ [HMI, §5.3.1] unramified outside Σ_p.

Fujiwara formulated his results in [F06] using a different Hecke algebra (out of a quaternion algebra D) related to $\mathbf{h}$ by the Jacquet-Langlands correspondence. We will describe the relation of $\mathbf{h}$ and Fujiwara's Hecke algebra in the last subsection §6.5.

Since we deal with several CM fields, we write $L_M^-(\psi)$ for $L_p^-(\psi)$ hereafter (emphasizing dependence on M). The assertion (L) proven in Corollary 6.15 is

$$h_{\mathfrak{i}}(M/F)L_M^-(\psi)|H(\varphi). \tag{6.11}$$

On the other hand, Fujiwara's result already quoted implies (see Corollary 6.23 and Remark 6.24):

$$H(\varphi) = h_{\mathfrak{i}}(M/F)\mathcal{F}_M^-(\psi) \text{ up to units in } \overline{W}[[\Gamma_M]]. \tag{B}$$

Thus we get:

$$L_M^-(\psi)|\mathcal{F}_M^-(\psi) \text{ in } \overline{W}[[\Gamma_M^-]]. \tag{C}$$

To conclude the theorem, we choose an appropriate totally real quadratic extension F'/F and put $E = F'M$. Since $\mathrm{Gal}(E/F) = \mathrm{Gal}(F'/F) \times \mathrm{Gal}(M/F) \cong (\mathbb{Z}/2\mathbb{Z})^2$, there exists a third quadratic extension M'/F in E/F. Note that M' is a CM field. In summary, we have again

$$\begin{array}{ccccc} & & M' & & \\ & \nearrow & & \searrow & \\ F & \hookrightarrow & F' & \hookrightarrow & E. \\ & \searrow & & \nearrow & \\ & & M & & \end{array}$$

We can arrange F'/F so that

(F1) for any prime $\mathfrak{r}$ of F ramifying in M, the inertia group of $\mathfrak{r}$ in $\mathrm{Gal}(E/F)$ is given by $\mathrm{Gal}(E/M')$ (so, any prime factor of $\mathfrak{r}$ in F' splits in E/F'),

(F2) for any prime $\mathfrak{q}$ inert in M/F in the conductor $\mathfrak{C}(\psi)$ of ψ, the decomposition group of $\mathfrak{q}$ in $\mathrm{Gal}(E/F)$ is given by $\mathrm{Gal}(E/M')$ (so, any prime factor of $\mathfrak{q}$ in F' splits in E/F'),

(F3) E/M ramifies at least one finite place outside p.

We consider the base-change $\psi_E = \psi \circ N_{E/M}$. Then by (F1–2), ψ_E has conductor whose prime factors all split in E/F' (so, (A1) for ψ_E is satisfied). Thus by the main theorem of [H06], we have $L^-_E(\psi_E) = \mathcal{F}^-_E(\psi_E)$ up to units in $\overline{W}[[\Gamma^-_E]]$. We have the restriction map Res : $\Gamma^-_E \to \Gamma^-_M$. By (F3), Res is a surjection. In [H09, §4], we gave a detailed argument proving $\mathrm{Res}(\mathcal{F}^-_E(\psi_E)) = \mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)$ for $\alpha = \left(\frac{E/M}{}\right)$ and $\mathrm{Res}(L^-_E(\psi_E)) = L^-_M(\psi)L^-_M(\psi\alpha)$ up to units in $\overline{W}[[\Gamma^-_M]]$. Thus

$$L^-_M(\psi)L^-_M(\psi\alpha) = \mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)$$

up to units in $\overline{W}[[\Gamma^-_M]]$. Note here that $\psi\alpha$ remains anticyclotomic (because α has order 2). By (C): $L^-_M(\psi)|\mathcal{F}^-_M(\psi)$ and $L^-_M(\psi\alpha)|\mathcal{F}^-_M(\psi\alpha)$, we conclude the individual identity:

$$L^-_M(\psi) = \mathcal{F}^-_M(\psi) \text{ and } L^-_M(\psi\alpha) = \mathcal{F}^-_M(\psi\alpha)$$

up to units in $\overline{W}[[\Gamma^-_M]]$. This finishes the reduction of the main conjecture to the split conductor case treated in [H06].

6.4. **Local deformation.** We now recall Fujiwara's result, which is not described in [H09] in details. We first prove the following lemma essentially due to Weil [W74] (and a similar result is given in [D96, page 141] for unramified $M_{\mathfrak{l}}/\mathbb{Q}_\ell$).

Lemma 6.17. *Let $\mathfrak{l}$ be a prime of F outside p such that $M_{\mathfrak{l}}$ is a quadratic extension field of $F_{\mathfrak{l}}$. Let $\overline{\varphi} = (\varphi \mod \mathfrak{m}_W)$ as a character of $\mathrm{Gal}(\overline{M}_{\mathfrak{l}}/F_{\mathfrak{l}})$ with values in $(W/\mathfrak{m}_W)^\times$. Assume that $\overline{\rho} := \mathrm{Ind}_M^F \overline{\varphi}$ is absolutely irreducible over $\mathrm{Gal}(\overline{M}_{\mathfrak{l}}/F_{\mathfrak{l}})$. Then every deformation $\rho : \mathrm{Gal}(\overline{M}_{\mathfrak{l}}/F_{\mathfrak{l}}) \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ for a local complete W-algebra A is of the form $\mathrm{Ind}_{M_{\mathfrak{l}}}^{F_{\mathfrak{l}}} \Phi$ for a character Φ congruent to $\overline{\varphi}$ modulo $\mathfrak{m}_A$.*

Though well known, we give a detailed proof. For a representation ? of $\mathrm{Gal}(\overline{M}_{\mathfrak{l}}/F_{\mathfrak{l}})$, write $\overline{M}_{\mathfrak{l}}^{\mathrm{Ker}(?)}$ (the splitting field) as $F_{\mathfrak{l}}(?)$. The inertia group of a Galois extension K/E of local fields is written as $I(K/E)$.

Proof. Write $G := \mathrm{Gal}(\widetilde{F}_{\mathfrak{l}}(\overline{\rho})/F_{\mathfrak{l}})$ and $G' := \mathrm{Gal}(\widetilde{F}_{\mathfrak{l}}(\overline{\rho})/M_{\mathfrak{l}})$ for the maximal p-profinite extension $\widetilde{F}_{\mathfrak{l}}(\overline{\rho})$ of $F_{\mathfrak{l}}(\overline{\rho})$. Then ρ_A factors through G. Let $\rho = \rho_A : G \to \mathrm{GL}_2(A)$ be a deformation of $\overline{\rho}$, and write $\pi\rho$ for the composite of ρ with the projection $\pi = \pi_A : \mathrm{GL}_2(A) \twoheadrightarrow \mathrm{PGL}_2(A)$. Then $\mathrm{Ker}(\pi\rho) \subset A^\times$, and hence the commutator $[\mathrm{Ker}(\pi\rho), \mathrm{Ker}(\pi\rho)]$ is contained in $\mathrm{Ker}(\rho)$. Therefore we may replace G by the maximal abelian quotient $D := G/[\mathrm{Ker}(\pi\rho), \mathrm{Ker}(\pi\rho)]$. Hereafter we regard ρ as a representation of D. We write D' for the image of G' in D and K for the fixed field of $\mathrm{Ker}(\pi\rho)$ in $\overline{F}_{\mathfrak{l}}$; so, $D = \mathrm{Gal}(K^{ab}/F_{\mathfrak{l}})$ and $D' = \mathrm{Gal}(K^{ab}/M_{\mathfrak{l}})$ for the maximal abelian extension K^{ab}/K inside $\widetilde{F}_{\mathfrak{l}}(\overline{\rho})$. In particular, $\mathrm{Ker}(\overline{\rho}) \hookrightarrow D \twoheadrightarrow \mathrm{Im}(\overline{\rho})$ and $\mathrm{Ker}(\pi\overline{\rho}) \hookrightarrow D \twoheadrightarrow \mathrm{Im}(\pi\overline{\rho})$ are extensions with p-profinite $\mathrm{Ker}(\overline{\rho})$. By the commutative diagram,

$$\begin{array}{ccc} \mathrm{Im}(\pi_{\mathbb{F}}\overline{\rho}) & \xrightarrow{\hookrightarrow} & \mathrm{PGL}_2(\mathbb{F}) \\ {\scriptstyle \pi_\rho}\uparrow & & \uparrow\!\uparrow {\scriptstyle \bmod \mathfrak{m}_A} \\ \mathrm{Im}(\pi_A\rho) & \xrightarrow[\hookrightarrow]{} & \mathrm{PGL}_2(A), \end{array}$$

the group $\mathrm{Ker}(\pi_\rho)$ is p-profinite.

For a homomorphism of a group X into D, we write $\widehat{X}$ for the closure of the image of X. Since D is almost p-profinite, $\widehat{X}$ is almost p-profinite. Then, by local class field theory, we have an exact sequences

$$\widehat{K}^\times \hookrightarrow D \twoheadrightarrow \mathrm{Gal}(K/F_{\mathfrak{l}}) \cong \mathrm{Im}(\pi\rho) \text{ and } \widehat{K}^\times \hookrightarrow D' \twoheadrightarrow \mathrm{Gal}(K/M_{\mathfrak{l}}). \tag{6.12}$$

Thus D is the completed Weil group relative to $K/F_{\mathfrak{l}}$. We also have exact sequences

$$\widehat{O}_K^\times \hookrightarrow I \twoheadrightarrow I(K/F_{\mathfrak{l}}) \text{ and } \widehat{O}_K^\times \hookrightarrow I' \twoheadrightarrow I(K/M_{\mathfrak{l}}) \tag{6.13}$$

for the inertia subgroup $I = I(K^{ab}/F_{\mathfrak{l}})$ and $I' = I(K^{ab}/M_{\mathfrak{l}})$. Since $O_K^\times$ is almost ℓ-profinite ($(\ell) = \mathfrak{l} \cap \mathbb{Z}$), $\widehat{O}_K^\times$ is almost ℓ-profinite. By the definition of G, $\widehat{O}_K^\times$ is also almost p-profinite; so, $\widehat{O}_K^\times$ is finite. This shows that I is a finite group.

By adding subscript t and w, we indicate the tame inertia group and the wild inertia group. Write $\pi_t : I \to I_t$ for the canonical projection and $\pi' : I \to \overline{\rho}(I)$ for the mod $\mathfrak{m}_A$ reduction map.

Since $\mathrm{Ker}(\pi_t) = I_w$ is ℓ-profinite and $\mathrm{Ker}(\pi')$ is p-profinite, we find $\mathrm{Ker}(\pi_t \times \pi') = \mathrm{Ker}(\pi_t) \cap \mathrm{Ker}(\pi')$ is trivial. Thus $I \hookrightarrow I_t \times \overline{\rho}(I)$. Since $\overline{\rho}(I') \subset \overline{\varphi}(I') \oplus \overline{\varphi}_c(I')$ is abelian with cyclic p-primary part inside I_t, I' is an abelian group. Writing I'_r for the r-primary part of I' for a prime r, I'_r for $r \neq p$ injects into $\overline{\rho}(I')$ which is a surjective image of the r-primary part of $O_{M_\mathfrak{l}}^\times$ as I' is the Galois group of an abelian fully ramified extension over $M_\mathfrak{l}$. If $r \neq \ell$, I'_r is the surjective image of $\mathbb{F}_Q^\times$ writing $O_{M_\mathfrak{l}}/\mathfrak{l} = \mathbb{F}_Q$; so, $|I'_r|\big|(Q-1)$. By the action of $\mathrm{Gal}(M_\mathfrak{l}/F_\mathfrak{l}) = \langle c \rangle$ via conjugation on I', I' fits into an exact sequence $I'^- \hookrightarrow I' \twoheadrightarrow I'^+$ for the "$-$"-eigenspace I'^- under the conjugation action of c.

Suppose that $M_\mathfrak{l}/F_\mathfrak{l}$ is unramified. Then $Q = q^2$ for $q = |O_{F_\mathfrak{l}}/\mathfrak{l}|$. Computing the "$\pm$"-eigenspaces of c on $\mathbb{F}_Q^\times$, we find $|I'^{\pm}_r|\big|(q \mp 1)$ for $r \neq p$ as I'^+_r is the Galois group of a fully ramified abelian extension over $F_\mathfrak{l}$.

If $M_\mathfrak{l}/F_\mathfrak{l}$ is ramified and $r \notin \{p, \ell\}$, $|I'_r|\big|(q-1)$ and $I'_r = I'^+_r$ as c acts trivially on $\mathbb{F}_q = O_{M_\mathfrak{l}}/\mathfrak{l} = O_{F_\mathfrak{l}}/\mathfrak{l}$. In any case, without assuming a ramification condition on $M_\mathfrak{l}/F_\mathfrak{l}$, a Frobenius lift ϕ in D acts on I_t by $\phi\sigma\phi^{-1} = \sigma^q$ for the order q of the residue field of $O_{F_\mathfrak{l}}$. Thus the action of ϕ on I'_r for $r \notin \{p, \ell\}$ coincides with the action of c.

Since A is p-profinite local and φ has order prime to p, we may and do identify φ with the Teichimüller lift to $A^\times$ of $\overline{\varphi}$. Thus we have a subgroup $\overline{D} \subset D$ with $D \cong \overline{D} \ltimes \mathrm{Ker}(\overline{\varphi})$ (resp. $\overline{D}' \subset D'$ with $D' \cong \overline{D}' \ltimes \mathrm{Ker}(\overline{\varphi}|_{D'})$) isomorphic to $\overline{\rho}(D)$ (resp. $\overline{\rho}(D')$), and $\overline{D}'$ acts on the representation space $V(\rho)$ with the distinct characters φ and φ_c. Note $\varphi \neq \varphi_c$ on $\overline{D}'$ as $\overline{\rho} = \mathrm{Ind}_{D'}^{D} \overline{\varphi}$ is absolutely irreducible. Thus the ?-eigenspace $V(?) \subset V(\rho)$ for $? = \varphi, \varphi_c$ has rank 1 over A and $V(\rho) = V(\varphi) \oplus V(\varphi_c)$ over $\overline{D}'$.

For $g \in I'$, on $\rho(g)V(\varphi)$, $\overline{I}' = \overline{D} \cap I$ acts by φ_g given by $\varphi_g(h) = \varphi(g^{-1}hg)$ for $h \in \overline{I}'$, Since I' is abelian, g preserves $V(\varphi)$; so, I' acts on $V(\varphi)$ by a character Φ extending φ. The group D' is generated by a Frobenius lift ϕ' in D' and I', and ϕ' normalizes I'; so, on $\rho(\phi')V(\Phi)$, I' acts either by Φ or Φ_c. If $M_\mathfrak{l}/F_\mathfrak{l}$ is unramified, $\Phi_c = \Phi_\phi$, and therefore $\rho(\phi)$ interchanges Φ and Φ_c. In particular ϕ' commutes with I' and D' is an abelian group. This implies $\phi'\sigma\phi'^{-1} = \sigma^Q = \sigma$ for I'_t; so, $|I'_t|\big|(Q-1)$ if $M_\mathfrak{l}/F_\mathfrak{l}$ is unramified, and hence $|I|\big|(Q-1)|\mathrm{Im}(\overline{\rho})|$ is bounded independently of ρ_A.

If $M_\mathfrak{l}/F_\mathfrak{l}$ is ramified, we can still argue as above, and find $V(\rho) = V(\Phi) \oplus V(\Phi_c)$ for a character Φ extending φ to I'. Since $\phi \in D'$ normalizes I', either (i) $\Phi_\phi = \Phi$ or (ii) $\Phi_\phi = \Phi_c$. The case (ii) implies $\varphi_c = \varphi_\phi$. Since $\phi \in D'$ and φ is a character of D', $\varphi_\phi = \varphi$; so, $\varphi_c = \varphi$, which is against absolute irreducibility of $\overline{\rho}$. Thus Case (i) occurs, and $\Phi_\phi = \Phi$ and hence Φ extends to D' (i.e., D' is abelian). Thus we find $\rho = \mathrm{Ind}_{D'}^{D} \Phi$. Since ϕ commutes with I', we find $|I'_t|\big|(q-1)$. In any case, $|I|\big|(Q-1)|\mathrm{Im}(\overline{\rho})|$ always. □

The following corollary of the proof of the above lemma is useful:

Corollary 6.18. *Let the notation and the assumptions be as in* Lemma 6.17. *If ρ_A is a deformation of $\overline{\rho} = \mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \overline{\varphi}$ for $\ell \neq p$, then the Artin conductor of ρ_A is equal to the Artin conductor of $\overline{\rho}$.*

Proof. The Artin conductor of the induced representation $\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi$ is given by $N_{M_\mathfrak{l}/F_\mathfrak{l}}(\mathfrak{C}(\xi))d(M_\mathfrak{l}/F_\mathfrak{l})$ for the conductor $\mathfrak{C}(\xi) = \mathfrak{C}(\xi|_{I'})$ of the character ξ and the relative discriminant $d(M_\mathfrak{l}/F_\mathfrak{l})$. Under the notation in the proof of Lemma 6.17, the p-primary part I'_p is a quotient of $\mathbb{F}_Q^\times$. Write $\rho_A = \mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \Phi$ by Lemma 6.17. Since $\Phi\varphi^{-1}|_{I'}$ factors through I'_p, If $\varphi|_{I'} \neq 1$, $\mathfrak{C}(\Phi) = \mathfrak{C}(\varphi) = \mathfrak{l}$. If φ is unramified, we get $\varphi = \varphi_c$ against absolute irreducibility of $\overline{\rho}$. This implies $\mathfrak{C}(\Phi) = \mathfrak{C}(\varphi)$, and hence the conductor of ρ_A and $\overline{\rho}$ match. □

6.5. $R = \mathbb{T}$ **theorem.** We recall here how the local ring attached to $\overline{\rho} = \mathrm{Ind}_M^F \overline{\varphi}$ of $\mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ can be identified with the universal Galois deformation ring, reducing the fact to [F06, §7]. This fact is used in §6.3. The Taylor–Wiles argument implemented by Fujiwara for quaternionic modular forms identify the universal Galois deformation ring with a local ring of a Hilbert modular Hecke algebra $\mathbf{h}_\kappa^{n.ord}(\mathfrak{N}_\chi, \chi; W)$ of weight $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]$ of diagonal torus $T = \mathbb{G}_m \times \mathbb{G}_m \subset \mathrm{GL}(2)_{/F}$ and a finite order Neben character set $\chi = (\chi_1, \chi_2, \chi_+)$ of $T(O_p) \times Cl_F(\mathfrak{N}'p^\infty)$. See Definition 6.5 in the text and (SA1–3) in [PAF, §4.3.1] for the weight and the Neben character set; in particular, κ satisfies $\kappa_1 + \kappa_2 = [\kappa]I$ with $[\kappa] \in \mathbb{Z}$ for $I = \sum_{\nu \in I_F} \nu$. Strictly speaking, Fujiwara works with quaternionic automorphic forms of a division algebra $D_{/F}$ producing Shimura variety of dimension ≤ 1. So here we describe how we can shift from quaternionic automorphic forms to Hilbert modular forms by the Jacquet–Langlands correspondence (which is perhaps not in the literature though obvious for some specialists).

We first study briefly a slight generalization of the deformation problems discussed in [EMI, §6.1.1]. In this section, W is a discrete valuation ring sufficiently large but a finite extension of $\mathbb{Z}_p$. We write $\mathbb{F} := W/\mathfrak{m}_W$.

Let $F(\overline{\rho})$ be the splitting field of $\overline{\rho} = \mathrm{Ind}_F^M \overline{\varphi}$; so, $\mathrm{Gal}(F(\overline{\rho})/F) \cong \mathrm{Im}(\overline{\rho})$. Let S be the set of prime factors of $\mathfrak{N}'$, and recall $S_{sc} \subset S$ which is the set of primes $\mathfrak{l}$ such that the image $\overline{\rho}(D_\mathfrak{l})$ is non-abelian; so, it falls in the normalizer of a subtorus of $\mathrm{GL}_2(\mathbb{F})$. In other words, S_{sc} is made up of super cuspidal primes $\mathfrak{v}$ for $\pi(\varphi)$ (or equivalently, the set of $\mathfrak{v}$ with absolutely irreducible $\mathrm{Ind}_M^F \varphi|_{D_\mathfrak{v}}$: $S_{sc} = \{\mathfrak{l}|\mathfrak{i}(\psi)\mathfrak{r}(\psi)\}$). Then $S = S_{ab} \sqcup S_{sc}$. Write $F^{(pS_{sc})}(\overline{\rho})$ for the maximal p-profinite extension of $F(\overline{\rho})$ unramified outside p and S_{sc}, and put $G := \mathrm{Gal}(F^{(pS_{sc})}/F)$ with $G' := \mathrm{Gal}(F^{(pS_{sc})}/M)$. We write $D_\mathfrak{l}$ (resp. $I_\mathfrak{l}$) for the decomposition (resp. inertia) group in G of a prime $\mathfrak{l}$ of F. Since φ has order prime to p (and $p > 2$) and $\overline{\rho} = \mathrm{Ind}_M^F \overline{\varphi}$, the image $\overline{\rho}(I_\mathfrak{l})$ of the $\mathfrak{l}$-inertia subgroup for $\mathfrak{l}|p\mathfrak{N}'$ is a non-trivial subgroup having values in the normalizer of a subtorus of $\mathrm{GL}_2(\mathbb{F})$.

Recall the Neben character set ϵ defined in Definition 6.7 for φ (i.e., $\epsilon_{j,\mathfrak{q}} = \epsilon^{?}_{j,\mathfrak{q}}$ for ? = min or new, and $\epsilon_{j,\mathfrak{q}} = \epsilon^{new}_{j,\mathfrak{q}}$ for $\mathfrak{q} \in S_{sc}$). In this subsection, the choice of "new" or "min" does not matter since the Hecke algebras produced are all isomorphic (as long as projecting algebra to the "new" part when we take ϵ^{new}_φ), since each principal series automorphic representation at $\mathfrak{q} \notin S_{nc} \cup \{\mathfrak{p}|p\}$ has a unique minimal vector (up to scalars) assigned by ϵ^{min}_φ and a unique new vector (up to scalars) at the same time. At $\mathfrak{p}|p$, we choose minimal branch character $\epsilon_\mathfrak{p} = \epsilon^{min}_{\varphi,\mathfrak{p}}$ (though as we have done above (6.6), we have chosen φ so that $\epsilon^{min}_\mathfrak{p} = \epsilon^{new}_\mathfrak{p}$). Our argument works even if $\epsilon^{min}_\mathfrak{p} \neq \epsilon^{new}_\mathfrak{p}$ (i.e., we do not make the specific choice of φ with $\varphi^- = \psi$).

Pick a Neben character set χ deforming ϵ. This means that at $\chi_{?,\mathfrak{q}} \equiv \epsilon_{?,\mathfrak{q}} \mod \mathfrak{m}_W$ $(? = 1, 2, +)$ for all $\mathfrak{q}$ outside p and χ is trivial outside S. We write the Hilbert modular Hecke algebra $\mathbf{h}_{\kappa,\chi} = \mathbf{h}^{n.ord}_\kappa(\mathfrak{N}_\chi, \chi; W)$ of weight $\kappa = (\kappa_1, \kappa_2)$ with a Neben character set $\chi = (\chi_1, \chi_2, \chi_+)$, which is a quotient of $\mathbf{h} = \mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ by an arithmetic point P of $W[[T(O_p) \times Cl_F(\mathfrak{N}'p^\infty]]$ associated to κ and χ. Here the minimal level $\mathfrak{N}_\chi$ is determined by χ as described just above Definition 6.7, and each automorphic representation π whose minimal vector induces a W-algebra homomorphism $\lambda_\pi : \mathbf{h}_{\kappa,\chi} \to W$ as Hecke eigenvalues has principal series component $\pi(\pi_{1,\mathfrak{q}}, \pi_{2,\mathfrak{q}})$ with $\chi_{j,\mathfrak{q}} = \pi_{j,\mathfrak{q}}$ for all primes $\mathfrak{q}$ outside $S_{sc} \cup \{\mathfrak{p}|p\}$. Then χ_j is given by, for all $x \in O^\times_{p\mathfrak{N}'}$

$$\chi_j(x)x_p^{\kappa_j} \equiv \epsilon_j(x) \mod \mathfrak{m}_W \ (j = 1, 2) \ \text{ and } \ \chi_+(x)x_p^{[\kappa]I} \equiv \epsilon_+(x) \mod \mathfrak{m}_W. \tag{6.14}$$

We write the character of the left-hand-side of (6.14) as $\chi_{\kappa,?}$ for ? = 1, 2, +. Note that $\mathfrak{N}_\chi = \mathfrak{N}(\mathfrak{C}(\chi_{1,p}) \cap \mathfrak{C}(\chi_{2,p}))$ (so, $\mathfrak{N}_\chi/\mathfrak{N}$ is a factor of a hight power of p).

At a prime $\mathfrak{l}|\mathfrak{i}(\psi)$ having odd exponent in $\mathfrak{I}(\psi)$, Fujiwara chooses for some reason a quaternion algebra D ramified only at such places and most ramified at archimedean places [F06, §3.7] (so, $D \otimes_\mathbb{Q} \mathbb{R}$ has at most one split factor $M_2(\mathbb{R})$). Main reasons for this choice of D are: Since he chooses the étale cohomology of a Shimura variety of an appropriate level group $K \subset G_D(F^{(\infty)}_\mathbb{A})$ $(G_D = D^\times)$ for the Taylor–Wiles ingredient M_Q of Hecke modules in (tw3) of [HMI, §3.2.3], to guarantee torsion-freeness of M_Q required by the Taylor–Wiles argument, he needs to shift to Shimura variety of dimension ≤ 1. In addition, as described in [F06, §3.7], a direct local-global compatibility in [GK80] of Hilbert modular Galois representations made out of D by Carayol in the indefinite case and by Taylor otherwise assures modular Galois deformations to have required local form at $\mathfrak{v}|\mathfrak{i}(\psi)\mathfrak{r}(\psi)$.

If $\mathfrak{I}(\psi)\mathfrak{R}(\psi)$ is trivial, everything we say here is exposed in [HMI, §3.2.4], where the Taylor-Wiles argument is done only using Hilbert modular forms. In [HMI, §3.2.4], M_Q is cut out from the p-adic Hecke algebra $\mathbf{h}^{n.ord}(\mathfrak{N}_\chi Q, \chi; W)$ of each extra level Q which is always torsion-free; so, there is no need to shift from $\mathrm{GL}(2)_{/F}$ to G_D; however, we do not reach the freeness of Fujiwara's M_Q over the corresponding local ring of the Hecke algebra [F06, Theorem 0.4]. In any case, the case where $\mathfrak{I}(\psi)\mathfrak{R}(\psi) \neq 1$ is not treated in [HMI] mainly to make the book short.

Fujiwara's Galois cohomological computation [F06, §3] similar to [HMI, §3.2.4] (to show that $\{M_Q\}_Q$ produces a Taylor–Wiles system) is nothing to do with the choice of G_D or $\mathrm{GL}(2)$; so, the only things necessary is to identify the level group in $\mathrm{GL}_2(\widehat{O})$ which produces the Hecke algebra containing the local ring parameterizing all modular deformations of a given type. Or else, as local–global compatibility and cohomological computation of the tangent spaces of local and global deformation rings supplied by Fujiwara, we can give ourselves M_Q using appropriate Hecke algebras similarly to [HMI] and repeat the argument as in [HMI], for which the exposition will be technical, long and tedious. So we describe in this subsection the deformation type Fujiwara used at primes $\mathfrak{v}|\mathfrak{i}(\psi)\mathfrak{r}(\psi)$, and we show that all Jacquet–Langlands image for GL(2) (from D) produces a unique local ring $\mathbb{T}_{\kappa,\chi}$ of $\mathbf{h}_{\kappa,\chi}$ with respect to the level data $(\mathfrak{N}_\chi, \chi)$, and patching $\{\mathbb{T}_{\kappa,\chi}\}_{\kappa,\chi}$ together, we produce a unique local ring $\mathbb{T}$ of $\mathbf{h}$ universal among nearly ordinary deformations of minimal level.

Plainly the following condition is satisfied:

(m) *for all primes $\mathfrak{l}$ outside p, ramification index of $\mathfrak{l}$ in $F(\overline{\rho})/F$ is prime to p.*

Let $\rho : G \to \mathrm{GL}_2(A)$ $(A \in CL_B)$ be a deformation of $\overline{\rho} = \mathrm{Ind}_M^F \overline{\varphi} : G \to \mathrm{GL}_2(\mathbb{F})$ acting on $V(\rho)$. We fix the ordinary quotient character $\overline{\delta} = \overline{\varphi}_c$ of $\overline{\rho}$ so that $\overline{\rho}|_{D_\mathfrak{p}} \cong \left(\begin{smallmatrix} \overline{\epsilon} & * \\ 0 & \overline{\delta}\end{smallmatrix}\right)$. Identifying the inertia group of maximal abelian extension of $F_\mathfrak{p}$ with $O_\mathfrak{p}^\times$, we say ρ is *p-ordinary* if

($\mathfrak{p}$) $\rho_A|_{D_\mathfrak{p}} \cong \begin{pmatrix} \delta_{1,\mathfrak{p}} & * \\ 0 & \delta_{2,\mathfrak{p}}\end{pmatrix}$ *for two characters $\delta_{j,\mathfrak{p}} : D_\mathfrak{p} \to A^\times$ with $\delta_{j,\mathfrak{p}}|_{I_\mathfrak{p}} \equiv \epsilon_{j,\mathfrak{p}} \mod \mathfrak{m}_A$ for $j = 1, 2$.*

Here $\epsilon_{j,\mathfrak{p}}$ is in the Neben character set made out of φ_c. By (A2), there is no ambiguity of the order of $\delta_{j,\mathfrak{p}}$. We call $\delta_{2,\mathfrak{p}}$ the *ordinarity* (quotient) character of ρ_A. The weight (κ, χ) condition is

($\mathfrak{p}_\kappa$) $\rho_A|_{D_\mathfrak{p}} \cong \begin{pmatrix} \delta_{1,\mathfrak{p}} & * \\ 0 & \delta_{2,\mathfrak{p}}\end{pmatrix}$ *for two characters $\delta_{j,\mathfrak{p}} : D_\mathfrak{p} \to A^\times$ with $\delta_{j,\mathfrak{p}}|_{I_\mathfrak{p}} = \chi_{\kappa,j}|_{I_\mathfrak{p}}$ for $j = 1, 2$.*

For $\mathfrak{l} \in S_{ab}$, in Definition 6.7, we have chosen $\epsilon_{j,\mathfrak{l}} : I_\mathfrak{l} \to W^\times$ as Neben characters defined by φ. By extending scalars $\mathbb{F}$, we assume that $\overline{\rho}|_{I_\mathfrak{l}} = \overline{\epsilon}_{1,\mathfrak{l}} \oplus \overline{\epsilon}_{2,\mathfrak{l}}$ for $\mathfrak{l} \in S_{ab}$ with $\overline{\epsilon}_{j,\mathfrak{l}} = (\epsilon_{j,\mathfrak{l}} \mod \mathfrak{m}_W)$.

We impose a "minimality" condition for $\mathfrak{q} \in S_{ab}$ (similar to ($\mathfrak{p}$)) :

($\mathfrak{q}_{ab}$) *For $\mathfrak{q} \in S_{ab}$, we have $\rho|_{I_\mathfrak{q}}$ mod $\mathfrak{m}_A \cong \begin{pmatrix} \varphi_\mathfrak{q} & 0 \\ 0 & \varphi_{c,\mathfrak{q}}\end{pmatrix}$.*

Here we regard φ as having values in A by the W-algebra structure $W \to A$. In the condition ($\mathfrak{q}_{ab}$), the right-hand-side is always diagonal by our choice of G (i.e., $\mathfrak{q}$ only can ramify in $F(\overline{\rho})/F$). For any deformation ρ_A on the bigger Galois group $\mathrm{Gal}(\overline{M}/F)$, ρ_A satisfying ($\mathfrak{q}_{ab}$) factors through G as the splitting field of ρ_A over $F(\overline{\rho})$ is p-profinite unramified. The following rigidity result at $\mathfrak{v} \in S_{sc}$ follows from Lemma 6.17

($\mathfrak{v}$) *For $\mathfrak{v} \in S_{sc}$ (i.e., $\mathfrak{v}|\mathfrak{i}(\psi)\mathfrak{r}(\psi)$), $\rho|_{D_\mathfrak{v}} = \mathrm{Ind}_{D'_\mathfrak{v}}^{D_\mathfrak{v}} \Phi_A$ for a character Φ_A of conductor $\mathfrak{C}(\varphi_\mathfrak{v})$ with $\Phi_A \equiv \varphi_A$ mod $\mathfrak{m}_A$,*

where $D'_\mathfrak{v}$ is the decomposition group at $\mathfrak{v}$ of G'. We always impose these two conditions ($\mathfrak{p}$) and ($\mathfrak{q}_{ab}$), while ($\mathfrak{v}$) is automatic by Corollary 6.18. If a p-adic Galois character $\xi : G \to \overline{\mathbb{Q}}_p^\times$ satisfies $\xi(x) = \chi_+(x)x_p^{[\kappa]I}$ for $x \in F^\times_{\mathbb{A}^{(\infty)}}$ as a Hecke character, the unitarization of the archimedean avatar of ξ is the archimedean avatar of χ_+.

We consider the following functors for a fixed absolutely irreducible representation $\overline{\rho} : G \to \mathrm{GL}_2(\mathbb{F})$ satisfying ($\mathfrak{p}$) and ($\mathfrak{q}_{ab}$). Recall $\mathcal{D}^\emptyset, \mathcal{D}, \mathcal{D}_\xi : \mathcal{C} \to SETS$ given by

$$\begin{aligned}
&\mathcal{D}^\emptyset(A) := \{\rho_A : G \to \mathrm{GL}_2(A) | \rho_A \bmod \mathfrak{m}_A = \overline{\rho}\}/\Gamma(\mathfrak{m}_A),\\
&\mathcal{D}^{ord}(A) = \{\rho_A \in \mathcal{D}^\emptyset(A) | (\mathfrak{p}) \text{ for all } \mathfrak{p}|p \text{ and } (\mathfrak{q}_{ab}) \text{ for all } \mathfrak{q} \in S_{ab}\},\\
&\mathcal{D}^{ord}_\kappa(A) = \{\rho_A \in \mathcal{D}^{ord}(A) | (\mathfrak{p}_\kappa) \text{ for all } \mathfrak{p}|p\},\\
&\mathcal{D}^{?,\xi}_*(A) = \{\rho_A \in \mathcal{D}^?(A) | \det\rho = \xi\} \quad \text{for } * = \text{nothing or } \kappa,
\end{aligned} \tag{6.15}$$

where $\Gamma(\mathfrak{m}_A) := \mathrm{Ker}(\mathrm{GL}_2(A) \to \mathrm{GL}_2(A/\mathfrak{m}_A))$. For the local ring $\mathbb{T}$ of $\mathbf{h}$ whose Galois representation $\rho_\mathbb{T}$ deforms $\overline{\rho}$, as a character of $F_\mathbb{A}^\times$, $\det\rho_\mathbb{T} = \widetilde{\varphi}|_{F_\mathbb{A}^\times}\chi_M$ for $\chi_M := \left(\frac{M/F}{}\right)$.

The deformation problem $\mathcal{D}^?_*$ in (6.15) requires (h2–3) and (Q4–6) in [HMI, §3.2.1] for primes outside $S_{sc} \cup \{p\}$, and at $\mathfrak{q} \in S_{sc}$, no restriction imposed; so, it is the type denoted by $\mathbf{u}$ (unrestricted) in [F06, §3.7]. Strictly speaking, our choice of the Galois group G automatically imposes ($\mathfrak{v}$) (and this is essentially the content of the condition $\mathbf{u}$ in [F06, §3.7]). The other conditions (h?) and (Q?) in [HMI, §3.2.1] are either automatic by our choice of G or does not apply for the induced representation $\overline{\rho} = \mathrm{Ind}_M^F \overline{\varphi}$ for φ of order prime to p. We record the following fact proven in [HMI, Theorem 3.25], [F06, Theorem 3.34] and [EMI, Theorem 6.1.1] covering (possibly) different cases:

Theorem 6.19. *Let the notation be as above. The functors $\mathcal{D}^?_*$ for the properties ? and $*$ in (6.15) is represented by a universal couple $(R^?_*, \boldsymbol{\rho}^?_*)$, respectively, so that $\mathcal{D}^?_*(A) \cong \mathrm{Hom}_\mathcal{C}(R^?_*, A)$ by $\rho \mapsto \varphi_\rho$ with $\varphi_\rho \circ \boldsymbol{\rho}^?_* \equiv \rho \mod \Gamma(\mathfrak{m}_A)$.*

Write $Cl_F^+(\mathfrak{x})$ for the strict class group modulo $\mathfrak{x}$. Decompose $Cl_F^+(\mathfrak{N}'p^\infty) = \Gamma_F^+ \times \Delta^+$ for the maximal torsion subgroup Δ^+. Then $R^?$ for $? = \emptyset, ord$ is a $W[\Delta^+]$-algebra by $\det(\boldsymbol{\rho}^?)|_{\Delta^+}$. Noting that ϵ_+ induces a character of Δ^+. Then we define

$$\mathcal{D}^{ord,\epsilon_+}(A) := \{\rho \in \mathcal{D}(A) |\det(\rho)|_{\Delta^+} = \varphi_\rho \circ \epsilon_+\}.$$

Plainly $\mathcal{D}^{ord,\epsilon_+}$ is represented by

$$R^{ord,\epsilon_+} := R^{ord} \otimes_{W[\Delta^+],\epsilon_+|_{\Delta^+}} W = R^{ord}/(\det(\boldsymbol{\rho}^{ord})(\delta) - \epsilon_+(\delta))_{\delta \in \Delta^+} \tag{6.16}$$

and the universal representation $\boldsymbol{\rho}^{ord,\epsilon_+}$ which the projection of $\boldsymbol{\rho}^{ord}$ to $\mathrm{GL}_2(R^{ord,\epsilon_+})$. The representations $\boldsymbol{\rho}^?_*$ is called *the universal representation* (of type $(?,*)$). By $\det(\boldsymbol{\rho}^{ord,\epsilon_+}) : \Gamma_F \to (R^{ord,\epsilon_+})^\times$ (resp. $\det(\boldsymbol{\rho}^{ord}) : Cl_F(\mathfrak{N}'p^\infty) \to R^\times$), R^{ord,ϵ_+} (resp. R) is a $W[[\Gamma_F]]$-algebra (resp. $W[[Cl_F(\mathfrak{N}'p^\infty)]]$-algebra).

Since $\kappa_1 + \kappa_2 = [\kappa]I$, the character $\chi_{\kappa,+}$ in (6.14) factors through $O_p^\times/\overline{O_+^\times}$ which is a subgroup of $Cl_F^+(p^\infty)$, where $O_+^\times \subset O^\times$ is the group of totally positive units. Here $\overline{O_+^\times}$ is the closure of $O_+^\times$ in the compact group $O_p^\times$. Since p is unramified in F and $p > 2$, $O_p^\times$ is uniquely a product of a torsion-free subgroup Γ_F^- and a maximal torsion subgroup Δ^-, ϵ_j can be regarded as a character of Δ^-. Then $\chi_\kappa^- := \chi_{\kappa,1}^{-1}\chi_{\kappa,2}$ as a character of $O_p^\times$ induces a character of $O_p^\times = \Gamma_F^- \times \Delta^-$ whose restriction to Δ^- is given by $\epsilon^-|_{O_p^\times}$. Also the image of $O_p^\times$ in the quotient Γ_F^+ of $Cl_F(p^\infty)$ is canonically a factor of $O_p^\times/\overline{O^\times}$. Thus we have a well defined $\chi_{\kappa,+}|_{\Gamma_F^+}$. Then, we define $\chi_{\boldsymbol{\Gamma}} : \boldsymbol{\Gamma} = \Gamma_F^+ \times \Gamma_F^- \to W$ by $\chi_\kappa^-\chi_{\kappa,+}$. Tracking down our construction, similarly to (6.16), we obtain

Corollary 6.20. *Let ξ be a Galois character of G given by $\xi(x) = \chi_+(x)x_p^{[\kappa]I}$ regarded as a Hecke character (so, $\xi|_{\Delta^+} = \epsilon_+|_{\Delta^+}$). Then $R_\kappa^\xi = R^{ord,\epsilon_+} \otimes_{W[[\boldsymbol{\Gamma}]],\chi_{\boldsymbol{\Gamma}}} W$, and $\boldsymbol{\rho}_\kappa^\xi$ is the projection of $\boldsymbol{\rho}^{ord,\epsilon_+}$.*

Let the notation be as in Corollary 6.20. Choose a weight κ such that $k = \kappa_1 - \kappa_2 + I \geq 2I$. Let ξ and χ be as in Corollary 6.20. For a suitably chosen level group $K = \prod_{\mathfrak{q}} K_{\mathfrak{q}} \subset G_D(F_{\mathbb{A}})$ (and a Neben character set) in [F06, §3.7] relative to $\overline{\rho}$, Fujiwara made a p-adic Hecke algebra $\mathbb{H}_{\kappa,\chi}$ of weight (κ,χ) with respect to the level K and identified $(R_\kappa^\xi, \boldsymbol{\rho}_\kappa^\xi)$ with $(\mathbb{T}_F^{\kappa,\chi}, \rho_{\mathbb{T}_F^{\kappa,\chi}})$ for a unique local ring $\mathbb{T}_F^{\kappa,\chi}$ of $\mathbb{H}_{\kappa,\chi}$ carrying the quaternionic modular deformation $\rho_{\mathbb{T}_F^{\kappa,\chi}}$ of $\overline{\rho}$. Since Fujiwara's quaternion algebra D splits at primes outside S_{sc}, identifying $D \otimes_F F_{\mathbb{A}^{(S_{sc}\infty)}}$ with $\mathrm{GL}_2(F_{\mathbb{A}^{(S_{sc}\infty)}})$, $K^{(sc)} = \prod_{\mathfrak{q} \notin S_{sc}} K_{\mathfrak{q}} = U_0(\mathfrak{N})^{(S_{sc})}$ (up to inner conjugation) and his Neben-character set is identical to χ outside S_{sc} (up to inner conjugation). The choice of $K_{sc} := \prod_{\mathfrak{q} \in S_{sc}} K_{\mathfrak{q}}$ and the character (identifying a unique vector in the representation space up to scalars) is described in [F06, §3.7] (via a result of Gérardin-Kutzko [GK80] making the local Langlands correspondence explicit for induced representations on the Galois side and the automorphic side). We do not specify K_{sc} but just mention that its level is given by the S_{sc}-part of $\mathfrak{N}_{sc}$ in (6.2). The main theorem of [F06, Theorem 0.2] include

Theorem 6.21 (K. Fujiwara). *Suppose $\kappa_1 - \kappa_2 \geq I$ (i.e., $\kappa_{1,\nu} - \kappa_{2,\nu} \geq 1$ for all $\nu \in I_F$). We have a canonical isomorphism $(R_\kappa^\xi, \boldsymbol{\rho}_\kappa^\xi) \cong (\mathbb{T}_F^{\kappa,\chi}, \rho_{\mathbb{T}_F^{\kappa,\chi}})$ under* (unr), (ord) *and* (A2–3). *In particular, $\mathbb{T}_F^{\kappa,\chi}$ is a relative local complete intersection over W.*

The local complete intersection property follows automatically from the Taylor–Wiles system argument Fujiwara used (cf. [EMI, Theorem 6.2.20]).

Corollary 6.22. *Let the notation and the assumption be as in* Theorem 6.21. *Then we have a canonical isomorphism $(\mathbb{T}_{\kappa,\chi}, \rho_{\mathbb{T}_{\kappa,\chi}}) \cong (\mathbb{T}_F^{\kappa,\chi}, \rho_{\mathbb{T}_F^{\kappa,\chi}})$ for the unique local ring $\mathbb{T}_{\kappa,\chi}$ of $\mathbf{h}_{\kappa,\chi}$ carrying a deformation of $\overline{\rho}$. Moreover, for the unique local ring $\mathbb{T}$ of $\mathbf{h} = \mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ ($\epsilon = \epsilon_\varphi^{min}$) specializing to $\mathbb{T}_{\kappa,\chi}$ by the control theorem (in* [PAF, Corollary 4.31]) *and its Galois representation $\rho_{\mathbb{T}}$, we have a canonical isomorphism $(\mathbb{T}, \rho_{\mathbb{T}}) \cong (R^{ord,\epsilon_+}, \boldsymbol{\rho}^{ord,\epsilon_+})$; in particular, $\mathbb{T}$ is a relative local complete intersection over $W[[\boldsymbol{\Gamma}]]$.*

By the local Langlands conjecture proven by Kutzko [K80] and the local-global compatibility (outside p) of p-adic Hilbert modular Galois representation actually made out of quaternion algebras, the local Artin conductor at a prime $\mathfrak{l} \nmid p$ of the p-adic Galois representation attached to a quaternionic holomorphic automorphic representation π of $G_D(F_{\mathbb{A}})$ is equal to the automorphic conductor of the Jacquet-Langlands image $JL(\pi)$ of π as an automorphic representation of $\mathrm{GL}_2(F_{\mathbb{A}})$.

As described in [F06, §3.7], the local-global compatibility follows from Carayol's (resp. Taylor's) construction [C86] (resp. [T89], [T95]) of the Galois representation if Fujiwara's quaternion algebra D is indefinite (resp. definite). For this, the choice of D is also forced (in appearance).

Proof. Simply, write $(T_F, \boldsymbol{\rho}_F)$ (resp. $(T, \boldsymbol{\rho})$) for $(\mathbb{T}_F^{\kappa,\chi}, \rho_{\mathbb{T}_F^{\kappa,\chi}})$ (resp. $(\mathbb{T}_{\kappa,\chi}, \rho_{\mathbb{T}_{\kappa,\chi}})$). Then T and T_F are reduced free of finite rank over W. Thus $T_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \prod_{E\in J} E$ and $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \prod_{E'\in J'} E'$ for finite extensions E, E' of $\mathrm{Frac}(W)$ with finite index set J, J'. Extending scalars W, we may assume that $O_E \cong O_{E'} \cong W$ for all E, E'. By the universality of (T_F, ρ_F), for each $\rho \in \mathcal{D}^{\xi}_{\kappa}(W)$, we have a unique holomorphic cuspidal automorphic representation π_D of $G_D(F_{\mathbb{A}})$ whose Galois representation ρ^D_π is isomorphic to ρ. Then π_D has a unique vector f_D (up to scalars) in the space of cusp forms $\mathcal{S}_D$ on Fujiwara's level group K and his Neben character. Thus ρ^D_π is isomorphic to the Galois representation $\rho_E : G \to \mathrm{GL}_2(O_E)$ attached to the component $E \in J$ uniquely. We thus write (π_D, f_D) as (π_E, f_E). By Jacquet–Langlands correspondence, we have a holomorphic cuspidal automorphic representation $\pi'_E = JL(\pi_E)$ of $\mathrm{GL}_2(F_{\mathbb{A}})$ whose Galois representation ρ'_E is isomorphic to ρ_E. By the multiplicity one theorem, π'_E is unique.

The representations ρ_E and ρ'_E have equal S_{sc}-part of the Artin conductor (equal to the S_{nc}-part of $d(M/F)N_{M/F}(\mathfrak{C}(\psi))$) by Corollary 6.18. By the identification of $K^{(sc)}$ with $U_0(\mathfrak{N})^{(S_{sc})}$ up to inner conjugations in G_D, π'_E has a minimal vector whose weight and Neben-character set is (κ, χ) on the level group $U_0(\mathfrak{N})$ outside p. Thus π'_E has a unique minimal vector f'_E in the space $\mathcal{S}$ of nearly ordinary modular forms of weight (κ, χ) and of level $U_0(\mathfrak{N}(\mathfrak{C}(\chi_{1,p}) \cap \mathfrak{C}(\chi_{2,p})))$. Thus we have a correspondence $\{E \in J|\rho_E \cong \rho\} \to \{E' \in J'|\rho_{E'} \cong \rho\}$ of subsets of J and J'. By multiplicity one, this set $\{E' \in J'|\rho_{E'} \cong \rho\}$ is a singleton, and by universality of (T_F, ρ_F), $\{E \in J|\rho_E \cong \rho\}$ is again a singleton. Thus we have an injection $J \hookrightarrow J'$. For any $E' \in J'$, $\rho_{E'} \in \mathcal{D}^{\xi}_{\kappa}(W)$ by definition, and hence $J \cong J'$. This implies $T_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as W-algebras by an isomorphism induced by the bijection: $E \cong E'$. Since the Jacquet–Langlands correspondence preserves Hecke operators outside S, by Chebotarev density, Hecke operators outside S generate T_F and T over W inside $T_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; so, we conclude $T \cong T_F$ as desired.

Once $(T_F, \boldsymbol{\rho}_F) \cong (T, \boldsymbol{\rho})$ is established with local complete intersection property over W, the identity $(\mathbb{T}, \rho_{\mathbb{T}}) \cong (R^{ord,\epsilon_+}, \boldsymbol{\rho}^{ord,\epsilon_+})$ (and local complete intersection property over $W[[\boldsymbol{\Gamma}]]$) follows from the control theorem (cf. [HMI, §3.2.8]) by a standard argument. □

Letting G act on $\mathfrak{sl}_2(A)$ by $x \mapsto \rho_A(\sigma)x\rho_A(\sigma)^{-1}$, we get the adjoint representation $Ad(\rho_A)$ out of $\rho_A \in \mathcal{D}^{\emptyset}(A)$. An immediate consequence of this corollary is (see [EMI, Theorem 6.2.21])

Corollary 6.23. *Let the notation and the assumption be as in* Corollary 6.22. *Then Pontryagin dual* $\mathrm{Sel}(Ad(\rho_A))^*$ *of the adjoint Selmer group* $\mathrm{Sel}(Ad(\rho_A))$ *for any deformation* $\rho_A \in \mathcal{D}^{ord,\epsilon_+}(A)$ *is canonically isomorphic to* $\Omega_{\mathbb{T}/W[[\boldsymbol{\Gamma}]]} \otimes_{W[[\boldsymbol{\Gamma}]],\varphi_{\rho_A}} A$, *where* $\Omega_{\mathbb{T}/W[[\boldsymbol{\Gamma}]]}$ *is the module of continuous* 1*-differentials over* $W[[\boldsymbol{\Gamma}]]$. *In particular, applying this to* $\rho := \mathrm{Ind}_M^F \widetilde{\varphi} \in \mathcal{D}^{ord,\epsilon_+}(\Lambda)$, *we have* $\mathrm{Sel}(Ad(\rho))^* \cong \Omega_{\mathbb{T}/W[[\boldsymbol{\Gamma}]]} \otimes_{W[[\boldsymbol{\Gamma}]],\varphi_\rho} \Lambda$, *which implies* $(H(\varphi)) = \mathrm{Fitt}_\Lambda(\mathrm{Sel}(Ad(\rho_A))^*)$ *for the Fitting ideal* $\mathrm{Fitt}_\Lambda(X)$ *of a finite* Λ*-module* X.

We have two remarks.

Remark 6.24. The identity of the differential module and the dual Selmer group is a general nonsense (see [EMI, (6.8)]), and the identity of the Fitting ideal is a consequence of the complete intersection property (see [EMI, Theorem 6.2.21]). Since $Ad(\mathrm{Ind}_M^F(\widetilde{\varphi})) \cong \chi_M \oplus \mathrm{Ind}_M^F \widetilde{\varphi}^-$ and $\mathrm{Sel}(\chi_F)^* \cong Cl_M^-(\mathfrak{i}(\psi)) \otimes_{\mathbb{Z}} W[[\Gamma_M^-]]$, we have $\mathrm{Sel}(\mathrm{Ind}_M^F \widetilde{\varphi})^* \cong (Cl_M^-(\mathfrak{i}(\psi)) \otimes_{\mathbb{Z}} W[[\Gamma_M^-]]) \oplus X^-[\varphi^-]$ for the anticyclotomic Iwasawa module $X^-[\varphi^-]$ as in [HMI, Theorem 5.33], where $Cl_M^-(\mathfrak{i}(\psi)) = \mathrm{Coker}(Cl_F(\mathfrak{i}(\psi)) \to Cl_M(\mathfrak{i}(\psi)))$. The identity $\mathrm{Sel}(\chi_F)^* \cong Cl_M^-(\mathfrak{i}(\psi)) \otimes_{\mathbb{Z}} W[[\Gamma_M^-]]$ follows from Lemma 6.17. In the above reference [HMI, Theorem 5.33], the Cl_M^- appears in place of the bigger $Cl_M^-(\mathfrak{i}(\psi))$ as we assumed $\mathfrak{i}(\psi)\mathfrak{r}(\psi) = 1$. Since the deformation problem Fujiwara constructed is unrestricted at $\mathfrak{v}|\mathfrak{i}(\psi)\mathfrak{r}(\psi)$, Lemma 6.17 implies that $\mathrm{Sel}(\chi_M)$ in this general case reflects deformation of the form $\mathrm{Ind}_M^F \Phi$ for a character Φ of $Cl_M(\mathfrak{i}(\psi))$ and is isomorphic to $Cl_M^-(\mathfrak{i}(\psi)) \otimes_{\mathbb{Z}} W[[\Gamma_M^-]]$ by the same argument as in [HMI]. This is an explanation of the decomposition $\mathrm{Fitt}_\Lambda(\mathrm{Sel}(\mathrm{Ind}_M^F \widetilde{\varphi})^*) = h_{\mathfrak{i}}(M/F)\,\mathrm{Fitt}_\Lambda(X^-[\psi])$.

The second remark is the product formula $H(\varphi) = h_{\mathfrak{i}}(M/F)L_p^-(\psi)$ in an algebraic manner.

Remark 6.25. Decomposing $Cl_M(\mathfrak{C}(\varphi)p^\infty)$ as $\Delta_M^{(p)} \times \Gamma_M^\circ$ for the p-profinite part Γ_M° with the prime-to-p part $\Delta_M^{(p)}$, we can extend φ to $\varphi^\circ : G \to W[[\Gamma_M^\circ]]$ by sending $g \in G$ to its image

$(\delta,\gamma) \in \Delta_M^{(p)} \times \Gamma_M^\circ$ in this order and giving the value $\varphi^\circ(g) := \varphi(\delta)\gamma \in W[[\Gamma_M^\circ]]$. Then $\mathrm{Ind}_M^F \varphi^\circ \in \mathcal{D}^{ord,\epsilon_+}(W[[\Gamma_M^\circ]])$, and hence we have the corresponding projection $\theta : \mathbb{T} \to W[[\Gamma_M^\circ]]$. Therefore we get a decomposition $\mathrm{Spf}(\mathbb{T}) = \mathrm{Spf}(W[[\Gamma_M^\circ]]) \cup \mathrm{Spf}(\mathbb{T}^\perp)$ for the complementary reduced closed formal subscheme $\mathrm{Spf}(\mathbb{T}^\perp)$. Similarly, we have a tautological surjection $\widetilde{\varphi} : W[[\Gamma_M^\circ]] \to \Lambda$, which produces $\mathrm{Spf}(\Lambda) \cup \mathrm{Spf}(W[[\Gamma_M^\circ]]^\perp)$ for the image $W[[\Gamma_M^\circ]]^\perp$ of $W[[\Gamma_M^\circ]]$ in $\mathrm{Ker}(\widetilde{\varphi}) \otimes_{W[[\mathbf{\Gamma}]]} \mathrm{Frac}(W[[\mathbf{\Gamma}]])$. Then we have an exact sequence $0 \to \mathrm{Ker}(\theta) \to R \to S \to 0$ for $R = \mathbb{T}$ and $S = W[[\Gamma_M^\circ]]$ [H13, §5]. Then for $C_0 := W[[\Gamma_M^\circ]] \otimes_{\mathbb{T}} \mathbb{T}^\perp$ and $C_M := W[[\Gamma_M^\circ]] \otimes_{W[[\Gamma_M^\circ]],\widetilde{\varphi}} \Lambda$. By [H13, Lemma 5.3], we have $(H(\varphi)) = \mathrm{Fitt}_\Lambda(C_0 \otimes_S \Lambda)\,\mathrm{Fitt}_\Lambda(C_M)$ and it is easy to see $\mathrm{Fitt}_\Lambda(C_M) = (h_{\mathrm{i}}(M/F))$. Thus the solution of the anticyclotomic main conjecture means $\mathrm{Fitt}_\Lambda(C_0 \otimes_S \Lambda) = (L_M^-(\psi))$.

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