NON-VANISHING OF INTEGRALS OF A MOD $p$ MODULAR FORM.

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Abstract. The proof of [H04, Theorem 3.2], [H07, Theorem 4.3] and [EAI, Theorem 8.31] is based on the assertion claiming that the Zariski closure in the Hilbert modular Shimura variety of an infinite set of CM points stable under the action of a CM torus contains an irreducible component of positive dimension with a CM point in the starting infinite set. A few years ago, Akshay Venkatesh pointed me out that this fact might not be true for a non-noetherian pro-variety like Shimura variety. I would like to present an argument proving this fact under an extra requirement on the starting infinite set of CM points. Thereby the assertion of [H04, Theorem 3.3], [H07, Theorem 4.3] and [EAI, Theorem 8.31] on non-vanishing modulo $p$ of Hecke $L$-values is valid for “Zariski dense” characters in the sense of these articles. In some special cases, non-vanishing is claimed for except finitely many characters in these articles, which is still an open question.

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Recall from [H04] the base totally real field $F$ with integer ring $O$ and the CM quadratic extension $M/F$ with integer ring $R$. We fix a prime $p > 2$ unramified in $F/\mathbb{Q}$ each of whose prime factor in $O$ splits in $M/F$ and a prime ideal $l$ of $O$ prime to $p$ with residual characteristic $\ell$. Let $R_n = O + l^n R$ (the order of conductor $l^n$) and put $Cl_n = \text{Pic}(R_n)$. Since $O \subset R_n$, we have a natural map $Cl_F := \text{Pic}(O) \to Cl_n$. We write $Cl_n^- := \text{Coker}(Cl_F \to Cl_n)$. Let $Cl_\infty := \lim \leftarrow n Cl_n$ and $Cl_\infty^- := \lim \leftarrow n Cl_n^-$.

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under natural projections. The group of fractional $R$-ideals prime to $I$ is naturally embedded into $Cl_{\infty}$ whose image in $Cl_{\infty}$ (resp. $Cl_{\infty}$) we write as $C_{1}^{\text{alg}} \subset Cl_{\infty}$ (resp. $C_{1}^{\text{alg}} \subset Cl_{\infty}$). Decompose $Cl_{\infty} = \Delta^{-} \times \Gamma$ for the maximal finite group $\Delta^{-}$ and $\mathbb{Z}_{M}$-free $\Gamma$. Since $Cl_{F}$ is finite, $\Gamma$ can be identified with the torsion-free part of $Cl_{\infty}$, and we have a decomposition $Cl_{\infty} = \Gamma \times \Delta$ with $\Delta$ surjecting down to $\Delta^{-}$ under the projection $Cl_{\infty} \rightarrow Cl_{\infty}$. Write $d = \text{rank}_{\mathbb{Z}_{M}} \Gamma$ and choose a basis $\gamma_{1}, \ldots, \gamma_{d}$ of $\Gamma$ over $\mathbb{Z}_{M}$. Let $F$ (resp. $\mathbb{Q}_{\ell}$) be an algebraic closure of $F_{p}$ (resp. $\mathbb{Q}_{\ell}$). We identify $\mu_{\infty}(F) = \mu_{\infty}(\mathbb{Q}_{\ell})$ as an $\ell$-divisible group, and just write as $\mu_{\infty}$.

For each projective fractional $R_{n}$-ideal $A$, we defined in [H04, §2.1 and §3.1] a CM abelian variety $X(A)$ of ordinary CM type $\Sigma$ and the associated CM point $x(A) = x_{\Sigma}(A)$ of the Shimura variety $Sh$ for $G = \text{Res}_{F/\mathbb{Q}} \text{GL}(2)$, which only depends on the ideal $A$ and a chosen $p$-ordinary CM type $\Sigma$. Choose suitably an irreducible component $V$ of the Shimura variety of prime-to-$p$ level defined over an algebraic closure $F = \overline{F}_{p}$ of $F_{p}$. We just fix a finite extension $W$ of $W(F)$ inside $\mathbb{C}_{p}$ and put $W = W_{p} = W \cap \overline{Q}$ for the algebraic closure $\overline{Q}$ of $\mathbb{Q}$. We embed $\overline{Q}$ into $\mathbb{C}$. We take a $U(1)$-eigenform $g_{/W}$ and put $f = \theta^{f}$ for the Ramanujan differential operator $\theta^{\kappa}$ given by $\prod_{k} \left( q_{o} \frac{dq_{o}}{dq_{o}} \right)^{\kappa_{\sigma}}$ with $\kappa_{\sigma} \geq 0$ for the $q$-expansion variables $q_{o} = \exp(2\pi i z_{\sigma})$. We use the same symbol $f$ also for $f_{/\mathfrak{f}} := f_{\mathfrak{f}}$ mod $m_{W}$ defined over $F$.

Write $\Gamma_{n}$ for the image of $\Gamma$ in $Cl_{n}$ (for small $n$, it can be just $\{1\}$). We fix a character $\psi: \Delta^{-} \rightarrow F^{\times}$ to project the measure on $Cl_{n}$ to $\Gamma_{n}$ (see Lemma 4.2 in the text). To define the measure, we need to replace $f(x(A))$ by $f([A]) := \lambda^{-1}(A) f(x(A))$ choosing a Hecke character of infinity type $k \Sigma + \kappa(1-c)$ and of conductor $\mathfrak{C}$ prime to $pk$ so that $f([A])$ only depends on the class $[A] \in Cl_{n}$ for all $n$ (see §3.1 for more details of the choice of $\lambda$). This allows us to define a “measure” $d\phi_{f,n} = d\phi_{f,n}$ on the finite group $Cl_{n}$ by $\int_{Cl_{n}} \phi_{f,n} = \sum_{[A] \in Cl_{n}} \phi([A]) f([A])$. If $f(U(1)) = a f$ with $a \neq 0$, $(\lambda(0) N(0)^{-1}) d\phi_{f,n}$ patches into a unique measure $d\phi_{f}$ on $Cl_{\infty}$, but if $f(U(1)) = 0$, this is just a collection of measures $\{d\phi_{f,n}\}_{n}$.

Let $F_{q}$ be the field of rationality of $f/\mathfrak{f}$, $\psi$ and $\lambda$ modulo $m_{W}$, and define an integer $r > 0$ such that $\ell$-Sylow subgroup of $F_{q}[\mu]_{\infty}$ has order $\ell^{r}$ (i.e., $\mu_{\infty}(F_{q}[\mu]) = \mu_{\infty}(F_{q}[\mu])$ and $\ell^{r} \parallel (q-1)$). Though the measure is defined in the earlier papers for $f$ with non-zero eigenvalue for $U(1)$, in this paper we define a measure on $Cl_{n}$ for each finite $n$ even for $f$ with $f(U(1)) = 0$, and the argument goes through even for $f$ killed by $U(1)$. The non-vanishing of the $U(1)$-eigenvalue is necessary to patch the measure on $Cl_{n}$ for each $n$ to get a measure on $Cl_{\infty}$, but this patching argument is not essential in the proof of non-vanishing results. Also if $f(U(1)) = 0$, $\int_{Cl_{n}} \chi \psi d\phi_{f,n} \neq 0$ can happen only for the minimal $n$ for which the integral is well defined. To project the measure $d\phi_{f,n}$ to $\Gamma_{n}$, we need to modify $f$ into a modular form $f_{\psi}$ and further to a function $f_{/\psi}: \bigcap_{n} Cl_{n} \rightarrow F$ which involve a transcendental operation depending on a choice of a finite subset $Q$ of $Cl_{\infty}/Cl_{1}^{\text{alg}}$ (see (4.7)) so that $\int_{Cl_{n}} \chi \psi d\phi_{f,n} = \int_{Cl_{n}} \chi \psi d\phi_{f,n}$ for all $n$ and all characters $\chi: \Gamma_{n} \rightarrow F^{\times}$. Indeed, we embed $\bigcap_{n} Cl_{n}$ into the product $V^{Q}$ of $Q$-copies of $V$, choose an infinite subset $\Xi$ of the disjoint union $\bigcup_{n} Cl_{n}$ and study the Zariski density in $V^{Q}$ of the embedded image $\Xi \hookrightarrow V^{Q}$.

We regard the set of continuous characters $\text{Hom}(\Gamma, \mu_{\infty})$ as a subset of $G_{m}^{d}(\overline{Q})$ by sending a character $\chi$ to $(\chi(\gamma_{1}), \ldots, \chi(\gamma_{d})) \in G_{m}^{d}(\overline{Q}) \subset G_{m}^{d}(\overline{Q})$. A subset $\mathcal{X}$ of $\text{Hom}(\Gamma, \mu_{\infty})$ is said to be Zariski dense if $\mathcal{X}$ is Zariski dense in $G_{m}^{d}$ over $\overline{Q}$. This notion of density is independent of the choice of the basis $\{\gamma_{j}\}_{j}$. Write $\text{cond}(\chi)$ for the conductor of $\chi$ which is a power of $\ell$.

Here is a new version of [H04, Theorem 3.3] and (a part of) [H07, Theorem 4.3]:

**Theorem 0.1.** Suppose that there exists $\xi \in F \cap O_{1}$ in each class $v \in (O_{1}/V O_{1})^{\times}$ for a sufficiently large $j \geq r$ (for a specific $r > 0$ defined in (4.16)) only dependent on $\Gamma$ (not $v$) such that the $q$-expansion coefficient $a(\xi, f_{\psi}) \neq 0$ in $F$ at an infinity cusp of $V$. Then the set of characters $\chi \in \text{Hom}(\Gamma, \mu_{\infty}(F))$ such that $\int_{Cl_{n}} \chi \psi d\phi_{f,n} \neq 0$ for $n$ given by $\text{cond}(\chi) = \Gamma^{n}$ is Zariski dense in $G_{m}^{d}(\overline{Q})$. If $\text{rank}_{\mathbb{Z}_{M}} \Gamma = 1$, $j$ can be taken to be equal to $r$.

For the Eisenstein series $g$ we took in [H04] and [H07], for any $v \in O/\mathfrak{f}$ and any $j \geq r$, the assumption of the theorem is satisfied except for a very rare case which satisfies conditions (M1–3) in [H07, Theorem 4.3]. For cusps forms, things are more complicated, and Hsieh [Hs14] uses Galois representations of the given cusp form as its traces is basically $q$-expansion coefficients. Of course, one needs to assume that the root number is not $-1$ in addition to some extra assumptions (as the square of the integral is the central critical values by Waldspurger).
Geometrically irreducible components of the Shimura variety of the level group \( \Gamma_0(\mathfrak{M}) \) are indexed by polarization (strict) ideal classes of \( F \). Then infinity cusps of a component \( V \) are indexed by equivalence classes of pairs \((a, b)\) of ideals with \((ab)^{-1}\) giving a polarization ideal of \( V \) (e.g., [PAF, §4.1.5]). The condition of the existence of \( \xi \) with \( a(\xi, f_\psi) \neq 0 \) does not depend on the choice of \( a, b \). If \( f(U(t)) = 0 \), \( \int_{C_\ell} \chi \psi d\nu f_{\nu,n} \neq 0 \) implies that \( l\)-conductor \( \Gamma' \) of \( \chi \) is exactly \( \Gamma' \) (i.e., \( \nu = n \)), while this non-vanishing holds for all \( n \geq \nu \) once it holds for \( n = \nu \) if \( f(U(t)) = af \) with \( a \neq 0 \).

Here are a more technical description of our method and the reason why I take up this problem again. First, we claimed in these papers [H04] and [H07] a stronger finiteness result of characters of vanishing integrals when \( \text{rank}_{\mathbb{Z}} \Gamma = 1 \) (e.g., [H04, Theorem 3.2]). As explained in §1.1, a few years ago, Akshay Venkatesh noticed a missing point (taken to be true in [H04] and [H07]) from the proof of [H04, Proposition 2.7]: positive dimensional irreducible components of the Zariski closure of an infinite set \( \Xi \) of closed points in a non-noetherian variety may not contain any points in the starting set \( \Xi \). For the proof of the above theorem, we need a Zariski density theorem of a thin infinite set \( \Xi \) of CM points \( V \) in the product of copies of an irreducible component of the prime-to-\( p \) Shimura variety \( Sh(p) \).

The first step we need is to show the Zariski closure \( X \) of \( \Xi \) contains a positive dimensional irreducible component having at least one point of \( \Xi \). This is because the density theorems Corollary 3.19 and Theorem 3.20 of [H10] we apply to show \( X = V \) require (as its starting hypothesis) existence of at least one positive dimensional component with non-trivial intersection with \( \Xi \). All the results of [H10] are valid and intact as the Zariski closure appearing in [H10] has at the onset the base point in the positive dimensional component. Unfortunately, under the setting of [H04] and [H07] and the present paper, the existence of a positive dimensional component with a point in \( \Xi \) is not evident a priori. In this paper, under some extra assumptions, we prove the existence of such positive dimensional components in Theorem 2.6, which is sufficient for a proof of Theorem 0.1. When \( \text{rank}_{\mathbb{Z}_l} \Gamma = 1 \), we obtain a slightly stronger result: Consider the sequence of vanishing integral:

\[
(*) := \{0 < n \in \mathbb{Z} | l^n \text{ is the conductor of } \chi \text{ with vanishing integral}\}.
\]

Then, under the condition in Theorem 0.1 on non-vanishing of \( q \)-expansion coefficients of \( f \) modulo \( p \), this sequence contains no infinite arithmetic progressions if \( \text{rank}_{\mathbb{Z}_l} \Gamma = 1 \) (see Corollary 5.3). This perhaps means that the natural density of \((*)\) is zero, though we will not touch this point in this article except for Conjecture 5.4. Since the description of \( \Xi \) is technical, we postpone it to Section 2 of the main text. Here we just say that \( \Xi \) is essentially the set of points in \( Sh(p) \) corresponding classes in \( \bigcup_\nu C_{\ell,\nu} \) which carries a character \( \chi \) with non-vanishing integral.

Note here that \( V^{\Xi} \) is a non-noetherian pro-variety, and hence the zero set of a modular form on \( V \) is infinite (of continuous cardinality) even if \( \dim V = 1 \) and \( |Q| = 1 \) as it contains the entire fiber of the infinite étale covering \( V \to V_K \) of the zeros of the modular form defined over the noetherian quotient \( V_K \). There is an example supplied by Venkatesh of a pro-curve in which any positive dimensional irreducible component of the Zariski closure of an infinite set \( \Xi \) is disjoint from \( \Xi \) (see §1.1). If \( g_\Xi \) (appearing in the proof of [H04, Theorem 3.2] denoted by \( g_\nu \) in the text: see (5.2)) had a non-zero eigenvalue for \( U(t) \), the sequence like \((*)\) associated to \( \{g_\Xi\}_{\Xi} \) would contain an infinite arithmetic progression (and thereby getting a contradiction). However it easy to see \( g_\Xi[U(t)] = 0 \); so, for the version of [H04, Theorem 3.2] and the part of [H07, Theorem 4.3] in the case where \( \text{rank}_{\mathbb{Z}_l} \Gamma = 1 \), we need to assume that \( \Xi \) contains an infinite arithmetic progression. The idea to reach the missing result is to show (under some extra assumptions) that \( \{1, 0, \bar{w} \} \) for \( \bar{w} \in N_{M/F}(R) \) generating powers of \( l^m \) for some \( 0 < m \in \mathbb{Z} \) act on the 0-dimensional irreducible components of \( X \) and \( X_K \) and all the orbits in \( X_K \) of this action are infinite. The noetherian scheme \( X_K \) cannot have infinite 0-dimensional components, and therefore, all components of \( X \) (and \( X_K \)) has positive dimension, as desired.

1. Irreducible components of Zariski closure

We study a general theory of Zariski closure in a pro-étale variety of an infinite set of close points. We start with a pathologic example.

1.1. An example. To motivate the reader to go through this article dealing with technical topics, we first discuss an example of an affine pro-scheme \( V = V_{\infty}/\mathcal{C} \) étale over the affine line \( V_0 = \text{Spec}(\mathbb{C}[X]) \) such that the Zariski closure of an infinite set \( \Xi \subset V(\mathbb{C}) \) does not have a single positive dimensional
irreducible component containing a point of $\Xi$. The example was supplied by Akshay Venkatesh in 2018 December.

For a (finite dimensional) scheme $S/k$ for an algebraically closed field $k$, we write $\text{Irr}(S)$ for the set of all irreducible components of $S$ and $\pi_0(S)$ for the set of all connected components of a scheme $S$. Put

$$\text{Irr}_d(S) := \{ I \in \text{Irr}(S) \mid \dim I = d \}.$$  

Thus $S = \bigcup_{Z \in \text{Irr}(S)} Z$ and $\text{Irr}(S) = \bigsqcup_{d \geq 0}^\dim S \text{Irr}_d(S)$. Set $\text{Irr}_+^k (S) = \bigsqcup_{d \geq 0}^\dim S \text{Irr}_d(S)$. If $S = \text{Spec}(A)$, we write $\text{Irr}(A) = \text{Irr}(\text{Spec}(A))$ and $\pi_0(A) = \pi_0(\text{Spec}(A))$. The set $\text{Irr}(A)$ is in bijection onto the set of minimal prime ideals of $A$, and we identify the two sets.

Take $k = \mathbb{C}$. Let $V_n := V_0 \times \mathbb{Z}/2^n\mathbb{Z}$ and the projection $\mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^m\mathbb{Z}$ for $m > n$ induces étale morphism $V_m \to V_n$. Let $P_j := (X - j) \subset \mathbb{C}[X]$ ($0 < j \in \mathbb{Z}$) and regard it as a closed point $j$ of $V_0$. We define $V := \varprojlim_n V_n \cong V_0 \times \mathbb{Z}_2$ and $\Xi = \{(j, 2^i) \in V \mid j = 1, 2, \ldots \}$. Write $(P_j, 2^i)_{n}$ for the maximal ideal of $V_n$ giving rise to the point $(j, 2^i \mod 2^n) \in V_n$. Therefore $(P_j, 2^i)_{\infty}$ for finite $j$ is the maximal ideal of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$ non-trivial equal to $P_j$ only at $2^i$-component of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$, and the prime ideal $(P_j, 2^i)_{n}$ is $P_j \oplus \bigoplus_{1 \neq 2^i \mod 2^n} \mathbb{C}[X]$. Write

$$\Xi_n := \{(j, 2^i)_{n} \in V_n \mid j = 1, 2, \ldots \}$$

for the image of $\Xi$ in $V_n$. Note that $V_n = \text{Spec}(\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X])$ and $V = \text{Spec}(\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X])$. We have $\bigcap_j (P_j, 2^i)_{\infty} = \{(0, 0) \in \bigoplus_{0 < j \in \mathbb{Z}} (P_j, 2^i)_{\infty}\}$, where $((0), 0)_{\infty}$ is the prime ideal of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$ equal to $(0)$ only at the 0-component of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$. Thus $\Xi = V_0 \sqcup \bigsqcup_{0 < j \in \mathbb{Z}} (P_j, 2^i)_{\infty} \subset V$, where $V_0$ is inserted as the 0-component. Thus only positive dimensional irreducible (and connected) component of the Zariski closure $\Xi$ in $V$ which does not contain any points of $\Xi$.

If we have a transitive action of a semi-group inside $\text{Aut}(V)$ on $\Xi$, we expect to be able to avoid such a pathologic example.

Though $\alpha : (v, z) \mapsto (v + 1, 2z)$ acts transitively on $\Xi$, $\alpha$ is not an automorphism of $V$ which is an automorphism of $V_0 \times \mathbb{Q}_2$ which is an in-do-pro-variety not a pro-variety. In the above example, we have

\begin{align*}
\text{Irr}_1(\Xi_n) &= \{(V_0 \times 0) \mid 0 \in \mathbb{Z}/2^n\mathbb{Z}\}, \\
\text{Irr}_0(\Xi_n) &= \{(j \times 2^i)_{n} \mid j = 1, \ldots, n - 1(2^i \neq 0 \in \mathbb{Z}/2^n\mathbb{Z})\}, \\
\text{Irr}_1(\Xi) &= \{(V_0 \times 0) \mid 0 \in \mathbb{Z}_2\} \quad \text{and} \\
\text{Irr}_0(\Xi) &= \{(j, 2^i)_{\infty} \mid 0 < j \in \mathbb{Z}, 2^i \in \mathbb{Z}_2\}.
\end{align*}

The action of any positive power of $\alpha$ brings some points in $\text{Irr}_0(\Xi_n)$ into a component in $\text{Irr}_1(\Xi_n)$ (non-stability of $\text{Irr}_0(\Xi_n)$ under $\alpha$ coming from the fact that $\alpha$ is not an automorphism of $V$). Writing $\pi_n : V \to V_n$, we can consider the reduced image $\pi_n(I) \subset V_n$ for $I \in \text{Irr}(\Xi)$. Let $\pi_{n,*}(\text{Irr}(\Xi)) = \{ \pi_{n,*}(I) \mid I \in \text{Irr}(\Xi) \}$ and $\pi_{n,*}(\text{Irr}_d(\Xi)) = \{ \pi_{n,*}(I) \mid I \in \text{Irr}_d(\Xi) \}$ as sets. Then

\begin{align*}
\text{Irr}_1(\Xi_n) &= \text{Irr}_1(\Xi_n), \quad \pi_{n,*}(\text{Irr}_0(\Xi_n)) \supset \text{Irr}_0(\Xi_n) \quad \text{with infinite} \pi_{n,*}(\text{Irr}_0(\Xi_n)) - \pi_0(\Xi_n) \quad \text{in} V_0 \times 0.
\end{align*}

\begin{align*}
\text{(ne) The image of} \{ (j, 2^i)_{\infty} \mid j \geq n \} \quad \text{lies in the one dimensional} \quad (V_0 \times 0) \quad \text{in} \text{Irr}_1(\Xi_n) \quad \text{and the} \\
0\text{-dimensional scheme} \quad (j, 2^i)_{\infty} \quad (j \geq n) \quad \text{is not étale over} \quad V_n.
\end{align*}

If we take a 2-unit $u \in \mathbb{Z}$ and consider $\Xi = \{(j, u^i)_{\infty} \mid j = 1, 2, \ldots \} \subset V$, one can show that $\text{Irr}(\Xi_n) = \{V_0 \times u^i | u^i \mod 2^n \}$ and $\text{Irr}(\Xi) = \{V_0 \times x | x \in \langle u \rangle \}$ for the subgroup $\langle u \rangle \subset \mathbb{Z}_2$ topologically generated by $u$. The action $[1] : (j, u^i) \mapsto (j + 1, u^{i+1})$ extends to an automorphism $[1] : (v, z) \mapsto (v + 1, uz)$. A similar morphism $\alpha(v, z) = (v + 1, 2z)$ for non-unit 2 in place of $u$ is not an automorphism of $V$.

Taking an infinite sequence of irreducible polynomials $X - a_j$ of $\mathbb{F}[X]$ with distinct $a_j \in \mathbb{F}$, we can make an example similar to (ne) also over $\mathbb{F}$ taking $V_0 := \text{Spec}(\mathbb{F}[X])$ and $\Xi = \{(P_j := ((X - a_j), 2^i))_{n} \mid j \in V_n = V_0 \times \mathbb{Z}/2^n\mathbb{Z}\}$. Then $\lim_{n} V_n = V_0 \times \mathbb{Z}_2$.

1.2. Geometry of irreducible components. We prepare some notion and geometric lemmas to prove the theorem. After the lemmas, in the following section, we study the correspondence action.

Let $\pi : V_{/ \mathbb{F}} \to V_{K/\mathbb{F}}$ be an affine étale Galois covering with $V = \text{Spec}_{\mathbb{Q}_K}(\mathcal{O}_V)$ (as a relative spectrum). Here $K = \text{Gal}(V/V_K)$ and $V = \varprojlim_{V \to \mathbb{F}} V_U$ for $U$ running over open subgroups of $K$ with $V_U = V/U$. In the following lemmas, assume that $V_K$ is noetherian (so, $V_U = V/U$ is also
noetherian for an open subgroup $U$ of $K$. Let $\Xi \subset V(\mathbb{F})$ be an infinite set of closed points with image $\Xi_K$ in $V_K(\mathbb{F})$.

**Lemma 1.1.** Regard $P' \in \Xi$ (resp. $P \in \Xi_K$) as a sheaf of $O_V$-ideal (resp. $O_{V_K}$-ideal) defining the point $P'$ (resp. $P$); so, for example $O_{V_K}/P \cong \mathbb{F}(P) = \mathbb{F}$ as a skyscraper sheaf supported by $P$. Let $X'$ (resp. $X$) be the Zariski closure of $\Xi$ (resp. $\Xi_K$) in $V$ (resp. $V_K$). Then

1. $X'$ and $X$ are reduced scheme, $X'/X$ is finite if $V/V_K$ is finite.
2. The projection $\pi_X : X' \to X$ is dominant inducing a surjection of $\mathbb{F}$-points: $X'(\mathbb{F}) \to X(\mathbb{F})$, and $X'$ is unramified over $X$.

As described in (ne), even if $\Xi \cong \Xi_K$ by the map induced by $\pi$, $\pi : X' \to X$ may not be étale.

**Proof.** By definition, we have $X' = \text{Spec}(O_V/\bigcap_{P \in \Xi} P')$ and $X = \text{Spec}(O_{V_K}/\bigcap_{P \in \Xi_K} P)$.

We prove the lemma first in the absolute affine case; so, we put $\mathcal{V}_K = \text{Spec}(A)$, $\mathcal{V} = \text{Spec}(A')$, $B = A/\bigcap_{P \in \Xi_K} P$ and $B' = A'/\bigcap_{P' \in \Xi} P'$. Since $B' \to \prod_{P' \in \Xi} A'/P'$ with the right-hand-side reduced, $B'$ is reduced. In the same way, $B$ is reduced.

If $A'/A$ is étale finite, we have $\Xi_K = \{P' \cap A | P' \in \Xi\}$; so, putting $b' := \bigcap_{P' \in \Xi} P'$ and $b := \bigcap_{P \in \Xi_K} P$, we have $b' \cap A = b$. Thus the induced map $B \to B'$ is injective. If $A'/A$ is not finite, we can write $A = \bigcup_i A_i$, with $A_i/A$ finite étale, we still get the injectivity. Therefore the projection $\text{Spec}(B') \to \text{Spec}(B)$ is dominant. Pick a maximal ideal $m \in \text{Spec}(B'(\mathbb{F}))$. Then by the going-up theorem [CRT, Theorem 9.3 (i)], we have a prime ideal $\mathfrak{p} \in \text{Spec}(B')$ with $\mathfrak{p}' \cap B = m$. Take a maximal ideal $m'$ containing $\mathfrak{p}'$, $m' \cap B = m$ is still a proper ideal as $B'/B$ is integral; so, $m' \cap B = m$. Thus $B'/m'$ is a finite extension of $B/m = \mathbb{F}$ which is algebraically closed, we conclude $B/m' = \mathbb{F}$ and $m' \in \text{Spec}(B'(\mathbb{F}))$; so, $\text{Spec}(B'(\mathbb{F})) \to \text{Spec}(B(\mathbb{F}))$ is onto.

Pick $m' \in \text{Spec}(B'(\mathbb{F}))$ and regard it as a maximal ideal of $A'$. Since $m' \supset b'$, $m := m' \cap A \supset b$; so, $m \in \text{Spec}(B(\mathbb{F}))$. We have the following commutative diagram of the completions at $m'$ and $m$:

$$
\begin{array}{ccc}
\widehat{A}_m & \longrightarrow & \widehat{A}'_m \\
\downarrow \text{onto} & & \downarrow \text{onto} \\
\widehat{B}_m & \longrightarrow & \widehat{B}'_m
\end{array}
$$

Since the top row composite: $\widehat{A}_m \to \widehat{A}'_m \to \widehat{A}'_{m'}$ is an isomorphism (as $A \to A'$ is étale), $p_m \circ i_m$ is onto. Therefore $B'/B$ is an unramified extension and is finite if $A'/A$ is finite. This proves (1) and (2) in the absolute affine case.

Now we treat the general relative affine case. We cover $V_K = \bigcup_A \text{Spec}(A)$ for affine open subscheme $\text{Spec}(A)$, and write $A' = \pi_A \mathcal{V} \mathcal{V}(\text{Spec}(A))$. Then $\text{Spec}(A')$ is an open subscheme of $V$ covering $\text{Spec}(A)$. Then we have $X' \cap \text{Spec}(A') = X' \times_{V_K} \text{Spec}(A) = \text{Spec}(B')$ and $X \cap \text{Spec}(A) = X \times_{V_K} \text{Spec}(A) = \text{Spec}(B)$ with $(A'/A, B'/B, \Xi \cap \text{Spec}(A'), \Xi_K \cap \text{Spec}(A))$ satisfying the assumption of Lemma 1.1. Since $B$ (resp. $B'$) depends on $A$, if needed, we write $B = B_A$ and $B' = B'_A$ to emphasize the dependence. By the above argument, $B'$ and $B$ are reduced algebra, and $B'$ is an unramified extension of $B$, $B'/B$ is finite if $A'/A$ is finite, and the projection $\text{Spec}(B') \to \text{Spec}(B)$ is dominant and the induced map: $\text{Spec}(B'(\mathbb{F})) \to \text{Spec}(B(\mathbb{F}))$ is surjective. Since $\text{Spec}(B'(\mathbb{F}))$ is the pull-back to $X'$ of $\text{Spec}(B)$ and $X = \bigcup_A \text{Spec}(B_A) = \bigcup_A \pi_A^{-1}(\text{Spec}(B_A))$ and $X = \bigcup_A \text{Spec}(B_A)$, the above proof in the affine case implies the assertion in the general case. \qed

Assume that $\Xi \cong \Xi_K$. We have another commutative diagram:

$$
\begin{array}{ccc}
B & \longrightarrow & \prod_{P \in \Xi_K} A/P \\
\pi_B \downarrow & & \downarrow \\
B' & \longrightarrow & \prod_{P' \in \Xi} A'/P'.
\end{array}
$$

The right vertical map is an isomorphism as $\Xi \cong \Xi_K$. Thus $\pi'_B$ is injective; so, again we see that $\text{Spec}(B') \to \text{Spec}(B)$ is dominant.

**Lemma 1.2.** Let the notation and the assumption be as in Lemma 1.1. Recall that $V_K$ is a noetherian scheme. Let $\pi_*(\text{Irr}(X')) := \{\pi(Z') | Z' \in \text{Irr}(X')\}$ for the set of the reduced image $\pi(Z') \subset X$. Then we have
(1) The image $\pi^*(\Irr(X'))$ contains $\Irr(X)$.

(2) For $Y \in \Irr(X)$, if $Y' \in \Irr(\pi^{-1}(Y))$ is contained in $X'$, we have $Y' \in \Irr(X')$, where $\pi^{-1}(Y) = Y \times_{V_K} V$.

(3) If $\Xi \cong \Xi_K$ under the projection $\nu \overset{\cong}{\to} V_K$, we have a unique section $\Irr_0(X) \to \Irr_0(X')$ of $\Irr_0(X') \mapsto \Im(\Irr_0(X')) \subset X$ and $\Irr_0(X') \subset \Xi$. Moreover writing $X'_U$ for the image of $X'$ in $V/U$ for an open subgroup $U$ of $K$, $\Irr_0(X') = \lim_U \Irr_0(X'_U)$ for $U$ running over all open subgroups of $K$.

(4) If $\dim Z = \dim X$ for $Z \in \Irr_{\dim X}(X')$, then $Z$ is in the image of $\Irr_{\dim X}(X')$ in $X$. In particular, $\Irr_{\dim X}(X') \neq \emptyset$.

Proof. Again we may assume that $V_K = \Spec(A)$, $V = \Spec(A')$, $X = \Spec(B)$ and $X' = \Spec(B')$ as in the proof of Lemma 1.1. Pick $p_Y \in \Irr(B)$ giving $Y \in \Irr(\Spec(B))$. Since $B'/B$ is integral, we find a prime $P' \in \Spec(B')$ such that $P' \cap B = p_Y$ by going-up theorem [CRT, Theorem 9.3 (i)]. For each $P' \in \Spec(B')$ with $P' \cap B = p_Y$ (i.e., $P' \in \pi^{-1}(Y) = \Spec(B'/p_YB')$, take a minimal prime $p' \subset P'$ (i.e., $p' \in \Irr(B')$). Then $p' \cap B$ is a prime ideal of $B$ and $p_Y \supset p' \cap B$; so, by minimality of $p_Y$, we have $p_Y = p' \cap B$. Thus $p_Y$ is in the image of $\Irr(B')$. This proves the assertion (1).

As $\nu \to \xi_K$ is étale, $\pi^{-1}(Y)$ is étale over $Y$; so, equi-dimensional. Suppose that $Y' \subset X'$ for $Y' \in \Irr(\pi^{-1}(Y))$. Then we find $Z' \in \Irr(X')$ such that $Z' \supset Y'$; so, $\pi(Z') \subset B$. We are going to show $Z' = Y'$. We have $X \supset \pi(Z') \supset Y$. Since $\pi(Z')$ is irreducible, $\pi(Z')$ containing $Y \in \Irr(X)$ implies $\pi(Z') = Y$. Thus $Z' \to Y$ is a integral dominant; so, dim $Z' = \dim Y' = \dim Y$. This shows $Z = Z' \in \Irr(X')$, as desired. Thus the assertion (2) follows.

To show the assertion (3) for $\Irr_0$, we first assume that $B'/B$ is finite. We regard $\Xi_K \subset \Spec(B)$. Pick $m \in \Irr_0(B)$. Then $B = B(m) \oplus B/m$ for some subring $B(m) \subset B$ as Spec($B/m$) is a connected component of Spec($B$). Thus $\Irr_0(B) = \{ Z : \pi_0(\Spec(B(m))) \mid \dim Z = 0 \}$. Since $B'/B$, the above decomposition induces an algebra direct sum $B' = B(m) \oplus B'/mB'$. Since $B'$ is finite over $B$, $B'/mB'$ has dimension $0$. By reducedness of $B'$, the direct summand $B'/mB'$ of $B'$ is a direct sum of fields. This means that $\pi$ induces a surjection of the upper row of the following diagram:

$$
\begin{array}{ccc}
\pi_0(\Spec(B'/mB')) & \longrightarrow & \pi_0(\Spec(B/m)) = \{ m \} \\
\downarrow & & \\
\Irr_0(B')
\end{array}
$$

for each $m \in \Irr_0(B) \subset \pi_0(B)$. Therefore $\pi^*(\Irr_0(B')) \supset \Irr_0(B)$. Pick $m \in \Irr_0(B)$. If $m \notin \Xi_K$, $\Xi_K \subset \Spec(B(m))$ as Spec($B = \Spec(B/m) \sqcup \Spec(B(m))$). This implies $B = A' \cap_{p \in \Xi_K} P$ is equal to $B(m)$, a contradiction. Thus $m \in \Xi_K$, and $\Irr_0(B) \subset \Xi_K$. Since $\Xi \cong \Xi_K$, $\pi_*$ has a unique section $\pi^* : \Irr_0(B) \to \Irr_0(B')$. If $B'/B$ is not finite, we can write $B' = \bigcup_j B_j$ for $B$-subalgebras $B_j \subset B' \subset B$ finite over $B$. We may assume that the index set is totally ordered so that $B_j \supset B_{j'}$ if $j' > j$. Let $X'_U = \Spec(B_U)$ for an open subgroup $U$ of $K$. Then $B_U/B$ is finite unramified. Then applying the above argument to finite $B_U/B$, we find natural inclusion $\Irr_0(B_U) \subset \pi_{U',U}(\Irr_0(B_U))$ for open subgroups $U' \subset U \subset K$ with a unique section $\pi_{U',U}^* : \Irr_0(B_U) \hookrightarrow \Irr_0(B_U)$. In particular, the injective limit of $\pi_{U',U}^*$ gives rise to the section $\pi^* : \Irr_0(B) \hookrightarrow \Irr_0(B')$ and $\Irr_0(B') = \lim_U \Irr_0(B_U)$. This proves the assertion (3).

Now suppose that $\dim B'/p = \dim B$ for $p \in \Irr(B)$. Such $p$ always exists as $B$ is noetherian. Since $B'/B$ is integral, $\dim B = \dim B'$. Then we take $p' \in \Spec(B'/B')$ such that $p' \cap B = p$. Such a prime exists as already remarked. Then $B'/p' \hookrightarrow B'/p'$ and hence $\dim B'/p' = \dim B/p = \dim B$ as $B'/p'$ is integral over $B/p$. Since $\dim B' = \dim B$, we conclude $p' \in \Irr_{\dim B}(B')$; so, $\Irr_{\dim B}(B') \neq \emptyset$. This proves the assertion (4).

Lemma 1.3. Suppose that $\pi_*(Z') = \pi(Z') \notin \Irr(X)$ for $Z' \in \Irr(X')$. Then there exists $Z_0 \in \Irr(X)$ such that $Z_0 \supset \pi_*(Z')$.

Proof. Again we may assume that $X = \Spec(B)$ and $X' = \Spec(B')$ as in the proof of Lemma 1.1. Write $Z' = \Spec(B'/p')$. By the assumption, $p' \cap B \notin \Irr(B)$; therefore $p' \cap B \supset p_0$ for a minimal prime ideal $p_0$ of $B$. By definition, $p_0 \in \Irr(B)$ and $p' \cap B \supset p_0$ means $p' \cap B \in \Spec(B/p_0)$. Thus $Z_0 = \Spec(B/p_0)$ does the job. □
Lemma 1.4. If $\Xi_{K,0}$ is a subset of $\Xi_K$ with finite $\Xi_K - \Xi_{K,0}$, then the Zariski closure $X$ of $\Xi_K$ in $V_K$ and that $X_0$ of $\Xi_{K,0}$ share irreducible components of positive dimension (i.e., $\text{Irr}_+(X) = \text{Irr}_+(X_0)$), and $\text{Irr}(X) - \text{Irr}(X_0)$ is a finite subset of $\Xi_K - \Xi_{K,0}$.

Proof. Again we may assume that $X = \text{Spec}(B)$ as in the proof of Lemma 1.1. Write $\Xi_K - \Xi_{K,0} = \{m_1, \ldots, m_n\}$ for maximal ideals $m_i$ of $A$ and put $a = \bigcap_i m_i$. Then for $b_0 = \bigcap_{i \in \Xi_{K,0}} P_i$ and $b = \bigcap_{P \in \Xi_K} P$, we have $b = b_0 \cap a$. For each $i$, either $m_i \supseteq b_0$ or $m_i + b_0 = A$ as $m_i$ is maximal. Thus we may assume that $\Xi_K - \Xi_{K,0} = \{m_i|m_i+b_0 = A\}$. Then $a + b_0 = A$ as $|\Xi_K - \Xi_{K,0}|$ is finite. Thus $A/b = A/b_0 \cap a = A/b_0 \oplus A/a$, and hence $X = X_0 \cup (\Xi_K - \Xi_{K,0})$ as desired. □

2. Zariski closure in Hilbert modular Shimura variety

Recall the CM quadratic extension $M_F$ with its integer ring $R$ from the introduction and their class groups $Cl_{\infty} = \lim_{\leftarrow n} Cl_n$ and $Cl_{\infty} = \lim_{\rightarrow n} Cl_n$, where $Cl_n$ is the ring class group of $R_n = O + \ell^n R$ and $Cl_{\infty} = Cl_n/Cl_f$. Writing $[\mathcal{A}]_n$ for the class of a proper $R_n$-ideal $\mathcal{A}$ in $Cl_n$. Let $Cl_{\text{alg}} := \{[x]_n = \lim_n [xR_n \cap M] \in Cl_{\infty}| x \in M_\delta^{\times} \text{ with } \ell \text{ max } 1 \} \subset Cl_{\infty}$, where $\hat{R}_n = R_n \otimes \hat{\mathbb{Z}}$ with $\hat{\mathbb{Z}} = \prod \mathbb{Z}_l$ (cf. [H04, page 755]). By multiplication, $Cl_{\text{alg}}$ acts on $Cl_{\infty}$ and $Cl_{\infty}$. Let $G := \text{Res}_{\mathbb{Q}/\mathbb{Q}} \text{GL}(2)$ and $Sh_{/\mathbb{Q}}$ be the Hilbert modular Shimura variety associated to $G$. Since $G(\mathbb{A}_{\text{alg}})$ acts on $Sh$ as automorphisms, we define the prime-to-$\ell$ level Shimura variety $Sh_{/\ell}$ by $Sh/G(\mathbb{Z}_\ell)$. The Shimura variety $Sh_{/\ell}$ extend canonically to a smooth pro-scheme over $\mathcal{W}$ (e.g. [PAF, Chapter 4]). Recall the irreducible component $V = V_{/\ell}$ of the Shimura variety $Sh_{/\ell}$ we fixed. By smoothness, $V_{/\ell} := V \times_{\mathcal{W}} \mathbb{F}$ is an irreducible component of $Sh_{/\ell}$.

Let $\mathcal{Q} \subset Cl_{\infty}$ be a finite subset independent modulo $C_{\text{alg}}$; i.e., $\delta C_{\text{alg}} \neq \delta' C_{\text{alg}}$ for any pair $(\delta, \delta') \in \mathbb{Q}^2$ with $\delta \neq \delta'$. Since $C_{\text{alg}}$ naturally contains $Cl_F$, for the image $Q^\perp$ of $Cl_{\infty}$, we have $\mathcal{Q} \cong Q^\perp$ and $\mathcal{Q}^\perp$ is still independent modulo $C_{\text{alg}}$. We often identify the two set $\mathcal{Q}$ and $\mathcal{Q}^\perp$. For a closed subgroup $K_{/\ell} \subset G(\mathbb{A}_{\text{alg}})$, we put $K = G(\mathbb{Z}_\ell) \times K_{/\ell}$ and write $V_K$ for the image of $V$ in $Sh_{/\ell}$. We set $V_{/\ell} := V_{/\ell}^\mathcal{Q}$ for $B = \bigcup_{\mathcal{Q}} W, \mathbb{F}$ (the product of $\mathcal{Q}$ copies of $V$) and $V_{K/\ell} := V_{K/\ell}^\mathcal{Q}$. We can embed $Cl_n$ into $V$ by $[\mathcal{A}] \mapsto x(\mathcal{A}) = x([\mathcal{A}]) := (x([\mathcal{A}] \delta))_{\delta \in \mathcal{Q}} \in V$, and write its image with $C_n$. Put $C^{(\infty)} = V \sqcup C_n \subset V$ as abelian variety sitting over $x(\mathcal{A})$ is uniquely determined by $[\mathcal{A}]$. Though the modular form $f$ is a function on $V$, we normalize it by multiplying a suitable Hecke character value later so that the normalized values at $x(\mathcal{A})$ and $x(\mathcal{A}')$ are identical if $[\mathcal{A}] = [\mathcal{A}']$ in $Cl_{\infty}$. Because of this normalization, we may regard $f$ as a function on $C^{(\infty)}$ modulo $Cl_F$. We fix an infinite subset $\Xi$ of $C^{(\infty)}$. When it is necessary to indicate the level group $K$ for which $x(\mathcal{A})$ resides in $V_K$ (or $V_K$), we write $x_K(\mathcal{A})$ in place of $x(\mathcal{A})$. Here $K$ can be a closed subgroup of $GL_2(F_\mathcal{L}^{(\infty)})$. Actually we only deal with the tower raising $l$-power level; so, $K$ can be a closed subgroup of $GL_2(O_l)$ which acts on $V$ and $V$.

We fix a CM type $\Sigma$ of $M$ and write $\Sigma_p$ for the set of $p$-adic places induced by the embedding in $\Sigma$ by the identification $\Xi \cong \mathbb{C}_p$ we fixed. We write $X$ (resp. $X_K$) for the Zariski closure of $\Xi$ (resp. $\Xi_K$). We recall two assumptions (unr) and (ord) in [H04, §2.1] for $p$ in addition to $\Xi \cong \Xi_K$ under the projection $V \to V_K$: 

\begin{align*}
\text{(ord)} \quad \Sigma & \text{ is $p$–ordinary: } \Sigma_p \cap \Sigma_p c = \emptyset \text{ for the generator } c \in \text{Gal}(M/F). \\
\text{(unr)} \quad p & \text{ is unramified in } F/\mathbb{Q}.
\end{align*}

2.1. Toric action. In this section, assuming the existence of an appropriate toric action on $\Xi$ induced by an infinite toric sub-semigroup $\mathcal{T}$ of $\text{Aut}(V)$, we prove that all irreducible components of $X$ has positive dimension; i.e., $\text{Irr}_+(X) = \text{Irr}_+(X_0)$ (see Theorem 2.6). The Zariski closure $X \subset V$ (resp. $X_K \subset V_K$) of $\Xi$ (resp. $\Xi_K$) forms a tower $\{X_0 \rightarrow X_K\}_K$ of varieties, and the tower induces a correspondence action on each noetherian layer $X_K$. If we have an appropriate action of a torus
\( \mathbf{T} \) in \( G(\mathbb{A}^{(p\infty)}) \) on \( \Xi \), by Lemma 1.2, the correspondence action on \( \lim_K \text{Irr}_0(X_K) = \text{Irr}_0(X) \subset \Xi \) coincides with the action of \( \mathbf{T} \). The idea of the proof is to show

\begin{enumerate}
\item \( \text{Irr}_0(X_K) \neq \emptyset \) for sufficiently small open \( K \) if \( \text{Irr}_0(X) \neq \emptyset \) (by Lemma 1.2 (3));
\item If \( \text{Irr}_0(X_K) \neq \emptyset \), by the action of \( \mathbf{T} \), \( \text{Irr}_0(X_K) \) has to be of infinite order, against the noetherian property of \( X_K \).
\end{enumerate}

We start with a list of conditions for proving the assertions (1)–(2) above under the correspondence action of \( \mathbf{T} \). After this, we state five lemmas about the action under these conditions before starting with supplying the missing argument/fact (stated as Theorem 2.6).

Supposing that \( K \) is an open compact subgroup of \( G(\mathbb{A}^{(\infty)}) \), \( \mathcal{V}_K \) is noetherian. On \( \mathcal{V} = V^Q \), \( \text{Aut}(V/F) \) diagonally acts. Let us denote by \( \Xi \) an infinite set of CM points in \( \mathcal{V} \) for which we would like to prove density in \( \mathcal{V} \). We suppose to have a semi-group \( \mathbf{T} \subset \text{Aut}(V/F) \) as in (T) below acting on \( \Xi \) under the diagonal action. The action of \( \mathbf{T} \) is supposed to come from the action of elements in \( G(\mathbb{A}^{(p\infty)}) \) on \( \text{Sh}(p) \). Since it is a semi-group action, \( \beta \in \mathbf{T} \) embeds \( \Xi \) into \( \Xi \); so, \( \beta(\Xi) \subset \Xi \) and \( \beta^{-1}(\Xi) \supset \Xi \), where \( \beta^{-1} \) may not be in \( \mathbf{T} \) but in \( \text{Aut}(V/F) \).

Let \( N := \{ g(u)|u \in O_1 \} \) for \( g(u) := (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) and \( B \) be the normalizer of \( N \) in \( \text{GL}_2(O_1) \) (i.e., \( B \) is the upper triangular Borel subgroup). We may regard \( N \) and \( B \) as group schemes over \( O_1 \); for example, \( N(A) = \{ (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) | u \in A \} \) for an \( O_1 \)-algebra \( A \). We consider the following conditions for \( K \):

\begin{enumerate}
\item \( K \) is closed of the form \( K^{(p)} \times K_p \times K_1 \) with \( K^{(p)} \subset \text{GL}_2(F_p^{(p\infty)}) \), \( \text{GL}_2(O_p) \subset K_p \subset \text{GL}_2(F_p) \) and \( N \subset K_1 \subset \hat{\Gamma}_0(1) \).
\item \( \pi : \mathcal{V} \twoheadrightarrow \mathcal{V}_K \) induces \( \Xi \iso \Xi_K \),
\end{enumerate}

where \( F_p^{(p\infty)} \) is the adele ring of \( F \) away from \( p\infty \), \( F_p = F \otimes_Q \mathbb{Q}_p \subset F_K \), and

\[ \hat{\Gamma}_0(\nu') = \{ g \in \text{GL}_2(O_1)|(g \mod \nu') \in B(O/\nu') \} \]

We put
\[ \hat{\Gamma}_1(\nu') = \{ g \in \text{GL}_2(O_1)|(g \mod \nu') \in N(O/\nu') \} \]

for the image \( \tilde{\gamma} \in \text{PGL}_2(O_1) \) of \( g \in \text{GL}_2(O_1) \). For general \( g \in \text{GL}_2(F_1) \), we write \( S^g := g^{-1}Sg \) for a subgroup \( S \subset G(\mathbb{A}^{(\infty)}) \). Decompose \( \hat{O} := \lim_{\to 0 \in N \in \Xi} O/NO = O_1 \times \hat{O}(1) \). In the application in [H04], we assumed \( K \) to be \( \hat{\Gamma}_0(1) \times \text{GL}_2(\hat{O}(1)) \).

We assume
\[ \text{T} = T \times \alpha^N \text{ for } \alpha^N = \{ \alpha^n|0 \leq n \in \mathbb{Z} \} \text{ and a group } \mathbf{T} \text{ acts on } \Xi, \]

where \( \alpha \in \text{GL}_2(F_1) \) is upper triangular and \( \alpha N \alpha^{-1} \supseteq N \). Here the semi-group \( \mathbf{T} \subset \text{Aut}(V/F) \) acts on \( \Xi \) under the diagonal action. The action of \( \mathbf{T} \) is basically multiplication by elements in \( C_{\text{alg}}^b \) (coming from the non-split torus \( M^\times \hookrightarrow G(\mathbb{A}^{(p\infty)}) \)) which permutes elements in \( C_{\text{alg}}^\alpha \) and is essential in the proof of [H10, Theorem 3.20] which shows that \( X = \mathcal{V} \) once we know \( \text{Irr}(X) = \text{Irr}_+^K(X) \).

In this article, the action of the semi-group \( \alpha^N \) plays the central role to prove \( \text{Irr}(X) = \text{Irr}_+^K(X) \). The condition \( \alpha N \alpha^{-1} \supseteq N \) implies that \( \alpha \in B(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}) \) for some \( m > 0 \) with a uniformizer \( \pi_1 \) of \( O_1 \), and if \( K_1 = \hat{\Gamma}_1(\nu') (\nu > 0) \), \( S = S_K := K \cap K^{\beta} \) is normalized by \( K \) and a representative set of \( S_K \setminus K \) can be chosen in \( N \). Note that \( \alpha N \alpha^{-1} = \bigcup_{\beta \in \alpha^N} N \beta N \subset \text{GL}_2(F_1) \) is a multiplicative semi-group.

Consider the following condition
\[ (\infty) \quad \text{every } \mathbf{T}-\text{orbit in } \Xi \text{ is infinite.} \]

This condition will be verified for our choice of \( \Xi \) in Proposition 2.10 for the above \( \alpha \) well chosen. Since \( \alpha \in B(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}) \) \( B(m > 0) \) does not have a fixed point in \( \mathcal{V} \), if one orbit \( \mathbf{T}(x) \) for \( x \in \Xi \) is infinite, every orbit is indeed infinite.

For simplicity, we assume hereafter \( K_1 = \hat{\Gamma}_1(\nu') \) or \( \hat{\Gamma}_1(\nu') \) with \( \nu > 0 \) and that \( K \) is open in \( G(\mathbb{A}) \) satisfying (K). Since \( \alpha \) is supposed to preserve the irreducible component \( V \) of \( \text{Sh}(p) \), we may assume that \( I^m = (\varpi) \) with \( \varpi = \varphi \varphi^c \) for some \( \varphi \in R \). Replacing \( m \) by a positive integer multiple of \( m \), we may further assume
\[ (2.1) \quad \text{for } a := \varpi^\nu / \varpi, \text{ elements } (\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}) \text{ and } (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \text{ in } G(\widehat{\mathbb{Z}}) \text{ belong to } K. \]

Indeed, by replacing \( m \) by \( mn \) and \( \varpi \) by \( \varpi^n \), \( a \) is replaced by \( a^n \) which is sufficiently close to 1. Hereafter, for simplicity, we assume that \( \alpha = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \) for \( \varpi = \varphi \varphi^c \) and write \( \beta \) for a general element.
in $N\alpha^NN$. We write $?(\Xi)_K$ (or $\beta, \beta^{-1}$) for the Zariski closure in $\mathcal{V}_K$ of the image $?(\Xi)_K = ?(\Xi_K)$ of $?(\Xi)$ in $\mathcal{V}_K$.

As we recall from [H04] in §3.1, the left action of $g \in G(\mathbb{A}^{(p, \infty)})$ on the point $x = (A, \eta) \in \mathcal{V}$ is given by $g(x) = \tau(g)^{-1}(x)$, where the right action $\tau(g)$ is by definition given by $\eta \to \eta \circ g$ for the level structure $\eta$ associated to the point $x$. If $\beta \in \alpha^N$, $K \supset N$ may not be normalized by $\beta$. Thus $\beta$ acts on $\mathcal{V}_K$ as a correspondence.

Let us explain this correspondence action in some details. Recall $S = S_K := K \cap \beta^{-1}K\beta = K \cap K^\beta$. By definition $\mathcal{S}_{\beta^{-1}} = K^{\beta^{-1}} \subset K \subset K$ and $\mathcal{S}_{\beta^{-1}} \subset K$ (so, $\mathcal{S}_{\beta^{-1}}$ satisfies the condition (K) while $S$ is not). Then $S$ is normalized by $N$ if $K_1 = \tilde{T}_1(\nu)$ but $S \supset N$. We have $N/\beta N = \bigcup_{u \in N} N/\beta(u)$ for a finite set $N = \{\beta(u)u \mod \nu\}$ for $0 < j$ given by $(\det(\beta)) = \nu$ (so, $m(j)$), and $K^\beta = \bigcup_{u \in N} K^\beta(u)$. Then we have the correspondence $U(\beta) \subset \mathcal{V}_K \times \mathcal{V}_K$ (with respect to the tower $\{\mathcal{V} \to \mathcal{V}_K\}_K$) defined by the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{V}_S & \xrightarrow{\mathcal{V}}^{v \to \beta(v) = \beta^{-1}} & \mathcal{V}_{\mathcal{S}_{\beta^{-1}}} \\
p_{S,K} & & p_{\mathcal{S}_{\beta^{-1}},K} \\
\mathcal{V}_K & \xrightarrow{\beta} & \mathcal{V}_K,
\end{array}
$$

where $U(\beta)$ is identified with a subvariety given by the diagonal image of $\mathcal{V}_S$ under the product of the projections $p_{S,K} \times (p_{\mathcal{S}_{\beta^{-1}},K} \circ \beta)$. It is easy to see $U(\beta^n) = U(\beta)^n$ under the correspondence action.

The correspondence $U(\beta)$ brings a point $x \in \mathcal{V}_K$ to a finite set $U(\beta)(x) := (p_{S_{\beta^{-1}},K} \circ \beta(p_{S_{\beta_{1}}}(x)))$. We assume, for $K$ satisfying (K),

(N) The action of $T$ on $\Xi$ extends to a correspondence action of the semi-group $NTN$ on $\Xi_K$.

If it is necessary to indicate the dependence of the level group $K$, we write $\Xi_K$ for the image of $\Xi$ in $\mathcal{V}_K$. We write $U(\beta^n)(\Xi_K) := \bigcup_{x \in \Xi_K} U(\beta^n)(x)$. The condition (N) means that $U(\beta)$ acts on $\Xi$ (i.e., $U(\beta)(\Xi_N) \subset \Xi_N$).

Since $\alpha_T \in \text{GL}_2(F)$, by (2.1), the correspondence $U(\beta)$ for $\beta \in \alpha^NS$ only depends on the double coset $\alpha^NS$. We need the following finiteness condition (which will be verified in Lemma 2.9 and (2.8)):

(F) $\Xi_N - U(\alpha^n)(\Xi_N)$ and $\alpha^{-n}(\Xi) - \Xi$ are finite for all $n > 0$.

Since $T$ is a group, (F) implies finiteness of $\Xi_N - U(\beta)(\Xi_N)$ and $\beta^{-1}(\Xi) - \Xi$ for all $\beta \in T$. We actually use only the finiteness of $\beta^{-1}(\Xi) - \Xi$ in the proof of the key result (Theorem 2.6).

Let $X = X_S$ (resp. $X_S = X_{\Xi,S}$) be the Zariski closure of $\Xi$ in $\mathcal{V}$ (resp. of the image $\Xi_S$ in $\mathcal{V}_S$) for a closed subgroup $S$ satisfying (K). Since $U(\beta)(\Xi) \subset \Xi$, we find $X_S \supset X_{U(\beta)(\Xi)} = \bigcup_{u \in N} X_{\beta(u)}(X_\Xi)$. Thus we have a tower $\{X_S\}_S$ of reduced schemes with projections $p_{S,K} : X_S \to X_S$ for $S' \subset S$ (which we write simply $p_S$ if $S$ is clear in the context). Therefore, we can think of the corresponding action of $\beta$ on $X_S$ with respect to the tower $\{X_S\}_S$.

If $S$ is open compact, $X_S$ is a reduced variety (i.e., reduced noetherian). The semi-group $NTN$ acts on $X$ sending $X = X_{\Xi}$ to $\beta(X) = X_{\beta(\Xi)}$ and also $U(\beta)(X) = X_{U(\beta)(\Xi)}$. For $\beta \in NTN$ and an open compact subgroup $K \subset G(\mathbb{A}^{(\infty)})$ satisfying (K), taking an open compact subgroup $S$ of $K$ such that $SS_{\beta^{-1}} \subset K$, we have a diagram

$$
\begin{array}{ccc}
X_S & \xrightarrow{\beta(X)} & X_{S_{\beta^{-1}}} \\
p_{S,K} & & p_{S_{\beta^{-1}},K} \\
X_K & \xrightarrow{C(\beta)} & X_K,
\end{array}
$$

where $C(\beta)$ is a subvariety given by the diagonal image of $X_S$ under $p_{S,K} \times p_{S_{\beta^{-1}},K} \circ \beta$. We regard $C(\beta)$ as a correspondence from $X_K$ into $X_K$. This correspondence is specifically on $X_K$ and its points and is possibly different from the operator $U(\beta)$ for the tower $\{\mathcal{V} \to \mathcal{V}_S\}_S$.

Lemma 2.1. Assume that $\mathcal{V}_S \to \mathcal{V}_K$ for $S = S_K := K \cap K^\beta$ is étale. Let $Y^S := p_{S,K}^{-1}(Z_K) = Z_K \times_{Y_K} \mathcal{V}_S$ for $Z_S \in \text{Irr}(X_S)$ and $Z_K := p_{S,K}(Z_S)$, and write $Y^S = \bigcup_{Z \in \text{Irr}(Y^S)} Z$. If $Z \neq Z'$ for
$Z, Z' \in \text{Irr}(Y^S)$, we have $Z \cap Z' = \emptyset$, so, $Y^S = \bigcup_{Z \in \text{Irr}(Y^S)} Z$. If $K_1 = \widehat{\Gamma}_1 \langle \ell' \rangle$ and $Z \in \text{Irr}(Y^S)$, then $K$ normalizes $S$ and for $u \in K/S$, either $u(Z) = Z$ or $u(Z) \cap Z = \emptyset$.

Proof. Note that $Y^S := p_{S,K}^{-1}(Z_K)$ is étale finite over $Z_K$ as $V_S \to V_K$ is étale. Thus $Y^S$ is equidimensional with dim $Z = \dim Y_S = \dim Z_S = \dim Z_K$ for $Z \in \text{Irr}(Y^S)$. If $\emptyset \neq Z \cap Z' \subseteq Z$ for $Z \neq Z'$, $(Z, Z' \in \text{Irr}(Y^S))$, $Z \to Z_K$ and $Z' \to Z_K$ are dominant by the equi-dimensionality. Thus by the étale property of $Y^S \to Z_K$, $Z(F) \to Z_K(F)$ and $Z'(F) \to Z_K(F)$ are onto. For $x \in (Z \cap Z')(F)$, $|((Z \cup Z') \times_k x_K)| < \deg(Z/Z_K) + \deg(Z/Z_K)$; so, $Z \cap Z'$ ramifies over $p_{S,K}(Z_S)$ since dim $Z = \dim Z'$, which is impossible as $Z \cup Z' \to Y^S \to Z_K$ is unramified by Lemma 1.1 (2). This shows the first assertion.

Suppose that $K_1 = \widehat{\Gamma}_1 \langle \ell' \rangle$ and $Z \in \text{Irr}(Y^S)$. Since $SN = K$ and $N$ normalizes $S = S_K = K \cap K^\beta$, $K$ normalizes $S$. If $K_1 = \widehat{\Gamma}_1 \langle \ell' \rangle$, $Y^S = \bigcup_{u \in K/S} u(Z)$; so, $u(Z)$ is still an irreducible component of $Y^S$, and $K/S$ acts on $\text{Irr}(Y^S)$. Thus the intersection is either empty or $u(Z) \cap Z = Z$. Since $u(Z) \cap Z = Z$, we have $Z \in u(Z)$. Since they are irreducible and have equal dimension, we conclude $Z = u(Z)$.

\textbf{Lemma 2.2.} Suppose (F). Then for $\beta \in \mathbf{T}$ and $K$ satisfying (K), we have $\text{Irr}_+(U(\beta)(X)_K) = \text{Irr}_+(X_K)$ and $\text{Irr}(X_K) - \text{Irr}(U(\beta)(X)_K) = (\text{Z}_K - U(\beta)(\Xi_K))$. Similarly we have $\text{Irr}_+(\beta^{-1}(X)_K) = \text{Irr}_+(X_K)$ and $\text{Irr}(\beta^{-1}(X)_K) - \text{Irr}(X_K) \subset (\beta^{-1}(\Xi) - \Xi)$.

Proof. As remarked after (F), $\Xi - U(\beta)(\Xi)$ is finite for all $\beta \in \mathbf{T}$. Since $U(\beta)(\Xi) \subset \Xi$, we have a closed immersion $U(\beta)(X)_K \subset X_K$. Since $U(\beta)(X)_K$ is the Zariski closure of $U(\beta)(\Xi_K)$, the finiteness of $\Xi - U(\beta)(\Xi)$ implies $\text{Irr}_+(X_K) = \text{Irr}_+(U(\beta)(X)_K)$ and $\text{Irr}(X_K) - \text{Irr}(U(\beta)(X)_K) \subset (\Xi - U(\beta)(\Xi))$ by Lemma 1.4. The last assertion follows from finiteness of $\beta^{-1}(\Xi) - \Xi$ assumed in (F). \qed

The semi-group element $\beta \in \mathbf{T}$ acts on $\pi_0(X)$ and $\text{Irr}(X)$ in the sense that $\beta$ sends $\pi_0(X)$ and $\text{Irr}(X)$ isomorphically onto $\pi_0(\beta(X))$ and $\text{Irr}(\beta(X))$, respectively. Therefore $\beta : x \mapsto \beta(x) = x\beta^{-1}$ induces an isomorphism $\beta : \text{Irr}(X) \cong \text{Irr}(\beta(X))$. Let $Z_S$ be an irreducible component of $X_S$ and write $\beta(Z_S) \in \text{Irr}(\beta(X)_{S_{\beta^{-1}}})$.

\textbf{Lemma 2.3.} Suppose that $S \subset K$ is a closed subgroup for an open compact subgroup $K = G(\mathbb{Z}_p) \times K^{(p)}$ in $G(K^{(\infty)})$. Take $Z_S \in \text{Irr}(X_S)$ with dim $Z_S = \dim X_S$ and write $Z_K$ for the image of $Z_S$ in $X_K$. Then $Z_K \in \text{Irr}(X_K)$, and there exists $x \in \Xi$ such that its image $x_K$ lies in an open subscheme of $Z_K$ made of smooth points of $Z_K$.

Proof. By Lemma 1.2, we have $Z_K \in \text{Irr}(X_K)$. Thus we prove the existence of the point $x \in \Xi$ as in the lemma. If $Z_K = X_K$, nothing to prove. We suppose that $Z_K \neq X_K$. Since $X_K$ is noetherian, the Zariski closure $Z_K$ of $Z_K - Z_K$ is a proper closed subscheme of $X_K$; so, by Zariski density of $Z_K$ in $X_K$, if $\Xi_K \subset Z_K$, we find $X_K = Z_K$, a contradiction. Therefore $(Z_K - Z_K) \cap \Xi_K \neq \emptyset$. For the Zariski closure $Z_K$ of $(Z_K - Z_K) \cap \Xi_K$ in $Z_K$, $Z_K \cup Z_K$ contains $((Z_K - Z_K) \cap \Xi_K) \cup (Z_K \cup Z_K) = \Xi_K$ as $X_K = Z_K \cup Z_K$. Thus $Z_K \cup Z_K = X_K = Z_K \cup Z_K$. Since $Z_K$ is a union of irreducible components of $X_K$ different from $Z_K$, this implies $X_K \subset Z_K$, and $(Z_K - Z_K) \cap \Xi_K$ is Zariski dense in $Z_K$. We can thus pick $x_K$ in the open subscheme $Z_K - Z_K$ in $Z_K$. Since the subscheme of smooth points of $Z_K - Z_K$ is non-empty and open in $Z_K$ [CRT, Theorem 24.4], we may assume that $x_K$ is a smooth point of $Z_K - Z_K$. \qed

For each reduced Zariski closed subset $Y$ of $V_S$, we put $\Xi^Y = Y \cap \Xi_S$.

\textbf{Lemma 2.4.} Suppose that $K$ is an open compact subgroup as in (K). Let $Z_K \in \text{Irr}(X_K)$. Then $\Xi^{Z_K}$ is dense in $Z_K$.

Proof. Since $\Xi_K \cap (Z_K - Z_K) = \Xi_K$ as seen in the proof of Lemma 2.3, $\Xi^{Z_K}$ containing $\Xi_K \cap (Z_K - Z_K)$ is dense in $Z_K$.

We can argue differently. For an irreducible component $Z_K$ of $X_K$, $Z_K - Z_K$ is an open subset of $X_K$; so, any open subset $Y' \subset (Z_K - Z_K)$, $Y' \cap \Xi_K \neq \emptyset$. Thus $\Xi^{Z_K} = \Xi_K \cap Z_K$ is dense in $Z_K$. \qed
Suppose $K_1 = \hat{1}(\mathbb{P})$. Assume that $\mathcal{V} \to \mathcal{V}_K$ is étale. Since the diagram

$$
\begin{array}{ccc}
Y^S & \xrightarrow{\sim} & \mathcal{V}_S \\
& \downarrow^{ps,K} & \downarrow^{\text{étale}} \\
Y_K & \xrightarrow{\sim} & \mathcal{V}_K
\end{array}
$$

is Cartesian, $Y^S \to Y_K$ is étale. Therefore, $Y^S$ is equi-dimensional with $\dim Y^S = \dim Y_K$. By Lemma 1.2, $\text{Irr}(Y^S) \cap \text{Irr}(X_S) \neq \emptyset$; so, we can define a non-empty subscheme $Y_S$ of $Y^S$ by

$$
Y_S := \bigcup_{Z \in \text{Irr}(X_S) \cap \text{Irr}(Y^S)} Z \overset{(*)}{=} \bigcup_{Z \in \text{Irr}_d(X_S) \cap \text{Irr}(Y^S)} Z \subset X_S,
$$

which is equi-dimensional with dimension $d := \dim Y_K$, and the identity $(*)$ follows from Lemma 2.1 under étaleness of $\mathcal{V} \to \mathcal{V}_K$. Thus $\text{Irr}(X_S) \cap \text{Irr}(Y^S) = \text{Irr}_d(X_S) \cap \text{Irr}(Y^S)$. Note that taking intersection $\text{Irr}(Y^S) \cap \text{Irr}(X_S) \neq \emptyset$ means that we can pick irreducible components of $X_S$ which dominates $Y_K$ (so, each member of $\text{Irr}(Y^S) \cap \text{Irr}(X_S) \neq \emptyset$ has dimension equal to $\dim Y_K$). By Lemma 2.1, $Y^S$ is a disjoint union of $Y_S$ and $\bigcup_{Z \in \text{Irr}(Y^S) \cap \text{Irr}(X_S)} X$, and hence $Y_S \to Y_K$ is étale finite dominant.

**Lemma 2.5.** Suppose that $K$ is an open compact subgroup as in (K) and pick $Y_K \in \text{Irr}_d(X_K)$ for $0 \leq d \leq \dim X$. Suppose $Y_S \to Y_K$ is étale. Then we have $Y^S = \bigcup_{Z \in \text{Irr}(Y^S)} Z$ and $Y_S = \bigcup_{Z \in \text{Irr}(Y^S) \cap \text{Irr}(X_S)} Z$. The set $\Xi^S$ is either empty or Zariski dense in $Z$ for $Z \in \text{Irr}(Y^S)$, and for each $x \in \Xi^S$, there is a unique irreducible component $Z \in \text{Irr}(Y_S)$ with $x \in Z$.

**Proof.** The first assertion is proven before the statement of the lemma. We prove the remaining assertion. If $Z \subset X_S$ for $Z \in \text{Irr}(Y^S)$, it is an irreducible component of $X_S$ by Lemma 1.2. Thus $\Xi^S$ is Zariski dense in $Z$ by Lemma 2.4. In other words, if $Z \not\subset X_S$, $\Xi^S$ is an empty set, and for each $x \in \Xi^S$, there is a unique irreducible component $Z \in \text{Irr}(Y_S)$ with $x \in Z$ as $Y_S$ is a disjoint union of $Z$. \qed

2.2. Modular correspondences acting on irreducible components of $X_K$. Pick an irreducible component $Y_K \in \text{Irr}_d(X_K)$ for $0 \leq d \leq \dim X_K$ with an open compact subgroup $K$ satisfying (K).

2.2.1. Definition of the correspondence. Choosing $x \in \Xi$ so that $x_S \in Y_S$ for $Y_S$ in (2.3), we have $\beta(x)_S := x_S \beta^{-1} \in \beta(Y_S) \subset X_{S^{-1}}$, and there is a unique irreducible component $Z$ of $Y_S$ containing $x_S$ by Lemma 2.5. Since $Y_S \overset{\sim}{\to} \beta(Y_S) \subset \beta(X)_{S^{-1}} \subset X_{S^{-1}}$, we have $\dim Y_K = \dim Y_S = \dim \beta(Y_S) = \dim \beta(Y_S)_K$ for the projection $\beta(Y_S)_K$ of $\beta(Y_S)$ in $X_K$.

For any pair of open compact subgroups $(K, S)$ with $K \supset SS^{-1}$ (so, $S \subset K \cap \beta^{-1}K\beta$), we have a diagram similar to (2.2):

$$
\begin{array}{ccc}
Y_S & \xrightarrow{\psi = \beta(v)} & \beta(Y_S) \\
& \downarrow^{ps,K} \text{finite} & \downarrow \\
Y_K & \xrightarrow{C_S(\beta)} & \beta(Y_S)_K := p_{S^{-1},K}(\beta(Y_S)) \overset{C}{\subset} X_K,
\end{array}
$$

for the correspondence $C_S(\beta)$ given by the reduced image $\text{Im}(ps,K \times p_{S^{-1},K} \circ \beta : Y_S \to \mathcal{V}_K \times \mathcal{V}_K)$ whose support is contained in $\mathcal{C}(\beta)$ in (2.2). Note that

$$
C_S(\beta) \text{ is independent of the choice of } S
$$

as $ps,K \times p_{S^{-1},K} \circ \beta = (ps,K \times p_{S^{-1},K} \circ \beta) \circ ps,S_K$ for $S_K = K \cap \beta K^{-1}$ (so, $C_S(\beta) = C_{S_K}(\beta)$).

As mentioned below (2.2), the correspondence $C_S(\beta)$ is with respect to the tower $\{X_S\}_S$ and is possibly different from $U(\beta)$ with respect to the tower $\{\mathcal{V}_K\}_K$.

Hereafter we choose $S$ to be $S_K$ and still write it as $S$ (so, the correspondence action of $C_S(\beta)$ on irreducible components we introduce in the proof of the following Theorem 2.6 only depends on $\beta$ (and $K$)). Note that $\beta(Y_S)_K = \bigcup_{u \in N_K} \beta u(Z)_K$ for some $u \in N \cong K/S_K$, where $\beta u(Z)_K$ is the image under $p_{S^{-1},K}$ of $\beta u(Z)$ for a component $Z \in \text{Irr}(Y_S)$ (cf. Lemma 2.1). Since $\beta : X_S \cong \beta(X)_{S^{-1}}$, we have

$$
\begin{array}{ccc}
Y_S & \xrightarrow{\psi = \beta(v)} & \beta(Y_S) \\
& \downarrow^{ps,K} \text{finite} & \downarrow \\
Y_K & \xrightarrow{C_S(\beta)} & \beta(Y_S)_K := p_{S^{-1},K}(\beta(Y_S)) \overset{C}{\subset} X_K,
\end{array}
$$

for the correspondence $C_S(\beta)$ given by the reduced image $\text{Im}(ps,K \times p_{S^{-1},K} \circ \beta : Y_S \to \mathcal{V}_K \times \mathcal{V}_K)$ whose support is contained in $\mathcal{C}(\beta)$ in (2.2). Note that

$$
C_S(\beta) \text{ is independent of the choice of } S
$$

as $ps,K \times p_{S^{-1},K} \circ \beta = (ps,K \times p_{S^{-1},K} \circ \beta) \circ ps,S_K$ for $S_K = K \cap \beta K^{-1}$ (so, $C_S(\beta) = C_{S_K}(\beta)$).
extended to an automorphism of $V$. By the above diagram with dominant $p_{S,K}$ and $p_{S^{d-1},K}$, we again find $\dim \beta(Y_S) = \dim Y_K$ as $\beta(Y_S) \subset p_{S^{d-1},K}(\beta(p_{S,K}(Y_K)))$.

2.3. Positive dimensionality of irreducible components of $X$. We now prove the following fact not described in [H04]:

**Theorem 2.6.** Suppose (unr) and (ord) at the beginning of Section 2 for $p$. Let $\Xi \subset V(\mathbb{F})$ be an infinite subset injecting into $V_K$ for any open compact subgroup $K$ satisfying (K) and (I). We assume that a semi-group $T \subset \text{Aut}(V(\mathbb{F})$ as in (T) embedded in $\text{Aut}(V(\mathbb{F})$ acts on $\Xi$, and assume (F) and (N).

(1) If the condition (\infty) is satisfied, all irreducible components of $X$ has positive dimension;

(2) If $T$ acts on $\Xi$ transitively, dim $X > 0$, $X$ is equi-dimensional, and the irreducible component containing a given $x \in \Xi$ is unique.

Under (\infty), we can replace $\Xi$ by an infinite orbit $T(x)$ and apply the result and conclude the Zariski closure of $T(x)$ is equidimensional of positive dimension; so, one of them contains $x$.

**Proof of (1).**\(^1\) We need to describe the correspondence action of $\beta$ on $Y_K \in \text{Irr}_d(X_K)$. First suppose that $d = \dim X$ as this is the easiest case. Then $\dim \beta(Y_S) = \dim X$, and hence $\beta(Z) \subset \text{Irr}(Y_S)$ for $Z \in \text{Irr}(Y_S)$. In this way, $\beta \in T$ acts on $Y_K \in \text{Irr}_d X(K)$ as correspondences (i.e., $Y_K$ is brought to a subset $\beta(Y_K) = \{\beta(Z) \mid Z \in \text{Irr}(X_S) \cap \text{Irr}(Y_S)\} \subset \text{Irr}_d X(K)$ whose member has equal dimension). Since $\text{Irr}(X_S) \cap \text{Irr}(Y_S)$ is made of $u(Z)$ for $u \in \text{N}$, for a finite subset $B$ of $\text{N} \cap \text{N}$, we have the correspondence action of $C(\beta)$ given by the image set $\beta(Y_K) := \bigcup_{\beta \in B} \beta(Y_K)$ under $C(\beta)$ on $\text{Irr}_d X(K)$.

Though we only need the result for $d = 0$, we give an argument for the intermediate dimension $0 < d < \dim X$ now as this introduces necessary notation for the case $d = 0$. Pick $Y_K \in \text{Irr}_d(X_K)$ and start with $Y_K \in \text{Irr}_d(X_K)$. As above to define the action of $C(\beta)$ on $\text{Irr}_d(X_K)$, we only need to give a good definition of the image set $\beta(Y_K)$ for a $\beta \in \text{N} \cap \text{N}$. For simplicity, write $S' = S^{d-1}$.

Let us recall a general notation: For an irreducible component $Y'_K$ of $X_K$, we define as before $Y'^{(d)} := p_{S,K}^{-1}(Y_K)$ and $Y_S'^{\beta} := \bigcup_{\beta \in B} \beta(Y_K)$ containing the base point $x_S \in \Xi_S$ chosen in \S 2.2.1 and we apply the above notation to the irreducible component $Y_K'$ of $X_K$ such that $\beta(Z) \subset Z'$ for an irreducible component $Z'$ of $Y_K'$ (so, $\beta(x_S) \in Z'$). To see the existence of an irreducible component $Y'_K$ of $X_K$ as above, we argue as follows. Since $\beta(Z)$ is an irreducible closed variety of $X_S'$, $p_{S,K}(\beta(Z))$ is an irreducible closed variety of $X_K$. Then there exists an irreducible component $Y_K'$ of $X_K$ containing $p_{S,K}(\beta(Z))$ of $X_K$ by Lemma 1.2 (1). Therefore $\beta(Z) \subset Y'_K$, which is contained in $Z' \subset \text{Irr}(Y'_K)$. So $\dim Z' = \dim Y_K' \geq d$ by Lemma 2.1. Replacing $(\beta, Y_K, S, K)$ by $(\beta^{-1}, Y_K', S', K)$, we apply the above argument. Note that $\beta^{-1}(Z') \subset \beta^{-1}(X_S)$; so, $\beta^{-1}(Z')' \subset \beta^{-1}(X_K)$.

By the choice of $Y_K$, Lemma 2.5 tells us that $Z$ is determined by the two conditions (i) $\beta^{-1}(X_K) \supset \beta^{-1}(Z')'$ and (ii) $x_S \in Z$. Since $\text{Irr}_+(\beta^{-1}(X_K)) = \text{Irr}_+(X_K)$ by Lemma 2.2 and $\beta^{-1}(Z')'$ is irreducible, we conclude from $\beta^{-1}(Z')' \supset Y_K$ that $\beta^{-1}(Z')' = Y_K$ (as $Y_K$ is an irreducible component of $\text{Irr}_+(\beta^{-1}(X_K)) = \text{Irr}_+(X_K)$); in particular, $\dim Z' = \dim Y_K = d$. So $Y_K' = \beta(Z)_K$ and that $\beta(Z)_K$ is an element in $\text{Irr}_d(X_K)$ (Lemma 2.2). Therefore, again $\beta \in T$ acts on $\text{Irr}_d(X_K)$ as correspondences (i.e., $Y_K$ is brought to a subset $\beta(Y_K) = \{\beta(Z) \mid Z \in \text{Irr}(X_S) \cap \text{Irr}(Y_S)\} \subset \text{Irr}_d(X_K)$ whose member has equal positive dimension).

Now suppose $d = 0$. Since the correspondence action preserves $\text{Irr}_+(X_K)$, it also preserves the complement $\text{Irr}_0(X_K)$. The following argument to see the correspondence action is really an action sending a point to a point also gives an alternative proof of the stability of $\text{Irr}_0(X_K)$ under the action of $T$. We proceed similarly to the case where $0 < d < \dim X$ using the same notation. Then $x_K = Y_K \in \text{Irr}_0(X_K)$ falls in the image $\Xi_K$ in $V_K$ of $\Xi$ by Lemma 1.2 (3). By (I), the projection $\pi : V \to V_K$ induces $\Xi \cong \Xi_K$; so, $p_{S,K}(x_K)$ is a finite set of points above $x_K$ and $\{x' \in p_{S,K}^{-1}(x_K) \mid x' \in X_S\}$ is a singleton by Lemma 1.2 (2-3). Thus $p_{S,K}^{-1}(Y_K) \cap X_S = p_{S,K}^{-1}(Y_K) \cap X_S$ is a singleton. Therefore $Y_K = \{Z := x_S\}$ is a singleton. Take an irreducible component $Y_K'$ of $X_K$ such that $\beta(Z) \subset Z'$ for an irreducible component $Z'$ of $Y_K'$ (so, $\beta(x_S) \in Z'$). Such a

---

\(^1\)In the proof, we use the existence of $\beta^{-1} \in \text{Aut}(V)$ essentially, while $\alpha : (v, z) \mapsto (v + 1, 2z)$ in \S 1.1 cannot be extended to an automorphism of $V$ there.
$Y_K'$ exists by Lemma 1.2 (1). So $\dim Z'=\dim Y_K' \geq 0$. We want to prove $\dim Y_K' = 0$. Since $\text{Irr}^{-1}(\beta^{-1}(X)_S) = \text{Irr}^{-1}(X)_S$ by Lemma 2.2 and (F), if $\dim Z' > 0$, we have $\dim \beta^{-1}(Z') > 0$ and $\beta^{-1}(Z)$ is an irreducible component of $X_S$. Since $\beta^{-1}(Z') \supset Z = x_S$ by construction and the two are irreducible components of $X_S$, we find that $\beta^{-1}(Z') = Z = x_S$, a contradiction against $\dim Z' > 0$. Hence $\dim Z' = 0$ and $Z' = \beta(Z) = \beta(x_S)$. This implies that $\beta$ brings $\text{Irr}_0(X_K)$ into $\text{Irr}_0(X_K)$. It is now clear that this is really an action (not a correspondence action) of $T$ on $\text{Irr}_0(X_K)$, and the action is compatible with the action of $T$ on $\Xi$ as $\text{Irr}_0(X_K) \subset \Xi_K \cong \Xi$. In particular, $\text{Irr}_0(X_K)$ contains $T(x_K)$ for each $x_K \in \text{Irr}_0(X_K) \subset \Xi_K$. Then by ($\infty$), $\text{Irr}_0(X_K)$ is infinite, a contradiction as $X_K$ is a noetherian scheme. Therefore $\text{Irr}_0(X_K) \cap T(x_K) = \emptyset$ for every open compact subgroup $K$ of $G(\mathbb{A}^+(p))$ satisfying (K) and $x_K \in \text{Irr}_0(X_K)$. This implies $\text{Irr}_0(X_K) = \emptyset$ for every open compact subgroup $K$ of $G(\mathbb{A}^+(p))$ satisfying (K), and therefore $\text{Irr}_0(X) = \emptyset$ by Lemma 1.2 (3). This shows that all irreducible components of $X$ have positive dimension.

**Proof of (2).** We have proven positive dimensionality of irreducible components of $X$. We need to prove equi-dimensionality of $X$ and the uniqueness of the component containing $x \in \Xi$ under equi-dimensionality. Since the smooth locus $X_K^{sm}$ of $X_K$ is open dense in $X_K$ by [CRT, Theorem 24.4], $\Xi_K := \Xi \cap X_K^{sm}$ is still dense in $X_K$. Since $\text{Irr}(X_K) = \pi_0(X_K^{sm})$, for each $x \in \Xi_K$, the irreducible component of $X_K^{sm}$ containing $x$ is unique. Since $T$ acts on $\text{Irr}(X_K) = \pi_0(X_K^{sm})$ as correspondence, for any $Z, Z' \in \text{Irr}(X_K)$, we find $x \in \Xi \cap Z^{sm}$ and $y \in \Xi \cap Z'^{sm}$.

Interchanging $Z$ and $Z'$ if necessary, by (T) and transitivity of the action, we can choose $\beta \in T$ with $\beta(x) = y$. Then $\beta(Z) \in \text{Irr}(X_K)$ and $y \in \beta(Z) \cap Z'$. Thus $y \in Z^{sing} = Z' - Z^{sm}$ (i.e., $y \in \beta(Z) \cap Z'$ different from any irreducible components of $\beta(Z)$) or $\beta(Z) \supset Z'$ or $\beta(Z) \subset Z'$ different from any irreducible components of $\beta(Z)$. Hence $\beta(Z) \subset Z'$ implies $Z = \dim \beta(Z) = \dim Z'$ and $Z' \in \text{Irr}(\beta(Z))$. Choosing one of $Z$ and $Z'$ to have maximal dimension $X$, the other has to have maximal dimension; so, $\text{Irr}(X_K) = \text{Irr}_{\dim X}(X_K)$; so, $X_K$ is equidimensional. This implies $X$ is equidimensional.

By the first fundamental sequence of differentials and unramifiedness of $X_S/X_K$ in Lemma 1.1, the projection induces a surjection:

$$\Omega_{X_K/F} \otimes_{O_{X_K}} F(x_K) \twoheadrightarrow \Omega_{X_S/F} \otimes_{O_{X_S}} F(x_S)$$

for $S \subset K$. By the proof of the equi-dimensionality, for $\dim O_{X_S,x_S} = \dim O_{X_K,x_K}$ for any point $x_S \in X_S$ with projection $x_K$ in $X_K$. Thus

$$\dim_{\mathbb{F}} \Omega_{X_K/F} \otimes_{O_{X_K}} F(x_K) \geq \dim_{\mathbb{F}} \Omega_{X_S/F} \otimes_{O_{X_S}} F(x_S) \geq \dim O_{X_S,x_S} = \dim O_{X_K,x_K}.$$ 

Here “$\dim_{\mathbb{F}}$” indicates dimension of an $\mathbb{F}$-vector space, and $\dim R$ for a ring $R$ means the Krull dimension of the ring $R$. Thus the singular locus

$$X_S^{sing} := \{x_S \in X_S | \dim_{\mathbb{F}} \Omega_{X_S/F} \otimes_{O_{X_S}} F(x_S) > \dim O_{X_S,x_S}\}$$

of $X_S$ is sent to $X_K^{sing}$, where $F(x)$ is the residue field of $x$. Thus $X_S^{sing} = \lim_{\rightarrow S} X_S^{sing}$, and hence $\dim X_S^{sing} < \dim X = \dim Y$ for any $Y \in \text{Irr}(X)$. Plainly $T$ preserves $X_S^{sing}$. If $x \in \Xi \cap X_S^{sing}$, then $\Xi = T(x) \subset X_S^{sing}$; so, $x = X_S^{sing}$, a contradiction. Thus $\Xi \cap X_S^{sing} = \emptyset$. Since $X_S^{sm} := X - X_S^{sing}$ is a dense open subscheme of $X$, $\pi_0(X_S^{sm}) \cong \text{Irr}(X_S^{sm}) \cong \text{Irr}(X)$ with $X_S^{sm} = \bigcup_{Y \in \text{Irr}(X)} Y_S^{sm}$. Thus for each given $x \in \Xi \subset X_S^{sm}$, $Y_S^{sm} \in \text{Irr}(X_S^{sm})$ containing $x$ is unique.

Since $X_K$ has positive dimension for an open compact level $K$ (as $|\Xi_K|$ is infinity; cf. Lemma 1.2 (4)), by the above proposition, all components of $X$ have positive dimension. Taking $x \in \Xi$ and an irreducible component of $X$ containing $x$, we get

**Corollary 2.7.** Let the notation and the assumption be as in Theorem 2.6 (1). Then $X$ contains an irreducible component $X_0$ of positive dimension with a point $x \in \Xi$. Moreover for each element $\xi$ of the stabilizer of $x$ in $T$, we have $\xi(x_0) = x_0$.

**Proof.** We need to prove the last assertion: $\xi(x_0) = x_0$. Since $\xi \in T$, $\xi(x_0)$ is another irreducible component of $X$ containing $x$. Taking a level group $K$ sufficiently small, $\xi(x_0) \cup x_0 \to X_K$ is unramified. Since $\xi(x_0) \cap x_0 \ni x$, unramifiedness and positive dimensionality of $X_0$ tells us that $\xi(x_0) = x_0$. 


There is another argument. Replacing \( \Xi \) by the orbit \( \Xi' := T(x) \), we may assume that \( T \) acts transitively on \( \Xi \). Then \( X_0 \) and \( \xi(X_0) \) are irreducible components of the Zariski closure \( X' \) of \( \Xi' \). Then we can apply Theorem 2.6 (2) to \( X' \) and \( \Xi' \). Since there is only one irreducible component containing \( x = \xi(x) \), we have \( X_0 = \xi(X_0) \).

**Remark 2.8.** Note that the stabilizer of \( x \in \Xi_n \) is given by \( T_x := (M^x \cap R_{n+1}^x \cap R_p^x)/(F^x \cap O_n^x \cap O_p^x) \) embedded into \( GL_2(F_{k(p)}^{(\infty)}) \subset GL_2(F_{k(p)}^{(\infty)}) \) which after \( p \)-adic completion contains a \( p \)-adically open subgroup. We can take \( T \) in (T) to be this group or the bigger group \( (M^x \cap R_{n+1}^x \cap R_p^x)/(F^x \cap O_n^x \cap O_p^x) \), and under this choice, we can apply Corollary 2.7 to \( \xi \in T_x \). The stability of \( X \) in Corollary 2.7 is a requirement of [H10, Corollary 3.19, Theorem 3.20], and the choice \( T_x \subset T \) is sufficient for this purpose. Since the central elements in \( F \) acts on \( \Xi \) trivially, we take \( T_x \) as above rather than \( M^x \cap R_{n+1}^x \cap R_p^x \).

2.4. Verification of (F) and (N) for infinite arithmetic progression. We briefly describe the choice of \( \Xi \subset V \). For the details of the definition of CM point \( x(A) \), see Section 3.

Let
\[
\mathcal{T} := (M^x \cap R_{n+1}^x \cap R_p^x)/(F^x \cap O_n^x \cap O_p^x),
\]
but for our convenience, we often shrink \( T \) slightly to a subgroup of finite index to define \( T \) as remarked in Remark 2.8. We write \( R_{(a)}^x \).

For each \( \xi \in R_{(p)}^x \) (in [H04, page 755] the symbol "\( \alpha \)" is used for the letter "\( \xi \)" here), we have \( x(A) := (X(A), \bar{X}(A), \eta(p)(A)) \overset{\xi}{\rightarrow} x(\xi(A)) \) as in the middle of [H04, page 755], and as seen in [H04, page 756], \( \rho_R(\xi(1))(B) = x([\xi(1)]B) \) for the class \( [\xi(1)] = [(\xi)] \) of the ideal \( (\xi) \) in \( Cl_\infty = \lim_n Cl_n \).

Recall \( C_n = \{ x(A): = x([A]\delta)\delta \in \mathcal{Q}, A \in Cl_n \} \) and \( C(\infty) = \bigcup_n C_n \subset V \). Thus \( \xi \in T \) acts on \( C(\infty) \) by \( [A] \mapsto [(\xi)] [A] \).

Let \( \varpi_1 \) be a prime element of \( \mathcal{O}_1 \). As specified in [H04, §2.1 and §3.1], for each proper fractional ideal \( A \) of \( \mathcal{O}_n \), we have a specific CM point \( x(A) \in Sh^{[p]}(\mathbb{F}) \). In our application, \( \Xi \) is made of the set of points of the form \( x(A) \). Note that (see (3.3))
\[
\left( \begin{array}{cc}
1 & 0 \\
\varpi_1 & 1 \\
\end{array} \right)
\left( \begin{array}{cc}
1 & 0 \\
0 & \varpi_1 \\
\end{array} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
\end{array} \right).
\]

By [H04, (3.2)] (see (3.2) in the text), writing \( \alpha_m := \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi_1 \end{array} \right) \) and \( \varrho(u) := \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \), we have
\[
\alpha_m \varrho(u)(x(N(R_n))) = \varrho(u \varpi_1)(\alpha_m(x(N(R_n))) = x(N(A) \ (0 < m \in \mathbb{Z})
\]
for \( A \) given by \( x(A) = x(R_n)/\mathcal{C}_u \) for a suitable subgroup \( \mathcal{C}_u \subset X(R_n) \) with \( \mathcal{C}_u \cong O/\mathfrak{m} \) depending on \( u \), \( x_N \) indicates the image of \( x \) in \( \mathcal{V}_N \), and \( A = R_{n+m} \) if \( u = 0 \). By (2.1), in (2.6), we can replace \( \varpi_1^m \) by \( \varpi \) by \( \varphi \varpi \) and \( x_N \) by \( x_K \), and the identity is valid on \( \mathcal{V}_K \) (in place of \( \mathcal{V}_N \)). Any \( A \in \text{Ker}(Cl_{(n+m)} \to Cl_m) \) with \( u > 0 \) can be written as in (2.6).

Set \( \Xi_j \) = \{ \{x(A) \in \mathcal{V}: A \in \text{Ker}(Cl_{n+m} \to Cl_j) \} \} for each \( n > j \geq 0 \) with a given \( j \). Since
\[
\Xi_j = \{ x([A]\delta) \delta \in \mathcal{Q}, A \in R_{(p)}^x \cap (1 + \mathfrak{m}R_l) \},
\]

defining \( T_j \subset \mathcal{T} \) (for \( T \) in (2.5)) by
\[
T_j := \{ \{ x \in (M^x \cap R_{n+1}^x \cap R_p^x) \mod \mathfrak{m} \} \in (R_{(p)}^x/\mathfrak{m}^j)\} / \mathfrak{O}_{(p)}^j,
\]
the group \( T_j \) acts transitively on \( \Xi_j \) for every \( n \) with \( n > j \geq 0 \). Here \( R_{(p)} \) and \( O_{(p)} \) are the localization at \( p \) of \( R \) and \( O \) not the completion, and note \( (T : T_j) < \infty \).

**Lemma 2.9.** Assume that \( \mathfrak{m} \) is generated by an element of \( N_{MF}/R \) and write \( \mathfrak{m} = (\varpi) \) with \( \varpi \in N_{MF}/R \). Define \( \alpha = \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi \end{array} \right) \) and let \( \mathcal{T} = T_{j,m} = T_j \times \bigcup_{k \geq 0} N_{\alpha^k N} \) as a semi-group. If \( \varpi \) is an infinite arithmetic progression of difference \( m \), for \( \Xi = \Xi_{\varpi} = \Xi_{\varpi,j} = \bigcup_{k \geq 0} \Xi_{\varpi,j}^{k + \mathfrak{m}} \), we have \( \Xi \supset U(\beta)(\Xi) \) for \( \beta \in T_{j,m} \) (which implies that the condition (N) is satisfied), and \( \Xi = U(\beta)(\Xi) \) is finite.

**Proof.** By (2.6), we have \( U(\alpha)(C_n) = C_{n+m} \) and \( U(\alpha)(\Xi_j) = x(\Xi_{j}^{n+m}) \). Thus the semi-group \( \bigcup_{k \geq 0} N_{\alpha^k N} \) acts on \( \Xi_{\varpi} \) for \( \varpi = \{ n_0 + im \mid i = 0, 1, 2 \ldots \} \) for \( U(\alpha^k) \ (0 \leq k \in \mathbb{Z}) \) sending \( \Xi_{\varpi,j}^{k + \mathfrak{m}} \) into
\[ \Xi_j^{n_0 + (i + k)m} \]. Any element of \( \Xi_j^{n_0 + (i + k)m} \) is an image of an element of \( \Xi_j^{n_0 + im} \) under the action of \( \beta \in N\alpha^k N \). Then we have
\[
\Xi_n - U(\alpha^k)(\Xi_n) = \bigcup_{i \geq 0} \Xi_j^{n_0 + im} - \bigcup_{i' \geq 0} \Xi_j^{n_0 + (k + i')m} = \bigcup_{i=0}^{k-1} \Xi_j^{n_0 + im}
\]
which is finite. Since \( T_j \) is a group acting transitively on \( \Xi_j^{n_0 + im} \), this implies \( \Xi - U(\beta)(\Xi) \) is finite for all \( \beta \in NNTN \) and hence we get (N) and (F) for \( U(\beta) \).

The point \( x(A) \) is given by identifying \( \hat{A}^{(p)} = A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(p)} \) with the prime-to-\( p \) Tate module of the corresponding CM abelian variety \( X(A) \); so, strictly speaking, it is more precise to write \( x(A^{(p)}) \) (or \( x(\hat{A}) \)) in place of \( x(A) \). Under this notation, \( \alpha(x(A^{(p)} \times R_{n,1})) = x(A^{(p)} \times R_{n+m,1}) \) and \( \alpha^{-1}(x(A^{(p)} \times R_{n,1})) = x(A^{(p)} \times R_{n-m,1}) \) as long as \( n > m \). See §5.3 for what happens when \( n \leq m \).

It appears that the map \( \alpha^{-1} \) is non-injective, but this comes from the fact that \( K \supset N \) (satisfying (K)) but \( S_K \) is not; in other words, \( ps_K \) is not injective but shrinking \( K \) to \( K' \) at 1 so that \( K' \cap N = \{1\} \) (with \( K/N \cong K'/\langle K' \cap N \rangle \)), the fiber of \( ps_K \) will be separated modulo \( S_K \cap K' \) (but the fiber of \( ps_{K'},K \) is non-trivial again). Thus \( \alpha^{-1}(C_n) = C_{n-m} \) as long as \( n > m \). This shows (F) for \( \alpha^{-i} \) and \( \Xi_{n,j} \).

(2.8) \[ \alpha^{-i}(\Xi) - \Xi \text{ is finite for all } i > 0 \]
as long as \( n \) is an infinite arithmetic progression of difference \( m \) as long as \( l^m \) is generated by an element of \( N_{M/F}(R) \) and \( \alpha \text{ diag } [1, \omega]^{-m} \in K \), writing \( \text{diag}[a, b] \) for the diagonal matrix with diagonal entries \( a, b \) from top to bottom. Therefore in this case the condition (F) is valid.

If necessary, we also write sometime \( x(\hat{A}) = A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \) for \( x(A) \) assuming \( A_p = R_p \). The group \( T \) acts on \( x(A) \) as follows: For \( \xi \in T \),
\[ x(A) = x(\hat{A}) \mapsto x(\xi^{-1}A) = x((\xi)A), \]
where \( \xi^{-1} \in M_k^\times \) is the finite idele with \( l \)-component equal to 1 and every component at finite place outside \( l \) is equal to \( \xi \).

In the idele class group \( I := M_k^\times/M_k^\times M_{\infty}^\times \), \( \xi \) is trivial but \( \xi^{-1} \) is not trivial; so, the action of \( \xi \) on \( Cl_n \) is non-trivial for any sufficiently large \( n \). Regard \( Cl_n = \text{Pic}(R_n) \) as a quotient of \( I \), and write \( (\xi) \) for the image of \( \xi^{-1} \) in \( Cl_n \). Since \( \text{Ker}(Cl_n \to Cl_0) \) is spanned by \( (\xi) \) running in \( T \), \( T \) acts transitively on \( \text{Ker}(Cl_n \to Cl_0) \). More generally, noting that \( T_c \subset T \) is the stabilizer of \( x(R_{n,c}) \) in \( Cl_c \), \( T \) acts transitively on \( \text{Ker}(Cl_n \to Cl_c) \) for all \( n \geq r \). From \( A \) with \( A_i = R_{n,1} \), we can create \( \hat{A}_i := A^{(i)} \times R_{n,1} \). Then even if \( A = \xi^{-1}R_n \) with \( \xi \in R_{n,1} \), \( \hat{A}_i \) is non-trivial in \( Cl_i \). In this way, the group \( T := R_{(p)}^x/O_{(p)}^x \) acts on \( C^{(p)}_{\text{des}} \) as in [H04, page 755].

Let \( \{0 < n_0 < n_1 < n_2 < \cdots < n_i < \cdots \} \) be an infinite sequence of integers such that \( \prod \) is generated by an elements in \( N_{M/F}(R) \). If \( m \) is an exponent such that \( \prod \) is generated by an elements in \( N_{M/F}(R) \), then any infinite arithmetic progression \( n = \{n_i = n_0 + im|0 \leq i \in \mathbb{Z}\} \) for an initial value \( 0 < n_0 \) satisfies this condition. Recall \( \Xi_j^{n_0} = \{(\xi)(A)\}_{\xi \in \Omega(A) \in V}[A] \in \text{Ker}(Cl_n \to Cl_j) \} \) for \( 0 < j < n_0 \) as in [H04, Proposition 2.7]. Define \( \Xi = \bigcup_{j=0}^{n_0} \Xi_j^{n_0} \subset V \). Since \( Cl_{n_0} \) and \( Cl_j \) is stable under the action of \( T_j \) and the projection \( Cl_{n_0} \to Cl_j \) is compatible with the action of \( T_j \), \( \Xi_{n_0} \) is stable under \( T_j \), and hence \( \Xi_{n_0} \) is also stable under \( T_j \). Thus we get

**Theorem 2.10.** Choose \( 0 < m \in \mathbb{Z} \) so that \( \prod \) is principal generated by \( \varpi = \varphi \varphi' \) with \( \varphi \in R \) and define \( \alpha \) as in Lemma 2.9. Suppose \( a \text{ diag } [1, \omega]^{-m} \in K \). If \( n \) is an infinite arithmetic progression (with initial value \( n_0 \) and difference \( m \) ), the semi-group \( T_{j,m} \) generated by the group \( T_j \) in (2.7) and \( \alpha = (1, 0) \) acts transitively on \( \Xi_{n, j} \) and satisfies (T), (N) and (F) (for \( T = T_j \)).

Theorem 2.6 combined with this result, Corollary 2.7 and [H10, Corollary 3.19, Theorem 3.20] gives

**Corollary 2.11.** If \( n \) contains an arithmetic progression, then \( \Xi_{n, j} \) for any \( j \geq r \) is Zariski dense in \( V^\mathbb{Q} \).
2.5. **Characteristic 0 version.** We consider $Sh_{/W}^{(p)}$ and its geometric irreducible component $V/W$ and define $V' = V_{/W}^{A}$ in the same manner as above. Consider $V_{/F} = V_{/W} \otimes W F$. Note that $V_{/W}$ is smooth over $W$ (see [PAF, Theorem 7.1]).

**Lemma 2.12.** Let $A$ be a smooth $W$-domain and $\Xi$ be a countable set of $W$-points of $\text{Spec}(A)$ and as a subscheme of $\text{Spec}(\mathbb{A})$, $\Xi$ is étale over $W$. Write $X := X \otimes_W \mathbb{F}$ for $X = A, A, \Xi$ as a subscheme of $\text{Spec}(\mathbb{A})$. Then if $\Xi$ is Zariski dense in $\text{Spec}(\mathbb{A})$, then the schematic closure of $\Xi$ in $\text{Spec}(A)$ is equal to $\text{Spec}(A)$ and $\Xi_\eta = \Xi \times W \eta$ for the generic point $\eta \in \text{Spec}(A)$ is Zariski dense in $\text{Spec}(A) \times W \eta$.

Order $\Xi = \{P_1, P_2, \ldots \}$ with $\Xi_n := \{P_1, \ldots, P_n\}$. Write $\hat{X} := \varprojlim_n X/m^\infty_n X$ for $X = A, A, \Xi, P_i$ (the formal completion along the special fiber).

**Proof.** Since $A$ is smooth over $W$, $\hat{A}$ (resp. $\Xi$) is smooth over $W$ (resp. $\mathbb{F}$); in particular, $\hat{A}$ is a domain. Since $\Xi_n$ is étale over $W$; so, is $\hat{A}_n$ over $W$. Thus $\Xi_n \cong \hat{A}_n \cong \Xi_n$ as point sets; hence $\Xi \cong \Xi$ as sets.

We have $A/j^n = \prod_j A/P_j = \prod_j A/P_j = \prod_j W$. Thus $A/j^n$ is $W$-flat. We have a short exact sequence $(\bigcap_{j=1}^n A_j) \otimes W F \to \hat{A} \to (A/j^n) \otimes W F$. For an $A$-ideal $a$ with $W$-flat quotient $\hat{A}/a$, we have an exact sequence $\hat{a} = a \otimes W F \to A \to (A/a) \otimes W F$. We identify $\hat{a} = a \otimes W F$ as an ideal of $\hat{A} = \hat{A} \otimes W F$. Take another ideal $b$ with $W$-flat $\hat{A}/b$. Then $\hat{A}/(a \cap b) \to \hat{A}/b \otimes a \Rightarrow \hat{A} \to (A/a \cap b) \otimes W F$. From the short exact sequence: $\hat{A}/(a \cap b) \to \hat{A}/a \cap \hat{A}/b \Rightarrow \hat{A}/(a + b)$ for the two ideals $a$ and $b$, we obtain a three term exact sequence $\hat{A}/a \cap b = (\hat{A}/a \cap \hat{A}/b) \otimes W F \Rightarrow \hat{A}/(a \cap b) \Rightarrow \hat{A}/(a + b)$. Hence $\hat{A}/a \cap b$ is Zariski dense in $\hat{A}/(a + b)$ and hence $\hat{A}/a \cap b$ is Zariski dense in $\hat{A}/(a + b)$. Therefore we conclude $P = (0)$. By Nakayama’s lemma for adically complete modules over a complete ring (e.g., [CAG, Exercise 7.2]), we conclude $P = (0)$.

The definition of $\Xi \subset V$ in Theorem 0.1 works well over $W$; so, we take a geometrically irreducible component $V$ of $Sh_{/W}^{(p)}$ with $x(R_n) \in V(W)$ for sufficiently large $n$ and define $V = V \otimes W$ and $\Xi \subset V$ as in Theorem 0.1.

**Proposition 2.13.** Assume $\Xi \otimes W F$ is Zariski dense in $V \otimes W F$. Then $\Xi \otimes \eta$ is Zariski dense in the generic fiber $V \otimes W \eta$.

**Proof.** Since $V \to V_K$ is affine, covering $V_K$ by open affine schemes $\text{Spec}(A_{K,i})$ and pulling them back to $\text{Spec}(A_{S,i}) \subset V_S$ for open subgroups $S \subset K$, we apply Lemma 2.12 to $\text{Spec}(A_{F}) \subset V$ for $A_i = \text{lim}_S A_{S,i}$ assuming Zariski density of $\Xi$ in the special fiber $V \otimes W F$ and conclude Zariski density in the generic fiber.

3. **Geometric modular forms and CM points**

The Hilbert modular Shimura variety $Sh_{/W}^{(p)}$ is the moduli (up to prime-to-$p$ $O$-linear isogeny) of triples $(X, \mathbb{A}, \eta)$ for an abelian variety $X$ of dimension $d = [F : \mathbb{Q}]$ with multiplication by $O$, an $O$-linear polarization class $\mathbb{A}$ up to multiplication by $(O_F)^{\times}$ (see [H04, §2.2]) and an $O$-linear level structure $\eta : V^{(p)}(X) = T(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \cong (F_K^{(p)})^2$ for the Tate module $T(X)$ of $X$.

For the Hilbert modular Shimura variety $Sh_{/W}^{(p)}$, we use the definition and notation introduced in [H04, Section 2]. See also [HMI, Section 4.3] for a more detailed description of the Shimura variety and modular forms. Geometric modular forms can be defined as global sections of weight $\kappa$ Hodge bundles over the Shimura variety, or equivalently a functorial rule assigning a value to classified abelian varieties with extra structure. Out of the assigned value at CM points, we create a distribution interpolating L-values in the next section Section 4.
3.1. CM points \( x(A) \). We recall the definition of the CM points \( x(A) \) from [H04]. Let \( G = \text{Res}_{\mathbb{Q}/Q} \text{GL}(2) \) (so, \( G(A) = \text{GL}_2(A \otimes Q F) \)). We write the left action: \( G(A)^{(p)} \times S_l(p) \to S_l(p) \) simply as \((g, x) \mapsto g(x) = : \tau(g)^{-1}(x) \). Here the action of \( \tau(g) \) is a right action induced by \( \eta \mapsto \eta g \) for the level structure. For each point \( x = (X, \overline{X}, \eta) \in Sh, \) we can associate a lattice \( \overline{L} = \eta^{-1}(\overline{T}(X)) \subset (F_\mathbb{A}^{(\infty)})^2 \). Then the level structure \( \eta \) is determined by the choice of a base \( w = (w_1, w_2) \) of \( \overline{L} \) over \( \overline{O} \). In view of the base \( w \), the inverted action \( x \mapsto g(x) \) is matrix multiplication: \( t \mapsto g^t w \), because \((\eta \circ g)^{-1}(T(X)) = g^{-1}(T(X)) = g\overline{L} \).

For each \( O \)-lattice \( A \), we recall a description of a CM point \( x(A) = (X, A, \eta(A)) \in Sh(p) \) from [H04, §2.1], where \( X(A), \eta \) is an abelian scheme of CM-type \((M, \Sigma) \) with \( H^1(X(A)(\overline{C}), \mathbb{Z}) = A \) in the sense \( X(A)(\overline{C}) = \mathbb{C}^n/A \mathbb{C}^n \) for \( \mathbb{A}^\Sigma = \{(a^\sigma, \ldots, a^{s_d}) \in \mathbb{C}^n | a \in A \} \) writing \( \Sigma = \{\sigma_1, \ldots, \sigma_d\} \). For the order \( R_A = \alpha \in M/\alpha A \subset A \) and an ideal \( a \) of \( R_A \), we write, as a finite flat group scheme over \( W \),

\[
X(A)[a] := \{x \in X(A) | ax = 0\} = \bigcap_{\alpha \in a} \ker(\alpha : X(A) \to A(A)).
\]

Recall the order \( R_n = 0 + 1^n R \subset M \) and the class groups \( C_l = \text{Coker}(\text{Pic}(O) \to \text{Pic}(R_n)) \) and \( C_l^\infty = \lim_{\rightarrow n} C_l^n \). By class field theory, \( C_l^n \) gives the Galois group of the maximal anticyclotomic class field in the ring class field of conductor \( l^n \). The ideal \( l_n = 1 + l^n R = 1 R_{n-1} \) is a prime ideal of \( R_n \), but is not proper (it is a proper ideal of \( R_{n-1} \)). Since \( X(R_n)[l_n] \cong R_n/l_n = O/1 \) and \( l_n R_{n-1} \subset R_n \), we find that \( X(R_n)[l_n] = R_{n-1}/R_n \) and \( X(R_n)/X(R_n)[l_n] \cong X(R_{n-1}) \). We pick a subgroup \( C \subset X(R_n)[l] \) isomorphic to \( O/1 \) but different from \( X(R_n)[l_n] \). We look into the quotient \( X(R_n)/C \). Take a lattice \( \mathfrak{A} \) so that \( X(R_n)/C = X(\mathfrak{A}) \cong \mathfrak{A}/R_n = C \). Since \( C \) is an \( O \)-submodule, \( \mathfrak{A} \) is an \( O \)-lattice of \( M \). Since \( IC = 0 \), we find \( I R_n \mathfrak{A} \subset \mathfrak{A} \). Thus \( \mathfrak{A} \) is \( R_{n+1} \) ideal, because \( R_{n+1} = 0 + 1 R_n \). Since \( C \) is not an \( R_n \) submodule, the ideal \( \mathfrak{A} \) is not \( R_n \) ideal; so, it is a \( R_{n+1} \) ideal. Since \( C \) generates over \( R_n \) all \( l \)-torsion points of \( X(R_n) \), we find \( R_n \mathfrak{A} = l^{-1} R_n \).

In this way, we have created \( l \) proper \( R_{n+1} \) ideals \( \mathfrak{A} \) with \( \mathfrak{A} R_n = l^{-1} R_n \).

We choose a base \( w = (w_1, w_2) \) of \( \overline{R} \) over \( \overline{O} \) in [H04, §2.1]: at \( p \), for the choice of the ordinary \( p \)-adic CM-type \( \Sigma = \{\mathfrak{p}/p\} \), writing \( R_\Sigma = \prod_{\mathfrak{p} \in \Sigma} R_\mathfrak{p} \) and \( R_{\Sigma^c} = \prod_{\mathfrak{p} \notin \Sigma} R_\mathfrak{p} \) for complex conjugation \( c \), \( R_p = R_{\Sigma^c} \oplus R_{\Sigma} \) and \( w = (w_1, w_2) = ((1, 0), (0, 1)) \).

Let \( A \) be an \( O \)-lattice and in our order \( R(A) := \alpha \in M/\alpha A \subset A \) has conductor \( f = (f(A)) \). Though we mainly deal with the case where \( f(A) = 1 \), we describe a general theory with arbitrary \( f \) prime to \( p \). We choose a “good” level structure \( \eta(A) \) of \( X(A) \) so that \( \eta(A)(\overline{O}^2) = \overline{A} \) in the following way. First we choose a representative set \( \{\mathfrak{A}_j\} \) of ideal classes of \( M \) (prime to \( pf \)). Then we can write \( \mathfrak{A}_j = a_j \overline{R} \) for an idele \( a_j \) with \( a_j = a_j^{(l^{\infty})} \) and choose \( \alpha \in M \) so that \( A R = \alpha \mathfrak{A}_j \). For an idele \( a \in F_\mathbb{A}^\infty \) (resp. an adele \( a \in F_\mathbb{A}^\infty \)) and an integral ideal \( \alpha \), \( a^{(\infty)} \) indicates \( a = 1 \) (resp. \( a = 0 \)) for each place \( v \) appearing in \( \infty \) or \( \mathbb{A} \). If \( f(A) = O \) (so, \( A \) is an \( R \)-ideal), we define the level structure \( \eta(A) \) by \((F_\mathbb{A}^{(\infty)})^2 \ni (a, b) \mapsto a a_j w_1 + b a_j w_2 \in M_\mathbb{A}^{(\infty)} = V(X(A)). \) When \( f(A) \neq O \), we first suppose that \( f = (\varphi^c) \) for \( \varphi \in M \). Take \( \alpha \in M \) such that \( A R = \alpha \mathfrak{A}_j \), and choose a base \( w(A) \) of \( \overline{A} \) so that \( w(A)^{(l)} = (\varphi a a_j w)^{(l)} \) and \( w(A)^{(l)} = a w_1 \cdot g \) for \( g \in GL_2(F_1) \) with \( \det(g^t) = \varphi^c \). Then we define \( \eta(A)(a, b) = a \cdot w_1(A) + b \cdot w_2(A) \in M_\mathbb{A}^{(\infty)} \). There is an ambiguity of the choice of \( \alpha \) and \( \varphi \) up to units in \( R \), but this does not cause any trouble later.

Suppose that \( f(A) \) is not generated by a norm from \( M \). We choose \( g \in G(\mathbb{A}^{(\infty)}) \) with \( g^{(l)} = 1 \) so that \( g(A) = a a_j w \cdot g \) gives a base over \( \overline{O} \) of \( \overline{A} \), and define \( \eta(A) \) by using \( w(A) \). In the above two cases, we choose \( g \) independent of the ideals in the proper ideal class of \( A \); in other words, we choose \( w(\beta A) = \beta a a_j w \cdot g \). We then define \( g(A) \in G(\mathbb{A}^{(\infty)}) \) by \( \eta(A) = \eta(\mathfrak{A}_j) \cdot g(A) \). We will later specify the choice of \( g \) precisely.

We introduce a representation \( \rho_A : M_\mathbb{A}^{(l^{(A)})} \to G(\mathbb{A}^{(\infty)}) \) by \( \alpha \eta(A) = (\alpha \eta(A) \cdot \rho_A(\alpha)) \). By our choice, we have \( \rho_A = \rho_R \) on \( M_\mathbb{A}^{(l^{(A)})} \), and

\[
\det(g(A)) \in F_\mathbb{A}^\infty \text{ if } f(A) \text{ is generated by a norm from } M.
\]
We choose a totally imaginary $\delta \in M$ with $\text{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$. Then the alternating form $(a, b) \mapsto (c(a)b - ac(b))/2a$ gives an identity $R \otimes \overline{R} = \mathfrak{c}$ for a fractional ideal $\mathfrak{c}$ of $F$. Here $\mathfrak{c} = \{x \in F|\text{Tr}_{F/Q}(x) \in \mathbb{Z}\} = \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ for the different $\mathfrak{d}$ of $F/Q$. Identifying $M \otimes \mathbb{R}$ with $\mathbb{C}^\Sigma$ by $m \otimes r \mapsto (\sigma(m)r)_{\sigma \in \Sigma}$, we find that $(a, ia) = \sqrt{\overline{a}a}$, $\mathfrak{c} = 0$ for $a \in M^\times$. Here the symbol “$\gg$” means total positivity. Thus $\text{Tr}_{F/Q} \circ (\cdot, \cdot)$ gives a Riemann form for the lattice $\Sigma(A)$, and therefore, a projective embedding of $\mathbb{C}^\Sigma/\Sigma(R)$ onto a projective abelian variety $X(A)/\mathbb{C}$. The complex abelian scheme $X(\mathfrak{A})$ extends to an abelian scheme over $\mathbb{W}$ (unique up to isomorphisms). In this way, we get a $\mathfrak{c}$-polarization $\Lambda(A) : X(\mathfrak{A})(\mathbb{C}) \otimes \mathfrak{c} \cong {}^tX(\mathfrak{A})(\mathbb{C})$ for the dual abelian scheme $^tX(\mathfrak{A}) = \text{Pic}^0_X(\mathfrak{A})/\mathbb{W}$. The same $\delta$ induces

$$\mathcal{R} \otimes \mathcal{R} = (O \otimes R) + \mathcal{I}^2(R \otimes R) = (f^{-1}\mathfrak{c})^*$$

and $\mathcal{A} \otimes \mathcal{A} = (N_{M/F}(\mathfrak{A}^{-1}) \otimes (\mathfrak{A}^{-1})^*)^*$, where the exterior product is taken over $O$. Hereafter we fix $\delta$ so that $\mathfrak{c}$ is prime to $p\mathfrak{p}A\mathfrak{d}$, and write $c(A)$ for $N_{M/F}(\mathfrak{A}^{-1}) \otimes (\mathfrak{A}^{-1})^*$ (so, $\mathfrak{c} = c(R)$). We can always choose such a $\delta$, since in this paper we only treat $\mathcal{A}$ with $\mathfrak{t}$-power conductor.

Since an isogeny defined over the field $\text{Frac}(\mathbb{W})$ of fractions of $\mathbb{W}$ between abelian schemes over $\mathbb{W}$ extends to the entire abelian scheme (e.g. [GME] Lemma 4.1.16), we have a well defined $c(A)$-polarization $\Lambda(A) : X(\mathfrak{A}) \otimes c(A) \cong {}^tX(\mathfrak{A})$. Replacing $X(\mathfrak{A})$ by an isomorphic $X(\alpha A)$ for $\alpha \in M$, we may assume that $\mathcal{A}_p = \mathcal{A}_\mathfrak{p}$. Then

$$X(\mathfrak{A})[\mathfrak{p}F] = X(\mathfrak{A})[\mathfrak{p}] \oplus X(\mathfrak{A})[\mathfrak{p}^2]$$

for $\mathfrak{p}_F = \mathfrak{p} \cap F$ is isomorphic by $\Lambda(A)$ to its Cartier dual. Since the Rosati-involution $a \mapsto a^* = \Lambda(A) \circ a \circ \Lambda(A)^{-1}$ is the complex conjugation $c$, $X(\mathfrak{A})(\mathfrak{p})/\mathbb{W}$ is multiplicative (étale locally) if and only if $X(\mathfrak{A})(\mathfrak{p}^2)$ is étale over $\mathbb{W}$.

We also specified the base of $\overline{R}_n$ to be $w_0$ in [H04, §2.1], because $\overline{R}_n = \overline{R}(0)$. To specify the base $w_1$ of $R_1$, we take $d \in O_1$ so that $R_1 = O_1(\sqrt{d}) \subset M_1$. We assume that $d$ is a $\mathfrak{t}$-adic unit if $\mathfrak{t}$ is unramified in $M/F$ and $d$ generates $\mathfrak{t}$ if $\mathfrak{t}$ ramifies in $M/F$. Then we choose $w_1(1, \sqrt{d})$.

Since the base of $R_n$ is given by $\alpha_n(1, \sqrt{d})$ for $\alpha_n = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ with a prime element $w_1$ of $O_1$, we find that $\alpha_n(x(R)) = x(R_n)$ and $\alpha_1(x(R_{n-1})) = x(R_n)$. Moreover, for a suitable $u \in O$

$$(3.2) \quad \varpi(x(R)) = \begin{pmatrix} 1 & \frac{u}{\sqrt{d}} \\ 0 & 1 \end{pmatrix} x(R_{n+1}) \quad \text{if} \quad \mathcal{A} = R_n/C \quad \text{for} \quad O/\mathfrak{t} \cong C \neq X(R_n)[\mathfrak{I}],$$

because the base of $\varpi \mathcal{U} \mathcal{A}_t$ satisfies

$$\varpi(x(R)) = \begin{pmatrix} 1 & \frac{u}{\sqrt{d}} \\ 0 & 1 \end{pmatrix} x(R_{n+1}) = \begin{pmatrix} 1 & \frac{u}{\sqrt{d}} \\ 0 & 1 \end{pmatrix} \alpha_{n+1} \left( \begin{pmatrix} 1 & \sqrt{d} \\ 0 & 1 \end{pmatrix} \right).$$

Here the action of $\mathfrak{w} : x(\mathfrak{A}) \mapsto \varpi(x(\mathfrak{A}))$ may bring $x(\mathfrak{A})$ on a geometrically irreducible component of $\mathcal{S}(\mathfrak{p})$ to a different one.

Now we consider $x(\mathfrak{A})$ in $V_K$ for an open subgroup $K \subset G(A^{(\infty)})$ containing $Z(\mathbb{Z})$. By repeating (3.2), if $x(\mathfrak{A}) = x(R_n)/C \cong O/\mathfrak{I}^m$ with $C \cap X(R_n)[\mathfrak{I}] = \{0\}$, then $\mathcal{A}$ is a proper $R_{n+1}$-ideal. If further $\mathfrak{I}^m$ is generated by an element $\varpi \otimes \mathfrak{F} \in F$, we get $x(\mathfrak{A}) = x(\mathfrak{A}) = x(\mathfrak{A}) \in V_K$ (because $\varpi/\mathfrak{I}^m \in K$) and

$$(3.3) \quad x(\mathfrak{A}) = \begin{pmatrix} 1 & \frac{u}{\sqrt{d}} \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & \varpi \\ 0 & 1 \end{pmatrix} \right) x(R_{n+m})$$

for a suitable $u \in O$.

The set $\{x(\mathfrak{A})[\mathcal{A}R_n] = \mathfrak{A}\}$ for $\mathfrak{A} \in C\mathfrak{I}$ running through ideal classes $\mathfrak{A}$ projecting down to a given ideal class $[\mathfrak{A}] \subset C\mathfrak{I}$ is in bijection with $O/\mathfrak{I}^m$ by associating $u$ to $\mathfrak{A}$ in (3.3) (see [H04, Proposition 4.2]).

### 3.2. Geometric modular forms.

Let $k$ be a weight of $T = \text{Res}_{O/\mathbb{Z}}^G \mathbb{G}_m$, that is, $k : T(A) = (A \otimes \mathbb{Z})^\times \to A^\times$ is a homomorphism given by $(a \otimes \xi)^k = \prod (a\xi)^{k_\sigma}$ for integers $k_\sigma$ indexed by field embeddings $\sigma : F \hookrightarrow \mathbb{C}$. Let $B$ be a base ring, which is a $\mathcal{W}$-algebra. We consider quadruples $(X, \overline{X}, \eta, \omega)/\mathcal{A}$ for a $B$-algebra $A$ with a differential $\omega$ generating $H^0(X, \Omega_{X/A})$ over $A \otimes \mathbb{W}$. We impose the following condition:

$$(3.4) \quad \eta^p(\overline{L}^p) = T(X) \otimes \mathbb{Z}^p$$

for $L_\mathfrak{c} = O \oplus \mathfrak{c}$ with a fixed $\mathfrak{c}$.

Under this condition, as seen in [H04, §2.3] and [HMR, §4.3.1], the classification up to prime-to-$p$ isogenies of the quadruples is equivalent to the classification up to isomorphisms. A modular form $f$ (integral over $B$) of weight $k$ is a functorial rule of assigning a value $f(X, \overline{X}, \eta, \omega)$ to $(A -$isomorphism class of) each quadruple $(X, \overline{X}, \eta, \omega)/\mathcal{A}$ (called a test object) defined over a $B$-algebra $A$. Here $\Lambda$ is a $c$-polarization which (combined with $\eta^p$) induces $L_\mathfrak{c} \otimes L_\mathfrak{c} \cong \mathfrak{c}$ given
by \(((a \oplus b), (a' \oplus b')) \mapsto ab' - a'b\). The Tate test object at the cusp \((a, b)\) for two fractional ideals with \(a^*b = c^*\) is an example of such test objects. The Tate semi-AVRM \(\text{Tate}_{\mu,k}(q)\) is defined over \(\mathbb{Z}[\{q^e\}_{e \in \{ab\}_k}\) and is given by the algebraization of the formal quotient \((\widehat{G}_m \otimes a^*)/q^b\) (see [HMI, §4.2.5] for details of this construction). The rule \(f\) is supposed to satisfy the following three axioms:

1. **G1** For a \(B\)-algebra homomorphism \(\phi: A \to A'\), we have
   \[ f((X, \overline{\lambda}, \eta^{(p)}, \omega) \times_{A, \phi} A') = \phi(f(X, \overline{\lambda}, \eta^{(p)}, \omega)). \]

2. **G2** \(f\) is finite at all cusps, that is, the \(q\)-expansion of \(f\) at every Tate test object does not have a pole at \(q = 0\).

3. **G3** \(f(X, \overline{\lambda}, \eta^{(p)}, \alpha \omega) = \xi^{-k} f(X, \overline{\lambda}, \eta^{(p)}, \omega)\) for \(\xi \in T(A)\).

We write \(G_k(c; B)\) for the space of all modular forms \(f\) satisfying (G1-3) for \(B\)-algebras \(A\). We put

\[
G_k(B) = \bigoplus_{c} G_k(c; B),
\]

where \(c\) prime to \(\ell p\) runs over a representative set of strict ideal classes of \(F\).

An element \(g \in G(\mathbb{A}^{(\infty)})\) fixing \(\widehat{L}_c\) acts on \(f \in G_k(c; B)\) by

\[ f|g(X, \overline{\lambda}, \eta^{(p)}, \omega) = f(X, \overline{\lambda}, \eta^{(p)} \circ g, \omega). \]

For a closed subgroup \(K \subset K_c = GL(\widehat{L}_c) \cap G_1(\mathbb{A}^{(\infty)})\), we write \(G_k(c; K; B)\) for the space of all \(K\)-invariant modular forms; thus,

\[ G_k(c; K; B) = H^0(K, G_k(c; B)). \]

Take an \(\mathfrak{N}\)–ideal \(\mathfrak{N}\) prime to \(\mathfrak{p} c\). Then the \(\mathfrak{N}\)–component of \(K_c\) is \(SL_2(O_{\mathfrak{N}})\). Let

\[ \Gamma_0(\mathfrak{N}) = \{(a b \ c \ d) \in SL_2(O_{\mathfrak{N}}) | c \in O_{\mathfrak{N}}\} \quad \text{and} \quad \Gamma_1(\mathfrak{N}) = \{(a b \ c \ d) \in \Gamma_0(\mathfrak{N}) | a \equiv b \equiv 1 \mod \mathfrak{N}O_{\mathfrak{N}}\}. \]

Assume that \(\mathfrak{N}\) is prime to \(\mathfrak{p} t\) and define for an open subgroup \(K_{\mathfrak{N}} \subset SL_2(O_{\mathfrak{N}})\)

\[ G_k(K_{\mathfrak{N}}; B) = \bigoplus_{c} G_k(c; K_{\mathfrak{N}} \times K_c^{(p\mathfrak{N})}; B). \]

A \(W\)-algebra \(B\) is called a \(p\)-adic algebra if \(B = \lim_{\longrightarrow \mathfrak{N}} B/p^\mathfrak{N}B\). We write \(\eta_{\text{ord}}\) for the pair of level structures \((\eta_{\text{ord}}^p : \mu_{p^\infty} \otimes \mathfrak{o}^{-1} \to X[p^\infty], \eta^{(p)})\). A \(p\)-adic modular form \(f\) over a \(p\)-adic \(W\)-algebra \(B\) is a functorial rule of assigning a value in \(A\) to triples \((X, \overline{\lambda}, \eta_{\text{ord}})/A\) with \(c\)-polarization class \(\overline{\lambda}\) satisfying an obvious version of (G1-2) for \(p\)-adic \(B\)-algebras \(A\) (not just \(B\)-algebras). In general, we do not impose (G3) on \(p\)-adic modular forms. See [HMI, §4.2.8] for more details about \(p\)-adic modular forms. We write \(V(c; B)\) for the space of \(p\)-adic modular forms defined over \(B\). We again define

\[
V(B) = \bigoplus_{c} V(c; B) \quad \text{and} \quad V(K_{\mathfrak{N}}; B) = \bigoplus_{c} V(c; K_{\mathfrak{N}} \times K_c^{(p\mathfrak{N})}; B),
\]

where \(V(c; K; B) = H^0(K, V(c; B))\). For \(f \in V(B)\), we write \(f_c \in V(c; B)\) for the \(c\)-component of \(f\), and we say that \(f\) is of level \(\mathfrak{N}\) if \(f\) in either in \(G_k(K_{\mathfrak{N}}; B)\) or in \(V(K_{\mathfrak{N}}; B)\) for \(K_{\mathfrak{N}} \subset SL_2(O_{\mathfrak{N}})\) with \(K_{\mathfrak{N}} = \Gamma_0(\mathfrak{N})\) or \(\Gamma_1(\mathfrak{N})\).

Since \(\eta_{\text{ord}}^p\) induces the identification \(\overline{\gamma}_{\text{ord}}^p : \widehat{G}_m \otimes O^* \cong \widehat{X}\) for the formal completion of \(X\) along the origin, by pushing forward the differential \(\frac{dt}{\mathfrak{o}}\), we can associate \((X, \overline{\lambda}, \eta^{(p)}, \overline{\eta}_{\text{ord}}^p, \overline{\eta}_{\text{ord}}^q)\) to a quadruple \((X, \overline{\lambda}, \eta_{\text{ord}}^p, \eta^{(p)})\). In this way, any modular form \(f\) satisfying (G1-3) can be regarded as a \(p\)-adic modular form by

\[ f(X, \overline{\lambda}, \xi \cdot \eta_{\text{ord}}^p, \eta^{(p)}) = \xi^{-k} f(X, \overline{\lambda}, \eta_{\text{ord}}^p, \eta^{(p)}) \quad \text{for} \quad \xi \in O_p^\infty. \]

Although we do not impose the condition (G3) on \(p\)-adic modular forms \(f\), we limit ourselves to the study of forms satisfying the following condition (G3') in order to define the modified value \(f(A)\) later at CM points \(x(A)\) truly independent of the choice of \(A\) in its proper ideal class. Here abusing our notation, \(x(A)\) is the quadruple \((X(A), \Lambda(A), \eta_{\text{ord}}(A), \omega(A))/\mathfrak{M}\) introduced in [H04,
§2.1). We consider the torus $T_M = \text{Res}_{R/\mathbb{Z}} \mathbb{G}_m$ and identify its character group $X^*(T_M)$ with the module $\mathbb{Z}[\Sigma \cup \Sigma_e]$ of formal linear combinations of embeddings of $M$ into $\overline{\mathbb{Q}}$. By the identity: $(X(\xi), \Lambda(\xi), \eta_{\text{ord}}(\xi) = \xi \eta_{\text{ord}}(\Lambda))/W \cong (X(\xi), \xi \Lambda(\xi), \eta_{\text{ord}}(\Lambda) \circ \rho_\Lambda(\xi))/W$, we may assume that for $k, \kappa \in \mathbb{Z}[\Sigma]$, $$(G^3)\ f(x(\xi)) = f(\rho_\Lambda(\xi)(x(\Lambda))) = \xi^{-k-\kappa}(1-c) f(x(\Lambda)) \text{ for } \xi \in T_M(\mathbb{Z}_\kappa).$$

It is known that for the $p$-adic differential operator $d_\alpha$ of Dwork-Katz ([K78] 2.5-6) corresponding to $\frac{\partial}{\partial x_0}$ for $\sigma \in \Sigma$, $\theta^\kappa \ f(\theta^\kappa = \prod_\alpha d_\alpha^\kappa)$ satisfies $(G^3)$ if $f \in G_k(B)$.

3.3. Hecke operators. Suppose that the $\Gamma_1$ of the level subgroup is equal to $\Gamma_0(\nu')$ ($\nu \geq 0$). Let $e_1 = \ell(1,0), e_2 = \ell(0,1)$ be the standard basis of $F^2 \otimes \mathbb{A}^{\infty}$. Then, under (3.4), for each triple $(X, X, \eta_{\text{ord}})/A$ with $\eta_{\text{ord}} = \eta_{p_{\text{ord}}} \times \eta_{(p)},$

$$C = \eta_{1}(1-\nu O_{\ell e_1} + O_{\ell e_2})/\eta((\ell)^2)$$
gives rise to an $A$-rational cyclic subgroup of $X$ of order $\nu'$, that is, a finite group sub scheme defined over $A$ of $X/\Lambda$ isomorphic to $O/\nu'O$ étale locally. Since $\Gamma_0(\nu')$ fixes $(1-\nu O_{\ell e_1} + O_{\ell e_2})/\eta((\ell)^2)$, the level $\Gamma_0(\nu')$ moduli problem is equivalent to the classification of quadruples $(X, X, C, \eta_{p_{\text{ord}}})/A$ for a subgroup $C$ of order $\nu'$ in $X$, where $\eta_{p_{\text{ord}}}(\ell)$ is the $(\ell$-ordinary) level structure outside $\ell$. Thus we may define for $f \in G_k(\Gamma_0(\nu'; \ell))$ the value of $f$ at $(X, X, C, \eta_{p_{\text{ord}}}(\ell))$, by $f(X, X, C, \eta_{p_{\text{ord}}}(\ell), \omega) = f(X, X, \eta_{p_{\text{ord}}}(\ell), \omega).$ When $f$ is a $p$-adic modular form, we replace the ingredient $\omega$ by the ordinary level structure $\eta_{p_{\text{ord}}}$ in order to define the value $f(X, X, C, \eta_{p_{\text{ord}}}(\ell)).$ We shall define Hecke operators $T(1, \nu')$ and $U(\nu')$ over $(p$-adic) modular forms of level $K$ (with $K = \Gamma_0(\nu'))$. The operator $U(\nu')$ is defined when $\nu > 0$, and $T(1, \nu')$ is defined when $\nu = 0$. Since $1$ is prime to $p$ and $B$ is an $\mathcal{W}$-algebra, any cyclic subgroup $C'$ of $X$ of order $\nu'$ is isomorphic to $O/\nu'O$ étale locally. We make the quotient $\pi: X \to X/C'$, and $\Lambda$, $\eta_{p_{\text{ord}}}$ and $\omega$ induce canonically a polarization $\pi_1, \Lambda$, a canonical level structure $\eta_{p_{\text{ord}}} = \pi \circ \eta_{p_{\text{ord}}}$, $\pi, \eta_{p_{\text{ord}}}(\ell)$ and a differential $(\pi^*)^{-1} \omega$ on $X/C'$. If $C' \cap C = \{0\}$ for the $\Gamma_0(\nu')$-structure $C$ (in this case, we call that $C'$ and $C$ are disjoint), $\pi_1(C) = C + C'/C'$ gives rise to the level $\Gamma_0(\nu')$-structure on $X/C'$. We write $X/C'$ for the new test object of the same level as the test object $X = (X, X, C, \eta_{p_{\text{ord}}}(\ell), \omega)$ we started with. When $f$ is $p$-adic, we suppose not to have $\omega$ in $X$, and when $f$ is classical, we ignore the ingredient $\eta_{p_{\text{ord}}}$ in $X$. Then we define (for $\nu > 0$)

$$(3.7) \quad f|U(\nu')(X) = \frac{1}{N(\nu')} \sum_{C'} f(X/C').$$

where $C'$ runs over all étale cyclic subgroups of order $\nu'$ disjoint from $C$. The newly defined $f|U(\nu')$ is a modular form of the same level as $f$ and $U(\nu') = U(\nu)$. Since the polarization ideal class of $X/C'$ is given by $\ell$ for the polarization ideal class $\ell$ of $X$, the operators $U(\nu')$ permute the components $f_\ell$. We recall some other isogeny actions on modular forms. For fractional ideals $\mathfrak{q}$ in $F$, we can think of the association $X \mapsto X \otimes \mathfrak{q}$ for each $\text{AVRM} X$. This operation will be made explicit in terms of the lattice $L = \pi_1(X)$ in $\text{Lie}(X)$. There are a natural polarization and a level structure on $X \otimes \mathfrak{q}$ induced by those of $X$. Writing $X(\Lambda, \eta) \otimes \mathfrak{q}$ for the triple made out of $(X(\Lambda, \eta)$ after tensoring $\mathfrak{q}$, we define $f|\mathfrak{q}(X, \Lambda, \eta) = f(X(\Lambda, \eta) \otimes \mathfrak{q})$ (see [PAF, §4.1.9] for more details of this definition, though $\mathfrak{q}$ here is $(\mathfrak{q})^{-1}$ in [PAF, §4.1.9]). For $X(\Lambda)$, we have $\langle \mathfrak{q}(X(\Lambda)) = \langle \mathfrak{q}(X)\rangle$.

$$(3.8) \quad \text{The effect of } \mathfrak{q} \text{ on the Fourier expansion at } (a, b) \text{ is given by that at } (a, \mathfrak{q}^{-1}b)$$

(e.g., by [PAF, §4.2.9], noting $\langle \mathfrak{q} \rangle$ here is $(\mathfrak{q})^{-1}$ in [PAF]).

Let $\mathfrak{q}$ be a prime ideal of $F$ outside $p$. For a test object $(X, X, C, \eta_{\mathfrak{q}(p)}(\ell), \omega)$ of level $\Gamma_0(\mathfrak{q})$, we can construct canonically its image under $\mathfrak{q}$-isogeny:

$$\langle \mathfrak{q}(X, X, C, \eta_{\mathfrak{q}(p)}(\ell), \omega) = (X', X, \pi_{\mathfrak{q}(p)} \times \eta_q, \pi^* \omega)$$

for the projection $\pi: X \to X' = X/C$, where $\eta_q = \eta_\mathfrak{q} \cdot GL_2(O_\mathfrak{q})$ for any level $\mathfrak{q}$-structure $\eta_\mathfrak{q}$ identifying $T_\mathfrak{q}(X')$ with $O_\mathfrak{q}$. Then we have a linear operator $\langle \mathfrak{q} : V(\Gamma_0(\mathfrak{q})); B \to V(\Gamma_0(\mathfrak{q})^\mathfrak{q}); B)$ given by $f|\mathfrak{q}(\langle X \rangle = f(\mathfrak{q}(\langle X \rangle).$ See [H04, (4.14)] for the description of this operator in terms of the lattice of $X$. 

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If \( q \) splits into \( \Omega \Omega \) in \( M/F \), choosing \( \eta_q \) induced by
\[
X(\mathcal{A})|q^{\infty} = M\Omega/R\Omega \times M\Omega/R\Omega \cong F_q/O_q \times F_q/O_q,
\]
we always have a canonical level \( q \)-structure on \( X(\mathcal{A}) \) dependent on the choice of the factor \( \Omega \). Then \( [q](X(\mathcal{A})) = X(\mathcal{A}[\Omega_q^{-1}] \) for \( \Omega_q \in CL_{\infty} \). When \( q \) ramifies in \( M/F \) as \( q = \Omega^2 \), \( X(\mathcal{A}) \) has a subgroup \( C = X(\mathcal{A})|\Omega_q^{-1} \) isomorphic to \( O/q \) for \( \Omega_q = \Omega \cap R_n \); so, we can still define \([q](X(\mathcal{A})) = X(\mathcal{A}\Omega_q^{-1}) = X(\mathcal{A}[\Omega_q^{-1}] \).

The effect on the \( q \)-expansion of the operator \([q] \) can be computed similarly to \( \langle \gamma \rangle \) (e.g. [DR80] 5.8; see also [PAF] 4.2.9), and the \( q \)-expansion of \( f|q \) at the cusp \( (a, b) \) is given by the \( q \)-expansion of \( f \) at the cusp \( (\eta_q, b) \).

These operators \([q] \) and \( \langle \gamma \rangle \) change polarization ideals (as we will see later in [H04, §4.2]); so, they permute components \( f_k \). By the \( q \)-expansion principle, \( f \mapsto [q] f \) and \( f \mapsto \langle \gamma \rangle f \) are injective.

4. Distribution attached to \( U(l) \)-eigenform

We recall notation and construction of a measure \( d\varphi_{f,n} \) on \( CL_{\infty}^{-} \) for a mod \( p \) modular form \( f \mid \mathbb{F} \) such that \( f(U(l)) = af \). If \( 0 \neq a \in \mathbb{F} \), we can patch together into a measure \( d\varphi_{f} \) on \( CL_{\infty} \). If \( a = 0 \), this is just a collection of infinitely many measures \( \{d\varphi_{f,n}\}_n \) (see Remark 4.1).

4.1. Anti-cyclotomic measure. Choose a \( U(l) \)-eigenform \( f \in V(T_1(l)) \) with a central character \( \chi \) for a \( p \)-adic ring \( A \) in which \( \chi \) is invertible. We suppose that \( f(U(l)) = (a/\chi(l)N(l))f \) for either a unit \( a \in A \) or \( a = 0 \). This \( f \) is an element of \( V(T_1(l)) \) defined over the non-connected Hilbert modular Shimura variety whose geometrically connected components are indexed by the strict class group \( CL_{\mathbb{F}}^+ \) of \( F \). Our geometrically irreducible component \( V \) carries \( x(A) \) for \( A \in CL_{\mathbb{F}}^+ \cap K_0 \) for \( K_0 := Ker(CL_{\infty} \to CL_0) \). Anyway \( f(x(A)) \) is well defined for all \( A \in CL_{\mathbb{F}}^+ \) possibly \( x(A) \) sitting in another geometrically connected component.

Choose a Hecke character \( \lambda \) of \( M \) such that

\( (f) \lambda \) has infinite type \( k + \kappa(1 - c) \) of conductor \( C \) which is a product of split primes over \( F \) \( (k, \kappa, \in \mathbb{Z}[\Sigma]) \).

Decompose \( C = \mathfrak{F} \mathfrak{F} C_\tau \) for integral ideals \( \mathfrak{F} \) and \( \mathfrak{F} \) such that \( \mathfrak{F} \mathfrak{F} \subset \mathfrak{F} C_\tau \), the Neben character of \( f \) as in [H07, (S1-3)] is given by \( (\lambda, \mathfrak{F}, \lambda, (\mathfrak{F} C_\tau)) \) for \( f \).

The existence of the satisfying \((f) \lambda \) implies \( k = k_\tau \) for any two embeddings \( \sigma, \tau \in \Sigma \); so, hereafter, often we identify \( k \) with the integer \( k_\tau \). It might appear strange to have the absolute value character \( (\lambda, \mathfrak{F} C_\tau) \) for \( f \). But when we extend a geometric modular form to an automorphic form on \( G(A) \), we multiply the factor \( |\det(g)|_A \) as the adellic Fourier expansion has the factor \( |\det(g)|_A \) in front of the Fourier expansion sum in [HMI, (2.3.15)]; so, the central action on a geometric modular form and the adellic one has this discrepancy. See [HMI, §2.3.2, §4.3.7] for more details on the relation of geometric Hilbert modular forms and adellic ones. Then by \((f) \lambda \) and \((G3)' \), \( f([|\lambda|]) = \lambda(A)^{-1}f(x(A)) \) for \( A \) prime to \( p \) depends only on the class of \( A \) in \( CL_n = CL_n/CL_F \). For the \( p \)-adic avatar \( \hat{\lambda}(x) = \lambda(x_R)x^{k+k(1-c)}_p \), we also have \( f([\lambda]) = \hat{\lambda}(A)^{-1}f(x(A)) \). This new definition is valid even for \( A \) with non-trivial common factor with \( p \). Then often we regard \( f \) as a function of \( C(\infty) \cup \bigcup_n C_n \) (embedded into \( Sh(l)_p \)) by \( 1_g \mathbb{F} \) by \( A \to x(A) \).

Writing \( X(\mathcal{A})/C = X(\mathcal{A}) \) for \( C \neq X(\mathcal{A})[l_n] \) for \( R_n \)-proper ideal \( A \) prime to \( l \), \( \mathfrak{A} \) is a proper \( R_{n+1} \)-ideal such that \( (R_n \mathfrak{A}) = A \). Since there are \( N(l) \) proper \( R_{n+1} \)-ideal such that \( (R_n \mathfrak{A}) = A \) if \( n > 0 \), we have
\[
(a/\lambda N(n))f([\lambda]) = (a/\lambda N(n))f(x(A)) = f(U(l))f(x(A)) = N(n)^{-1} \sum_{\mathfrak{A}: (R_n \mathfrak{A}) = A} f([\mathfrak{A}])
\]
\[
= N(n)^{-1} \lambda(\mathfrak{A}) \sum_{\mathfrak{A}: (R_n \mathfrak{A}) = A} f([\mathfrak{A}]) = N(n)^{-1} \lambda(\mathfrak{A}) \sum_{\mathfrak{A}: (R_n \mathfrak{A}) = A} f([\mathfrak{A}]) \quad \text{if } n > 0.
\]

Since \( f([\mathfrak{A}]) \) only depends on the class of \( CL_{n+1}^{-} \), this implies

(1) \( a \cdot f([\mathfrak{A}_n]) = \sum_{[\mathfrak{B}]_{n+1} \in CL_{n+1}^{-} \exists [\mathfrak{B}]_{n+1} 

2 \not\equiv [\mathfrak{A}]_n \} f([\mathfrak{B}])_{n+1} \),

(2) \( f(U(l))([\mathfrak{A}]) = \lambda N(n)^{-1} \sum_{[\mathfrak{B}]_{n+1} \in CL_{n+1}^{-} \exists [\mathfrak{B}]_{n+1} 

2 \not\equiv [\mathfrak{A}]_n \} f([\mathfrak{B}])_{n+1} \).
We can rewrite the above relation (1) as

\[(4.1) \quad a \cdot f([A]_n) = \sum_{[A]_{n+1} \in \text{Cl}_{n+1}} f([B]_{n+1}) \quad \text{if } n > 0.\]

More generally as seen in [H04, (3.8)], we get, for integers \(n > m \geq 1\),

\[(4.2) \quad \sum_{[A]_{n-1} \in \text{Cl}_{n-1}} f([B]_{n-1}) = a^{n-m} f([A]_m),\]

where \(\mathcal{A}\) runs over all elements in \(\text{Cl}_{n-1}\) which project down to \(\mathfrak{m}^{n-n}\mathfrak{A} \in \text{Cl}_{n-1}'\). The second relation (2) can be written

\[(C1) \quad f(U(n)([A]_n)) = \lambda(n)^{-1} f([B]_{n+1}) \quad \text{if } f([A]_{n+1}) = f([B]_{n+1}) \quad \text{for any } [B]_{n+1} \text{ with } [R_nB]_n = [A]_n,\]

\[(C2) \quad f(U(n')(m)([A]_n)) = \lambda N(n')^{-m} \sum_{[A]_{n+m} \in \text{Cl}_{n+m}} f([B]_{n+m}).\]

For each function \(\phi : \text{Cl}_{\infty} \to A\) factoring through \(\text{Cl}_{n-1}'\), assuming \(a \in A^\times\), we define

\[(4.3) \quad \int_{\text{Cl}_{\infty}} \phi d\varphi_f = a^{-n} \sum_{A \in \text{Cl}_{n-1}'} \phi(A^{-1}) f([A]).\]

Then for \(n > m \geq 1\), assuming \(a \in A^\times\), we find

\[a^{-n} \sum_{A \in \text{Cl}_{n-1}'} \phi(A^{-1}) f([A]) = a^{-m} \sum_{A \in \text{Cl}_{n-1}'} \phi(A^{-1}) a^{m-n} \sum_{A \in \text{Cl}_{n-1}'} f([A])\]

\[= a^{-m} \sum_{A \in \text{Cl}_{n-1}'} \phi(A^{-1}) a^{m-n} f(U(n)([A])) = \int_{\text{Cl}_{\infty}} \phi(x) d\varphi_f(x).\]

Thus \(\varphi_f\) gives an \(A\)-valued distribution on \(\text{Cl}_{\infty}'\) well defined independently of the choice of \(m\) for which \(\phi\) factors through \(\text{Cl}_{n-1}'\), because \(U(n') = U(n/m)\).

**Remark 4.1.** The assumption that \(a \in A^\times\) is not essential. If \(a = 0\), we just define for each finite \(n\) and a function \(\phi : \text{Cl}_{n-1}' \to A\)

\[(4.4) \quad \int_{\text{Cl}_{n-1}'} \phi d\varphi_f = \sum_{A \in \text{Cl}_{n-1}'} \phi(A^{-1}) f([A])\]

without dividing by \(a\). Though we lose the distribution relation (4.4) above, we have well defined value \(\int_{\text{Cl}_{n-1}'} \phi d\varphi_f\) dependent on \(n\). Changing \(\infty\) by \(n\), all the formulas independent of the distribution relation holds even when \(a = 0\). So hereafter we allow the case where \(a = 0\), and as a convention, we use \(n\) in place of \(\infty\). If \(a \in A^\times\), we can replace \(n\) by \(\infty\) since \(\int_{\text{Cl}_{n-1}'} = \int_{\text{Cl}_{n-1}'}\) as long as the integral factors through \(\text{Cl}_{n-1}'\). Thus if \(a = 0\), by (4.1), \(\int_{\text{Cl}_{n-1}'} \phi d\varphi_f \neq 0\) happens for a unique \(n > 0\). This \(n\) is a minimal \(n\) for which \(\phi\) factors through \(\text{Cl}_{n-1}'\). To write formulas uniform, we define \(a = 1\) if \(a = 0\) and \(a = a\) if \(a \neq 0\) in \(\mathbb{F}\).

Classical modular forms can be defined over the integer ring of a number field; so, we assume that \(f\) is defined over a discrete valuation ring \(\mathcal{V}\) (of residual characteristic \(p\)) in a number field \(E\). We assume that \(E\) is the smallest field containing \(M'\) for the reflex \((M', \Sigma')\) of \((M, \Sigma)\) and the values \(\lambda(\mathfrak{A})\) for all \(M\)-fractional ideals \(\mathfrak{A}\). We write \(\mathcal{P}|p\) for the prime ideal of the \(p\)-integral closure \(\mathcal{V}\) of \(\mathcal{V}\) in \(\overline{\mathbb{Q}}\) corresponding to \(i_p : \mathbb{Q} \to \overline{\mathbb{Q}}\). More generally, if \(f = \theta^g\) for a classical modular form \(g\) integral over \(\mathcal{V}\), the value \(f([A])\) is algebraic, abelian over \(M'\) and \(\mathcal{P}\)-integral over \(\mathcal{V}\) by a result of Shimura and Katz (see [EAI, §§2.1] and [K78]).

Let \(f = \theta^g\) for \(g \in G_{\mathfrak{b}}(\Gamma_0(1)|\mathcal{V})\). Suppose \(f(U(n)) = (a/\lambda(n)) N(n) f\) for \(a\) giving a unit of \(\mathcal{V}/\mathcal{P}\). For the moment, let \(\varphi\) be the measure associated to \(f\) with values in \(A = \mathcal{V}\). We have a well defined measure \(\varphi\) mod \(\mathcal{P}\). Let \(E_f\) be the field of rationality of \(x(A)\) for all \([A] \in \text{Cl}^{alg}\) over \(E[\mu_{\infty}]\). Then \(E_f/E\) is an abelian extension unramified outside \(\ell\), and we have the Frobenius element \(\sigma_{\mathfrak{b}} \in \text{Gal}(E_f/E)\) (that is, the image of \(\mathfrak{b}\) under the Artin reciprocity map) for each ideal \(\mathfrak{b}\) of \(E\) prime to \(\ell\). By Shimura’s reciprocity law ([ACM, 26.8], writing \((M', \Sigma')\) for the reflex CM type of \((M, \Sigma)\), we find for \(\sigma = \sigma_{\mathfrak{b}}, x(A)^* = x(N(\mathfrak{b}^{-1} \Sigma') A)\) for the norm \(N : E \to M'\). As for \(\eta_{ord}(A)\), we find \(\sigma \circ \eta_{ord}(A) = u_{ord}\) for \(u \in R_{\Sigma'}^{\times}\). Since \(A_p \cong R_p\), we have \(X(R)[p^{\infty}] \cong X(A)[p^{\infty}]\) as a Galois
module. Thus we conclude $u = \psi_E(b)$ for the Hecke character $\psi_E$ of $E^\times / E^\times$ giving rise to the zeta function of $X(R)$. From this, we see $f([A])^\sigma = f([N(b)^{-r}%27]}_A)$ for any ideal $b$, since $\psi_E(b) \in M$ generates the ideal $N(b)^{\Sigma} \subset M$ ([ACM] Sections 13 and 19) and hence $\psi_E(b)^{k+e(1-c)} = \lambda(N(b)^{\Sigma})$.

We then have

\begin{equation}
\sigma \cdot \left( \int_{Cl_n} \phi(x) d\varphi_f(x) \right) = \int_{Cl_n} \sigma \circ \phi(N(b)^{\Sigma}) x d\varphi_f(x),
\end{equation}

where $N(b)$ is the norm of $b$ over $M'$. Writing $\mathbb{F}_q$ for $q := p^\nu$ for the residue field of $E \cap \mathcal{P}$, any modular form defined over $\mathbb{F}_q$ is a reduction modulo $\mathcal{P}$ of a classical modular form defined over $\mathcal{V}$ of sufficiently high weight. Since $\xi^{\Sigma} \in M'$ for $\xi \in M$ as the reflex of $\Sigma$ is a sub-CM-type of $\Sigma$, we have $\mathbb{F}_q \subset \mathbb{F}_q'$. Thus the above identity is valid for $\sigma = \Phi^s (s \in \mathbb{Z})$ for the Frobenius element $\Phi \in \text{Gal}(\mathbb{F}/\mathbb{F}_q)$. In this case, $N(b)$ is a power of a prime ideal $|p|p$ in $M'$.

We now assume that $A = F = \tilde{V}/\mathcal{P}$ and regard the measure $\varphi_f$ as having values in $F$. Then (4.5) shows that if $\phi$ is a character of $Cl_n^-$ with arbitrary $n > 0$, for $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}_q)$,

\begin{equation}
\int_{Cl_n} \chi(x) d\varphi_f(x) = 0 \iff \int_{Cl_n} \sigma \circ \chi(x) d\varphi_f(x) = 0.
\end{equation}

Let $\mathbb{F}_q[\mu_\ell]$ be the finite subfield of $F$ generated by all $\ell$-th roots of unity over $\mathbb{F}_q$; so, it is the field of rationality of $\lambda$, $f$, and $\mu_\ell$ over the residue field of $M' \cap \mathcal{P}$.

### 4.2. Measure projected to $\Gamma$ and $\Gamma_n$. Recall $\Gamma_n$ which is the image of $\Gamma$ in $Cl_n^-$. Since each fractional $R$-ideal $\mathfrak{a}$ prime to $l$ defines a class $[\mathfrak{a}]$ in $Cl_{\infty}$, we can embed the ideal group of fractional ideals prime to $l$ into $Cl_{\infty}$. We write $C^{alg}$ for its image. Thus the projection of $[\mathfrak{a}]$ in $Cl_n^-$ is $[\mathfrak{a}]_n$ as specified for the integral ideal $\mathfrak{a}$ above. Then $\Delta^{alg} = \Delta^- \cap C^{alg}$ is generated by prime ideals of $M$ ramified over $F$. We choose a complete representative set for $\Delta^{alg}$ made of product of prime ideals in $M$ ramified over $F$ prime to $p$. We may choose this set as $\{ \mathfrak{a}^{-1} \in \mathfrak{a} \in \mathfrak{R} \}$, where $\mathfrak{R}$ is made of square-free product of primes non-principal outside $l$ in $F$ ramifying in $M/F$, and $\mathfrak{R}$ is a unique ideal in $M$ with $\mathfrak{R}^2 = \mathfrak{r}$. Note that $\{ \mathfrak{r} \in \mathfrak{R} \}$ is a complete representative set for $\mathfrak{r}$-torsion elements in the quotient $Cl_0^-$. In [H04] and [H07], we used $Cl_n$ in place of $Cl_n^-$; so, we had to choose a complete representative set $\mathcal{S}$ of the image $\overline{Cl}_F$ of $Cl_F$ in $Cl_n$, which is not necessary. Indeed, since $f([A]) = f([\mathfrak{a}A])$ for an $O$-ideal $\mathfrak{a}$ by our choice of $\lambda$, we have $\bar{h}f([A]) = \sum_{\mathfrak{a} \in \mathcal{S}} f([\mathfrak{a}A])$ for $\bar{h} := [\overline{Cl}_F]$, and if we make our choice of $\lambda$, this implies the triviality of the measure if $p|\bar{h}$. To avoid this, we do not sum over $\mathcal{S}$.

We fix a character $\psi : \Delta^- \to \mathbb{F}^\times$, and define

\begin{equation}
f_\psi = \sum_{\mathfrak{r} \in \mathfrak{R}} \lambda \psi^{-1}(\mathfrak{r}) f[\mathfrak{r}],
\end{equation}

In [H04] and [H07], $f_\psi$ is defined by

\begin{equation}
\sum_{\mathfrak{r} \in \mathfrak{R}} \lambda \psi^{-1}(\mathfrak{r}) \left( \sum_{\mathfrak{a} \in \mathcal{S}} \psi \lambda^{-1}(\mathfrak{a}) f[\mathfrak{a}] \right) \left[ [\mathfrak{r}] \right],
\end{equation}

and we do not follow this definition.

Choose a complete representative set $\mathcal{Q}$ for $Cl_{\infty}^- / \Gamma \Delta^{alg}$ made of primes $\mathfrak{Q}$ of $M$ split over $F$ outside $pl$ except for the trivial element $1$ representing $1 \in Cl_{\infty}^- / \Gamma \Delta^{alg}$. Thus $q := N_M/F(\mathfrak{Q})$ is a prime ideal of $O$ if $\mathfrak{Q} \neq R$ (and $q = O$ if $\mathfrak{Q} = R$). We choose $\eta_{\mathfrak{Q}}^{(p)}$ out of the base $(w_1, w_2)$ of $\tilde{R}_R$ so that at $q = \mathfrak{Q} \cap F$, $w_{1, q} = (1, 0) \in R_\mathfrak{Q} \times R_\mathfrak{Q} = R_q$ and $w_{2, q} = (0, 1) \in R_\mathfrak{Q} \times R_\mathfrak{Q} = R_q$. Since all operators $\{ \mathfrak{a} \}$, $[\mathfrak{r}]$ and $[\mathfrak{r}]$ commute with $U(l)$, $f_\psi[\mathfrak{r}]$ is still an eigenform of $U(l)$ with the same eigenvalue as $f$. Thus in particular, we have a measure $\varphi_{f_\psi[q]}$. We then define another measure $\varphi_{f_\psi}$ on $\Gamma$ by

\begin{equation}
\int_{\Gamma_N} \phi d\varphi_{f_\psi} = \sum_{\mathfrak{Q} \in \mathcal{Q}} \lambda \psi^{-1}(\mathfrak{Q}) \int_{\Gamma_N} \phi \Delta \varphi_{f_\psi[q]},
\end{equation}

where $\phi \Delta \varphi_{f_\psi} = \phi(y) \varphi_{f_\psi}[\mathfrak{r}]$ for the projection $[\mathfrak{Q}]_\mathfrak{r}$ in $\Gamma$ of the class $[\mathfrak{Q}] \in Cl_{\infty}^-$. As already remarked, $\phi \mapsto \phi \Delta \varphi$ is a transcendental action unless $\mathfrak{Q} = R$. If $\mathfrak{Q} = R$, $\phi \Delta \varphi = \phi$ and $f \varphi_\psi[q] = f$. 


Lemma 4.2. If $\chi : \Gamma_n \to F^\times$ and $\psi : \Delta^- \to F^\times$ are characters, we have

$$\int_{\Gamma_n} \chi d\varphi_j^\psi = \int_{C\Gamma_n} \psi d\varphi_j.$$

Here recall the image $\Gamma_n$ of $\Gamma$ in $C\Gamma_n$.

Proof. For a proper $R_n$–ideal $\mathcal{A}$, by the above definition of these operators,

$$f([\tau],[q]|(\mathcal{A})) = \lambda(\mathcal{A})^{-1} f(x(\Omega^{-1} \ell^{-1} \mathcal{A})).$$

Since $\chi = \psi$ on $\Delta^-$, we have

$$\int_{\Gamma_n} \chi d\varphi_j^\psi = \sum_{\Omega \in \mathcal{Q}} \sum_{\tau \in \mathcal{R}} (\lambda^{-1} \chi^{-1} \psi^{-1})(\Omega \mathcal{R}) \sum_{\mathcal{A} \in \Gamma_n} \chi(\mathcal{A}) f([\tau],[q]|(\mathcal{A}))$$

$$= \sum_{\mathcal{A}, \Omega, \tau} \chi(\Omega^{-1} \mathcal{R}^{-1} \mathcal{A}) f((\Omega^{-1} \mathcal{R}^{-1} \mathcal{A})) = \int_{C\Gamma_n} \psi d\varphi_j,$$

because $C\Gamma_n = \bigsqcup_{\mathcal{Q}} \mathcal{Q}^{-1} \Gamma_n$. \hfill $\square$

We write $F_q := F_q[v] \subset F$ for the field of rationality of $\psi$ over $F_q$. Then $\sigma \in \text{Gal}(F/F_q)$ preserves $d\varphi_j^\psi$. Then (4.6) shows that if $\chi$ is a character of $\Gamma$, for $\sigma \in \text{Gal}(F/F_q)$,

$$\int_{\Gamma_n} \chi(x) d\varphi_j^\psi(x) = 0 \iff \int_{\Gamma_n} \sigma \circ \chi(x) d\varphi_j^\psi(x) = 0.$$

4.3. Trace relation. For any finite extensions $\kappa/\kappa'/F_q[\mu_\ell]$, we consider the trace map: $\text{Tr}_{\kappa/\kappa'}(\xi) = \sum_{\sigma \in \text{Gal}(\kappa/\kappa')} \sigma(\xi)$ for $\xi \in \kappa$. Recall the image $\Gamma_n$ of $\Gamma$ in $C\Gamma_n$. Define

$$f^\psi(\mathcal{A}) = \sum_{\Omega \in \mathcal{Q}} \psi(\Omega^{-1} f_{\psi}(\mathcal{A}\Omega^{-1}|[\Omega]_F)$$

for the projection $[\mathcal{Q}]_F \in \Gamma \in [\mathcal{Q}]$.

Let $\chi : \Gamma_n \to F^\times$ be a character. Suppose that $\text{Im}(\chi) \cap F_q[\mu_\ell]^\times$ has order $\ell^r$ and that $\chi$ has order $\ell^s$. Note that $1 \leq r \in \mathbb{Z}$. Fix $j \geq r$, and write

$$\Phi = \Phi_n := \Gamma_n \cap \chi^{-1}(F_q[\mu_\ell]^\times)$$

and $[\mathcal{A}_y] = [\mathcal{A}_y]_n$ for the image of $y \in \Gamma$ in $\Gamma_n$. By (3.3), we have an isomorphism of $O$-modules:

$$O/\mathcal{U} \cong \Phi_n \text{ by } u \mapsto \varphi(u/\mathcal{U})x(R_n+j).$$

Note that $[R_n-1\mathcal{A}_y]_n = [\mathcal{A}_y]_n-1$ for all $n$. Recall $a \in F^\times$ defined in Remark 4.1. If $\nu \geq j$, for

$$d = [F_q[\chi] : F_q[\mu_\ell]] = [\text{Im}(\chi) : \text{Im}(\chi) \cap F_q[\mu_\ell]^\times] = \ell^{r-j},$$

$$\int_{\Gamma_n} \text{Tr}_{F_q[\chi]/F_q[\mu_\ell]} \circ \chi(y^{-1} x) d\varphi_j^\psi(x)$$

$$= \frac{d}{a^\nu} \sum_{\mathcal{A} \in \mathcal{A}_y^{-1} \Phi_n} \chi(y^{-1} \mathcal{A}) f^\psi(\mathcal{A}) = \frac{d}{a^\nu} \sum_{\mathcal{A} \in \Phi_n} \chi(\mathcal{A}) f^\psi(\mathcal{A}[\mathcal{A}_y]),$$

because for an $\ell$–power root of unity and a finite extension $\kappa/F_q[\mu_\ell]$, $\zeta \in \mu_\ell^r - \mu_\ell^j$,

$$\text{Tr}_{\kappa[\mu_\ell]/\kappa[\mu_\ell^j]}(\xi) = \begin{cases} \ell^{r-j} \xi^s & \text{if } \xi \in \kappa[\mu_\ell] \text{ and } \kappa[\mu_\ell] \cap \mu_\ell^s = \mu_\ell^j \\ 0 & \text{otherwise.} \end{cases}$$

Thus by (4.9), we have

$$\sum_{[\mathcal{A}] \in \Phi_n} \chi(\mathcal{A}) f^\psi(\mathcal{A}[\mathcal{A}_y]) = 0 \text{ if } \int_{C\Gamma_n} \psi d\varphi_j = 0.$$

Let $\mathcal{F}(\Phi_n[\mathcal{A}_y], F)$ be the space of functions $\phi : \Phi_n[\mathcal{A}_y] \to F$. Consider the linear form $\ell_\chi : \mathcal{F}(\Phi_n[\mathcal{A}_y], F) \to F$ given by $\ell_\chi(\phi) = \sum_{[\mathcal{A}] \in \Phi_n} \chi([\mathcal{A}] \phi([\mathcal{A}][\mathcal{A}_y]))$. Since the orthogonal complement of the space spanned by $\{ \ell_\chi \}_{\sigma \in \text{Gal}(\mathbb{Q}[\mu_\ell]/\mathbb{Q})}$ in $\mathcal{F}(\Phi_n, F)$ under the pairing

$$\langle \phi, \phi' \rangle = \sum_{[\mathcal{A}] \in \Phi_n} \phi([\mathcal{A}][\mathcal{A}_y]) \phi'([\mathcal{A}])$$

...
is spanned by characters of order $\leq \ell^{-1}$. If $\varepsilon = 1$, the orthogonal complement is made of constant functions on $\Phi_n$. Thus assuming that the integral (4.13) vanishes with $\Phi_n \equiv \mu$ and that $\text{Gal}(\mathbb{F}_q^{|\mu|}/\mathbb{F}_q) = \text{Gal}(\mathbb{Q}^{|\mu|}/\mathbb{Q})$, $[A] \mapsto f^Q_\psi([A][A_y])$ is a constant function of $[A]$ whose value is $f([A_y])$, for $\alpha = \text{diag}(1, w_1^\omega)$, we have

$$\ell f^Q_\psi([A_y]_n) = \sum_{[B] \in \Phi_{n,u}} f^Q_\psi([B]_n) - \sum_{u \equiv 0 \mod \ell} f^Q_\psi((g(u/w_1^\omega)\alpha_1)([A_y]_n-1)) = a f^Q_\psi([A_y]_n-1).$$

This is easy to see if we choose $A_y$ prime to $l$ (i.e., $A_{y,1} = R_{n,1}$). Hereafter exclusively the latter $r$ for the integer defined by

$$\mu_r \cap \mathbb{F}_q^{|\mu|} = \mu_r.$$

5. PROOF OF THEOREM 0.1

Write $\hat{G}_m/\mathbb{Z}_\ell$ for the formal completion over $\mathbb{Z}_\ell$ at the origin of $G_m(\mathbb{F}_\ell)$. Let $\text{Hom}(\Gamma, \mu_{\ell\infty})$ embed into $G_m^d/\mathbb{Z}_\ell$ and $\hat{G}_m^d/\mathbb{Z}_\ell$ by choosing a basis $(\gamma_1, \ldots, \gamma_d)$ of $\Gamma$ over $\mathbb{Z}_\ell$ and sending $\chi \in \text{Hom}(\Gamma, \mu_{\ell\infty})$ to $(\chi(\gamma_1), \ldots, \chi(\gamma_d))$. A subset $S$ of $\text{Hom}(\Gamma, \mu_{\ell\infty})$ therefore has its Zariski closure $\overline{S}$ (resp. $\hat{S}$) in $G_m^d(\mathbb{Q}_\ell)$ (resp. $\hat{G}_m^d(\mathbb{Q}_\ell)$). Since $\text{Aut}(\hat{G}_m^d) = \text{GL}_d(\mathbb{Z}_\ell)$, the isomorphism class of $\hat{S}$ is independent of the choice of the basis. As we will see later for our choice of $S$ that $\dim \hat{S} = \dim \overline{S}$, and hence $\overline{S}$ being a proper Zariski closed set is independent of the choice of basis.

Fix a character $\psi: \Delta \to \mathbb{F}^\times$. Let

$$X = X_\psi := \{ \chi \in \text{Hom}(\Gamma, \mu_{\ell\infty}) \mid \int_{[1]} \chi \psi d\varphi \neq 0 \text{ for some } n \}.$$

If $a = 0$, as seen in Remark 4.1, $\int_{[1]} \chi \psi d\varphi \neq 0$ for one value $n$; in other words, for $n$ given by $\text{cond}(\chi) = l^n$, the integral is not defined over $C_{l,n}$ for $n' < n$ and the integral vanishes for $n' > n$. On the contrary, if $a \neq 0$, the vanishing (and non-vanishing) of the integral is independent of $n$ as long as it is well defined.

Assume the following condition:

$$\text{(5.1) The Zariski closure } \overline{X}_\psi \text{ in } G_m^d(\mathbb{Q}_\ell) \text{ of the set } X_\psi \text{ has dimension } < d,$$

and we are going to deduce absurdity.

5.1. Proof. We prepare a lemma. Let $\mathbb{C}_\ell$ be the $\ell$-adic completion of $\mathbb{Q}_\ell$. Let $W$ be a discrete valuation ring finite over the Witt vector ring $W(\mathbb{Q}_\ell)$ inside $\mathbb{C}_\ell$ for an algebraic closure $\mathbb{F}_\ell$ of $W(\mathbb{Q}_\ell)$, and write $\mathcal{K}$ for its quotient field. For a formal subscheme $X$ of $\hat{G}_m/W$, we write $X(\mathbb{C}_\ell) := X(\mathbb{Q}_\ell)$ for the integral closure $\mathbb{W}$ of $W$ in $\mathbb{C}_\ell$. The map $t \mapsto t^{p^n}$ is an automorphism of $\mu_{\ell^n}$ for $\mathbb{F}_\ell = (\mathbb{Z}/p^n\mathbb{Z})^\times$. Take a sequence of $z_n \in \mathbb{Z}$ lifting $z_n$ and assuming $\{z_n\}$ converges to $z \in \mathbb{Z}_\ell^\times$. Then $\zeta \mapsto \zeta^{p^n}$ gives rise to an automorphism $z \in \mathbb{Z}_\ell^\times$ of $\mu_{\ell^n}$. In this way, $\ell$-adic unit $z$ acts on $\mu_{\ell^n}$. If $z \in \mathbb{Q} \cap \mathbb{Z}_\ell^\times$, it is of the form $t^{p^n}$ for some $n$. We can find such a large $p^n$-power $P = p^n$ with an $r$-multiple $j = r^n$ and a positive integer $N$ such that there exists a sequence of subsets $\{Y_n\}_{n=N}^\infty$ outside $\mathcal{X}(\mathbb{Q}_\ell)$ such that

$$Y_n = \left\{ \left( \frac{p^k e_1}{p^n+a_1} \cdots \frac{p^k e_d}{p^n+a_d} \right) \mod \mathbb{Z}_\ell^d \bigg| (k_i) \in \mathbb{Z}_d^d \right\}$$

if we choose a base $\{e_i\}$ of $\mathbb{Z}_\ell^d$ suitably.

Proof. We choose the $p^n$-power $P$ so that $P \equiv 1 \mod l$. Let $\Gamma_P = P^Z_{2l} \subset \mathbb{Z}_\ell^\times$, which is an open subgroup of $1 + \ell \mathbb{Z}_\ell$. Let $\mathcal{X}(\mathbb{Q}_\ell) := \mathcal{X}(\mathbb{Q}_\ell) \cap \mu_{\ell^n}(\mathbb{Q}_\ell)$. Since $\mathcal{X}(\mathbb{Q}_\ell) \subset \mathcal{X}$, we have $\mathcal{X}(\mathbb{Q}_\ell) \in \mathcal{X}(\mathbb{Q}_\ell)$; so, the Zariski closure of $\mathcal{X}(\mathbb{Q}_\ell)$ is stable under $t \mapsto t^{p^n}$. We may replace $\mathcal{X}$ by the Zariski closure of $\mathcal{X}(\mathbb{Q}_\ell)$ as the lemma only concerns about $\mathcal{X}(\mathbb{Q}_\ell) \cap \mu_{\ell^n}(\mathbb{Q}_\ell)$, and after the replacement, the stability
Consider the projection $Q \times X$ for $\alpha$ as in $G$. Write $W$ be the $t$-adic integer ring of $K$ with maximal ideal $m_W$. Let $\overline{X}/W$ be the schematic closure of $\overline{X}_K$ in $G^d_{m/W}$. Writing $G^d_{m/W} = \text{Spec}(R)$ for $R = W[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ and $X = \text{Spec}(R \otimes W)$ with an ideal $\alpha$ of $R \otimes W$, $\overline{X}/W = \text{Spec}(R/\alpha)$ for $\alpha = a \cap R$. Thus $\overline{X}/W$ is flat over $W$. Let $m = m_W + (t_1 - 1, \ldots, t_d - 1) \subset W[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ and $\widehat{\mathfrak{m}}$ be the $m$-adic closure of $\mathfrak{a}$. We write $\widehat{\mathfrak{m}}$ for $m$ and $\mathfrak{a}$ for $\mathfrak{m}$. Define $\mathfrak{a} = \mathfrak{a} \cap R$. Since $A = \text{Spec}(R)$ for $A := W[[T_1, \ldots, T_d]]/\mathfrak{a}$, $\widehat{\mathfrak{a}}$ is flat over $\mathfrak{a}$. Since $\dim(A) = \dim \text{gr}(A) = \dim \text{gr}(R/\mathfrak{a}) = \dim(R/\mathfrak{a})$ (e.g., [CRT, Theorem 15.7]), we find $\dim \widehat{\mathfrak{a}} = \dim R < d$.

Since $\widehat{X} = \bigcup_{\mathfrak{m} \in X} \mathfrak{m} \mathfrak{a} \subset \widehat{\mathfrak{m}}$, we have $\widehat{X} \subset \widehat{\mathfrak{m}}$. Thus $\widehat{X}$ is stable under an open subgroup $U$ of $\mathfrak{m}$. Here an element $s \in 1 + \mathfrak{m}$ acts on $\widehat{G}^d_{m}$ by $t \mapsto t^s$. Since $\widehat{X}$ is noetherian, it has finitely many geometrically irreducible components, and $U$ permutes them. Thus replacing $U$ by an open subgroup, we may assume that $U$ fixes each geometrically irreducible component. By extending scalars, we may assume that each geometrically irreducible component is defined over $W$. Then by Lemma 5.5 below, $\widehat{X} = \bigcup_{\ell \in Z, s \in T_{\ell, s}, \text{ where } T_{\ell, s} \text{ is a formal subtorus of } \widehat{G}^d_{m/W}}$ and $Z$ is a finite subset of $\mu^d_{\infty}(W)$. We first assume that $Z = \{1\}$. By this assumption, $\widehat{X}$ is a union of subtori $\{T_{\ell, s}\}_{\ell, s}$ with $|J| < \infty$ and $\dim T_{\ell, s} < d$. Thus we have its $t$-adic Tate module $TT_{\ell, s} = \lim_n T_{\ell, s}^n$. Put $TT_{\widehat{X}} := \bigcup_{\ell} TT_{\ell, s}$. We identify $\mu^d_{\infty} = t^{-n}T/T = T/t^nT$; so, $\mu^d_{\infty} = \mathbb{Q}_l t^n$. In particular, $\overline{X}(\mathbb{Q}_l^\infty) = \overline{X} \cap \mu^d_{\infty}$ is the image of $\bigcup_{\ell} \mathbb{Q}_l TT_{\ell, s}$ in $\mu^d_{\infty} = t^{-n}T/T$. Then we can choose a base $\{e_1, \ldots, e_n\}$ of $T$ over $\mathbb{Z}_l$, such that $\mathbb{Z}_l e \cap T \mathbb{X} = \{0\}$ for $e = e_1 + e_2 + \cdots + e_d$. Then the $t$-adic distance from the $\mathbb{Q}_l$-span $\mathbb{Q}_l T \mathbb{X} = \bigcup_{\ell} \mathbb{Q}_l TT_{\ell, s}$ to the point $x$ is larger than or equal to $c t^n$ for a positive constant $c$ independent of $n$. Thus we can find sufficiently large power $P$ of $t$ (adically very close to 1) so that $U_n = \Gamma P \mathbb{Q}_l^{\infty} + \cdots + \Gamma P \mathfrak{m}^{\infty}$ for $\Gamma = P^{\infty}$ gives rise to an open neighborhood of $x$ disjoint from $\mathbb{Q}_l T \mathbb{X}$. Then the image $Y_n$ of $U_n$ in $\mu^d_{\infty}$ is disjoint from $\widehat{X}(\mathbb{Q}_l^\infty)$ and hence from $\overline{X}$ for all $n \geq 1$.

When $Z \neq \{1\}$, we consider the subgroup $\langle Z \rangle$ of $\mu^d_{\infty}$ generated by $Z$. The group $\langle Z \rangle$ is finite. Consider the projection $\pi : \widehat{G}^d_{m} \rightarrow \widehat{G}^d_{m}/\langle Z \rangle$. The image of $\pi(\mathbb{X})$ under $\pi$ is a union of subtori and hence stable under scalar multiplication by elements in $\mathbb{Z}_l$. Using the result proven under the condition $Z = \{1\}$ applied to $\pi$, we write $Y_n$ for the sets constructed for $\pi(\mu^d_{\infty}) = \mu^d_{\infty}/\langle Z \rangle$. Then we find that for $n > N$ any $\Gamma P^{\infty}$-orbit of an element in the pull-back image $Y_n$ gives a desired set $T_n \subset \mu^d_{\infty}(\mathbb{Q}_l^\infty)$. This finishes the proof.

Choose a $\mathbb{Z}_l$-basis $\gamma_1, \ldots, \gamma_d$ for $d = \text{rank}_{\mathbb{Z}_l} \Gamma$. Then identify $\text{Hom}(\Gamma, \mu^d_{\infty})$ with $\mu^d_{\infty}$ by $\chi \mapsto (\chi(\gamma_1), \ldots, \chi(\gamma_d))$. Here is a more accurate version of Theorem 0.1.

**Theorem 5.2.** Suppose that for a given class $v \in (O/W)^\times$ with a sufficiently large $j \geq r > 0$ for $r$ as in Theorem 0.1 and a cusp $(a, b)$, there exists $\epsilon \in ab \cap -v$ such that $a(\epsilon, f_\epsilon) \neq 0$ in $F$. Then the set of characters $\chi \in \text{Hom}((\Gamma, \mu^d_{\infty})$ with $\chi(\epsilon) = v$ and non-vanishing $\int_{\mathbb{Q}_l^\infty} \psi d\phi \neq 0$ for $n > 0$ given by cond$(\chi) = P^n$ is Zariski dense in $G^d_{m/W}$. If rank$_{\mathbb{Z}_l} \Gamma = 1$, we can take $j = r$ for $r$ as in (4.16).

Though the minimal possible $r$ depends on $l$, the assumption in the theorem is in appearance weaker than

(h) There exists a strict ideal class $c$ of $F$ such that $c(\mathbb{Q}^{-1} R^{-1} s)$ is in $c$ for some $(\mathbb{Q}, R, s) \in \mathbb{Q} \times S \times R$ and for any given integer $j \geq r > 0$, the $N(\mathbb{Q})^j$ modular forms $f_{v, c}(\epsilon(\frac{1}{\mathbb{Q}} v))$ for $u \in \mathbb{V}^{-1}/O$ are linearly independent,
which is assumed in [H04, Theorem 3.3].

Proof. Let

$$\mathcal{X} = \{ \chi \in \text{Hom}(\Gamma, \mu_{\ell^\infty}) \rightarrow \mathbb{G}^d_{m/\mathcal{O}} \mid \int_{\Gamma} \chi d\nu^\phi \neq 0 \text{ and } v(\chi) = v \}$$

and $\hat{\mathcal{X}}$ (resp. $\hat{\mathcal{F}}$) be the formal Zariski (resp. Zariski) closure of $\mathcal{X}$ in $\mathbb{G}^d_{m/\mathcal{W}}$ (resp. $\mathbb{G}^d_{m}$). Note that $\hat{\mathcal{F}}^\varphi \subset \hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^\varphi \subset \hat{\mathcal{X}}$. Suppose $\hat{\mathcal{F}}$ is a proper Zariski closed subset of $\mathbb{G}^d_{m}$ and get a contradiction.

First suppose $d = 1$. Then $\hat{\mathcal{X}}$ is a finite set. Take $j := r$ in (4.11) and (4.12). So there exists $B > 0$ such that if the conductor of a character $\chi$ is $l^0$ with $n > B$, by (4.15), identifying $\Phi_n = O_l/(O_l)^r$ as in (4.12) by $x(A_u) = g(u/\ell^r)x(R_{n+r}) \mapsto O_l/(O_l)^r$,

$$\int_{\Gamma} \chi d\nu^\phi = \sum_{[A] \in \Phi_n} \chi(A) f^\phi_\nu([g(u/\omega^r)])([A]A_u) = \sum_u \sum_{[A] \mod \nu^r} \xi^\mu_u f^\nu_\varphi([g(u/\ell^r)])([A]A_u) = 0.$$ 

Here $[A]_u$ is any element in $\Gamma_n$. Let $\Xi_n = \{ x([A]) \mid [A] \in \Gamma_n \}$ and $\Xi := \bigcup_{n \geq B} \Xi_n \cap V$. Then $\Xi$ is associated to an infinite arithmetic progression of difference $\nu$ (for minimal exponent $\tau$ is generated by $N_{M_{n/F}}(R)$).

Since $\chi|_{\Phi} : O\ell^r \rightarrow \mathbb{F}^\times$ is an arbitrary character of order $\ell^r$, we may fix a character $\chi_v(u) = \xi^\mu_u$ for $v \in (O/\ell^r)^\times$ independent of $n > B$ as an additive character of $O/\ell^r$. Writing

$$(5.2) g_v := \sum_{u \in O/\ell^r} \chi_v(u)f_\nu([g(u/\omega^r)])$$

we find $\sum_{[A] \in \Omega} \psi^{-1}(\Omega) g_v([q]([A][\Omega]_r]) = 0$ for all $[A] \in \Xi$. By Corollary 2.11, $\Xi$ is Zariski dense in $\mathbb{V} = V\Omega$, and hence we conclude $g_v([q] = 0$. Since the $q$-expansion of a modular form $h([q]$ at $(a, b)$ is given by the $q$-expansion of $h$ at $(qa, b)$, so, by $q$-expansion principle, $g_v([q] = 0 \Leftrightarrow g_v = 0$ (e.g., [H10, (5.10)])).

Note $a(\xi, g_v) = N(l)^r a(\xi, f_\nu)$ as long as $\Xi \equiv -v \mod \ell^r$. Since $v$ is arbitrary, we can choose $v$ so that $\xi$ as in the theorem satisfies $\Xi \equiv -v \mod \ell^r$; so, $g_v \neq 0$, a contradiction.

We now assume $d \geq 2$. Take a base $\gamma_1, \ldots, \gamma_d$ of $\Gamma$ of $\mathbb{Z}_\ell$, which gives rise to an identification $\text{Hom}(\Gamma, \mu_{\ell^\infty}) = \mu_{\ell^\infty}$ by $\chi \mapsto (\chi(\gamma_1), \ldots, \chi(\gamma_d))$. Regard $\mu_{\ell^\infty} \subset \mathbb{G}^d_{m/\mathcal{O}}$ and apply Lemma 5.1 to $\mathcal{F} \subset \mathbb{G}^d_{m}$. Thus we have the base $e_1, \ldots, e_d$ as in Lemma 5.1 of the Tate module $T\Omega_\text{mod}(\Gamma, \mu_{\ell^\infty}) = \lim_{\leftarrow} \Omega_\text{mod}(\Gamma, \mu_{\ell^\infty})$. We rewrite the corresponding basis of $\Gamma$ as $\gamma_1, \ldots, \gamma_d$; so, the $\mathbb{Z}_\ell$-module $\gamma_1^{\mathbb{Z}_\ell}$ is sent isomorphically onto $\mathbb{Z}_\ell e_i$ for each $i$. Recall $Cl^{\infty} = \lim_{\leftarrow} Cl_n$ and $Cl^{\infty} = \Gamma \times \Delta$ for a finite group $\Delta$. Pick the smallest integer $0 < a \in \mathbb{Z}$ so that $\text{Ker}(Cl^{\infty} \rightarrow Cl_a) \subset \Gamma$. Choose $a_1, \ldots, a_d$ so that $\prod_{i \leq a_i} = \text{Ker}(Cl^{\infty} \rightarrow Cl_{a+n})$ for $n \geq 0$. Let $p^r > j$ for $j \in \mathbb{Z}$ as in Lemma 5.1.

Suppose $\tau$ is principal generated by $\varphi = \varphi^\nu$ for $\varphi \in R$. Then $\Upsilon = \bigcup_{\ell \geq N} \Upsilon_\ell$ is disjoint from $\overline{\mathcal{X}}$ by Lemma 5.1 for some positive integer $N$. Put $\Xi_{a+i} = \{ x([A]) \mid A \in \text{Ker}(Cl_{a+i} \rightarrow Cl_a) \}$, replacing $m$ by a positive multiple so that $\ell + N > a$. Define an infinite arithmetic progression $n := \{ a + im \mid i = 1, 2, \ldots \}$. Then $T_{a,m}$ acts transitively on $\Xi$, and by Theorem 2.6 and the proof of [H04, Proposition 2.8]. $\Xi$ embedded in $V\Omega$ by $A \mapsto x([A]) := (x([A][\Omega]_r])_{\Omega \in \Omega}$ is Zariski dense.

For each $\chi \in \Upsilon$, 

$$\sum_\Omega \psi^{-1}(\Omega) \sum_{A \in \chi^{-1}(\nu^\phi)} \chi(A) f_\nu([A][\Omega]_r) = 0$$ 

holds by (4.15) (see also [H04, page 770]) for $A$ with $x([A]) \in \Xi$.

Identify again $\Phi_n = O_l/(O_l)^r$. Let $g_v := \sum_{u \in O/\ell^r} \chi_v(u)f_\nu([g(u/\omega^r)])$ for $\chi_v(u) = \xi^\mu_u$ for $\Upsilon := \text{Tr}_{O_l/\mathcal{L}_u}$. Then

$$(5.3) \sum_\Omega \psi^{-1}(\Omega) \sum_{A \in \Phi_n} g_v([q]([A][\Omega]_r]) = 0 \text{ for } x([A]) \in \Xi.$$ 

By Zariski density of $\Xi$ in $V\nu$, we conclude $g_v([q] = 0$. Since $[q] \in \text{Aut}(S_h^{(n)})$, we conclude $g_v = 0$.

For a chosen class $v \in (O/\ell^r)$, we find $\xi$ such that $\xi \equiv -v$ and $a(\xi, f_\nu) \neq 0 \Leftrightarrow a(\xi, g_v) \neq 0$, and from this we conclude contradiction against $a(\xi, f_\nu) \neq 0$.

Here is an obvious corollary of the proof of Theorem 5.2:
Corollary 5.3. Let the notation be as in Theorem 5.2. Suppose \( d = 1 \) and \( a(\xi, f) \neq 0 \) for some \( \xi \in -v \) for a given \( v \in (\mathbb{Q}/\mathbb{Z})^\times \). For a character \( \chi \in \text{Hom}(\Gamma, \mu_n(\mathbb{F})) \), define \( n(\chi) \) by Ker\( (\chi) = \text{Ker}(\Gamma \to Cl_n(\chi)) \). Define a subset of \( \mathbb{Z} \) by

\[
\mathfrak{n}_n := \{ n(\chi) | v(\chi) = v \text{ and } \int_{Cl_n(\chi)} \chi \psi d\varphi_T = 0 \}.
\]

Then \( \mathfrak{n}_n \) cannot contain any infinite arithmetic progression.

We can interpret heuristically the above corollary into a natural density 0 result. Let

\[
\mathfrak{n} = \{ 0 < n_0 < n_1 < n_2 < \cdots < n_i < \cdots \}
\]

be an infinite sequence of integers. We define the density of \( \mathfrak{n} \) by

\[
D(\mathfrak{n}) := \lim_{|x| \to \infty} \frac{|\{ j | n_j \leq |x| \}|}{|x|}.
\]

Consider the function \( \phi = \phi_n : j \mapsto n_j \) defined on the set of natural numbers \( N := \{ n \in \mathbb{Z} | n > 0 \} \).

We study \( D(\mathfrak{n}) \) in terms of \( \phi \). Suppose

(E) \( \mathfrak{n} \) does not contain any arithmetic progression.

Let \( \Delta \phi(x) = \phi(x + 1) - \phi(x) \). Suppose that \( \Delta \phi(x) \) is bounded by an integer \( B > 0 \). Then the map \( \mathbb{Z} \ni x \mapsto \phi(x) \mod B \) has a fiber \( F \) over \( a \in [0, B) \cap \mathbb{Z} \) with infinitely many elements by the pigeon hole principle. Arrange the set \( F' := \{ m | a + mB \in F \} \) in increasing order, if \( F' \) contains an additive subgroup, then \( \mathfrak{n} \) contains an arithmetic progression, a contradiction to (E). Thus \( \Delta \phi(x) \) is unbounded. Therefore \( \lim_{x \to \infty} \phi(x)/Bx = \infty \) for all \( B > 0 \), and we have \( |\Delta \phi(x)| \leq B \) for \( x \gg 0 \). This implies \( D(\mathfrak{n}) = 0 \) if \( \mathfrak{n} \) does not contains arithmetic progression. Thus it is perhaps reasonable to expect

Conjecture 5.4. Let the notation and the assumption be as in Corollary 5.3. Then \( D(\mathfrak{n}_n) = 0 \).

5.2. A rigidity lemma. We study formal subschemes of \( \hat{G} := \hat{G}_{m/W}^n \) stable under the action of \( t \mapsto t^z \) for all \( z \) in an open subgroup \( U \) of \( \mathbb{Z}_t^\times \). We recall with a proof the following result used in the proof of Theorem 0.1 from [H14, Lemma 4.1]:

Lemma 5.5. Let \( K \) be a finite extension of \( \text{Frac}(W(\mathbb{F}_p)) \) and \( W \) be the integral closure of \( W(\mathbb{F}_p) \) in \( K \). Let \( T = \text{Spf}(T) \) be a closed formal subscheme of \( \hat{G} := \hat{G}_{m/W}^n \) flat geometrically irreducible over \( W \) (i.e., \( T \cap \mathbb{T}_\ell = W \)). Suppose there exists an open subgroup \( U \) of \( \mathbb{Z}_t^\times \) such that \( T \) is stable under the action \( \hat{G} \ni t \mapsto t^u \in \hat{G} \) for all \( u \in U \). If \( T \) contains a Zariski dense subset \( \Omega \subset T(\mathbb{C}_\ell) \cap \mu_{p^n}(\mathbb{C}_\ell) \), then we have \( \omega \in \Omega \) and a formal subtorus \( T \) such that \( T = T_\omega \).

A similar assertion is not valid for a formal group \( \hat{G}_{m/K}^2 = \text{Spec}(K[[T, T']]) \) over a characteristic 0 field \( K \). Writing \( t = t + T \) and \( t' = 1 + T' \) for multiplicative variables, the formal subscheme \( Z \) defined by \( t\log(t') = 1 \) is not a formal torus, but it is stable under \( (t, t') \mapsto (t^m, t'^m) \) for any \( m \in \mathbb{Z} \). See [C02, Remark 6.6.1 (iv)] for an optimal expected form of the assertion similar to the above lemma.

Proof. Let \( T_s \) be the singular locus of the associated scheme \( T_{sh} = \text{Spec}(T) \) over \( W \), and put \( T_0 = T_{sh} \setminus T_s \). The scheme \( T_s \) is a closed formal subscheme of \( T \) with \( \text{dim} \ T_s < \text{dim} \ T \) as \( T \) is excellent [CRT, §32]. To see this, we note, by the structure theorem of complete noetherian ring, that \( T \) is finite over a power series ring \( W[[X_1, \ldots, X_d]] \subset T \) for \( d = \text{dim} \ W \) (cf. [CRT, §29]). The sheaf of continuous differentials \( \Omega_T/\text{Spf}(W[[X_1, \ldots, X_d]]) \) is a torsion \( T \)-module, and \( T_s \) is the support of the formal sheaf of \( \Omega_T/\text{Spf}(W[[X_1, \ldots, X_d]]) \) (which is a closed formal subscheme of \( T \)). The regular locus \( T_0 \) of \( T \) is open dense in the generic fiber \( T_{sh}^h : T_{sh} \times W K \) of \( T_{sh} \). Then \( \Omega^0 := T_0 \cap \Omega \) is Zariski dense in \( T_{sh}^h \).

In this proof, by extending scalars, we always assume that \( W \) is sufficiently large so that for \( \zeta \in \Omega \) we focus on, we have \( \zeta \in \hat{G}(W) \) and that we have a plenty of elements of infinite order in \( T(W) \) and in \( T^\circ(K) \cap T(W) \), which we simply write as \( T^\circ(W) := T^\circ(K) \cap T(W) \).

Note that the stabilizer \( U_\zeta \) of \( \zeta \in \Omega \) in \( U \) is an open subgroup of \( U \). Indeed, if the order of \( \zeta \) is equal to \( \ell^k \), then \( U_\zeta = U \cap (1 + \ell^k \mathbb{Z}_\ell) \). Thus making a variable change \( t \mapsto t^{\zeta^{-1}} \) (which commutes with the action of \( U_\zeta \)), we may assume that the identity \( 1 \) of \( \hat{G} \) is in \( \Omega^0 \).
Let $\mathcal{G}^{an}$, $T_{an}$ and $T^{an}$ be the rigid analytic spaces associated to $T$ and $T^{an}$ (in Berthelot’s sense in [395, §7]). We put $T_{an} = T_{an} \setminus T^{an}$, which is an open rigid analytic subspace of $T_{an}$. Then we apply the logarithm map $\log : \mathcal{G}^{an}(\mathbb{C}_\ell) \to \mathbb{C}_\ell^\times$ sending $(t)_i \in \mathcal{G}^{an}(\mathbb{C}_\ell)$ (the $\ell$-adic open unit ball centered at $1 = (1, 1, \ldots, 1)$) to $(\log((t)_i))_i \in \mathbb{C}_\ell^\times$ for the $\ell$-adic Iwasawa logarithm map $\log_\ell : \mathbb{C}_\ell^\times \to \mathbb{C}_\ell$. Then for each smooth point $x \in T^\infty(W)$, taking a small analytic open neighborhood $V_x$ of $x$ (isomorphic to an open ball in $W_d$ for $d = \dim W$) in $T^\infty(W)$, we may assume that $V_x = G_x \cap T^\infty(W)$ for an $n$-dimensional open ball $G_x \in \mathbb{G}(\mathbb{W})$ centered at $x \in \mathbb{G}(\mathbb{W})$. Since $\Omega^\circ \neq 0$, $\log(T^\times(W))$ contains the origin $0 \in \mathbb{C}_\ell^\times$. Take $\zeta \in \Omega^\circ$. Write $T_\zeta$ for the Tangent space at $\zeta$ of $T$. Then $T_\zeta \cong W^d$ for $d = \dim W T$. The space $T_\zeta \otimes W \mathbb{C}_\ell$ is canonically isomorphic to the tangent space $T_0$ of $\log(V_x)$ at $0$.

If $\dim W T = 1$, there exists an infinite order element $t_1 \in T(W)$. We may (and will) assume that $U = (1 + \ell^m \mathbb{Z}_\ell) / (\zeta)$ for $0 < m \in \mathbb{Z}$. Then $T$ is the (formal) Zariski closure $\overline{t_1 T}$ of

$$t_1^U = \{ t_1^{1 + \ell^m} | z \in \mathbb{Z}_\ell \} = t_1 \{ t_1^{1 + \ell^m} | z \in \mathbb{Z}_\ell \},$$

which is a coset of a formal subgroup $Z$. The group $Z$ is the Zariski closure of $\{ t_1^{1 + \ell^m} | z \in \mathbb{Z}_\ell \}$; in other words, regarding $t_1^U$ as a $W$-algebra homomorphism $t_1^U : T \to \mathbb{C}_\ell$, we have $t_1 Z = \text{Spf}(\mathbb{Z})$ for $Z = T / \bigcap_{u \in U} \ker(t_1^U)$. Since $t_1^U$ is an infinite set, we have $\dim W Z > 0$. From geometric irreducibility and $\dim W T = 1$, we conclude $T = t_1 Z$ and $Z \cong \hat{G}^\circ_m$. Since $T$ contains roots of unity $\zeta = \zeta_t \in \Omega(\hat{G})$, we confirm that $T = \zeta Z$ for $\zeta \in \mathbb{C} / \mu_{m'}^{\infty}$, for $m' \gg 0$. Replacing $t_1$ by $t_1^{1+m}$ for $m$ as above if necessary, we have the translation $T_\zeta \ni s \mapsto t_1 s \in Z$ of one parameter subgroup $Z_t \ni s \mapsto t_1^U$. Thus we have $\log(t_1) = \frac{dt_1}{ds}|_{s=0} \in T_\zeta$, which is sent by “$\log : \hat{G} \to \mathbb{C}_\ell^\times$” to $\log(t_1) \in T_0$. This implies that $\log(t_1) \in T_0$ and hence $\log(t_1) \in T_\zeta$. So, we have $\log(t_1) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ (under the identification of the tangent space at any $x \in \hat{G}$ with $\text{Lie}^\times(\hat{G})$). Therefore $T_\zeta$ over $\zeta \in \Omega^\circ$ can be identified canonically. This is natural as $Z$ is a formal torus, and the tangent bundle on $Z$ is constant, giving $\text{Lie}^\times(Z)$.

Suppose that $d = \dim W T > 1$. Consider the Zariski closure $Y$ of $t^U$ for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^\circ$). Since $V_U$ permutes finitely many geometrically irreducible components, each component of $Y$ is stable under an open subgroup of $U$. Therefore $Y = \bigcup \zeta' T_{\zeta'}$ is a union of formal subtori $T_{\zeta'}$ of dimension $\leq 1$, where $\zeta'$ runs over a finite set inside $\mu_{m'}^{\infty}(\mathbb{C}_\ell) \cap \mathbb{T}(\mathbb{C}_\ell)$. Since $\dim W Y = 1$, we can pick $T_{\zeta'}$ of dimension 1 which we denote simply by $T$. Then $T$ contains $t^U$ for some $u \in U$. Applying the argument in the case of $\dim W T = 1$, we find $u \log(t) = \log(t^U) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ and $t \in V_\zeta$. Summarizing our argument, we have found

(t) The Zariski closure of $t^U$ in $T$ for an element $t \in V_\zeta$ of infinite order contains a coset $\xiT$ of one dimensional subtorus $T$, $\xi^{m'} = 1$ and $t^{m'} \in T$ for some $m' > 0$;

(D) Under the notation as above, we have $\log(t) \in T_\zeta$ for all $\zeta \in \Omega^\circ$.

Moreover, the image $\overline{V_\zeta}$ of $V_\zeta$ in $\hat{G}(\mathbb{W}) / T$ is isomorphic to $(d - 1)$-dimensional open ball. If $d > 1$, therefore, we can find $\overline{t} \in \overline{V_\zeta}$ of infinite order. Pulling back $\overline{t} \to t' \in V_\zeta$, we find $\log(t'), \log(t'^U) \in T_\zeta$, and $\log(t')$ and $\log(t'^U)$ are linearly independent in $T_\zeta$. Inductively arguing this way, we find infinite order elements $t_1, \ldots, t_d$ in $V_\zeta$ such that $\log(t_1)$ span over the quotient field $K$ of $W$ the tangent space $T_{\zeta/K} = T_\zeta \otimes W K \to T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{1/K} \subset T_0$ with $T_{\zeta/K} \subset T_0$. Thus the tangent bundle over $T_{a/K}$ is constant as it is constant over the Zariski dense subset $\Omega^\circ$. Therefore $T_{a/K}$ is close to an open dense subscheme of a coset of a formal subgroup. We pin-down this fact.

Take $t_i \in V_\zeta$ as above ($i = 1, 2, \ldots, d$) which give rise to a basis $\{ \partial_i \log(t_i) \}_i$ of the tangent space of $T_{\zeta/K} = T_{1/K}$. Note that $t_1^U \in T$ and $u \partial_i \log(t_1^U) = u \log(t_i) \in T_{1/K}$ for $u \in U$. The embedding $\log : V_\zeta \to T_\zeta \subset \text{Lie}(\hat{G}(\mathbb{W}) / W)$ is surjective onto an open neighborhood of $0 \in T_0$ (by extending scalars if necessary). For $t \in V_\zeta$, if we choose $t$ closer to $\zeta$, $\log(t)$ getting closer to $0$. Thus by replacing $t_1, \ldots, t_d$ inside $V_\zeta$ to elements in $V_\zeta$ closer to $\zeta$, we may assume that $\log(t) \pm \log(t_i)$ for all $i \neq j$ is in $\log(V_\zeta)$.

So, for each pair $i \neq j$, we can find $t_{i,j} \in V_\zeta$ such that $\log(t_i) + \log(t_j) = \log(t_{i,j})$. The element $\log(t_{i,j})$ is uniquely determined in $\log(\mathcal{G}^{an}(\mathbb{C}_\ell)) \cong \mathcal{G}^{an}(\mathbb{C}_\ell) / \mu_{m'}^{\infty}(\mathbb{C}_\ell)$. Thus we conclude $\zeta_{i,j} = t_{i,j}^{1 + \ell^m} \in \mathbb{C}_\ell^{1 + \ell^m}$ for sufficiently large $N$. Replacing $T$ by its image under the $\ell$-power isogeny $\hat{G} \to \hat{G}^\circ$ and $t_i$ by $t_i^{1 + \ell^N}$, we may assume that $t_{i,j}^{1 + \ell^m} = t_{i,j}$ all in $T$. Since $t_1^U \subset T$, by (t), for a sufficiently large $n' \in \mathbb{Z}$, we find $t_1^U \subset T$, by (t), for a sufficiently large $m' \in \mathbb{Z}$, we find a one dimensional subtorus $H_i$
containing \( t_i^{e_i} \) such that \( \zeta_i \hat{H} \subset T \) with some \( \zeta_i \in \mu_{p^{e_i}} \) for all \( i \). Thus again replacing \( T \) by the image of the \( \ell \)-power isogeny \( \hat{G} \ni t \mapsto t^{e_i} \in \hat{G} \), we may assume that the subgroup \( \hat{H} \) (Zariski) topologically generated by \( t_1, \ldots, t_d \) is contained in \( T \). Since \( \{ \log(t_i) \}_i \) is linearly independent, we conclude \( \dim_W \hat{H} \geq d = \dim_W T \), and hence \( T \) must be the formal subgroup \( \hat{H} \) of \( \hat{G} \). Since \( T \) is geometrically irreducible, \( \hat{H} = T \) is a formal torus. Pulling it back by the \( \ell \)-power isogenies we have used, we conclude \( T = \zeta_i^2 \hat{H} \) for the original \( T \) and \( \zeta_i \in \mu_{p^{e_i}}(W) \). Since \( \Omega \) is Zariski dense in \( T \), we may assume that \( \zeta_i \in \Omega \). This finishes the proof.

5.3. Semi-group action. Though we do not need it, we add here an explicit determination of the action of \( \alpha_n \) and \( \alpha_m^{-1} \) on the point \( x(A) \) defined in \([H04, \S 2.1]\). More generally we consider a pair \((L, \eta : \hat{O}^2 \cong \hat{L})\) of an \( O \)-lattice \( L \) of \( M \) and an \( O \)-linear isomorphism \( \eta : (F_k^{(p,\infty)})^2 \cong \hat{L} \otimes O F_k^{(p,\infty)} \) with \( \eta((\hat{O}^p)^2) = \hat{L}^p \). We suppose that \( L_p = R_p \). We define \( L_g = \Im(\eta \circ g(\hat{O}^2)) \cap M \) and \((L, \eta) g = (L_g, \eta \circ g)\). The pair gives rise to a point \( x(L) \in Sh^p(W) \).

Choose a prime element \( \varpi_1 \) of \( O_1 \) and if \( l \) ramifies in \( R \), we suppose that \( R_l = O_l + \sqrt{\varpi_1} O_l \). Recall \( R_n = O + nR \). If \( \ell \) is odd or \( \ell \) does not split in \( R \), we write \( R_l = O_l + \delta_l O_l \) so that \( \delta = \sqrt{-\varpi_1} \) if \( l \) ramifies in \( R \) and \( \delta = \sqrt{\varpi_1} \) for \( d \in O \) if \( l \) is unramified (\( d = \delta^2 \) is square if \( l \) is \( \mathfrak{L} \)-linear and \( d = (\delta, -\delta) \in R_2 \times R_{2c} = R_1 \)). If \( \ell = 2 \) and \( l \) splits in \( R \), we define \( R_1' = \{x \in R_1 \mid x \equiv x^2 \mod 2 \} \) and we start with this order, which has basis \((1, -1)\) and \((1, 1)\) in \( O_1 \times O_1 = R_1 \). We note in this case \( R_1 = R_1' \cap \bigcap_{l \neq p} R_1 \) for primes \( q \in O_1 \) and we put \( \delta = (1, -1) \in R_1, l \) (so, we start with non-maximal order \( R_1 \). Then we put \( \alpha_l = (\frac{1}{l}, \frac{0}{l}) \in \Gamma L_2(O_1) \). We often regard \( \alpha_l \in \Gamma \) so that its component at a prime \( q \neq l \) is equal to \( 1 \). We simply write \( R_n \) for the pair \((R_n, \eta_n)\) with \( \eta_n(a, b) = a + \varpi_1 b \) at \( l \) and outside \( l \), we choose the basis given in \([H04, \text{page 741}]\) and define \( \eta \) accordingly.

Then we put \( \alpha_l^{\pm 1}(x(R_n)) = x(R_n, l^{\pm 1}) \) under the action defined above. This action depends only on local component at \( l \). As seen in \([H04, \text{page 760}]\), we have

\[(5.4) \quad \alpha_l(x(R_n)) = x(R_{n+1}) \quad \text{and} \quad \alpha_l^{-1}(x(R_n)) = x(R_{n-1}) \quad \text{if} \ n > 0.\]

Note

\[\alpha_l^{-n} \left[ \frac{1}{\delta} \right] = \left[ \frac{1}{\varpi_1^{-n} \delta} \right] = \varpi_1^{-n} \delta \left[ \frac{\varpi_1^{-n} \delta^{-1}}{1} \right]\]

Let \( \alpha_l \) act on \( \eta_n \), we need to change the original \( \eta_n \) to \( \eta_n' \) given by \( \eta_n'(a, b) = \varpi_1^{-n} \delta^{-1}(a \varpi_1^n \delta^{-1} b) \) at \( l \) and outside \( l \), the choice is the same as \( \eta_n \). The lattice will change as follows

- (unr) \( R_l \ni \Gamma^{-n} R_{n,l} \) with \( R_0 = R \) if \( l \) remains prime or \( l \) is odd and split in \( R \);
- (ram) \( R_l \ni \Gamma^{-n} 2R_{n,l} \) with \( R_0 = R \) if \( l = 2 \) in \( R \);
- (sp2) \( R_{0,l} \ni \Gamma^{-n} R_{n,l} \) with \( R_0 = R_1 \) if \( l \mid 2 \) and \( l \) splits in \( R \).

Denote \( x'(A) = (A, \eta_n) \) with \( A \) equal to the ideal as in (unr), (ram) and (sp2). Since

\[C_{|n} = \left\{ \text{fractional projective } R_n \text{-ideals} \right\} \backslash \left\{ \text{principal } R_n \text{-ideals} \right\},\]

we may allow \( R_n \)-ideals not prime to \( l \). For an \( R_n \)-fractional ideal \( A \) prime to \( l \), we denote \( A_n \) (resp. \( A'_n \)) by the \( R_n \)-fractional ideal \( A_n \) (resp. \( A'_n \)) with \( A_{n,l} = R_{n,l} \) (resp. \( A'_{n,l} \) given as in (unr), (ram) and (sp2)), and outside \( l \), it is equal to the given \( A \). We have the following effect of the \( \alpha_l^{\pm 0} \) on the points \( x(A_n) \) and \( x'(A'_n) \):

\[\begin{align*}
(+) \quad & \alpha_l^m(x(A_n)) = x(A_{n+m}) \quad \text{and} \quad \alpha_l^{-m}(x'(A'_n)) = x'(A'_{n+m}) \quad \text{if} \ n > 0 \quad \text{and} \ m \geq 0;
(0) \quad & \alpha_l^m(x'(A'_n)) = x(A_{m-n}) \quad \text{and} \quad \alpha_l^{-m}(x(A_n)) = x'(A'_{m-n}) \quad \text{if} \ m \geq n;
(-) \quad & \alpha_l^m(x'(A'_n)) = x'(A'_{n-m}) \quad \text{and} \quad \alpha_l^{-m}(x(A_n)) = x(A_{n-m}) \quad \text{if} \ n > m.
\end{align*}\]

6. A KEY STEP IN THE PROOF OF ANTICYCLOTOMIC MAIN CONJECTURE IN [H06]

A key step towards the proof of the anticyclotomic main conjecture is the following divisibility in the introduction of \([H06]\):

\[(L) \quad (h(M)/h(F)) L_p^W(\psi^-) | H(\psi^-) \text{in} \ W[[\Gamma_M]].\]

Here \( h(M) \) (resp. \( h(F) \)) is the class number of \( M \) (resp. \( F \)), and \( \Gamma_M \) is the Galois group over \( M \) of the composite of all \( \mathbb{Z}_p \)-extensions of \( M \) and \( \Gamma_M \) is its anti-cyclotomic projection. In \([H06]\), \( H(\psi^-) \) is written as \( H(\psi) \), though it depends essentially only on \( \psi^- \) (strictly speaking, its is defined for \( \psi \) with minimal conductor giving a fixed \( \psi^- \)). All the ingredients in the above formula are described
in the introduction of [H06]. In particular, $L_p^\psi(\psi^-)$ is the anti-cyclotomic Katz $p$-adic L-function with branch character $\psi^-$, $H(\psi^-)$ is a congruence power series associated to the $p$-adic analytic family $\theta(\psi)$ of modular form containing the theta series of $\psi$ with anticyclotomic projection $\psi^-$ (see [H06] for precise definition). In [H06], this is attributed to [H07, Corollary 5.6], whose proof relies on the stronger version [H07, Theorem 4.7] of Corollary 5.3 asserting finiteness of $p_n$, (actually the density 0 expectation in Conjecture 5.4 is sufficient). In [H06], this corollary was quoted as Corollary 5.5, but it became Corollary 5.6 after publication of [H07] one year after the publication of [H06]. This stronger version is still an open question. However the proof of (L) is valid if the analytic density 0 result in Conjecture 5.4 holds. In any case, we can give two different proofs of the anticyclotomic main conjecture in the following ways (without assuming any conjecture).

Here is the first on relying on the vanishing of the $\mu$-invariant of Katz $p$-adic L-functions [H11]. Indeed, in [HT93, Theorem 1], (L) is proven under the vanishing of the $\mu$-invariant of the Katz $p$-adic L functions, which was proven in [H11] 18 years later.

The second argument is a modification of the argument in [H07]. For a Hecke character $\varphi$ of $M$ of type $A_0$, regarding it as a character of $\text{Gal}(\overline{\mathbb{Q}}/M)$ by class field theory, we write $\varphi_0(\sigma) = \varphi(\sigma c)$ for complex conjugation $c$ and $\varphi^- = \varphi/\varphi_c$. Following the technique of [HT93], the following formula was proven in [H07, Theorem 5.5]:

\[(K0) \quad \frac{L}{H(\psi^-)} = \frac{L_p(\psi^{-1}\varphi)L_p(\psi^{-1}\varphi_c)}{(h(M)/h(F)L_p^\psi(\psi^-)),}\]

Here $\psi$ is a given finite order branch character of $M$ with conductor made of split primes of $M/F$ for which we want to prove (L) and $\varphi$ is a character of order $\ell$-power of conductor $\ell$-power. The numerator $L \in \mathcal{W}[[\Gamma_M]]$ interpolating Rankin product of the two CM families $\theta(\psi)$ and $\theta(\varphi)$. The denominator of the left-hand-side of the product of the two Katz $p$-adic L-functions $L_p(\psi^{-1}\varphi)$ and $L_p(\psi^{-1}\varphi_c)$ with branch characters $\psi^{-1}\varphi$ and $\psi^{-1}\varphi_c$, respectively. If the stronger version of Corollary 5.3 (or Conjecture 5.4) is valid, choosing $I$ so that rank $\mathbb{Z}_p^\bigcap I = 1$, we can arrange the two Katz $p$-adic L-functions $L_p(\psi^{-1}\varphi)$ and $L_p(\psi^{-1}\varphi_c)$ to be units in $\mathcal{W}[[\Gamma_M]]$ and (L) follows. Whether the actual Corollary 5.3 is sufficient for this argument is not clear. However there is a way-out. We choose one more CM quadratic extension $M_1/F$ disjoint from $M$. For a Hecke character $\varphi$ of $X = M, M_1$ and the composite $K = MM_1$, write $\hat{\varphi} := \varphi \circ N_{K/X}$ as a Hecke character of $K$.

Adjusting the notation to the formula (K0), in [H09, Theorem 3.5] (where slightly different notation was used), the following formula generalizing (K0) is proven:

\[(K1) \quad \frac{L}{H(\hat{\psi}^-)} = \frac{L_p(\hat{\psi}^{-1}\hat{\xi})}{(h(M)/h(F)L_p^\psi(\psi^-))},\]

where $\xi$ is a branch Hecke character of $M_1$ whose conductor is an $\ell$-power for a split prime $I$ of $M_1/F$, $L \in \mathcal{W}[[\Gamma_M \times \Gamma_{M_1}]]$ and $L_p^\psi(\hat{\psi}^{-1}\hat{\xi})$ is the Katz $p$-adic L-function in $\mathcal{W}[[\Gamma_K]]$ projected to $\mathcal{W}[[\Gamma_M \times \Gamma_{M_1}]]$ by $N_{K/M} \times N_{K_1/M_1}$. The formula in [H09] is more general including the case where the conductor of $\psi$ can have inert or ramified primes, and in such a case, there is an extra factor in the denominator of the right-hand-side. By Theorem 0.1 applied to Hilbert modular Eisenstein series for the maximal real subfield of $K$, one can find $\xi$ for which $L_p(\hat{\psi}^{-1}\hat{\xi})$ is a unit in $\mathcal{W}[[\Gamma_K]]$ and is a unit in $\mathcal{W}[[\Gamma_M \times \Gamma_{M_1}]]$ after the projection. This shows (L) also.

**References**

Books

NON-VANISHING OF INTEGRALS OF A MOD \( p \) MODULAR FORM


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