NON-VANISHING OF INTEGRALS OF A MOD p MODULAR FORM, A PRELIMINARY VERSION.

HARUZO HIDA

Abstract. The proof of [H04, Theorem 3.2] and [H07, Theorem 4.3] is based on the assertion claiming that the Zariski closure in the Hilbert modular Shimura variety of an infinite set of CM points stable under the action of a CM torus contains an irreducible component of positive dimension with a CM point in the starting infinite set. A few years ago, Akshay Venkatesh pointed me out that this fact might not be true for a non-noetherian pro-variaty like Shimura variety. I would like to present an argument proving this fact under an extra requirement on the starting infinite set of CM points. Thereby the assertion of [H04, Theorem 3.3] and [H07, Theorem 4.3] on non-vanishing modulo p of Hecke L-values is valid for "Zariski dense" characters in the sense of these articles. In some special cases, non-vanishing is claimed for except finitely many characters in these articles, which is still an open question.

Contents

Recall from [H04] the base totally real field F with integer ring O and the CM quadratic extension $M_{/F}$ with integer ring R. We fix a prime $p > 2$ unramified in $F/\mathbb{Q}$ each of whose prime factor in O splits in M/F and a prime ideal $\mathfrak{l}$ of O prime to p with residual characteristic ℓ. Let $R_n = O + \mathfrak{l}^n R$ (the order of conductor $\mathfrak{l}^n$) and put $Cl_n = \mathrm{Pic}(R_n)$. Since $O \subset R_n$, we have a natural map $Cl_F := \mathrm{Pic}(O) \to Cl_n$. We write $Cl_n^- := \mathrm{Coker}(Cl_F \to Cl_n)$. Let $Cl_\infty := \varprojlim_n Cl_n$ and $Cl_\infty^- := \varprojlim_n Cl_n^-$ under natural projections. The group of fractional R-ideals prime to $\mathfrak{l}$ is naturally embedded into Cl_∞ whose image in Cl_∞ (resp. Cl_∞^-) we write as $Cl^{alg} \subset Cl_\infty$ (resp, $C^{alg} \subset Cl_\infty^-$).

Date: August 27, 2022.

The author is partially supported by the NSF grant: DMS 1464106.

Decompose $Cl^-_\infty = \Delta^- \times \Gamma$ for the maximal finite group Δ^- and $\mathbb{Z}_\ell$-free Γ. Since Cl_F is finite, Γ can be identified with the torsion-free part of Cl_∞ and we have a decomposition $Cl_\infty = \Gamma \times \Delta$ with Δ surjecting down to Δ^- under the projection $Cl_\infty \twoheadrightarrow Cl^-_\infty$. Write Γ_n for the image of Γ in Cl_n (for small n, it can be just $\{1\}$). We identify Γ_n with the image of Γ in Cl^-_n. Write $d = \mathrm{rank}_{\mathbb{Z}_\ell}\Gamma$ and choose a basis $\gamma_1, \dots, \gamma_d$ of Γ over $\mathbb{Z}_\ell$. Let $\mathbb{F}$ (resp. $\overline{\mathbb{Q}}_\ell$) be an algebraic closure of $\mathbb{F}_p$ (resp. $\mathbb{Q}_\ell$). We identify $\mu_{\ell^\infty}(\mathbb{F}) = \mu_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$ as an ℓ-divisible group, and write it just as μ_{ℓ^∞}. We regard the set of continuous characters $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ as a subset of $\mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$ by sending a character χ to $(\chi(\gamma_1), \dots, \chi(\gamma_d)) \in \mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell) \subset \mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell)$. A subset $\mathcal{X}$ of $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ is said to be Zariski dense if $\mathcal{X}$ is Zariski dense in $\mathbb{G}^d_m$ over $\overline{\mathbb{Q}}_\ell$.

For each projective fractional R_n-ideal $\mathcal{A}$, we defined in [H04, §2.1 and §3.1] a CM abelian variety $X(\mathcal{A})$ of ordinary CM type Σ and the associated CM point $x(\mathcal{A}) = x_\Sigma(\mathcal{A})$ of the Shimura variety Sh for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)$, which only depends on the ideal $\mathcal{A}$ and a chosen p-ordinary CM type Σ. Choose suitably an irreducible component V of the Shimura variety of prime-to-p level defined over an algebraic closure $\mathbb{F} = \overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. We just fix a finite extension W of $W(\mathbb{F})$ inside $\mathbb{C}_p$ and put $\mathcal{W} = \mathcal{W}_p = W \cap \overline{\mathbb{Q}}$ for the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}_p$. We embed $\overline{\mathbb{Q}}$ into $\mathbb{C}$. We take a $U(\mathfrak{l})$-eigenform $g_{/\mathcal{W}}$ and put $f = \theta^\kappa f$ for the Ramanujan differential operator θ^κ given by $\prod_\sigma \left(q_\sigma \frac{d_\sigma}{dq_\sigma}\right)^{\kappa_\sigma}$ with $\kappa_\sigma \geq 0$ for the q-expansion variables $q_\sigma = \exp(2\pi i z_\sigma)$. We use the same symbol f also for $f_{/\mathbb{F}} := f \bmod \mathfrak{m}_W$ defined over $\mathbb{F}$. We fix a character $\psi : \Delta^- \to \mathbb{F}^\times$ (see Lemma 3.2 in the text). To define the measure, we need to replace $f(x(\mathcal{A}))$ by $f([\mathcal{A}]) := \lambda^{-1}(\mathcal{A})f(x(\mathcal{A}))$ choosing (and fixing) a Hecke character of infinity type $k\Sigma + \kappa(1-c)$ and of conductor $\mathfrak{C}$ prime to $p\ell$ such that $f([\mathcal{A}])$ only depends on the class $[\mathcal{A}] \in Cl^-_n$ for all n (see §3.4 for more details of the choice of λ). This allows us to define a "measure" $d\varphi_f = d\varphi_{f,n}$ on the finite group Cl^-_n by $\int_{Cl^-_n} \phi d\varphi_{f,n} = \sum_{[\mathcal{A}] \in Cl^-_n} \phi([\mathcal{A}])f([\mathcal{A}])$. If $f|U(\mathfrak{l}) = af$ with $a \neq 0$, $(\lambda(\mathfrak{l})N(\mathfrak{l})a^{-1})^n d\varphi_{f,n}$ patches into a unique measure $d\varphi_f$ on Cl^-_∞, but if $F|U(\mathfrak{l}) = 0$, this is just a collection of measures $\{d\varphi_{f,n}\}_n$.

Let $\mathbb{F}_\mathbf{q}$ be the field of rationality of $f_{/\mathbb{F}}$, ψ and λ modulo $\mathfrak{m}_W$, and define an integer $r > 0$ such that ℓ-Sylow subgroup of $\mathbb{F}_\mathbf{q}[\mu_\ell]^\times$ has order ℓ^r (i.e., $\mu_{\ell^\infty}(\mathbb{F}_\mathbf{q}[\mu_\ell]) = \mu_{\ell^r}(\mathbb{F}_\mathbf{q}[\mu_\ell])$ and $\ell^r \parallel (\mathbf{q}-1)$). Though the measure is defined in the earlier papers for f with non-zero eigenvalue for $U(\mathfrak{l})$, in this paper we define a measure on Cl^-_n for each finite n even for f with $f|U(\mathfrak{l}) = 0$, and the argument goes through even for f killed by $U(\mathfrak{l})$. The non-vanishing of the $U(\mathfrak{l})$-eigenvalue is necessary to patch the measure on Cl^-_n for each n to get a measure on Cl^-_∞, but this patching argument is not essential in the proof of non-vanishing results. Also if $f|U(\mathfrak{l}) = 0$, $\int_{Cl^-_n} \chi\psi d\varphi_{f,n} \neq 0$ can happen only for the minimal n for which the integral is well defined. To project the measure $d\varphi_{f,n}$ to Γ_n, we need to modify f into a modular form f_ψ and further to a function $f^{\mathcal{Q}}_\psi : \bigsqcup_n Cl^-_n \to \mathbb{F}$ which involve a transcendental operation depending on a choice of a representative set $\mathcal{Q}$ of $Cl^-_\infty/\Gamma\Delta^-[2]$ (see (3.15)) so that $\int_{\Gamma_n} \chi d\varphi_{f^{\mathcal{Q}}_\psi, n} = \int_{Cl_n} \chi\psi d\varphi_{f,n}$ for all n and all characters $\chi : \Gamma_n \to \mathbb{F}^\times$, where $\Delta^-[2] = \{\delta \in \Delta^- | \delta^2 = 1\}$. Write $\mathrm{cond}(\chi)$ for the conductor of χ.

Here is a new version of [H04, Theorem 3.3] and (a part of) [H07, Theorem 4.3]:

Theorem 0.1. *Suppose that there exists $\xi \in F \cap O_\mathfrak{l}$ in each class $v \in (O/\mathfrak{l}^j)^\times$ for sufficiently large $j \geq r$ depending on $\mathfrak{l}$ such that the q-expansion coefficient $a(\xi, f_\psi) \neq 0$ in $\mathbb{F}$ at an infinity cusp of V. Then the set of characters $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}(\mathbb{F}))$ such that $\int_{Cl^-_n} \chi\psi d\varphi_{f,n} \neq 0$ for n given by* $\mathrm{cond}(\chi) = \mathfrak{l}^n$ *is Zariski dense. If* $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma = 1$, *$j$ can be taken to be equal to r.*

For the Eisenstein series g we take in [H04] and [H07], for any $v \in O/\mathfrak{l}^j$ and any $j \geq r$, the assumption of the theorem is satisfied except for a very rare case which satisfies conditions (M1–3) in [H07, Theorem 4.3]. For cusp forms, things are more complicated, and Hsieh [Hs14] uses Galois representations of the given cusp form as its traces is basically q-expansion coefficients (and of course, one need to assume that the root number is not -1 (as the square of the integral is the central critical values by Waldspurger) in addition to extra assumptions).

Geometrically irreducible components of the Shimura variety of level group $\Gamma_0(\mathfrak{N})$ are indexed by polarization (strict) ideal classes of F. Then infinity cusps of a component V is indexed by a pair $(\mathfrak{a}, \mathfrak{b})$ of ideals with $(\mathfrak{a}\mathfrak{b})^{-1}$ giving a polarization ideal of V (e.g., [PAF, §4.1.5]). The condition of the existence of ξ with $a(\xi, f_\psi) \neq 0$ does not depend on the choice of $\mathfrak{a}, \mathfrak{b}$. If $f|U(\mathfrak{l}) = 0$, $\int_{Cl_n} \chi\psi d\varphi_{f,n} \neq 0$ implies that $\mathfrak{l}$-conductor $\mathfrak{l}^\nu$ of χ is exactly $\mathfrak{l}^n$ (i.e., $\nu = n$), while this non-vanishing holds for all $n \geq \nu$ once it holds for $n = \nu$ if $f|U(\mathfrak{l}) = af$ with $a \neq 0$.

Here are a more technical description of our method and the reason why I take up this problem again. First, we claimed in these papers [H04] and [H07] a stronger finiteness result of characters of vanishing integrals when $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma = 1$ (e.g., [H04, Theorem 3.2]). As explained in §1.1, a few years ago, Akshay Venkatesh noticed a missing point (taken to be true in [H04] and [H07]) from the proof of [H04, Proposition 2.7]: positive dimensional irreducible components of the Zariski closure of an infinite set Ξ of closed points in a non-noetherian variety may not contain any points in the starting set Ξ. For the proof of the above theorem, we need a density theorem of a thin infinite set Ξ of CM points in the prime-to-p Shimura variety $Sh^{(p)}$. The density theorems Corollary 3.19 and Theorem 3.20 of [H10] requires existence of at least one positive dimensional component with non-trivial intersection with Ξ. All the results of [H10] are valid and intact as the Zariski closure appearing in [H10] has at the onset the base point in the positive dimensional component. Under some extra assumptions in the setting of [H04] and [H07], we prove the existence of such positive dimensional components in Theorem 2.6, which results the above Theorem 0.1 and shows that when $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma = 1$, χ's with vanishing integral make a very thin subset (something similar to density 0 sets but could be infinite; Corollary 4.3).

To describe Ξ, write $\mathbb{F}_{\mathbf{q}}$ with $\mathbf{q} := |\mathbb{F}_{\mathbf{q}}|$ for the minimal field of rationality of λ, ψ and f. Define $r > 0$ by $\mu_{\ell^r}(\mathbb{F}_{\mathbf{q}}[\mu_\ell]) = \mu_{\ell^\infty}(\mathbb{F}_{\mathbf{q}}[\mu_\ell])$. Put $\Phi_n = \Phi_{n,j} := \{\gamma \in \Gamma | \gamma^{\ell^j} = 1\}$ for $j \geq r$. We define $\varepsilon \geq r$ so that $\Phi_{n,j} \cong (\mathbb{Z}/\ell^j\mathbb{Z})^d$ if and only if $n \geq \varepsilon$. We fix a local generator of the ideal $\mathfrak{l}O_{\mathfrak{l}}$. As seen in [H04, (3.1)], for $c \in \Phi_n$ ($n \geq \varepsilon$), we have a unique class $u \in O_{\mathfrak{l}}/\mathfrak{l}^j O_{\mathfrak{l}}$ and a proper ideal $\mathcal{A}$ such that $X(\mathcal{A}) \cong X(R_{n-j})/C$ for a subgroup $C \cong O/\mathfrak{l}^j$ such that $x(\mathcal{A}) = \varrho(u/\varpi_{\mathfrak{l}}^j)(x(R_n))$ for a good choice $\mathcal{A} \in c$ for $\varrho(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. In this way, we can identify Φ_n with $O/\mathfrak{l}^j$ by $\mathcal{A} := \mathcal{A}_u \leftrightarrow u \in O/\mathfrak{l}^j$. We fix a generator $\zeta = \zeta_j$ of $\mu_{\ell^j}(\mathbb{F})$ so that $\varprojlim_n \zeta_n \in \mathbb{Z}_\ell(1)$ is a generator of $\mathbb{Z}_\ell(1)$. Then for any character $\chi : \Gamma \to \mathbb{F}^\times$ with $\mathrm{Ker}(\chi) = \Gamma_n$ (for $n \geq \varepsilon$), we can find $v = v(\chi) = v_j(\chi) \in (O_{\mathfrak{l}}/\mathfrak{l}^j O_{\mathfrak{l}})^\times$ such that $\chi([\mathcal{A}_u]) = \zeta_j^{\mathrm{Tr}_{F_{\mathfrak{l}}/\mathbb{Q}_\ell}(uv)}$ for all $u \in O_{\mathfrak{l}}$.

Let $\underline{n} = \{0 < n_0 < n_1 < \dots\}$ be an infinite sequence of integers. Define an infinite set $\Xi = \Xi_{\underline{n}}$ of CM points by

$$(0.1)\quad \Xi = \Xi_{\underline{n}} = \Xi_{\underline{n},m} := \bigcup_j \Xi_{n_j} \quad \text{with} \quad \Xi_{n_j} = \{x(\mathcal{A}) \in V | \mathcal{A} \in K_{n_j,m}\} \quad \text{for } K_{n,m} := \mathrm{Ker}(\Gamma_n \to \Gamma_m)$$

with a fixed integer $0 < m \leq n_0$. Note that $K_{n,m}$ is equal to $\mathrm{Ker}(Cl_n^- \to Cl_m^-)$. Actually we embed $\Xi_{\underline{n}}$ into a product $V^{\mathcal{Q}}$ of copies of V indexed by a finite subset $\mathcal{Q}$ of Cl_∞ independent modulo Cl^{alg} as in [H04, page 755] by $x(\mathcal{A}) \mapsto (x(\mathcal{A}\delta))_{\delta \in \mathcal{Q}} \in V^{\mathcal{Q}}$ (see below for more details of this embedding). In the proof of Theorem 0.1, we take $\underline{n}$ given by $K_{n_j,m} = \mathrm{Ker}(\chi)$ for χ with $v_j(\chi) = v$ and $\int_{\Gamma_n} \chi d\varphi_{f_\psi^{\mathcal{Q}},n} = 0$ for all $n \geq j$ and a suitable choice of m. The number ℓ^{n_j} is the order of χ if the relative class number of $M_{/F}$ is prime to ℓ.

In this note, out of Theorem 2.6, we prove Theorem 0.1 which is an equivalent version of [H04, Theorem 3.3] and the part of [H07, Theorem 4.3] claiming Zariski density of non-vanishing L-values. After this, we prove a weaker version Corollary 4.3 of [H04, Theorem 3.2] and the part of [H07, Theorem 4.3] in the case where $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma = 1$. We find the desired positive dimensional component in the proof of Theorem 2.6 assuming that $\underline{n}$ contains an infinite arithmetic progression. This is enough to finish the proof of [H04, Theorem 3.3] and the part of [H07, Theorem 4.3] claiming Zariski density of non-vanishing L-values.

Note here that $V^{\mathcal{Q}}$ is a non-noetherian pro-variety, and hence the zero set of a modular form on V is quite likely infinite (of continuous cardinality) even if $\dim V = 1$ and $|\mathcal{Q}| = 1$. There is an example supplied by Venkatesh of a pro-curve in which any positive dimenional irreducible component of the Zariski closure of an infinite set Ξ is disjoint from Ξ (see §1.1). If $g_{\mathfrak{Q}}$ (appearing in the proof of [H04, Theorem 3.2]) had a non-zero eigenvalue for $U(\mathfrak{l})$, the sequence associated to $\{g_{\mathfrak{Q}}\}_{\mathfrak{Q}}$ would contain an infinite arithmetic progression. However it is easy to see $g_{\mathfrak{Q}}|U(\mathfrak{l}) = 0$; so, for the version of [H04, Theorem 3.2] and the part of [H07, Theorem 4.3] in the case where $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma = 1$, we need to assume that $\underline{n}$ contains an infinite arithmetic progression. The idea to reach the missing result is to show (under some extra assumptions) that powers of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ for $\varpi \in N_{M/F}(R)$ generating a power $\mathfrak{l}^m$ act on the 0-dimensional irreducible components of X_K and all the orbits of this action are infinite. The noetherian scheme X_K cannot have infinite 0-dimensional components, and therefore, all components of X_K has positive dimension. Taking the limit, all components of X has positive dimension as desired.

1. Irreducible components of Zariski closure

We study a general theory of Zariski closure in a pro-étale variety of an infinite set of close points. We start with a pathologic example.

1.1. **An example.** To motivate the reader to go through this article dealing with technical topics, we first discuss an example of an affine pro-scheme $V = V_{\infty/\mathbb{C}}$ étale over the affine line $V_0 = \mathrm{Spec}(\mathbb{C}[X])$ such that the Zariski closure of an infinite set $\Xi \subset V(\mathbb{C})$ does not have a single positive dimensional irreducible component containing a point of Ξ. The example was supplied by Akshay Venkatesh in 2018 December.

For a (finite dimensional) scheme $S_{/k}$ for an algebraically closed field k, we write $\mathrm{Irr}(S)$ for the set of all irreducible components of S and $\pi_0(S)$ for the set of all connected components of a scheme S. Put

$$\mathrm{Irr}_d(S) := \{I \in \mathrm{Irr}(S) | \dim I = d\}.$$

Thus $S = \bigcup_{Z \in \mathrm{Irr}(S)} Z$ and $\mathrm{Irr}(S) = \bigsqcup_{d=0}^{\dim S} \mathrm{Irr}_d(S)$. Set $\mathrm{Irr}_+(S) = \bigsqcup_{d>0}^{\dim S} \mathrm{Irr}_d(S)$. If $S = \mathrm{Spec}(A)$, we write $\mathrm{Irr}(A) = \mathrm{Irr}(\mathrm{Spec}(A))$ and $\pi_0(A) = \pi_0(\mathrm{Spec}(A))$. The set $\mathrm{Irr}(A)$ is in bijection onto the set of minimal prime ideals of A, and we identify the two sets.

Take $k = \mathbb{C}$. Let $V_n := V_0 \times \mathbb{Z}/2^n\mathbb{Z}$ and the projection $\mathbb{Z}/2^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2^n\mathbb{Z}$ for $m > n$ induces étale morphism $V_m \twoheadrightarrow V_n$. Let $P_j := (X - j) \subset \mathbb{C}[X]$ $(0 < j \in \mathbb{Z})$ and regard it as a closed point j of V_0. We define $V := \varprojlim_n V_n \cong V_0 \times \mathbb{Z}_2$ and $\Xi = \{(j, 2^j)_\infty \in V | j = 1, 2, \dots\}$. Write $(P_j, j)_n$ for the maximal ideal of V_n giving rise to the point $(j, 2^j \bmod 2^j) \in V_n$. Therefore $(P_j, 2^j)_\infty$ for finite j is the maximal ideal of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$ non-trivial equal to P_j only at 2^j-component of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$, and the prime ideal $(P_j, 2^j)_n$ is $P_j \oplus \bigoplus_{i \not\equiv 2^j \mod (2^n)} \mathbb{C}[X]$. Write

$$\Xi_n := \{(j, 2^j)_n = (j, 2^j)_n \in V_n | j = 1, 2, \dots\}$$

for the image of Ξ in V_n. Note that $V_n = \mathrm{Spec}(\bigoplus_{\mathbb{Z}/2^n\mathbb{Z}} \mathbb{C}[X])$ and $V = \mathrm{Spec}(\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X])$. We have $\bigcap_j (P_j, 2^j)_\infty = ((0), 0)_\infty \oplus \bigcap_{0<j\in\mathbb{Z}} (P_j, 2^j)_\infty$, where $((0), 0)_\infty$ is the prime ideal of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$ equal to (0) only at the 0-component of $\bigoplus_{\mathbb{Z}_2} \mathbb{C}[X]$. Thus $\overline{\Xi} = V_0 \sqcup \bigsqcup_{0<j\in\mathbb{Z}} (P_j, 2^j)_\infty \subset V$, where V_0 is inserted as the 0-component. Thus only positive dimensional irreducible (and connected) component of the Zariski closure $\overline{\Xi}$ in V is V_0 which does not contain any points of Ξ.

> *If we have a transitive action of a semi-group inside* $\mathrm{Aut}(V)$ *on* $\overline{\Xi}$*, we expect to be able to avoid such a pathologic example.*

Though $\alpha : (v, z) \mapsto (v + 1, 2z)$ acts transitively on Ξ, α is not an automorphism of V. It is an automorphism of $V_0 \times \mathbb{Q}_2$ which is an indo-pro-variety not a pro-variety. In the above example, we have

$$\begin{aligned} &\mathrm{Irr}_1(\overline{\Xi}_n) = \{(V_0 \times 0) | 0 \in \mathbb{Z}/2^n\mathbb{Z}\},\ \mathrm{Irr}_0(\overline{\Xi}_n) = \{(j \times 2^j)_n | j = 1, \dots, n-1 (2^j \neq 0 \in \mathbb{Z}/2^n\mathbb{Z})\}, \\ &\mathrm{Irr}_1(\overline{\Xi}) = \{(V_0 \times 0) | 0 \in \mathbb{Z}_2\} \ \text{ and } \ \mathrm{Irr}_0(\overline{\Xi}) = \{(j, 2^j)_\infty | 0 < j \in \mathbb{Z}, 2^j \in \mathbb{Z}_2\}. \end{aligned} \tag{1.1}$$

The action of any positive power of α brings some points in $\mathrm{Irr}_0(\overline{\Xi}_n)$ into a component in $\mathrm{Irr}_1(\overline{\Xi}_n)$ (non-stability of $\mathrm{Irr}_0(\overline{\Xi}_n)$ under α coming from the fact that α is not an automorphism of V). Writing $\pi_n : V \to V_n$, we can consider the reduced image $\pi_n(I) \subset V_n$ for $I \in \mathrm{Irr}(\overline{\Xi})$. Let $\pi_{n,*}(\mathrm{Irr}(\overline{\Xi})) = \{\pi_n(I) | I \in \mathrm{Irr}(\overline{\Xi})\}$ and $\pi_{n,*}(\mathrm{Irr}_j(\overline{\Xi})) = \{\pi_n(I) | I \in \mathrm{Irr}_j(\overline{\Xi})\}$ as sets. Then

$$\pi_{0,*} : \mathrm{Irr}_1(\overline{\Xi})) \cong \mathrm{Irr}_1(\overline{\Xi}_0)),\ \pi_{0,*}(\mathrm{Irr}_0(\overline{\Xi})) \supset \mathrm{Irr}_0(\overline{\Xi}_n) \ \text{ with infinite } \pi_{0,*}(\mathrm{Irr}_0(\overline{\Xi})) - \mathrm{Irr}_0(\overline{\Xi}_n) \text{ in } V_0 \times 0. \tag{1.2}$$

(ne) The image of $\{(j, 2^j)_\infty \in \Xi | j \geq n\}$ lies in the one dimensional $(V_0 \times 0) \in \mathrm{Irr}_1(\overline{\Xi}_n)$ and the 0-dimensional scheme $(j, 2^j)_\infty$ $(j \geq n)$ is not étale over V_n.

If we take a 2-unit $u \in \mathbb{Z}$ and consider $\Xi = \{(j, u^j)_\infty | j = 1, 2, \dots\} \subset V$, one can show that $\mathrm{Irr}(\overline{\Xi}_n) = \{V_0 \times u^j | u^j \mod 2^n\}$ and $\mathrm{Irr}(\overline{\Xi}) = \{V_0 \times x | x \in \langle u \rangle\}$ for the subgroup $\langle u \rangle \subset \mathbb{Z}_2^\times$ topologically generated by u. The action $[1] : (j, u^j) \mapsto (j + 1, u^{j+1})$ extends to an automorphism $[1] : (v, z) \mapsto (v + 1, uz)$. A similar morphism $\alpha(v, z) = (v + 1, 2z)$ for non-unit 2 in place of u is not an automorphism of V.

Taking an infinite sequence of irreducible polynomials $X-a_j$ of $\mathbb{F}[X]$ with distinct $a_j \in \mathbb{F}$, we can make an example similar to (ne) also over $\mathbb{F}$ taking $V_0 := \mathrm{Spec}(\mathbb{F}[X])$ and $\Xi = \{(P_j := ((X-a_j), 2^j)\}_j$ with $V_n = V_0 \times \mathbb{Z}/2^n\mathbb{Z}$. Then $\varprojlim_n V_n = V_0 \times \mathbb{Z}_2$.

1.2. **Geometry of irreducible components.**

We prepare some notation and geometric lemmas to prove the theorem. After the lemmas, in the following section, we study the correspondence action.

Let $\pi : \mathcal{V}_{/\mathbb{F}} \to \mathcal{V}_{K/\mathbb{F}}$ be an affine étale Galois covering with $\mathcal{V} = \mathrm{Spec}_{\mathcal{O}_{\mathcal{V}_K}}(\mathcal{O}_{\mathcal{V}})$ (as a relative spectrum). Here $K = \mathrm{Gal}(\mathcal{V}/\mathcal{V}_K)$ and $\mathcal{V} = \varprojlim_{U \triangleleft K} \mathcal{V}_U$ for U running over open subgroups of K with $\mathcal{V}_U = \mathcal{V}/U$. In the following lemmas, assume that $\mathcal{V}_K$ is noetherian (so, $\mathcal{V}_U = \mathcal{V}/U$ is also noetherian for an open subgroup U of K. Let $\Xi \subset \mathcal{V}(\mathbb{F})$ be an *infinite* set of closed points with image Ξ_K in $\mathcal{V}_K(\mathbb{F})$.

Lemma 1.1. *Regard $P' \in \Xi$ (resp. $P \in \Xi_K$) as a sheaf of $\mathcal{O}_{\mathcal{V}}$-ideal (resp. $\mathcal{O}_{\mathcal{V}_K}$-ideal) defining the point P' (resp. P); so, for example $\mathcal{O}_{\mathcal{V}_K}/P \cong \mathbb{F}(P) = \mathbb{F}$ as a skyscraper sheaf supported by P. Let X' (resp. X) be the Zariski closure of Ξ (resp. Ξ_K) in $\mathcal{V}$ (resp. $\mathcal{V}_K$). Then*

1. *X' and X are reduced scheme, X'/X is finite if $\mathcal{V}/\mathcal{V}_K$ is finite.*
2. *The projection $\pi_X : X' \to X$ is dominant inducing a surjection of $\mathbb{F}$-points: $X'(\mathbb{F}) \twoheadrightarrow X(\mathbb{F})$, and X' is unramified over X.*

As described in (ne), even if $\Xi \cong \Xi_K$ by the map induced by π, $\pi : X' \to X$ may not be étale.

Proof. By definition, we have $X' = \mathrm{Spec}(\mathcal{O}_{\mathcal{V}}/\bigcap_{P' \in \Xi} P')$ and $X = \mathrm{Spec}(\mathcal{O}_{\mathcal{V}_K}/\bigcap_{P \in \Xi_K} P)$.

We prove the lemma first in the absolute affine case; so, we put $\mathcal{V}_K = \mathrm{Spec}(A)$, $\mathcal{V} = \mathrm{Spec}(A')$, $B = A/\bigcap_{P \in \Xi_K} P$ and $B' = A'/\bigcap_{P' \in \Xi} P'$. Since $B' \hookrightarrow \prod_{P' \in \Xi} A'/P$ with the right-hand-side reduced, B' is reduced. In the same way, B is reduced.

If A'/A is étale finite, we have $\Xi_K = \{P' \cap A | P' \in \Xi\}$; so, putting $\mathfrak{b}' := \bigcap_{P' \in \Xi} P'$ and $\mathfrak{b} := \bigcap_{P \in \Xi_K} P$, we have $\mathfrak{b}' \cap A = \mathfrak{b}$. Thus the induced map $B \xrightarrow{i} B'$ is injective. If A'/A is not finite, we can write $A = \bigcup_i A_i$ with A_i/A finite étale, we still get the injectivity. Therefore the projection $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is dominant. Pick a maximal ideal $\mathfrak{m} \in \mathrm{Spec}(B)(\mathbb{F})$. Then by the going-up theorem [CRT, Theorem 9.3 (i)], we have a prime ideal $\mathfrak{p} \in \mathrm{Spec}(B')$ with $\mathfrak{p}' \cap B = \mathfrak{m}$. Take a maximal ideal $\mathfrak{m}'$ containing $\mathfrak{p}'$, $\mathfrak{m}' \cap B \supset \mathfrak{m}$ is still a proper ideal as B'/B is integral; so, $\mathfrak{m}' \cap B = \mathfrak{m}$. Thus $B'/\mathfrak{m}'$ is a finite extension of $B/\mathfrak{m} = \mathbb{F}$ which is algebraically closed, we conclude $B/\mathfrak{m}' = \mathbb{F}$ and $\mathfrak{m}' \in \mathrm{Spec}(B')(\mathbb{F})$; so, $\mathrm{Spec}(B')(\mathbb{F}) \to \mathrm{Spec}(B)(\mathbb{F})$ is onto.

Pick $\mathfrak{m}' \in \mathrm{Spec}(B')(\mathbb{F})$ and regard it as a maximal ideal of A'. Since $\mathfrak{m}' \supset \mathfrak{b}'$, $\mathfrak{m} := \mathfrak{m}' \cap A \supset \mathfrak{b}$; so, $\mathfrak{m} \in \mathrm{Spec}(B)(\mathbb{F})$. We have the following commutative diagram of the completions at $\mathfrak{m}'$ and $\mathfrak{m}$:

$$\begin{tikzcd}
\widehat{A}_{\mathfrak{m}} \arrow[r, "\hookrightarrow"] \arrow[d, "\text{onto}"'] & \widehat{A}'_{\mathfrak{m}} \arrow[r, "\twoheadrightarrow"] \arrow[d, "\text{onto}"] & \widehat{A}'_{\mathfrak{m}'} \arrow[d, "\text{onto}"] \\
\widehat{B}_{\mathfrak{m}} \arrow[r, "i_{\mathfrak{m}}"] & \widehat{B}'_{\mathfrak{m}} \arrow[r, "p_{\mathfrak{m}'}", "\twoheadrightarrow"'] & \widehat{B}'_{\mathfrak{m}'}.
\end{tikzcd}$$

Since the top row composite: $\widehat{A}_{\mathfrak{m}} \hookrightarrow \widehat{A}'_{\mathfrak{m}} \twoheadrightarrow \widehat{A}'_{\mathfrak{m}'}$ is an isomorphism (as $A \hookrightarrow A'$ is étale), $p_{\mathfrak{m}'} \circ i_{\mathfrak{m}}$ is onto. Therefore B'/B is an unramified extension and is finite if A'/A is finite. This prove (1) and (2) in the absolute affine case.

Now we treat the general relative affine case. We cover $\mathcal{V}_K = \bigcup_A \mathrm{Spec}(A)$ for affine open subscheme $\mathrm{Spec}(A)$, and write $A' = \pi_*\mathcal{O}_{\mathcal{V}}(\mathrm{Spec}(A))$. Then $\mathrm{Spec}(A')$ is an open subscheme of $\mathcal{V}$ covering $\mathrm{Spec}(A)$. Then we have $X' \cap \mathrm{Spec}(A') = X' \times_{\mathcal{V}_K} \mathrm{Spec}(A) = \mathrm{Spec}(B')$ and $X \cap \mathrm{Spec}(A) = X \times_{\mathcal{V}_K} \mathrm{Spec}(A) = \mathrm{Spec}(B)$ with $(A'/A, B'/B, \Xi \cap \mathrm{Spec}(A'), \Xi_K \cap \mathrm{Spec}(A))$ satisfying the assumption of Lemma 1.1. Since B (resp. B') depends on A, if needed, we write $B = B_A$ and $B' = B'_A$ to emphasize the dependence. By the above argument, B' and B are reduced algebra, and B' is an unramified extension of B, B'/B is finite if A'/A is finite, and the projection $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is dominant and the induced map: $\mathrm{Spec}(B')(\mathbb{F}) \to \mathrm{Spec}(B)(\mathbb{F})$ is surjective. Since $\mathrm{Spec}(B')$ is the pull-back to X' of $\mathrm{Spec}(B)$ and $X' = \bigcup_A \mathrm{Spec}(B'_A) = \bigcup_A \pi^{-1}(\mathrm{Spec}(B_A))$ and $X = \bigcup_A \mathrm{Spec}(B_A)$, the above proof in the affine case implies the assertion in the general case. $\square$

Assume that $\Xi \cong \Xi_K$. We have another commutative diagram:

$$\begin{tikzcd}
B \arrow[r, "\hookrightarrow"] \arrow[d, "\pi_B^*"'] & \prod_{P\in\Xi_K} A/P \arrow[d, "\wr"'] \\
B' \arrow[r, "\hookrightarrow"'] & \prod_{P\in\Xi} A'/P'.
\end{tikzcd}$$

The right vertical map is an isomorphism as $\Xi \cong \Xi_K$. Thus π_B^* is injective; so, again we see that $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is dominant.

Lemma 1.2. *Let the notation and the assumption be as in* Lemma 1.1. *Recall that* $\mathcal{V}_K$ *is a noetherian scheme. Let* $\pi_*(\mathrm{Irr}(X')) := \{\pi(Z')|Z' \in \mathrm{Irr}(X')\}$ *for the set of the reduced image* $\pi(Z') \subset X$. *Then we have*

1. *The image* $\pi_*(\mathrm{Irr}(X'))$ *contains* $\mathrm{Irr}(X)$,
2. *For* $Y \in \mathrm{Irr}(X)$, *if* $Y' \in \mathrm{Irr}(\pi^{-1}(Y))$ *is contained in* X', *we have* $Y' \in \mathrm{Irr}(X')$, *where* $\pi^{-1}(Y) = Y \times_{\mathcal{V}_K} \mathcal{V}$.
3. *If* $\Xi \cong \Xi_K$ *under the projection* $\mathcal{V} \xrightarrow{\pi} \mathcal{V}_K$, *we have a unique section* $\mathrm{Irr}_0(X) \to \mathrm{Irr}_0(X')$ *of* $\mathrm{Irr}_0(X') \twoheadrightarrow \mathrm{Im}(\mathrm{Irr}_0(X')) \subset X$ *and* $\mathrm{Irr}_0(X') \subset \Xi$. *Moreover writing* X'_U *for the image of* X' *in* $\mathcal{V}/U$ *for an open subgroup* U *of* K, $\mathrm{Irr}_0(X') = \varinjlim_U \mathrm{Irr}_0(X'_U)$ *for* U *running over all open subgroups of* K.
4. *If* $\dim Z = \dim X$ *for* $Z \in \mathrm{Irr}_{\dim X}(X)$, *then* Z *is in the image of* $\mathrm{Irr}_{\dim X'}(X')$ *in* X. *In particular,* $\mathrm{Irr}_{\dim(X)}(X') \neq \emptyset$.

Proof. Again we may assume that $\mathcal{V}_K = \mathrm{Spec}(A)$, $\mathcal{V} = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$ as in the proof of Lemma 1.1. Pick $\mathfrak{p}_Y \in \mathrm{Irr}(B)$ giving $Y \in \mathrm{Irr}(\mathrm{Spec}(B))$. Since B'/B is integral, we find a prime $P' \in \mathrm{Spec}(B')$ such that $P' \cap B = \mathfrak{p}_Y$ by going-up theorem [CRT, Theorem 9.3 (i)]. For each $P' \in \mathrm{Spec}(B')$ with $P' \cap B = \mathfrak{p}_Y$ (i.e., $P' \in \pi^{-1}(Y) = \mathrm{Spec}(B'/\mathfrak{p}_Y B')$), take a minimal prime $\mathfrak{p}' \subset P'$ (i.e., $\mathfrak{p}' \in \mathrm{Irr}(B')$). Then $\mathfrak{p}' \cap B$ is a prime ideal of B and $\mathfrak{p}_Y \supset \mathfrak{p}' \cap B$; so, by minimality of $\mathfrak{p}_Y$, we have $\mathfrak{p}_Y = \mathfrak{p}' \cap B$. Thus $\mathfrak{p}_Y$ is in the image of $\mathrm{Irr}(B')$. This proves the assertion (1).

As $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is étale, $\pi^{-1}(Y)$ is étale over Y; so, equi-dimensional. Suppose that $Y' \subset X'$ for $Y' \in \mathrm{Irr}(\pi^{-1}(Y))$. Then we find $Z' \in \mathrm{Irr}(X')$ such that $Z' \supset Y'$; so, $\pi(Z') \subset X$. We are going to show $Z' = Y'$. We have $X \supset \pi(Z') \supset Y$. Since $\pi(Z')$ is irreducible, $\pi(Z')$ containing $Y \in \mathrm{Irr}(X)$ implies $\pi(Z') = Y$. Thus $Z' \twoheadrightarrow Y$ is a integral dominant; so, $\dim Z' = \dim Y' = \dim Y$. This shows $Z = Z' \in \mathrm{Irr}(X')$, as desired. Thus the assertion (2) follows.

To show the assertion for Irr_0, we first assume that B'/B is finite. We regard $\Xi_K \subset \mathrm{Spec}(B)$. Pick $\mathfrak{m} \in \mathrm{Irr}_0(B)$. Then $B = B^{(\mathfrak{m})} \oplus B/\mathfrak{m}$ for a subring $B^{(\mathfrak{m})} \subset B$ as $\mathrm{Spec}(B/\mathfrak{m})$ is a connected component of $\mathrm{Spec}(B)$. Thus $\mathrm{Irr}_0(B) = \{Z \in \pi_0(\mathrm{Spec}(B))|\dim Z = 0\}$. Since $B' \supset B$, the above decomposition induces an algebra direct sum $B' = B'^{(\mathfrak{m})} \oplus B'/\mathfrak{m}B'$. Since B' is finite over B, $B'/\mathfrak{m}B'$ has dimension 0. By reducedness of B', the direct summand $B'/\mathfrak{m}B'$ of B' is a direct sum of fields. This means that π induces a surjection of the upper row of the following diagram:

$$\begin{tikzcd}
\pi_0(\mathrm{Spec}(B'/\mathfrak{m}B')) \arrow[r, "\twoheadrightarrow", "\pi_*"'] \arrow[d, "\cap"'] & \pi_0(\mathrm{Spec}(B/\mathfrak{m})) = \{\mathfrak{m}\} \\
\mathrm{Irr}_0(B') &
\end{tikzcd}$$

for each $\mathfrak{m} \in \mathrm{Irr}_0(B) \subset \pi_0(B)$. Therefore $\pi_*(\mathrm{Irr}_0(B')) \supset \mathrm{Irr}_0(B)$. Pick $\mathfrak{m} \in \mathrm{Irr}_0(B)$. If $\mathfrak{m} \notin \Xi_K$, $\Xi_K \subset \mathrm{Spec}(B^{(\mathfrak{m})})$ as $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}) \sqcup \mathrm{Spec}(B^{(\mathfrak{m})})$. This implies $B = A/\bigcap_{P\in\Xi_K} P$ is equal to $B^{(\mathfrak{m})}$, a contradiction. Thus $\mathfrak{m} \in \Xi_K$, and $\mathrm{Irr}_0(B) \subset \Xi_K$. Since $\Xi \cong \Xi_K$, π_* has a unique section $\pi^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(B')$. If B'/B is not finite, we can write $B' = \bigcup_j B_j$ for B-subalgebras $B_j \subset B'$ finite over B. We may assume that the index set is totally ordered so that $B_{j'} \supset B_j$ if $j' > j$. Let $X'_U = \mathrm{Spec}(B_U)$ for an open subgroup U of K. Then B_U/B is finite unramified. Then applying the above argument to finite B_U/B, we find natural inclusion $\mathrm{Irr}_0(B_U) \subset \pi_{U',U,*}(\mathrm{Irr}_0(B_{U'}))$ for open subgroups $U' \subset U \subset K$ with a unique section $\pi^*_{U',U} : \mathrm{Irr}_0(B_U) \hookrightarrow \mathrm{Irr}_0(B_{U'})$. In particular, the injective limit of $\pi^*_{U',U}$ gives rise to the section $\pi^* : \mathrm{Irr}_0(B) \hookrightarrow \mathrm{Irr}_0(B')$ and $\mathrm{Irr}_0(B') = \varinjlim_U \mathrm{Irr}_0(B_U)$. This proves the assertion (3).

Now suppose that $\dim B/\mathfrak{p} = \dim B$ for $\mathfrak{p} \in \mathrm{Irr}(B)$. Such $\mathfrak{p}$ always exists as B is noetherian. Since B'/B is integral, $\dim B = \dim B'$. Then we take $\mathfrak{p}' \in \mathrm{Spec}(B')$ such that $\mathfrak{p}' \cap B = \mathfrak{p}$. Such a prime exists as already remarked. Then $B/\mathfrak{p} \hookrightarrow B'/\mathfrak{p}'$ and hence $\dim B'/\mathfrak{p}' = \dim B/\mathfrak{p} = \dim B$ as $B'/\mathfrak{p}'$ is integral over $B/\mathfrak{p}$. Since $\dim B' = \dim B$, we conclude $\mathfrak{p}' \in \mathrm{Irr}_{\dim B'}(B')$; so, $\mathrm{Irr}_{\dim B'}(B') \neq \emptyset$. This proves the assertion (4). $\square$

Lemma 1.3. *Suppose that* $\pi_*(Z') = \pi(Z') \notin \mathrm{Irr}(X)$ *for* $Z' \in \mathrm{Irr}(X')$*. Then there exists* $Z_0 \in \mathrm{Irr}(X)$ *such that* $Z_0 \supset \pi_*(Z')$*.*

Proof. Again we may assume that $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$ as in the proof of Lemma 1.1. Write $Z' = \mathrm{Spec}(B'/\mathfrak{p}')$. By the assumption, $\mathfrak{p}' \cap B \notin \mathrm{Irr}(B)$; therefore $\mathfrak{p}' \cap B \supsetneq \mathfrak{p}_0$ for a minimal prime ideal $\mathfrak{p}_0$ of B. By definition, $\mathfrak{p}_0 \in \mathrm{Irr}(B)$ and $\mathfrak{p}' \cap B \supset \mathfrak{p}_0$ means $\mathfrak{p}' \cap B \in \mathrm{Spec}(B/\mathfrak{p}_0)$. Thus $Z_0 = \mathrm{Spec}(B/\mathfrak{p}_0)$ does the job. $\square$

Lemma 1.4. *If* $\Xi_{K,0}$ *is a subset of* Ξ_K *with finite* $\Xi_K - \Xi_{K,0}$*, then the Zariski closure* X *of* Ξ_K *in* $\mathcal{V}_K$ *and that* X_0 *of* $\Xi_{K,0}$ *share irreducible components of positive dimension* (*i.e.,* $\mathrm{Irr}_+(X) = \mathrm{Irr}_+(X_0)$), *and* $\mathrm{Irr}(X) - \mathrm{Irr}(X_0)$ *is a finite subset of* $\Xi_K - \Xi_{K,0}$*.*

Proof. Again we may assume that $X = \mathrm{Spec}(B)$ as in the proof of Lemma 1.1. Write $\Xi_K - \Xi_{K,0} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_h\}$ for maximal ideals $\mathfrak{m}_i$ of A and put $\mathfrak{a} = \bigcap_i \mathfrak{m}_i$. Then for $\mathfrak{b}_0 = \bigcap_{P \in \Xi_{K,0}} P$ and $\mathfrak{b} = \bigcap_{P \in \Xi_K} P$, we have $\mathfrak{b} = \mathfrak{b}_0 \cap \mathfrak{a}$. For each i, either $\mathfrak{m}_i \supset \mathfrak{b}_0$ or $\mathfrak{m}_i + \mathfrak{b}_0 = A$ as $\mathfrak{m}_i$ is maximal. Thus we may assume that $\Xi_K - \Xi_{K,0} = \{\mathfrak{m}_i | \mathfrak{m}_i + \mathfrak{b}_0 = A\}$. Then $\mathfrak{a} + \mathfrak{b}_0 = A$ as $|\Xi_K - \Xi_{K,0}|$ is finite. Thus $A/\mathfrak{b} = A/\mathfrak{b}_0 \cap \mathfrak{a} = A/\mathfrak{b}_0 \oplus A/\mathfrak{a}$, and hence $X = X_0 \sqcup (\Xi_K - \Xi_{K,0})$ as desired. $\square$

2. Zariski closure in Hilbert modular Shimura variety

Recall $Cl_\infty = \varprojlim_n Cl_n$ and $Cl^-_\infty = \varprojlim_n Cl^-_n$. Writing $[\mathcal{A}]_n$ for the class of a proper R_n-ideal $\mathcal{A}$ in Cl_n. Let

$$Cl^{alg} := \{[x]_\infty = \varprojlim_n [x\widehat{R}_n \cap M] \in Cl_\infty | x \in M_{\mathbb{A}}^\times \text{ with } x_{\mathfrak{l}\infty} = 1\} \subset Cl_\infty,$$

where $\widehat{R}_n = R_n \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ with $\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$ (cf. [H04, page 755]).

Let $G := \mathrm{Res}_{O/\mathbb{Z}}\mathrm{GL}(2)$ and $Sh_{/\mathbb{Q}}$ be the Hilbert modular Shimura variety associated to G. Since $G(\mathbb{A}^{(\infty)})$ acts on Sh as automorphisms, we define the prime-to-p level Shimura variety $Sh^{(p)}$ by $Sh/G(\mathbb{Z}_p)$. The Shimra variety $Sh^{(p)}$ extend canonically to a smooth pro-scheme over $\mathcal{W}$ (e.g. [PAF, Chapter 4]). Recall the irreducible component $V = V^{(p)}$ of the Shimura variety $Sh^{(p)}_{/\overline{\mathbb{Q}}}$ we fixed. By smoothness, $V_{/\mathbb{F}} := V \times_{\mathcal{W}} \mathbb{F}$ is an irreducible component of $Sh^{(p)}_{/\mathbb{F}}$.

Let $\mathcal{Q} \subset Cl^-_\infty$ be a finite subset independent modulo C^{alg}; i.e., $\delta C^{alg} \neq \delta' C^{alg}$ for any pair $(\delta, \delta') \in \mathcal{Q}^2$ with $\delta \neq \delta'$. For a closed subgroup $K^{(p)} \subset G(\mathbb{A}^{(p\infty)})$, we put $K = G(\mathbb{Z}_p) \times K^{(p)}$ and write V_K for the image of V in $Sh^{(p)}_K = Sh/K$. We set $\mathcal{V}_{/B} := V^{\mathcal{Q}}_{/B}$ for $B = \overline{\mathbb{Q}}, \mathcal{W}, \mathbb{F}$ (the product of $\mathcal{Q}$ copies of V) and $\mathcal{V}_{K/B} := V^{\mathcal{Q}}_{K/\mathbb{F}}$. We can embed Cl_n into $\mathcal{V}$ by $[\mathcal{A}] \mapsto \mathbf{x}(\mathcal{A}) = \mathbf{x}([\mathcal{A}]) := (x([\mathcal{A}]\delta))_{\delta \in \mathcal{Q}} \in \mathcal{V}$, and write its image with C_n. Put $C^{(\infty)} = \bigsqcup_n C_n \subset \mathcal{V}$ as abelian variety sitting over $\mathbf{x}(\mathcal{A})$ is uniquely determined by $[\mathcal{A}]$. We fix an infinite subset Ξ of $C^{(\infty)}$.

We fix a CM type Σ of M and write Σ_p for the set of p-adic places induced by the embedding in Σ by the identification $\mathbb{C} \cong \mathbb{C}_p$ we fixed. We write X (resp. X_K) for the Zariski closure of Ξ (resp. Ξ_K). We recall two assumptions (unr) and (ord) in [H04, §2.1] for p in addition to $\Xi \cong \Xi_K$ under the projection $\mathcal{V} \to \mathcal{V}_K$:

(ord) Σ *is p–ordinary:* $\Sigma_p \cap \Sigma_p c = \emptyset$ *for the generator c of* $\mathrm{Gal}(M/F)$*.*

Such a CM type Σ is called a *p–ordinary* CM type. The existence of a p–ordinary CM type is equivalent to the fact that all prime factors of p in F split into a product of two distinct primes in M. We suppose

(unr) *p is unramified in* $F/\mathbb{Q}$*.*

In this section, assuming the existence of an appropriate toric action on Ξ induced by an infinite toric sub-semigroup $\mathbf{T}$ of $\mathrm{Aut}(\mathcal{V}_{/\mathbb{F}})$, we prove that all irreducible components of X has positive dimension; i.e., $\mathrm{Irr}(X) = \mathrm{Irr}_+(X)$ (see Theorem 2.6). The idea of the proof is to show

(1) $\mathrm{Irr}_0(X_K) \neq \emptyset$ if $\mathrm{Irr}_0(X) \neq \emptyset$;
(2) By the action of $\mathbf{T}$, $\mathrm{Irr}_0(X_K)$ has to be of infinite order, against the noetherian property of X_K.

At the end, we take Ξ to be $\Xi_{\underline{n},m}$ as in (0.1), but for the moment, the set is a general infinite subset of $C^{(\infty)}$.

2.1. **Toric action.** We now describe a list of conditions for proving the items (1)-(2) above under the correspondence action of $\mathbf{T}$. After this, we state five lemmas about the action under these conditions before starting with supplying the missing argument/fact (stated as Theorem 2.6).

We suppose that K is an open compact subgroup of $G(\mathbb{A}^{(\infty)})$; so, $\mathcal{V}_K$ is noetherian. On $\mathcal{V} = V^{\mathcal{Q}}$, $\mathrm{Aut}(V/\mathbb{F})$ diagonally acts. We suppose to have a semi-group $\mathbf{T} \subset \mathrm{Aut}(V/\mathbb{F})$ acting on Ξ under the diagonal action. The action of $\mathbf{T}$ is supposed to come from the action of elements in $G(\mathbb{A}^{(p\infty)})$ on $Sh^{(p)}$. Since it is a semi-group action, $\beta \in \mathbf{T}$ embeds Ξ into Ξ; so, $\beta(\Xi) \subset \Xi$ and $\beta^{-1}(\Xi) \supset \Xi$, where β^{-1} may not be in $\mathbf{T}$ but in $\mathrm{Aut}(\mathcal{V}/\mathbb{F})$.

Let $N := \{\varrho(u) | u \in O_{\mathfrak{l}}\}$ for $\varrho(u) := \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ and B be the normalizer of N in $\mathrm{GL}_2(O_{\mathfrak{l}})$ (i.e., B is the upper triangular Borel subgroup). We may regard N as a group scheme over $O_{\mathfrak{l}}$ such that $N(A) = \{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) | u \in A\}$ for an $O_{\mathfrak{l}}$-algebra A. We consider the following conditions for K:

(K) K is closed of the form $K^{(p\mathfrak{l})} \times K_p \times K_{\mathfrak{l}}$ with $\mathrm{GL}_2(O_p) \times N \subset K_{p\mathfrak{l}} \subset \mathrm{GL}_2(F_p) \times \widehat{\Gamma}_0(\mathfrak{l})$,
(I) $\pi : \mathcal{V} \twoheadrightarrow \mathcal{V}_K$ induces $\Xi \cong \Xi_K$,

where $\widehat{\Gamma}_0(\mathfrak{l}^n) = \{g \in GL_2(O_{\mathfrak{l}}) | (g \bmod \mathfrak{l}^n) \in B(O/\mathfrak{l}^n)\}$. We put

$$\widehat{\Gamma}_1(\mathfrak{l}^n) = \{g \in GL_2(O_{\mathfrak{l}}) | (\overline{g} \bmod \mathfrak{l}^n) \in N(O/\mathfrak{l}^n)\}$$

for the image $\overline{g} \in \mathrm{PGL}_2(O_{\mathfrak{l}})$ of $g \in \mathrm{GL}_2(O_{\mathfrak{l}})$. For general $g \in \mathrm{GL}_2(F_{\mathfrak{l}})$, we write $U^g := g^{-1}Ug$ for a subgroup U of $G(\mathbb{A}^{(\infty)})$. In the application in [H04], we assumed K to be $\widehat{\Gamma}_0(\mathfrak{l}) \times \mathrm{GL}_2(\widehat{O}^{(\mathfrak{l})})$. We assume

$$\text{(T)} \qquad \mathbf{T} = \mathcal{T} \times \alpha^{\mathbb{N}} \text{ for } \alpha^{\mathbb{N}} = \{\alpha^n | 0 \leq n \in \mathbb{Z}\} \text{ and a group } \mathcal{T},$$

where $\alpha \in \mathrm{GL}_2(F_{\mathfrak{l}})$ is upper triangular and $\alpha N \alpha^{-1} \supsetneq N$. The condition $\alpha N \alpha^{-1} \supsetneq N$ implies that $\alpha \in B \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right) B$ for some $m > 0$ with a uniformizer $\varpi_{\mathfrak{l}}$ of $O_{\mathfrak{l}}$, and if $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$ $(\nu > 0)$, $U_K := K \cap K^\beta$ is normalized by K and a representative set of $U_K \backslash K$ can be chosen in N. Note that $N\alpha^{\mathbb{N}}N := \bigcup_{\beta \in \alpha^{\mathbb{N}}} N\beta N \subset \mathrm{GL}_2(F_{\mathfrak{l}})$ is a multiplicative semi-group. Consider the following condition

$$(\infty) \qquad \text{every } \mathbf{T}\text{-orbit in } \Xi \text{ is infinite.}$$

This condition will be verified if $\underline{n}$ is an infinite arithmetic progression (Proposition 2.10) for a suitable choice of α and $\mathcal{T}$. Since $\alpha \in B \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right) B$ $(m > 0)$ does not have a fixed point in $\mathcal{V}$, if one orbit $\mathbf{T}(x)$ for $x \in \Xi$ is infinite, every orbit is infinite. For simplicity, we assume hereafter $K_{\mathfrak{l}} = \widehat{\Gamma}_0(\mathfrak{l}^\nu)$ with $\nu > 0$. Since α is supposed to preserve the irreducible component V of $Sh^{(p)}$, we may assume that $\mathfrak{l}^m = (\varpi)$ with $\varpi = \varphi\varphi^c$ for some $\varphi \in R$. Replacing m by a positive integer multiple of m, we may further assume

$$\text{(2.1)} \qquad \text{for } a := \varpi_{\mathfrak{l}}^m/\varpi, \text{ elements } \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \text{ in } G(\widehat{\mathbb{Z}}) \text{ belong to } K.$$

Indeed, by replacing m by mn and ϖ by ϖ^n, a is replaced by a^n which is sufficiently close to 1. Hereafter, for simplicity, we assume that $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ for $\varpi = \varphi\varphi^c$ and write β for a general element in $N\alpha^{\mathbb{N}}N$.

As we recall from [H04] in §3.1, the left action of $g \in G(\mathbb{A}^{(p\infty)})$ on the point $x = (A, \eta) \in \mathcal{V}$ is given by $g(x) = \tau(g)^{-1}(x)$, where the right action $\tau(g)$ is given by $\eta \mapsto \eta \circ g$ for the level structure η giving the point x. If $\beta \in \alpha^{\mathbb{N}}$, $K \supset N$ may not be normalized by β. Thus β acts on $\mathcal{V}_K$ as a correspondence.

Let us explain this correspondence action in some details. Let $U := K \cap \beta^{-1}K\beta = K \cap K^\beta$. By definition $U^{\beta^{-1}} = K^{\beta^{-1}} \cap K \subset K$ and $U^{\beta^{-1}}U \subset K$ (so, $U^{\beta^{-1}}$ satisfies the condition (K) while U is not). Then U is normalized by N if $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$ but $U \supsetneq N$. We have $N\beta N = \bigsqcup_{u \in \mathcal{N}} N\beta\varrho(u)$ for a finite set $\mathcal{N} = \left\{\varrho(u) | u \mod \mathfrak{l}^j\right\}$ for $0 < j$ given by $(\det(\beta)) = \mathfrak{l}^j$ (so, $m|j$), and $K\beta K = \bigsqcup_{u \in \mathcal{N}} K\beta\varrho(u)$. Then we have the correspondence $U(\beta) \subset \mathcal{V}_K \times \mathcal{V}_K$ defined by the following commutative diagram

$$\begin{tikzcd}
\mathcal{V}_U \arrow[r, "v\mapsto\beta(v)=v\beta^{-1}", "\sim"'] \arrow[d, "p_{U,K}"'] & \mathcal{V}_{U^{\beta^{-1}}} \arrow[d, "p_{U^{\beta^{-1}},K}"'] \\
\mathcal{V}_K \arrow[r, "U(\beta)"'] & \mathcal{V}_K,
\end{tikzcd}$$

where $U(\beta)$ is identified with a subvariety given by the diagonal image of $\mathcal{V}_U$ under the product of the projections $p_{U,K}\times(p_{U^{\beta^{-1}},K}\circ\beta)$. It is easy to see $U(\beta^n)=U(\beta)^n$ under the correspondence action. The correspondence $U(\beta)$ brings a point $x\in\mathcal{V}_K$ to a finite set $U(\beta)(x):=(p_{U^{\beta^{-1}},K}\circ\beta)(p_{U,K}^{-1}(x))$. We assume, for K satifying (K),

(N) The action of $\mathbf{T}$ on Ξ extends to a correspondence action of the semi-group $N\mathbf{T}N$ on Ξ_K.

If it is necessary to indicate the dependence of the level group K, we write Ξ_K for the image of Ξ in $\mathcal{V}_K$. We write $U(\beta^n)(\Xi_K):=\bigcup_{x\in\Xi_K}U(\beta^n)(x)$. The condition (N) means that $U(\beta)$ acts on Ξ (i.e., $U(\beta)(\Xi_N)\subset\Xi_N$).

Since $\alpha_{\mathfrak{l}}\in\mathrm{GL}_2(F_{\mathfrak{l}})$, by (2.1), the correspondence $U(\beta)$ for $\beta\in U\alpha^{\mathbb{N}}U$ only depends on the double coset $N\beta N$. We need the following finiteness assertion (which will be verified in Lemma 2.9 and (2.8)):

(F) $\Xi_N-U(\alpha^n)(\Xi_N)$ and $\alpha^{-n}(\Xi)-\Xi$ are finite for all $n>0$.

Since $\mathcal{T}$ is a group, (F) implies finiteness of $\Xi_N-U(\beta)(\Xi_N)$ and $\beta^{-1}(\Xi)-\Xi$ for all $\beta\in\mathbf{T}$. We actually use only the finiteness of $\beta^{-1}(\Xi)-\Xi$ in the proof of the key result (Theorem 2.6).

Let $X=X_\Xi$ (resp. $X_U=X_{\Xi,U}$) be the Zariski closure of Ξ in $\mathcal{V}$ (resp. of the image Ξ_U in $\mathcal{V}_U$) for a closed subgroup U satisfying (K). Since $U(\beta)(\Xi)\subset\Xi$, we find $X_\Xi\supset X_{U(\beta)(\Xi)}=\bigcup_{u\in\mathcal{N}}\beta\varrho(u)(X_\Xi)$. Thus we have a tower $\{X_U\}_U$ of reduced schemes with projections $p_{U',U}:X_{U'}\to X_U$ for $U'\subset U$ (which we write simply $p_{U'}$ if U is clear in the context). If U is open compact, X_U is a reduced variety (i.e., reduced noetherian). The semi-group $N\mathbf{T}N$ acts on X sending $X=X_\Xi$ to $\beta(X)=X_{\beta(\Xi)}$ and also $U(\beta)(X)=X_{U(\beta)(\Xi)}$. For $\beta\in N\mathbf{T}N$ and an open compact subgroup $K\subset G(\mathbb{A}^{(\infty)})$ satisfying (K), taking an open compact subgroup U of K such that $UU^{\beta^{-1}}\subset K$, we have a diagram

$$\begin{tikzcd}
X_U \arrow[r, "v\mapsto\beta(v)"] \arrow[d, "p_{U,K}"'] & \beta(X)_{U^{\beta^{-1}}}\subset X_{U^{\beta^{-1}}} \arrow[d, "p_{U^{\beta^{-1}},K}"'] \\
X_K \arrow[r, "C(\beta)"'] & \beta(X)_K\subset X_K,
\end{tikzcd}\tag{2.2}$$

where $C(\beta)$ is a subvariety given by the diagonal image of X_U under $p_{U,K}\times p_{U^{\beta^{-1}},K}\circ\beta$.

Lemma 2.1. *Assume that $\mathcal{V}_U\to\mathcal{V}_K$ for $U=U_K:=K\cap K^\beta$ is étale. Let $Y^U:=p_{U,K}^{-1}(Z_K)=Z_K\times_{\mathcal{V}_K}\mathcal{V}_U$ for $Z_U\in\mathrm{Irr}(X_U)$ and $Z_K:=p_{U,K}(Z_U)$, and write $Y^U=\bigcup_{Z\in\mathrm{Irr}(Y^U)}Z$. If $Z\neq Z'$ for $Z,Z'\in\mathrm{Irr}(Y^U)$, we have $Z\cap Z'=\emptyset$; so, $Y^U=\bigsqcup_{Z\in\mathrm{Irr}(Y^U)}Z$. If $K_{\mathfrak{l}}=\widehat{\Gamma}_1(\mathfrak{l}^\nu)$ and $Z\in\mathrm{Irr}(Y^U)$, then K normalizes U and for $u\in K/U$, either $u(Z)=Z$ or $u(Z)\cap Z=\emptyset$.*

Proof. Note that $Y^U:=p_{U,K}^{-1}(Z_K)$ is étale finite over Z_K as $\mathcal{V}_U\to\mathcal{V}_K$ is étale. Thus Y^U is equi-dimensional with $\dim Z=\dim Y_U=\dim Z_U=\dim Z_K$ for $Z\in\mathrm{Irr}(Y^U)$. If $\emptyset\neq Z\cap Z'\subsetneq Z$ for $Z\neq Z'$ $(Z,Z'\in\mathrm{Irr}(Y^U))$, $Z\to Z_K$ and $Z'\to Z_K$ are dominant by the equi-dimensionality. Thus by the étale property of $Y^U\to Z_K$, $Z(\mathbb{F})\to Z_K(\mathbb{F})$ and $Z'(\mathbb{F})\to Z_K(\mathbb{F})$ are onto. For $x\in(Z\cap Z')(\mathbb{F})$, $|((Z\cup Z')\times_{Z_K}x_K)(\mathbb{F})|<\deg(Z'/Z_K)+\deg(Z/Z_K)$; so, $Z\cup Z'$ ramifies over $p_{U,K}(Z_U)$ since $\dim Z=\dim Z'$, which is impossible as $Z\cup Z'\hookrightarrow Y^U\to Z_K$ is unramified by Lemma 1.1 (2). This shows the first assertion.

Suppose that $K_{\mathfrak{l}}=\widehat{\Gamma}_1(\mathfrak{l}^\nu)$ and $Z\in\mathrm{Irr}(Y^U)$. Since $U_KN=K$ and N normalizes U_K, K normalizes U_K. If $K_{\mathfrak{l}}=\widehat{\Gamma}_1(\mathfrak{l}^\nu)$, $Y^U=\bigcup_{u\in K/U_K}u(Z)$; so, $u(Z)$ is still an irreducible component of Y^U, and K/U_K acts on $\mathrm{Irr}(Y^U)$. Thus the intersection is either empty or $u(Z)\cap Z=Z$. If $u(Z)\cap Z=Z$, we have $Z\subset u(Z)$. Since they are irreducible and have equal dimension, we conclude $Z=u(Z)$. □

Lemma 2.2. *For a scheme S, recall* $\mathrm{Irr}_+(S) := \bigsqcup_{d>0} \mathrm{Irr}_d(S)$. *Suppose* (F). *Then for* $\beta \in \mathbf{T}$ *and* K *satisfying* (K), *we have* $\mathrm{Irr}_+(U(\beta)(X)_K) = \mathrm{Irr}_+(X_K)$ *and* $\mathrm{Irr}(X_K) - \mathrm{Irr}(U(\beta)(X)_K) \subset (\Xi_K - U(\beta)(\Xi_K))$. *Similarly we have* $\mathrm{Irr}_+(\beta^{-1}(X)_K) = \mathrm{Irr}_+(X_K)$ *and* $\mathrm{Irr}(\beta^{-1}(X)_K) - \mathrm{Irr}(X_K) \subset (\beta^{-1}(\Xi) - \Xi)$.

Proof. As remarked after (F), $\Xi - U(\beta)(\Xi)$ is finite for all $\beta \in \mathbf{T}$. Since $U(\beta)(\Xi) \subset \Xi$, we have a closed immersion $U(\beta)(X_K) \subset X_K$. Since $U(\beta)(X)_K$ is the Zariski closure of $U(\beta)(\Xi_K)$, the finiteness of $\Xi - U(\beta)(\Xi)$ implies $\mathrm{Irr}_+(X_K) = \mathrm{Irr}_+(U(\beta)(X)_K)$ and $\mathrm{Irr}(X_K) - \mathrm{Irr}(U(\beta)(X)_K) \subset (\Xi - U(\beta)(\Xi))$ by Lemma 1.4. The last assertion follows from finiteness of $\beta^{-1}(\Xi) - \Xi$ assumed in (F). □

The semi-group element $\beta \in \mathbf{T}$ acts on $\pi_0(X)$ and $\mathrm{Irr}(X)$ in the sense that β sends $\pi_0(X)$ and $\mathrm{Irr}(X)$ isomorphically onto $\pi_0(\beta(X))$ and $\mathrm{Irr}(\beta(X))$, respectively. Therefore $\beta : x \mapsto \beta(x) = x\beta^{-1}$ induces an isomorphism $\beta_* : \mathrm{Irr}(X_U) \cong \mathrm{Irr}(\beta(X)_{U^{\beta^{-1}}})$. Let Z_U be an irreducible component of X_U and write $\beta(Z_U) \in \mathrm{Irr}(\beta(X)_{U^{\beta^{-1}}})$.

Lemma 2.3. *Suppose that* $U \subset K$ *is a closed subgroup for an open compact subgroup* $K = G(\mathbb{Z}_p) \times K^{(p)}$ *in* $G(\mathbb{A}^{(\infty)})$. *Take* $Z_U \in \mathrm{Irr}(X_U)$ *with* $\dim Z_U = \dim X_U$ *and write* Z_K *for the image of* Z_U *in* X_K. *Then* $Z_K \in \mathrm{Irr}(X_K)$, *and there exists* $x \in \Xi$ *such that its image* x_K *lies in an open subscheme of* Z_K *made of smooth points of* Z_K.

Proof. By Lemma 1.2, we have $Z_K \in \mathrm{Irr}(X_K)$. If $Z_K = X_K$, nothing to prove. We suppose that $Z_K \neq X_K$. Since X_K is noetherian, the Zariski closure $Z_K^\perp$ of $X_K - Z_K$ is a proper closed subscheme of X_K; so, by Zariski density of Ξ_K in X_K, if $\Xi_K \subset Z_K^\perp$, we find $X_K = Z_K^\perp$, a contradiction. Therefore $(Z_K - Z_K^\perp) \cap \Xi_K \neq \emptyset$. For the Zariski closure Z'_K of $(Z_K - Z_K^\perp) \cap \Xi_K$ in Z_K, $Z'_K \cup Z_K^\perp$ contains $((Z_K - Z_K^\perp) \cap \Xi_K) \cup (Z_K^\perp \cap \Xi_K) = \Xi_K$ as $X_K = Z_K \cup Z_K^\perp$. Thus $Z'_K \cup Z_K^\perp = X_K = Z_K \cup Z_K^\perp$. Since $Z_K^\perp$ is a union of irreducible components of X_K different from Z_K, this implies $Z_K \subset Z'_K$, and $(Z_K - Z_K^\perp) \cap \Xi_K$ is Zariski dense in Z_K. We can thus pick x_K in the open subscheme $Z_K - Z_K^\perp$ in Z_K. Since the subscheme of smooth points of $Z_K - Z_K^\perp$ is non-empty and open in Z_K [CRT, Theorem 24.4], we may assume that x_K is a smooth point of $Z_K - Z_K^\perp$. □

For each reduced Zariski closed subset $\mathcal{Y}$ of $\mathcal{V}_U$, we put $\Xi^{\mathcal{Y}} = \mathcal{Y} \cap \Xi_U$.

Lemma 2.4. *Suppose that* K *is an open compact subgroup as in* (K). *Let* $Z_K \in \mathrm{Irr}(X_K)$. *Then* Ξ^{Z_K} *is dense in* Z_K.

Proof. Since $\Xi_K \cap (Z_K - Z_K^\perp)$ is dense in Z_K as seen in the proof of Lemma 2.3, Ξ^{Z_K} containing $\Xi_K \cap (Z_K - Z_K^\perp)$ is dense in Z_K.

We can argue differently. For an irreducible component Z_K of X_K, $Z_K - Z_K^\perp$ is an open subset of X_K; so, any open subset $Y' \subset (Z_K - Z_K^\perp)$, $Y' \cap \Xi_K \neq \emptyset$. Thus $\Xi^{Z_K} = \Xi_K \cap Z_K$ is dense in Z_K. □

Take $x \in \Xi$ and $\beta \in \mathbf{T}$ and fix the open compact subgroup K satisfying (K). Suppose $\mathcal{V} \to \mathcal{V}_K$ is étale. Let $U = U_K = K \cap K^\beta \subset K$ such that $UU^{\beta^{-1}} \subset K$. Take $Y_K \in \mathrm{Irr}_d(X_K)$ with $Y_K \ni x_K$. Let $Y^U := p_{U,K}^{-1}(Y_K)$ for the projection $p_{U,K} : \mathcal{V}_U \to \mathcal{V}_K$. By Lemma 2.1, $Y^U = \bigsqcup_{Z \in \mathrm{Irr}(Y^U)} Z$. By $\Xi \cong \Xi_K$, $\Xi^{Y^U} \cong \Xi^{Y_K}$. We have a partition $\Xi^{Y^U} = \bigsqcup_{Z \in \mathrm{Irr}(Y^U)} \Xi^Z$ for $\Xi^Z = \Xi_U \cap Z$.

Suppose $K_{\mathfrak{l}} = \widehat{\Gamma}_1(\mathfrak{l}^\nu)$. Assume that $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is étale. Since the diagram

$$\begin{tikzcd}
Y^U \arrow[r, "\hookrightarrow"] \arrow[d] & \mathcal{V}_U \arrow[d, "p_{U,K}"', "\text{étale}"] \\
Y_K \arrow[r, "\hookrightarrow"] & \mathcal{V}_K
\end{tikzcd}$$

is cartesian, $Y^U \to Y_K$ is étale. Therefore, Y^U is equi-dimensional with $\dim Y^U = \dim Y_K$. By Lemma 1.2, $\mathrm{Irr}(Y^U) \cap \mathrm{Irr}(X_U) \neq \emptyset$; so, we can define a non-empty subscheme Y_U of Y^U by

$$Y_U := \bigcup_{Z \in \mathrm{Irr}(X_U) \cap \mathrm{Irr}(Y^U)} Z \overset{(*)}{=} \bigsqcup_{Z \in \mathrm{Irr}_d(X_U) \cap \mathrm{Irr}(Y^U)} Z \subset X_U, \tag{2.3}$$

which is equi-dimensional with dimension $d := \dim Y_K$, and the identity $(*)$ follows from Lemma 2.1 under étaleness of $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$. Thus $\mathrm{Irr}(X_U) \cap \mathrm{Irr}(Y^U) = \mathrm{Irr}_d(X_U) \cap \mathrm{Irr}(Y^U)$. Note that taking intersection $\mathrm{Irr}(Y^U) \cap \mathrm{Irr}(X_U) \neq \emptyset$ means that we can pick irreducible components of X_U which dominates Y_K (so, each member of $\mathrm{Irr}(Y^U) \cap \mathrm{Irr}(X_U) \neq \emptyset$ has dimension equal to $\dim Y_K$). By

Lemma 2.1, Y^U is a disjoint union of Y_U and $\bigsqcup_{Z\in\mathrm{Irr}(Y^U)-\mathrm{Irr}(Y_U)} X$, and hence $Y_U \to Y_K$ is étale finite dominant.

Lemma 2.5. *Suppose that K is an open compact subgroup as in* (K) *and pick $Y_K \in \mathrm{Irr}_d(X_K)$ for $0 \le d \le \dim X$. Suppose $\mathcal{V}_U \to \mathcal{V}_K$ is étale. Then we have $Y^U = \bigsqcup_{Z\in\mathrm{Irr}(Y^U)} Z$ and $Y_U = \bigsqcup_{Z\in\mathrm{Irr}(Y^U), Z\subset X_U} Z$. The set Ξ^Z is either empty or Zariski dense in Z for $Z \in \mathrm{Irr}(Y^U)$, and for each $x \in \Xi^{Y_U}$, there is a unique irreducible component $Z \in \mathrm{Irr}(Y_U)$ with $x \in Z$.*

Proof. The first assertion is proven before the statement of the lemma. We prove the remaining assertion. If $Z \subset X_U$ for $Z \in \mathrm{Irr}(Y^U)$, it is an irreducible component of X_U by Lemma 1.2. Thus Ξ^Z is Zariski dense in Z by Lemma 2.4. In other words, if $Z \not\subset X_U$, Ξ^Z is an empty set, and for each $x \in \Xi^{Y_U}$, there is a unique irreducible component $Z \in \mathrm{Irr}(Y_U)$ with $x \in Z$ as Y_U is a disjoint union of Z. □

2.2. **Modular correspondence acting on irreducible components.** Pick an irreducible component $Y_K \in \mathrm{Irr}_d(X_K)$ for $0 \le d \le \dim X_K$ with an open compact subgroup K satisfying (K).

2.2.1. *Definition of the correspondence.* Choosing $x \in \Xi$ so that $x_U \in Y_U$ for Y_U in (2.3), we have $\beta(x)_U := x_U\beta^{-1} \in \beta(Y_U) \subset X_{U^{\beta^{-1}}}$, and there is a unique irreducible component Z of Y_U containing x_U by Lemma 2.5. Since $Y_U \xrightarrow[\beta]{\sim} \beta(Y_U) \subset \beta(X)_{U^{\beta^{-1}}} \subset X_{U^{\beta^{-1}}}$, we have $\dim Y_K = \dim Y_U = \dim \beta(Y_U) = \dim \beta(Y_U)_K$ for the projection $\beta(Y_U)_K$ of $\beta(Y_U)$ in X_K.

For any pair of open compact subgroups (K, U) with $K \supset UU^{\beta^{-1}}$ (so, $U \subset K \cap \beta^{-1}K\beta$), we have a diagram similar to (2.2):

$$\begin{tikzcd}
Y_U \arrow[r, "v\mapsto\beta(v)", "\sim"'] \arrow[d, "p_{U,K}"', "\text{finite}"] & \beta(Y_U) \arrow[r, "\subset"] \arrow[d] & X_{U^{\beta^{-1}}} \arrow[d, "p_{U^{\beta^{-1}},K}"', "\text{finite}"] \\
Y_K \arrow[r, "C_U(\beta)"'] & \beta(Y_U)_K := p_{U^{\beta^{-1}},K}(\beta(Y_U)) \arrow[r, "\subset"'] & X_K,
\end{tikzcd}$$

for the correspondence $C_U(\beta)$ given by the reduced image $\mathrm{Im}(p_{U,K} \times p_{U^{\beta^{-1}},K} \circ \beta : Y_U \to \mathcal{V}_K \times \mathcal{V}_K)$ whose support is contained in $C(\beta)$ in (2.2). Note that

$$C_U(\beta) \textit{ is independent of the choice of } U \tag{2.4}$$

as $p_{U,K} \times p_{U^{\beta^{-1}},K} \circ \beta = (p_{U_K,K} \times p_{U_K^{\beta},K} \circ \beta) \circ p_{U,U_K}$ for $U_K = K \cap \beta K \beta^{-1}$ (so, $C_U(\beta) = C_{U_K}(\beta)$).

Hereafter we choose U to be U_K and still write it as U (so, the correspondence action of $C_U(\beta)$ on irreducible components we introduce in the proof of the following Theorem 2.6 only depends on β (and K)). Note that $\beta(Y_U)_K = \bigcup_u \beta u(Z)_K$ for some $u \in \mathcal{N} \cong K/U_K$, where $\beta u(Z)_K$ is the image under $p_{U^{\beta^{-1}},K}$ of $\beta u(Z)$ for a component $Z \in \mathrm{Irr}(Y_U)$ (cf. Lemma 2.1). Since $\beta : X_U \cong \beta(X)_{U^{\beta^{-1}}}$, $\mathrm{Irr}(\beta(Y_U)) \subset \mathrm{Irr}_d(\beta(X)_{U^{\beta^{-1}}})$. By the above diagram with dominant $p_{U,K}$ and $p_{U^{\beta^{-1}},K}$, we again find $\dim \beta(Y_U)_K = \dim Y_K$ as $\beta(Y_U)_K \subset p_{U^{\beta^{-1}},K}(\beta(p_{U,K}^{-1}(Y_K)))$.

2.3. **Positive dimensionality of irreducible components of X.** We now prove the following fact not described in [H04]:

Theorem 2.6. *Suppose* (unr) *and* (ord) *in* [H04, §2.1] *for p. Let $\Xi \subset \mathcal{V}(\mathbb{F})$ be an infinite subset injecting into $\mathcal{V}_K$ for any open compact subgroup K satisfying* (K) *and* (I). *We assume that a semi-group $\mathbf{T} \subset \mathrm{Aut}(V/\mathbb{F})$ as in* (T) *embedded in $\mathrm{Aut}(\mathcal{V}/\mathbb{F})$ acts on Ξ, and assume* (F) *and* (N).

1. *If the condition (∞) is satisfied, all irreducible components of X has positive dimension;*
2. *If $\mathbf{T}$ acts on Ξ transitively, $\dim X > 0$, X is equi-dimensional, and the irreducible component containing a given $x \in \Xi$ is unique.*

Under (∞), we can replace Ξ by an infinite orbit $\mathbf{T}(x)$ and apply the result and conclude the Zariski closure of $\mathbf{T}(x)$ is equidimensional of positive dimension; so, one of them contains x.

Proof of (1).[1] We need to describe the correspondence action of β on $Y_K \in \mathrm{Irr}_d(X_K)$. First suppose that $d = \dim X$ as this is the easiest case. Then $\dim \beta(Y_U)_K = \dim X$, and hence $\beta(Z)_K \in$

---

[1] In the proof, we use the existence of $\beta^{-1} \in \mathrm{Aut}(V)$ essentially, while $\alpha : (v, z) \mapsto (v + 1, 2z)$ in §1.1 canot be extended to an automorphism of V there.

$\mathrm{Irr}_{\dim X}(X_K)$ for $Z \in \mathrm{Irr}(Y_U)$. In this way, $\beta \in \mathbf{T}$ acts on $Y_K \in \mathrm{Irr}_{\dim X}(X_K)$ as correspondences (i.e., Y_K is brought to a subset $\beta(Y_K) = \{\beta(Z)_K | Z \in \mathrm{Irr}(X_U) \cap \mathrm{Irr}(Y^U)\} \subset \mathrm{Irr}_{\dim X}(X_K)$ whose member has equal dimension). Since $\mathrm{Irr}(X_U) \cap \mathrm{Irr}(Y^U)$ is made of $u(Z)$ for $u \in N$, for a finite subset B of $N\beta N$, we have the correspondence action of $C(\beta)$ given by the image set $\beta(Y_K) := \bigcup_{\beta' \in B}\{\beta'(Y_K)\}$ under $C(\beta)$ on $\mathrm{Irr}_{\dim X}(X_K)$.

Though we only need the result for $d = 0$, we give an argument in the case $0 < d < \dim X$ now as this introduces necessary notation for the case $d = 0$. Pick $Y_K \in \mathrm{Irr}_d(X_K)$ and start with $Y_K \in \mathrm{Irr}_d(X_K)$. As above to define the action of $C(\beta)$ on $\mathrm{Irr}_d(X_K)$, we only need to give a good definition of the image set $\beta(Y_K)$ for a $\beta \in N\mathbf{T}N$. For simplicity, write $U' := U^{\beta^{-1}}$. Let us recall a general notation: For an irreducible component Y'_K of X_K, we define as before $Y'^{U'} := p^{-1}_{U',K}(Y'_K)$ and $Y'_{U'} := \bigsqcup_{Z' \in \mathrm{Irr}_+(X_{U'}) \cap \mathrm{Irr}_+(Y'^{U'})} Z'$ (by (2.3)). Now recall the irreducible component $Z \in \mathrm{Irr}_+(Y_U)$ containing the base point $x_U \in \Xi_U$ chosen in §2.2.1 and we apply the above notation to the irreducible component Y'_K of X_K such that $\beta(Z) \subset Z'$ for an irreducible component Z' of $Y'_{U'}$ (so, $\beta(x_U) \in Z'$). To see the existence of an irreducible component Y'_K of X_K as above, we argue as follows. Since $\beta(Z)$ is an irreducible closed variety of $X_{U'}$, $p_{U',K}(\beta(Z))$ is an irreducible closed variety of X_K. Then there exists an irreducible component Y'_K containing $p_{U',K}(\beta(Z))$ of X_K by Lemma 1.2 (1). Therefore $\beta(Z) \subset Y'_{U'}$ which is contained in $Z' \in \mathrm{Irr}(Y'_{U'})$. So $\dim Z' = \dim Y'_K \geq d$ by Lemma 2.1. Replacing (β, Y_K, U, K) by $(\beta^{-1}, Y'_K, U', K)$, we apply the above argument. Note that $\beta^{-1}(Z') \subset \beta^{-1}(X_{U'})$; so, $\beta^{-1}(Z')_K \subset \beta^{-1}(X)_K$. By the choice of Y'_K, Lemma 2.5 tells us that Z is determined by the two conditions (i) $\beta^{-1}(X)_K \supset \beta^{-1}(Z')_K \supset Y_K$ and (ii) $x_U \in Z$. Since $\mathrm{Irr}_+(\beta^{-1}(X)_K) = \mathrm{Irr}_+(X_K)$ by Lemma 2.2 and $\beta^{-1}(Z')_K$ is irreducible, we conclude from $\beta^{-1}(Z')_K \supset Y_K$ that $\beta^{-1}(Z')_K = Y_K$ (as Y_K is an irreducible component of $\mathrm{Irr}_+(\beta^{-1}(X)_K) = \mathrm{Irr}_+(X_K)$); in particular, $\dim Z' = \dim Y_K = d$. So, $Y'_K = \beta(Z)_K$ and that $\beta(Z)_K$ is an element in $\mathrm{Irr}_d(X_K)$ (Lemma 2.2). Therefore, again $\beta \in \mathbf{T}$ acts on $\mathrm{Irr}_d(X_K)$ as correspondences (i.e., Y_K is brought to a subset $\beta(Y_K) = \{\beta(Z)_K | Z \in \mathrm{Irr}(X_U) \cap \mathrm{Irr}(Y^U)\} \subset \mathrm{Irr}_d(X_K)$ whose member has equal positive dimension).

Now suppose $d = 0$. Since the correspondence action preserves $\mathrm{Irr}_+(X_K)$, it also preserves the complement $\mathrm{Irr}_0(X_K)$. The following argument to see the correspondence action is really an action sending a point to a point also gives an alternative proof of the stability of $\mathrm{Irr}_0(X_K)$ under the action of $\mathbf{T}$. We proceed similarly to the case where $0 < d < \dim X$ using the same notation. Then $x_K = Y_K \in \mathrm{Irr}_0(X_K)$ falls in the image Ξ_K in $\mathcal{V}_K$ of Ξ by Lemma 1.2 (3). By (I), the projection $\pi : \mathcal{V} \twoheadrightarrow \mathcal{V}_K$ induces $\Xi \cong \Xi_K$; so, $p^{-1}_{U,K}(x_K)$ is a finite set of points above x_K and $\{x' \in p^{-1}_{U,K}(x_K) | x' \in X_U\}$ is a singleton by Lemma 1.2 (2–3). Thus $p^{-1}_{U,K}(Y_K) \cap X_U = p^{-1}_{U,K}(x_K) \cap X_U$ is a singleton. Therefore $Y_U = \{Z := x_U\}$ is a singleton. Take an irreducible component Y'_K of X_K such that $\beta(Z) \subset Z'$ for an irreducible component Z' of $Y'_{U'}$ (so, $\beta(x_U) \in Z'$). Such a Y'_K exists by Lemma 1.2 (1). So $\dim Z' = \dim Y'_K \geq 0$. We want to prove $\dim Y'_K = 0$. Since $\mathrm{Irr}_+(\beta^{-1}(X)_U) = \mathrm{Irr}_+(X_U)$ by Lemma 2.2, if $\dim Z' > 0$, we have $\dim \beta^{-1}(Z') > 0$ and $\beta^{-1}(Z')$ is an irreducible component of X_U. Since $\beta^{-1}(Z') \supset Z = x_U$ by construction and the two are irreducible components of X_U, we find that $\beta^{-1}(Z') = Z = x_U$, a contradiction against $\dim Z' > 0$. Hence $\dim Z' = 0$ and $Z' = \beta(Z) = \beta(x_U)$. This implies that β brings $\mathrm{Irr}_0(X_K)$ into $\mathrm{Irr}_0(X_K)$. It is now clear that this is really an action (not a correspondence action) of $\mathbf{T}$ on $\mathrm{Irr}_0(X_K)$, and the action is compatible with the action of $\mathbf{T}$ on Ξ as $\mathrm{Irr}_0(X_K) \subset \Xi_K \cong \Xi$.

In particular, $\mathrm{Irr}_0(X_K)$ contains $\mathbf{T}(x_K)$ for each $x_K \in \mathrm{Irr}_0(X_K) \subset \Xi_K$. Then by (∞), $\mathrm{Irr}_0(X_K)$ is infinite, a contradiction as X_K is a noetherian scheme. Therefore $\mathrm{Irr}_0(X_K) \cap \mathbf{T}(x_K) = \emptyset$ for every open compact subgroup K of $G(\mathbb{A}^{(p)})$ satisfying (K) and $x_K \in \mathrm{Irr}_0(X_K)$. This implies $\mathrm{Irr}_0(X_K) = \emptyset$ for every open compact subgroup K of $G(\mathbb{A}^{(p)})$ satisfying (K), and therefore $\mathrm{Irr}_0(X) = \emptyset$ by Lemma 1.2 (3). This shows that all irreducible components of X have positive dimension.

Proof of (2). We have proven positive dimensionality of irreducible components of X. We need to prove equi-dimensionality of X and the uniqueness of the component containing $x \in \Xi$ under equi-dimensionality. Since the smooth locus X^{sm}_K of X_K is open dense in X_K by [CRT, Theorem 24.4], $\Xi^{sm}_K := \Xi \cap X^{sm}_K$ is still dense in X_K. Since $\mathrm{Irr}(X_K) = \pi_0(X^{sm}_K)$, for each $x \in \Xi^{sm}_K$, the irreducible component of X^{sm}_K containing x_K is unique. Since $\mathbf{T}$ acts on $\mathrm{Irr}(X_K) = \pi_0(X^{sm}_K)$ as correspondence, for any $Z, Z' \in \mathrm{Irr}(X_K)$, we find $x \in \Xi \cap Z^{sm}$ and $y \in \Xi \cap Z'^{sm}$.

Interchanging Z and Z' if necessary, by (T) and transitivity of the action, we can choose $\beta \in \mathbf{T}$ with $\beta(x) = y$. Then $\beta(Z) \in \mathrm{Irr}(X_K)$ and $y \in \beta(Z) \cap Z'$. Thus $y \in Z'^{sing} = Z' - Z^{sm}$ (i.e., $y \in \beta(Z) \cap Z'$ with Z' different from any irreducible components of $\beta(Z)$) or $\beta(Z) \supset Z'$ or $\beta(Z) \subset Z'$. The case: $y \in Z'^{sing} = Z' - Z^{sm}$ does not occur as we have chosen $y \in Z'^{sm}$. Since the correspondence action of $\mathbf{T}$ preserves $\mathrm{Irr}_d(X_K)$ for any given $d > 0$, the remaining cases $\beta(Z) \supset Z'$ or $\beta(Z) \subset Z'$ imply $\dim Z = \dim \beta(Z) = \dim Z'$ and $Z' \in \mathrm{Irr}(\beta(Z))$. Choosing one of Z and Z' to have maximal dimension $\dim X$, the other has to have maximal dimension; so, $\mathrm{Irr}(X_K) = \mathrm{Irr}_{\dim X}(X_K)$; so, X_K is equidimensional. This implies X is equidimensional.

By the first fundamental sequence of differentials and unramifiedness of X_U/X_K in Lemma 1.1, the projection induces a surjection:

$$\Omega_{X_K/\mathbb{F}} \otimes_{\mathcal{O}_{X_K}} \mathbb{F}(x_K) \twoheadrightarrow \Omega_{X_U/\mathbb{F}} \otimes_{\mathcal{O}_{X_U}} \mathbb{F}(x_U)$$

for $U \subset K$. By the proof of the equi-dimensionality, for $\dim \mathcal{O}_{X_U,x_U} = \dim \mathcal{O}_{X_K,x_K}$ for any point $x_U \in X_U$ with projection x_K in X_K. Thus

$$\dim_{\mathbb{F}} \Omega_{X_K/\mathbb{F}} \otimes_{\mathcal{O}_{X_K}} \mathbb{F}(x_K) \geq \dim_{\mathbb{F}} \Omega_{X_U/\mathbb{F}} \otimes_{\mathcal{O}_{X_U}} \mathbb{F}(x_U) \geq \dim \mathcal{O}_{X_U,x_U} = \dim \mathcal{O}_{X_K,x_K}.$$

Here "$\dim_{\mathbb{F}}$" indicates dimension of an $\mathbb{F}$-vector space, and $\dim R$ for a ring R means the Krull dimension of the ring R. Thus the singular locus

$$X_U^{sing} := \{x_U \in X_U \,|\, \dim_{\mathbb{F}} \Omega_{X_U/\mathbb{F}} \otimes_{\mathcal{O}_{X_U}} \mathbb{F}(x_U) > \dim \mathcal{O}_{X_U,x_U}\}$$

of X_U is sent to X_K^{sing}, where $\mathbb{F}(x)$ is the residue field of x. Thus $X^{sing} = \varprojlim_U X_U^{sing}$, and hence $\dim X^{sing} < \dim X = \dim Y$ for any $Y \in \mathrm{Irr}(X)$. Plainly $\mathbf{T}$ preserves X^{sing}. If $x \in \Xi \cap X^{sing}$, then $\Xi = \mathbf{T}(x) \subset X^{sing}$; so, $X = X^{sing}$, a contradiction. Thus $\Xi \cap X^{sing} = \emptyset$. Since $X^{sm} := X - X^{sing}$ is a dense open subscheme of X, $\pi_0(X^{sm}) \cong \mathrm{Irr}(X^{sm}) \cong \mathrm{Irr}(X)$ with $X^{sm} = \bigsqcup_{Y \in \mathrm{Irr}(X)} Y^{sm}$. Thus for each given $x \in \Xi \subset X^{sm}$, $Y^{sm} \in \mathrm{Irr}(X^{sm})$ containing x is unique. □

Since X has positive dimension (as Ξ is infinity; cf. Lemma 1.2 (4)), by the above proposition, all components have positive dimension. Taking $x \in \Xi$ and an irreducible component of X containing x, we get

Corollary 2.7. *Let the notation and the assumption be as in* Theorem 2.6. *Then X contains an irreducible component of positive dimension with a point $x \in \Xi$. Moreover for each element ξ of the stabilizer of x in $\mathrm{GL}_2(F_{\mathbb{A}}^{(p\infty)})$, we have $\xi(X) = X$.*

Proof. We need to prove the last asertion: $\xi(X) = X$. Taking a level group K sufficiently small, $\xi(X) \cup X \to X_K$ is unramified. Since $\alpha(X) \cap X \ni x$, unramifiedness tells us that $\xi(X) = X$. □

Remark 2.8. Note that the stabilizer of $x \in \Xi_n$ is given by $M^\times \cap R_{n,\mathfrak{l}}^\times$ embedded into $\mathrm{GL}_2(F_{\mathbb{A}}^{(\mathfrak{l}p\infty)}) \subset \mathrm{GL}_2(F_{\mathbb{A}}^{(p\infty)})$ which after p-adic completion contains a p-adically open subgroup. Thus the stability of X in Corollary 2.7 is a requirement of [H10, Corollary 3.19, Theorem 3.20] we apply here to X.

2.4. **Verification of (F) and (N) for an infinite arithmetic progression.** Let

$$\mathcal{T} := R_{(p\mathfrak{l})}^\times / O_{(p\mathfrak{l})}^\times. \tag{2.5}$$

For each $\xi \in R_{(p)}^\times$ (in [H04, page 755] the symbol "α" is used for the letter "ξ" here), we have $x(\mathcal{A}) := (X(\mathcal{A}), \overline{\Lambda}(\mathcal{A}), \eta^{(p)}(\mathcal{A})) \xrightarrow[\xi]{\sim} x(\xi\mathcal{A})$ as in the middle of [H04, page 755], and as seen in [H04, page 756], $\rho_R(\xi^{(\mathfrak{l})})(x(\mathcal{B})) = x([\xi^{(\mathfrak{l})}]\mathcal{B})$ for the class $[\xi^{(\mathfrak{l})}] = [(\xi)]$ of the ideal (ξ) in $Cl_\infty = \varprojlim_n Cl_n$. Recall $C_n = \{\mathbf{x}(\mathcal{A}) := (x([\mathcal{A}]\delta))_{\delta \in \mathcal{Q}} | \mathcal{A} \in Cl_n\}$ and $C^{(\infty)} = \bigsqcup_n C_n \subset \mathcal{V}$. Thus $\xi \in \mathcal{T}$ acts on $C^{(\infty)}$ by $[\mathcal{A}] \mapsto [(\xi)][\mathcal{A}]$.

Let $\varpi_{\mathfrak{l}}$ be a prime element of $O_{\mathfrak{l}}$. As specified in [H04, §2.1 and §3.1], for each proper fractional ideal $\mathcal{A}$ of R_n, we have a specific CM point $x(\mathcal{A}) \in Sh^{(p)}(\mathbb{F})$. In our application, Ξ is made of the set of points of the form $x(\mathcal{A})$. Note that

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & \frac{u}{\varpi_{\mathfrak{l}}^m} \\ 0 & 1 \end{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right).$$

By [H04, (3.2)] (see (3.2) in the text), writing $\alpha_m := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^m \end{smallmatrix}\right)$ and $\varrho(u) := \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$\alpha_m \varrho(u)(x_N(R_n)) = \varrho(\frac{u}{\varpi_{\mathfrak{l}}^m})\alpha_m(x_N(R_n)) = x_N(\mathcal{A}) \quad (0 < m \in \mathbb{Z}) \tag{2.6}$$

for $\mathcal{A}$ given by $x(\mathcal{A}) = x(R_n)/C$ for a suitable subgroup $C \subset X(R_n)$ with $C \cong O/\mathfrak{l}^m$ depending on u, x_N indicates the image of x in $\mathcal{V}_N$, and $\mathcal{A} = R_{n+m}$ if $u = 0$. By (2.1), in (2.6), we can replace $\varpi_{\mathfrak{l}}^m$ by $\varpi = \varphi\varphi^c$ and x_N by x_K, and the identity is valid on $\mathcal{V}_K$ (in place of $\mathcal{V}_N$). Any $\mathcal{A}$ in $\mathrm{Ker}(Cl_{n+m} \to Cl_n)$ with $n > 0$ can be written as in (2.6).

Set $\Xi_j^n = \{\mathbf{x}(\mathcal{A}) \in \mathcal{V} | \mathcal{A} \in \mathrm{Ker}(Cl_n \to Cl_r)\}$ for $n > j \geq 0$. Since

$$\Xi_j^n = \{\mathbf{x}([\mathcal{A}]\delta) | \mathcal{A} = \xi R_n \text{ with } \xi \in R_{(p\mathfrak{l})}^\times \cap (1 + \mathfrak{l}^j R_{\mathfrak{l}})\},$$

$$\mathcal{T}_j := \{\xi \in \mathcal{T} | (\xi \bmod \mathfrak{l}^j) \in (R_{(p\mathfrak{l})}/\mathfrak{l}^j)^\times\}/O_{(p\mathfrak{l})}^\times \quad (\text{for } \mathcal{T} \text{ in (2.5)}) \tag{2.7}$$

acts transitively on Ξ_j^n for every $n > j \geq 0$. Here $R_{(p\mathfrak{l})}$ and $O_{(p\mathfrak{l})}$ are the localization at $p\mathfrak{l}$ of R and O not the completion.

Lemma 2.9. *Assume that* $\mathfrak{l}^m$ *is generated by an element of* $N_{M/F}(R)$ *and write* $\mathfrak{l}^m = (\varpi)$ *with* $\varpi \in N_{M/F}(R)$. *Define* $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ *and let* $\mathbf{T} = \mathbf{T}_{j,m} = \mathcal{T}_j \times \bigcup_{k \geq 0} N\alpha^k N$ *as a semi-group. If* $\underline{n}$ *is an infinite arithmetic progression of difference* m, *for* $\Xi = \Xi_{\underline{n},j}$, *we have* $\Xi \supset U(\beta)(\Xi)$ *for* $\beta \in \mathbf{T}_{j,m}$ (*which implies that the condition* (N) *is satisfied*), *and* $\Xi - U(\beta)(\Xi)$ *is finite.*

Proof. By (2.6), we have $U(\alpha)(C_n) = C_{n+m}$ and $U(\alpha)(\Xi_j^n) = x(\Xi_j^{n+m})$. Thus the semi-group $\bigcup_{k \geq 0} N\alpha^k N$ acts on $\Xi_{\underline{n}} = \bigsqcup_{i \geq 0} \Xi_j^{n_0+im}$ for $\underline{n} = \{n_0 + im | i = 0, 1, 2 \ldots\}$ for $U(\alpha^k)$ $(0 \leq k \in \mathbb{Z})$ sending $\Xi_j^{n_0+im}$ into $\Xi_j^{n_0+(i+k)m}$. Any element of $\Xi_j^{n_0+(i+k)m}$ is an image of an element of $\Xi_j^{n_0+im}$ under the action of $\beta \in N\alpha^k N$. Then we have

$$\Xi_{\underline{n}} - U(\alpha^k)(\Xi_{\underline{n}}) = \bigsqcup_{i \geq 0} \Xi_j^{n_0+im} - \bigsqcup_{j \geq 0} \Xi_j^{n_0+(k+i)m} = \bigsqcup_{i=0}^{k} \Xi_j^{n_0+im}$$

which is finite. Since $\mathcal{T}_j$ is a group acting transitively on $\Xi_j^{n_0+im}$, this implies $\Xi - U(\beta)(\Xi)$ is finite for all $\beta \in N\mathbf{T}N$ and hence we get (N) and (F) for $U(\beta)$. $\square$

The point $x(\mathcal{A})$ is given by identifying $\widehat{\mathcal{A}}^{(p)} = \mathcal{A} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ with the prime-to-p Tate module of the corresponding CM abelian variety $X(\mathcal{A})$; so, strictly speaking it is more precise to write $x(\widehat{\mathcal{A}}^{(p)})$ (or $x(\widehat{\mathcal{A}})$) in place of $x(\mathcal{A})$. Under this notation, $\varrho(x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n,\mathfrak{l}})) = x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n+m,\mathfrak{l}})$ and $\alpha^{-1}(x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n,\mathfrak{l}})) = x(\widehat{\mathcal{A}}^{(p\mathfrak{l})} \times R_{n-m,\mathfrak{l}})$ as long as $n > m$. See §4.3 for what happens when $n \leq m$.

It appears that the map α^{-1} is non-injective, but this comes from the fact that $K \supset N$ (satisfying (K)) but U_K is not; in other words, $p_{U,K}$ is not injective but shrinking K to K' at $\mathfrak{l}$ so that $K' \cap \mathcal{N} = \{1\}$ (but with $K/N \cong K'/(K' \cap N)$), the fiber of $p_{U_K,K}$ will be separated modulo $U_K \cap K'$ (but the fiber of $p_{U_{K'},K'}$ is non-trivial again). Thus $\alpha^{-1}(C_n) = C_{n-m}$ as long as $n > m$. This shows (F) for α^{-i} and $\Xi_{\underline{n},j}$:

$$\alpha^{-i}(\Xi) - \Xi \text{ is finite for all } i > 0 \tag{2.8}$$

as long as $\underline{n}$ is an infinite arithmetic progression of difference m as long as $\mathfrak{l}^m$ is generated by an element of $N_{M/F}(R)$ and $\alpha\,\mathrm{diag}[1, \varpi_{\mathfrak{l}}]^{-m} \in K$, denoting by $\mathrm{diag}[a, b]$ for the diagonal matrix with diagonal entries a, b from top to bottom. Therefore in this case the condition (F) is valid.

If necessary, we also write sometime $x(\widehat{\mathcal{A}})$ $(\widehat{\mathcal{A}} = \mathcal{A} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ for $x(\mathcal{A})$ assuming $\mathcal{A}_p = R_p$. The group $\mathcal{T}$ acts on $x(\mathcal{A})$ as follows: For $\xi \in \mathcal{T}$,

$$x(\mathcal{A}) = x(\widehat{\mathcal{A}}) \mapsto x(\xi^{(\mathfrak{l})}\widehat{\mathcal{A}}) = x((\xi)\mathcal{A}),$$

where $\xi^{(\mathfrak{l})} \in M_{\mathbb{A}}^\times$ is the finite idele with $\mathfrak{l}$-component equal to 1 and every component at finite place outside $\mathfrak{l}$ is equal to ξ.

In the idele class group $\mathcal{I} := M_{\mathbb{A}}^\times / M^\times M_\infty^\times$, ξ is trivial but $\xi^{(\mathfrak{l})}$ is not trivial; so, the action of ξ on Cl_n is non-trivial. Regard $Cl_n = \mathrm{Pic}(R_n)$ as a quotient of $\mathcal{I}$, and write $(\xi) = (\xi)_n$ for the image of $\xi^{(\mathfrak{l})}$ in Cl_n. Since $\mathrm{Ker}(Cl_n \to Cl_0)$ is spanned by $(\xi)_n$ with ξ running in $\mathcal{T}$, $\mathcal{T}$ acts transitively on $\mathrm{Ker}(Cl_n \to Cl_0)$. More generally, noting that $\mathcal{T}_r \subset \mathcal{T}$ is the stabilizer of $x(R_r)$ in Cl_r, $\mathcal{T}_r$ acts transitively on $\mathrm{Ker}(Cl_n \to Cl_r)$ for all $n \geq r$. From $\widehat{\mathcal{A}}$ with $\mathcal{A}_{\mathfrak{l}} = R_{n,\mathfrak{l}}$, we can create $\widehat{\mathcal{A}}_i := \widehat{\mathcal{A}}^{(\mathfrak{l})} \times R_{i,\mathfrak{l}}$. Then even if $\mathcal{A} = \xi R_n$ with $\xi_{\mathfrak{l}} \in R_{n,\mathfrak{l}}$ (i.e., $\mathcal{A}$ is trivial in Cl_n), for $i > n$ with $\xi_{\mathfrak{l}} \notin R_{i,\mathfrak{l}}$, $\widehat{\mathcal{A}}_i$ is non-trivial in Cl_i. In this way, the group $\mathcal{T} := R_{(p\mathfrak{l})}^\times / O_{(p\mathfrak{l})}^\times$ acts on C^{alg} as in [H04, page 755].

Let $\underline{n} = \{0 < n_0 < n_1 < n_2 < \cdots < n_i < \cdots\}$ be an infinite sequence of integers such that $\mathfrak{l}^{n_i}$ is generated by an elements in $N_{M/F}(R)$. If m is an exponent such that $\mathfrak{l}^m$ is generated by an elements in $N_{M/F}(R)$, then any infinite arithmetic progression $\underline{n} = \{n_i = n_0 + im | 0 \leq i \in \mathbb{Z}\}$ for an initial value $0 < n_0$ satisfies this condition. Recall $\Xi_j^{n_i} = \{(x([\mathcal{A}]\delta))_{\delta \in \mathcal{Q}} \in \mathcal{V} | [\mathcal{A}] \in \mathrm{Ker}(Cl_{n_i} \to Cl_j)\}$ for $0 < j \leq n_0$ as in [H04, Proposition 2.7]. Define $\Xi = \Xi_{\underline{n},j} = \bigsqcup_i \Xi_j^{n_i} \subset \mathcal{V}$. Since Cl_{n_i} and Cl_j is stable under the action of $\mathcal{T}_j$ and the projection $Cl_{n_i} \to Cl_j$ is compatible with the action of $\mathcal{T}_j$, Ξ_{n_i} is stable under $\mathcal{T}_j$, and hence $\Xi_{\underline{n},j}$ is also stable under $\mathcal{T}_j$. Thus we get

Theorem 2.10. *Choose $0 < m \in \mathbb{Z}$ so that $\mathfrak{l}^m$ is principal generated by $\varpi = \varphi\varphi^c$ with $\varphi \in R$ and define α as in* Lemma 2.9. *Suppose $\alpha\,\mathrm{diag}[1, \varpi_{\mathfrak{l}}]^{-m} \in K$. If $\underline{n}$ is an infinite arithmetic progression (with initial value n_0 and difference m), the semi-group $\mathbf{T}_{j,m}$ generated by the group $\mathcal{T}_j$ in* (2.7) *and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ acts transitively on $\Xi_{\underline{n},j}$ and satisfies* (T), (N) *and* (F) *(for $\mathcal{T} = \mathcal{T}_j$).*

Theorem 2.6 combined with this result, Corollary 2.7 and [H10, Corollary 3.19, Theorem 3.20] gives

Corollary 2.11. *If $\underline{n}$ contains an arithmetic progression, then $\Xi_{\underline{n},j}$ for any $j \geq r$ is Zariski dense in $V^{\mathcal{Q}}$.*

2.5. **Characteristic** 0 **version.** We consider $Sh^{(p)}_{/\mathcal{W}}$ and its geometric irreducible component $V_{/\mathcal{W}}$ and define $\mathcal{V} = V^{\mathcal{Q}}_{/\mathcal{W}}$ in the same manner as above. Consider $\mathcal{V}_{/\mathbb{F}} = \mathcal{V}_{/\mathcal{W}} \otimes_{\mathcal{W}} \mathbb{F}$. Since $V_{/\mathcal{W}}$ is smooth over $\mathcal{W}$ (see [PAF]).

Lemma 2.12. *Let A be a smooth $\mathcal{W}$-domain and Ξ be a countable set of $\mathcal{W}$-points of* $\mathrm{Spec}(A)$ *and as a subscheme of* $\mathrm{Spec}(A)$*, Ξ is étale over $\mathcal{W}$. Write $\overline{X} := X \otimes_{\mathcal{W}} \mathbb{F}$ for $X = A, A_i, \Xi$ as a subscheme of* $\mathrm{Spec}(\overline{A})$*. Then if $\overline{\Xi}$ is Zariski dense in* $\mathrm{Spec}(\overline{A})$*, then the schematic closure of Ξ in* $\mathrm{Spec}(A)$ *is equal to* $\mathrm{Spec}(A)$ *and $\Xi_\eta = \Xi \times_{\mathcal{W}} \eta$ for the generic point $\eta \in$* $\mathrm{Spec}(A)$ *is Zariski dense in* $\mathrm{Spec}(A) \times_{\mathcal{W}} \eta$*.*

Order $\Xi = \{P_1, P_2, \ldots\}$ with $\Xi_n := \{P_1, \ldots, P_n\}$. Write $\widehat{X} := \varprojlim_n X/\mathfrak{m}^n_{\mathcal{W}} X$ for $X = A, A_i, \Xi_i, P_i$ (the formal completion along the special fiber).

Proof. Since A is smooth over $\mathcal{W}$, $\widehat{A}$ (resp. $\overline{A}$) is smooth over W (resp. $\mathbb{F}$); in particular, $\overline{A}$ is a domain. Since Ξ_n is étale over $\mathcal{W}$; so, is $\widehat{\Xi}_n$ over W. Thus $\Xi_n \cong \widehat{\Xi}_n \cong \overline{\Xi}_n$ as point sets; hence $\Xi \cong \overline{\Xi}$ as sets.

We have $A/\bigcap_{j=1}^n P_j \hookrightarrow \prod_j A/P_j = \prod_{j=1}^n \mathcal{W}$. Thus $A/\bigcap_{j=1}^n P_j$ is $\mathcal{W}$-flat. In the same manner, $\widehat{A}/\bigcap_{j=1}^n \widehat{P}_j$ is W-flat. We have a short exact sequence $(\bigcap_{j=1}^n \widehat{P}_j) \otimes_W \mathbb{F} \hookrightarrow \overline{A} \twoheadrightarrow (\widehat{A}/\bigcap_{j=1}^n \widehat{P}_j) \otimes_W \mathbb{F}$. For an $\widehat{A}$-ideal $\mathfrak{a}$ with W-flat quotient $\widehat{A}/\mathfrak{a}$, we have an exact sequence $\overline{\mathfrak{a}} = \mathfrak{a} \otimes_W \mathbb{F} \hookrightarrow \overline{A} \twoheadrightarrow (A/\mathfrak{a}) \otimes_W \mathbb{F}$. We identify $\overline{\mathfrak{a}} = \mathfrak{a} \otimes_W \mathbb{F}$ as an ideal of $\overline{A}$ and $\overline{A}/\overline{\mathfrak{a}}$ with $(A/\mathfrak{a}) \otimes_W \mathbb{F}$. Take another ideal $\mathfrak{b}$ with W-flat $\widehat{A}/\mathfrak{b}$. Then $\widehat{A}/(\mathfrak{a} \cap \mathfrak{b}) \hookrightarrow \widehat{A}/\mathfrak{b} \oplus \widehat{A}/\mathfrak{b}$ implies $\widehat{A}/(\mathfrak{a} \cap \mathfrak{b})$ is W-flat. From the short exact sequence: $\widehat{A}/(\mathfrak{a} \cap \mathfrak{b}) \hookrightarrow \widehat{A}/\mathfrak{a} \oplus \widehat{A}/\mathfrak{b} \twoheadrightarrow \widehat{A}/(\mathfrak{a} + \mathfrak{b})$ for the two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we obtain a three term exact sequence $\overline{A}/\overline{\mathfrak{a} \cap \mathfrak{b}} = (\widehat{A}/\mathfrak{a} \cap \mathfrak{b}) \otimes_W \mathbb{F} \xrightarrow{\bar{i}} \overline{A}/\overline{\mathfrak{a}} \oplus \overline{A}/\overline{\mathfrak{b}} \twoheadrightarrow \overline{A}/(\overline{\mathfrak{a}} + \overline{\mathfrak{b}})$. Thus $\mathrm{Im}(\bar{i}) \cong \overline{A}/(\overline{\mathfrak{a}} \cap \overline{\mathfrak{b}})$ and $\mathrm{Coker}(\bar{i}) \cong \overline{A}/(\overline{\mathfrak{a}} + \overline{\mathfrak{b}})$ which implies $\overline{\mathfrak{a} \cap \mathfrak{b}} \subset \overline{\mathfrak{a}} \cap \overline{\mathfrak{b}}$. By induction on n, we thus have $\overline{\bigcap_{j=1}^n P_j} \subset \bigcap_{j=1}^n \overline{P}_j$ and hence $(\bigcap_{P \in \Xi} \widehat{P}) \otimes_W \mathbb{F} = \overline{\bigcap_{P \in \Xi} P} \subset \bigcap_{P \in \Xi} \overline{P}_j$, whose right-hand-side is (0) by Zariski-density. For $\mathbf{P} := \bigcap_{P \in \Xi} \widehat{P}$, we have $\widehat{\mathbf{P}} \otimes_W \mathbb{F} = \mathbf{P} \otimes_W \mathbb{F}$. Therefore we conclude $\widehat{\mathbf{P}} \otimes_W \mathbb{F} = (0)$. By Nakayama's lemma for adically complete modules over a complete ring (e.g., [CAG, Exercise 7.2]), we conclude $\mathbf{P} = (0)$. Since $\bigcap_{P \in \Xi} P \subset \bigcap_{P \in \Xi} \widehat{P} \subset \mathbf{P} = (0)$, we conclude $\bigcap_{P \in \Xi} P = 0$. Thus Ξ is schematically dense in $\mathrm{Spec}(A)$. Since $K = \mathrm{Frac}(\mathcal{W})$ is flat over $\mathcal{W}$, we have $\bigcap_{P \in \Xi} (P \otimes_{\mathcal{W}} K) = (\bigcap_{P \in \Xi} P) \otimes_{\mathcal{W}} K = 0$; so, $\Xi \otimes_{\mathcal{W}} \eta$ is Zariski dense in $\mathrm{Spec}(A) \times_{\mathcal{W}} \eta = \mathrm{Spec}(A \otimes_{\mathcal{W}} K)$. □

The definition of $\Xi \subset \mathcal{V}$ in Theorem 0.1 works well over $\mathcal{W}$; so, we take a geometrically irreducible component V of $Sh^{(p)}_{/\mathcal{W}}$ with $x(R_n) \in V(\mathcal{W})$ for sufficiently large n and define $\mathcal{V} = V^{\mathcal{Q}}$ and $\Xi \subset \mathcal{V}$ as in Theorem 0.1.

Proposition 2.13. *Assume $\Xi \otimes_{\mathcal{W}} \mathbb{F}$ is Zariski dense in $\mathcal{V} \otimes_{\mathcal{W}} \mathbb{F}$. Then $\Xi \otimes \eta$ is Zariski dense in the generic fiber $\mathcal{V} \otimes_{\mathcal{W}} \eta$.*

Proof. Since $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is affine, covering $\mathcal{V}_K$ by open affine schemes $\mathrm{Spec}(A_{K,i})$ and pulling them back to $\mathrm{Spec}(A_{U,i}) \subset \mathcal{V}_U$ for open subgroups $U \subset K$, we apply Lemma 2.12 to $\mathrm{Spec}(A_i) \subset \mathcal{V}$ for $A_i = \lim_U A_{U,i}$ assuming Zariski density of Ξ in the special fiber $\mathcal{V} \otimes_{\mathcal{W}} \mathbb{F}$ and conclude Zariski density in the generic fiber. $\square$

3. Distribution attached to $U(\mathfrak{l})$-eigenform

We recall notation and construction of a measure $d\varphi_{f,n}$ on Cl_n^- for a mod p modular form $f_{/\mathbb{F}}$ such that $f|U(\mathfrak{l}) = af$. If $0 \neq a \in \mathbb{F}$, we can patch together into a measure $d\varphi_f$ on Cl_∞^-. If $a = 0$, this is lust a collection of infinitely many measures $\{d\varphi_{f,n}\}_n$. As for the Hilbert modular Shimura variety $Sh^{(p)}$ which is the moduli (up to prime-to-p O-linear isogeny) of triples $(X, \overline{\Lambda}, \eta)$ for an abelian variety X of dimension $d = [F : \mathbb{Q}]$ with multiplication by O, an O-linear polarization class $\overline{\Lambda}$ up to multiplication by $(O_+^{(p)})^\times$ (see [H04, §2.2]) and an O-linear level structure $\eta : V^{(p)}(X) = \mathcal{T}(X) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(p)} \cong (F_{\mathbb{A}}^{(p)})^2$ for the Tate module $\mathcal{T}(X)$ of X. For the Hilbert modular Shimura variety $Sh^{(p)}$, we use the definition and notation introduced in [H04, Section 2].

3.1. CM points $x(\mathcal{A})$.

We recall the definition of the CM points $x(\mathcal{A})$ from [H04]. We write the left action: $G(\mathbb{A}^{(p\infty)}) \times Sh^{(p)} \to Sh^{(p)}$ simply as $(g, x) \mapsto g(x) := \tau(g)^{-1}(x)$. Here the action of $\tau(g)$ is a right action induced by $\eta \mapsto \eta \circ g$ for the level structure . For each point $x = (X, \overline{\Lambda}, \eta) \in Sh$, we can associate a lattice $\widehat{L} = \eta^{-1}(\mathcal{T}(X)) \subset (F_{\mathbb{A}}^{(\infty)})^2$. Then the level structure η is determined by the choice of a base $w = (w_1, w_2)$ of $\widehat{L}$ over $\widehat{O}$. In view of the base w, the inverted action $x \mapsto g(x)$ is matrix multiplication: ${}^t w \mapsto g\,{}^t w$, because $(\eta \circ g^{-1})^{-1}(\mathcal{T}(X)) = g\eta^{-1}(\mathcal{T}(X)) = g\widehat{L}$.

For each O-lattice $\mathcal{A}$, we recall a description of a CM point $x(\mathcal{A}) = (X(\mathcal{A}), \Lambda(\mathcal{A}), \eta(\mathcal{A})) \in Sh^{(p)}$ from [H04, §2.1], where $X(\mathcal{A})_{/\mathcal{W}}$ is an abelian scheme of CM-type (M, Σ) with $H^1(X(\mathcal{A})(\mathbb{C}), \mathbb{Z}) = \mathcal{A}$ in the sense $X(\mathcal{A})(\mathbb{C}) = \mathbb{C}^\Sigma / \mathcal{A}^\Sigma$ for $\mathcal{A}^\Sigma = \{(a^{\sigma_1}, \dots, a^{\sigma_d}) \in \mathbb{C}^\Sigma | a \in \mathcal{A}\}$ writing $\Sigma = \{\sigma_1, \dots, \sigma_d\}$. For the order $R_{\mathcal{A}} := \{\alpha \in M | \alpha\mathcal{A} \subset \mathcal{A}\}$ and an ideal $\mathfrak{a}$ of $R_{\mathcal{A}}$, we write, as a finite flat group scheme over $\mathcal{W}$,

$$X(\mathcal{A})[\mathfrak{a}] := \{x \in X(\mathcal{A}) | \mathfrak{a}x = 0\} = \bigcap_{\alpha \in \mathfrak{a}} \mathrm{Ker}(\alpha : X(\mathcal{A}) \to X(\mathcal{A})).$$

Recall the order $R_n = O + \mathfrak{l}^n R \subset M$ and the class groups $Cl_n^- = \mathrm{Coker}(\mathrm{Pic}(O) \to \mathrm{Pic}(R_n))$ and $Cl_\infty^- = \varprojlim_n Cl_n^-$. By class field theory, Cl_n^- gives the Galois group of the maximal anticyclotomic class field in the ring class field of conductor $\mathfrak{l}^n$ over M. The ideal $\mathfrak{l}_n = \mathfrak{l} + \mathfrak{l}^n R = \mathfrak{l}R_{n-1}$ is a prime ideal of R_n but is not proper (it is a proper ideal of R_{n-1}). Since $X(R_n)[\mathfrak{l}_n] \cong R_n/\mathfrak{l}_n = O/\mathfrak{l}$ and $\mathfrak{l}_n R_{n-1} \subset R_n$, we find that $X(R_n)[\mathfrak{l}_n] = R_{n-1}/R_n$. In other words, we have $X(R_n)/X(R_n)[\mathfrak{l}_n] \cong X(R_{n-1})$. We pick a subgroup $C \subset X(R_n)[\mathfrak{l}]$ isomorphic to $O/\mathfrak{l}$ but different from $X(R_n)[\mathfrak{l}_n]$. We look into $X(R_n)/C$. Take a lattice $\mathfrak{A}$ so that $X(R_n)/C = X(\mathfrak{A}) \Leftrightarrow \mathfrak{A}/R_n = C$. Since C is an O–submodule, $\mathfrak{A}$ is an O–lattice of M. Since $\mathfrak{l}C = 0$, we find $\mathfrak{l}R_n\mathfrak{A} \subset \mathfrak{A}$. Thus $\mathfrak{A}$ is R_{n+1}–ideal, because $R_{n+1} = O + \mathfrak{l}R_n$. Since C is not an R_n–submodule, the ideal $\mathfrak{A}$ is not R_n–ideal; so, it is a proper R_{n+1}–ideal. Since C generates over R_n all $\mathfrak{l}$–torsion points of $X(R_n)$, we find $R_n\mathfrak{A} = \mathfrak{l}^{-1}R_n$. In this way, we have created ℓ proper R_{n+1}–ideals $\mathfrak{A}$ with $\mathfrak{A}R_n = \mathfrak{l}^{-1}R_n$.

We choose a base $w = (w_1, w_2)$ of $\widehat{R}$ over $\widehat{O}$ in [H04, §2.1]: at p, for the choice of the ordinary p-adic CM-type $\Sigma = \{\mathfrak{P}|p\}$, writing $R_\Sigma = \prod_{\mathfrak{P} \in \Sigma} R_{\mathfrak{P}}$ and $R_{\Sigma^c} = \prod_{\mathfrak{P} \in \Sigma} R_{\mathfrak{P}^c}$ for complex conjugation c, $R_p = R_{\Sigma^c} \oplus R_\Sigma$ and $w = (w_1, w_2) = ((1, 0), (0, 1))$.

Let $\mathcal{A}$ be an O-lattice in M whose order $R(\mathcal{A}) := \{\alpha \in M | \alpha\mathcal{A} \subset \mathcal{A}\}$ has conductor $\mathfrak{f} = \mathfrak{f}(\mathcal{A})$. Though we mainly deal with the case where $\mathfrak{f}(\mathcal{A}) = \mathfrak{l}^n$, we describe a general theory with arbitrary $\mathfrak{f}$ prime to p. We choose a "good" level structure $\eta(\mathcal{A})$ of $X(\mathcal{A})$ so that $\eta(\mathcal{A})(\widehat{O}^2) = \widehat{\mathcal{A}}$ in the following way. First we choose a representative set $\{\mathfrak{A}_j\}$ of ideal classes of M (prime to $p\mathfrak{f}$). Then we can write $\widehat{\mathfrak{A}}_j = a_j \widehat{R}$ for an idele a_j with $a_j = a_j^{(\mathfrak{f}p\infty)}$ and choose $\alpha \in M$ so that $\mathcal{A}R = \alpha\mathfrak{A}_j$. If $\mathfrak{f}(\mathcal{A}) = O$ (so, $\mathcal{A}$ is an R–ideal), we define the level structure $\eta(\mathcal{A})$ by $(F_{\mathbb{A}}^{(\infty)})^2 \ni (a, b) \mapsto a\alpha a_j w_1 + b\alpha a_j w_2 \in M_{\mathbb{A}}^{(\infty)} = V(X(\mathcal{A}))$. When $\mathfrak{f}(\mathcal{A}) \neq O$, we first suppose that $\mathfrak{f} = (\varphi\varphi^c)$ for $\varphi \in M$. Take $\alpha \in M$ such that $\mathcal{A}R = \alpha\mathfrak{A}_j$, and choose a base $w(\mathcal{A})$ of $\widehat{\mathcal{A}}$ so that $w(\mathcal{A})^{(\mathfrak{f})} = (\varphi\alpha a_j w)^{(\mathfrak{f})}$ and $w(\mathcal{A})_{\mathfrak{f}} = \alpha w_{\mathfrak{f}} \cdot g$ for $g \in GL_2(F_{\mathfrak{f}})$ with $\det(g_{\mathfrak{f}}) = \varphi\varphi^c$. Then we define $\eta(\mathcal{A})(a, b) = a \cdot w_1(\mathcal{A}) + b \cdot w_2(\mathcal{A}) \in M_{\mathbb{A}}^{(\infty)}$. There is an ambiguity of the choice of α and φ up to units in R, but this does not cause any trouble later.

Suppose that $\mathfrak{f}(\mathcal{A})$ is not generated by a norm from M. Let $G = \mathrm{Res}_{F/\mathbb{Q}}GL(2)$ (so, $G(A) = GL_2(A \otimes_{\mathbb{Q}} F)$). We choose $g \in G(\mathbb{A}^{(\infty)})$ with $g^{(\mathfrak{f})} = 1$ so that $w(\mathcal{A}) = \alpha a_j w \cdot g$ gives a base over $\widehat{O}$ of $\widehat{\mathcal{A}}$, and define $\eta(\mathcal{A})$ by using $w(\mathcal{A})$. In the above two cases, we choose g independent of the ideals in the proper ideal class of $\mathcal{A}$; in other words, we choose $w(\beta\mathcal{A}) = \beta\alpha a_j w \cdot g$. We then define $g(\mathcal{A}) \in G(\mathbb{A}^{(\infty)})$ by $\eta(\mathcal{A}) = \eta(\mathfrak{A}_j) \cdot g(\mathcal{A})$. We will later specify the choice of g precisely.

We introduce a representation $\rho_{\mathcal{A}} : M_{\mathbb{A}}^{\times} \to G(\mathbb{A}^{(\infty)})$ by $\alpha\eta(\mathcal{A}) = \eta(\mathcal{A}) \cdot \rho_{\mathcal{A}}(\alpha)$. By our choice, we have $\rho_{\mathcal{A}} = \rho_R$ on $M_{\mathbb{A}}^{(\mathfrak{f}(\mathcal{A}))\times}$, and

$$\det(g(\mathcal{A})) \in F_{+}^{\times} \text{ if } \mathfrak{f}(\mathcal{A}) \text{ is generated by a norm from } M. \tag{3.1}$$

Regarding Σ as a set of p-adic places (i.e., field embeddings of M into $\overline{\mathbb{Q}}_p$) and composing with $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ we fixed, we may regard Σ as a set of complex embeddings. We write $\Sigma(\mathcal{A}) := \{(\sigma(a))_{\sigma\in\Sigma} \in \mathbb{C}^{\Sigma} | a \in \mathcal{A}\}$ as a lattice in $\mathbb{C}^{\Sigma} := \prod_{\sigma\in\Sigma}\mathbb{C}$.

We choose a totally imaginary $\delta \in M$ with $\mathrm{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$. Then the alternating form $(a, b) \mapsto (c(a)b - ac(b))/2\delta$ gives an identity $R \wedge_O R = \mathfrak{c}^*$ for a fractional ideal $\mathfrak{c}$ of F. Here $\mathfrak{c}^* = \{x \in F | \mathrm{Tr}_{F/\mathbb{Q}}(x\mathfrak{c}) \subset \mathbb{Z}\} = \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ for the different $\mathfrak{d}$ of $F/\mathbb{Q}$. Identifying $M \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^{\Sigma}$ by $m \otimes r \mapsto (\sigma(m)r)_{\sigma\in\Sigma}$, we find that $(a, ia) = \frac{\sqrt{-1}}{\delta}a\overline{a} \gg 0$ for $a \in M^{\times}$. Thus $\mathrm{Tr}_{F/\mathbb{Q}} \circ (\cdot, \cdot)$ gives a Riemann form for the lattice $\Sigma(\mathcal{A})$, and therefore, a projective embedding of $\mathbb{C}^{\Sigma}/\Sigma(R)$ onto a projective abelian variety $X(\mathcal{A})_{/\mathbb{C}}$. The complex abelian scheme $X(\mathcal{A})$ extends to an abelian scheme over $\mathcal{W}$ (unique up to isomorphisms). In this way, we get a $\mathfrak{c}$–polarization $\Lambda(\mathcal{A}) : X(\mathcal{A})(\mathbb{C}) \otimes \mathfrak{c} \cong {}^tX(\mathcal{A})(\mathbb{C})$ for the dual abelian scheme ${}^tX(\mathcal{A}) = \mathrm{Pic}^0_{X(\mathcal{A})/\mathcal{W}}$. The same δ induces

$$\mathcal{R} \wedge \mathcal{R} = \mathfrak{f}(O \wedge R) + \mathfrak{f}^2(R \wedge R) = (\mathfrak{f}^{-1}\mathfrak{c})^* \text{ and } \mathcal{A} \wedge \mathcal{A} = (N_{M/F}(\mathcal{A})^{-1}\mathfrak{f}(\mathcal{A})^{-1}\mathfrak{c})^*,$$

where the exterior product is taken over O. Hereafter we fix δ so that $\mathfrak{c}$ is prime to $p\mathfrak{f}(\mathcal{A})\mathfrak{d}$, and write $\mathfrak{c}(\mathcal{A})$ for $N_{M/F}(\mathcal{A})^{-1}\mathfrak{f}(\mathcal{A})^{-1}\mathfrak{c}$ (so, $\mathfrak{c} = \mathfrak{c}(R)$). We can always choose such a δ, since in this paper we only treat $\mathcal{A}$ with $\mathfrak{l}$–power conductor.

Since a generically defined isogeny between abelian schemes over $\mathcal{W}$ extends to the entire abelian scheme (e.g. [GME] Lemma 4.1.16), we have a well defined $\mathfrak{c}(\mathcal{A})$–polarization $\Lambda(\mathcal{A}) : X(\mathcal{A}) \otimes \mathfrak{c}(\mathcal{A}) \cong {}^tX(\mathcal{A})$. Replacing $X(\mathcal{A})$ by an isomorphic $X(\alpha\mathcal{A})$ for $\alpha \in M$, we may assume that $\mathcal{A}_p = R_p$. Then

$$X(\mathcal{A})[\mathfrak{p}_F] = X(\mathcal{A})[\mathfrak{p}] \oplus X(\mathcal{A})[\mathfrak{p}^c]$$

for $\mathfrak{p}_F = \mathfrak{p} \cap F$ is isomorphic by $\Lambda(\mathcal{A})$ to its Cartier dual. Since the Rosati-involution $a \mapsto a^* = \Lambda(\mathcal{A}) \circ {}^ta \circ \Lambda(\mathcal{A})^{-1}$ is the complex conjugation c, $X(\mathcal{A})[\mathfrak{p}]_{/\mathcal{W}}$ is multiplicative (étale locally) if and only if $X(\mathcal{A})[\mathfrak{p}^c]$ is étale over $\mathcal{W}$.

We also specified the base of $\widehat{R}_n^{(\mathfrak{l})}$ to be $w^{(\mathfrak{l})}$ there, because $\widehat{R}_n^{(\mathfrak{l})} = \widehat{R}^{(\mathfrak{l})}$. To specify the base $w_{\mathfrak{l}}$ of $R_{\mathfrak{l}}$, we take $d \in O_{\mathfrak{l}}$ so that $R_{\mathfrak{l}} = O_{\mathfrak{l}}[\sqrt{d}] \subset M_{\mathfrak{l}}$. We assume that d is a $\mathfrak{l}$–adic unit if $\mathfrak{l}$ is unramified in M/F and d generates $\mathfrak{l}$ if $\mathfrak{l}$ ramifies in M/F. Then we choose $w_{\mathfrak{l}} = (1, \sqrt{d})$.

Since the base of $R_{n,\mathfrak{l}}$ is given by $\alpha_n{}^t(1, \sqrt{d})$ for $\alpha_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^n \end{smallmatrix}\right)$ with a prime element $\varpi_{\mathfrak{l}}$ of $O_{\mathfrak{l}}$, we find that $\alpha_n(x(R)) = x(R_n)$ and $\alpha_1(x(R_{n-1})) = x(R_n)$. Moreover, for a suitable $u \in O$

$$\varpi_{\mathfrak{l}}(x(\mathcal{A})) = \begin{pmatrix} 1 & \frac{u}{\varpi_{\mathfrak{l}}} \\ 0 & 1 \end{pmatrix}(x(R_{n+1})) \text{ if } \mathcal{A} = R_n/C \text{ for } O/\mathfrak{l} \cong C \neq X(R_n)[\mathfrak{l}_n], \tag{3.2}$$

because the base of $\varpi_{\mathfrak{l}}\mathcal{A}_{\mathfrak{l}}$ is given by $\begin{pmatrix} 1+\varpi_{\mathfrak{l}}^n u\sqrt{d} \\ \varpi_{\mathfrak{l}}^{n+1}\sqrt{d} \end{pmatrix} = \begin{pmatrix} 1 & \frac{u}{\varpi_{\mathfrak{l}}} \\ 0 & 1 \end{pmatrix}\alpha_{n+1}\begin{pmatrix} 1 \\ \sqrt{d} \end{pmatrix}$. Here the action of $\varpi_{\mathfrak{l}} : x(\mathcal{A}) \mapsto \varpi_{\mathfrak{l}}(x(\mathcal{A}))$ may bring $x(\mathcal{A})$ on a geometrically irreducible component of $Sh^{(p)}$ to a different one.

Now we consider $x(\mathcal{A})$ in V_K for an open subgroup $K \subset G(\mathbb{A}^{(\infty)})$ containing $Z(\widehat{\mathbb{Z}})$. By repeating (3.2), if $x(\mathcal{A}) = x(R_n)/C$ for $C \cong O/\mathfrak{l}^m$ with $C \cap X(R_n)[\mathfrak{l}_n] = \{0\}$, then $\mathcal{A}$ is a proper R_{n+m}–ideal. If further $\mathfrak{l}^m$ is generated by an element $\varpi \in F$, we get $x(\mathcal{A}) = x(\varpi\mathcal{A}) = \varpi_{\mathfrak{l}}^m(x(\mathcal{A}))$ in V_K (because $\varpi/\varpi_{\mathfrak{l}}^m \in K$) and

$$x(\mathcal{A}) = \begin{pmatrix} 1 & \frac{u}{\varpi_{\mathfrak{l}}^m} \\ 0 & 1 \end{pmatrix}(x(R_{n+m})) = \left(\begin{smallmatrix} 1 & \frac{u}{\varpi} \\ 0 & 1 \end{smallmatrix}\right)(x(R_{n+m})) \text{ for a suitable } u \in O. \tag{3.3}$$

The set $\{x(\mathcal{A}) | [\mathcal{A}R_n] = [\mathfrak{A}]\}$ for $\mathcal{A} \in Cl_{n+m}$ running through ideal classes $\mathcal{A}$ projecting down to a given ideal class $[\mathfrak{A}] \in Cl_n$ is in bijection with $O/\mathfrak{l}^m$ by associating u to $\mathcal{A}$ in (3.3) (see [H04, Proposition 4.2]).

3.2. **Geometric modular forms.** Let k be a weight of $T = \mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$, that is, $k : T(A) = (A \otimes_{\mathbb{Z}} O)^\times \to A^\times$ is a homomorphism given by $(a \otimes \xi)^k = \prod (a\xi^\sigma)^{k_\sigma}$ for integers k_σ indexed by field embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$. Let B be a base ring, which is a $\mathcal{W}$–algebra. We consider quadruples $(X, \overline{\Lambda}, \eta^{(p)}, \omega)_{/A}$ for a B–algebra A with a differential ω generating $H^0(X, \Omega_{X/A})$ over $A \otimes_{\mathbb{Z}} O$. We impose the following condition:

$$\eta^{(p)}(\widehat{L}_{\mathfrak{c}}^{(p)}) = \mathcal{T}(X) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)} \quad \text{for } L_{\mathfrak{c}} = O \oplus \mathfrak{c}^* \text{ with a fixed } \mathfrak{c}. \tag{3.4}$$

Under this condition, as seen in [H04, §2.3], the classification up to prime-to-p isogenies of the quadruples is equivalent to the classification up to isomorphisms. A modular form f (integral over B) of weight k is a functorial rule of assigning a value $f(X, \overline{\Lambda}, \eta^{(p)}, \omega) \in A$ to (the A–isomorphism class of) each quadruple $(X, \overline{\Lambda}, \eta^{(p)}, \omega)_{/A}$ (called a *test object*) defined over a B–algebra A. Here Λ is a $\mathfrak{c}$–polarization which (combined with $\eta^{(p)}$) induces $L_{\mathfrak{c}} \wedge L_{\mathfrak{c}} \cong \mathfrak{c}^*$ given by $((a \oplus b), (a' \oplus b')) \mapsto ab' - a'b$. The Tate test object at the cusp $(\mathfrak{a}, \mathfrak{b})$ for two factional ideals with $\mathfrak{a}^*\mathfrak{b} = \mathfrak{c}^*$ is an example of such test objects. The Tate semi-AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is defined over $\mathbb{Z}[[q^\xi]]_{\xi \in (\mathfrak{ab})_+}$ and is given by the algebraization of the formal quotient $(\widehat{\mathbb{G}}_m \otimes \mathfrak{a}^*)/q^{\mathfrak{b}}$ (see [HMI, §4.2.5] for details of this construction). The rule f is supposed to satisfy the following three axioms:

(G1) For a B–algebra homomorphism $\phi : A \to A'$, we have
$$f((X, \overline{\Lambda}, \eta^{(p)}, \omega) \times_{A,\phi} A') = \phi(f(X, \overline{\Lambda}, \eta^{(p)}, \omega)).$$

(G2) f is finite at all cusps, that is, the q–expansion of f at every Tate test object does not have a pole at $q = 0$.

(G3) $f(X, \overline{\Lambda}, \eta^{(p)}, \alpha\omega) = \xi^{-k} f(X, \overline{\Lambda}, \eta^{(p)}, \omega)$ for $\xi \in T(A)$.

We write $G_k(\mathfrak{c}; B)$ for the space of all modular forms f satisfying (G1-3) for B–algebras A. We put

$$G_k(B) = \bigoplus_{\mathfrak{c}} G_k(\mathfrak{c}; B), \tag{3.5}$$

where $\mathfrak{c}$ prime to ℓp runs over a representative set of strict ideal classes of F.

An element $g \in G(\mathbb{A}^{(\infty)})$ fixing $\widehat{L}_{\mathfrak{c}}$ acts on $f \in G_k(\mathfrak{c}; B)$ by
$$f|g(X, \overline{\Lambda}, \eta^{(p)}, \omega) = f(X, \overline{\Lambda}, \eta^{(p)} \circ g, \omega).$$

For a closed subgroup $K \subset K_{\mathfrak{c}} = GL(\widehat{L}_{\mathfrak{c}}) \cap G_1(\mathbb{A}^{(\infty)})$, we write $G_k(\mathfrak{c}; K; B)$ for the space of all K–invariant modular forms; thus,
$$G_k(\mathfrak{c}; K; B) = H^0(K, G_k(\mathfrak{c}; B)).$$

Take an O–ideal $\mathfrak{N}$ prime to $p\mathfrak{c}$. Then the $\mathfrak{N}$–component of $K_{\mathfrak{c}}$ is $SL_2(O_{\mathfrak{N}})$. Let
$$\Gamma_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(O_{\mathfrak{N}}) \middle| c \in \mathfrak{N}O_{\mathfrak{N}} \right\} \quad \text{and} \quad \Gamma_1(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N}) \middle| a \equiv b \equiv 1 \mod \mathfrak{N}O_{\mathfrak{N}} \right\}.$$

Assume that $\mathfrak{N}$ is prime to $p\mathfrak{l}$ and define for an open subgroup $K_{\mathfrak{l}\mathfrak{N}} \subset SL_2(O_{\mathfrak{l}\mathfrak{N}})$
$$G_k(K_{\mathfrak{l}\mathfrak{N}}; B) = \bigoplus_{\mathfrak{c}} G_k(\mathfrak{c}; K_{\mathfrak{l}\mathfrak{N}} \times K_{\mathfrak{c}}^{(p\mathfrak{l}\mathfrak{N})}; B).$$

A W–algebra B is called a p–adic algebra if $B = \varprojlim_n B/p^nB$. We write η_{ord} for the pair of level structures $(\eta_p^{ord} : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \hookrightarrow X[p^\infty], \eta^{(p)})$. A p–adic modular form f over a p–adic W–algebra B is a functorial rule of assigning a value in A to triples $(X, \overline{\Lambda}, \eta_{ord})_{/A}$ with $\mathfrak{c}$–polarization class $\overline{\Lambda}$ satisfying an obvious version of (G1-2) for p–adic B–algebras A (not just B–algebras). In general, we do not impose (G3) on p–adic modular forms. We write $V(\mathfrak{c}; B)$ for the space of p–adic modular forms defined over B. We again define

$$V(B) = \bigoplus_{\mathfrak{c}} V(\mathfrak{c}; B) \quad \text{and} \quad V(K_{\mathfrak{l}\mathfrak{N}}; B) = \bigoplus_{\mathfrak{c}} V(\mathfrak{c}; K_{\mathfrak{l}\mathfrak{N}} \times K_{\mathfrak{c}}^{(p\mathfrak{l}\mathfrak{N})}; B), \tag{3.6}$$

where $V(\mathfrak{c}; K; B) = H^0(K, V(\mathfrak{c}; B))$. For $f \in V(B)$, we write $f_{\mathfrak{c}} \in V(\mathfrak{c}; B)$ for the $\mathfrak{c}$–component of f, and we say that f is of *level* $\mathfrak{N}$ if f in either in $G_k(K_{\mathfrak{l}\mathfrak{N}}; B)$ or in $V(K_{\mathfrak{l}\mathfrak{N}}; B)$ for $K_{\mathfrak{l}\mathfrak{N}} \subset SL_2(O_{\mathfrak{l}\mathfrak{N}})$ with $K_{\mathfrak{N}} = \Gamma_0(\mathfrak{N})$ or $\Gamma_1(\mathfrak{N})$.

Since η_p^{ord} induces the identification $\widehat{\eta}_p^{ord} : \widehat{\mathbb{G}}_m \otimes O^* \cong \widehat{X}$ for the formal completion of X along the origin, by pushing forward the differential $\frac{dt}{t}$, we can associate $(X, \overline{\Lambda}, \eta^{(p)}, \widehat{\eta}_{p,*}^{ord}\frac{dt}{t})$ to a quadruple

$(X,\overline{\Lambda},\eta_p^{ord},\eta^{(p)})$. In this way, any modular form f satisfying (G1-3) can be regarded as a p–adic modular form by

$$f(X,\overline{\Lambda},\eta_{ord})=f(X,\overline{\Lambda},\eta^{(p)},\widehat{\eta}_{p,*}^{ord}\frac{dt}{t}).$$

By the q–expansion principle (cf. [HMI, Corollary 4.16] or [PAF, Corollary 4.23]), we thus have a canonical embedding of $G_k(B)$ into $V(B)$ which keeps the q–expansion. A p–adic modular form associated to a modular form in $G_k(B)$ satisfies the following replacement of (G3):

(g3) $f(X,\overline{\Lambda},\xi\cdot\eta_p^{ord},\eta^{(p)})=\xi^{-k}f(X,\overline{\Lambda},\eta_p^{ord},\eta^{(p)})$ for $\xi\in O_p^{\times}$.

Although we do not impose the condition (G3) on p–adic modular forms f, we limit ourselves to the study of forms satisfying the following condition (G3$'$) in order to define the modified value $f([\mathcal{A}])$ later at CM points $x(\mathcal{A})$ truly independent of the choice of $\mathcal{A}$ in its proper ideal class. Here abusing our notation, $x(\mathcal{A})$ is the quadruple $(X(\mathcal{A}),\Lambda(\mathcal{A}),\eta_{ord}(\mathcal{A}),\omega(\mathcal{A}))_{/\mathcal{W}}$ introduced in [H04, §2.1]). We consider the torus $T_M=\mathrm{Res}_{R/\mathbb{Z}}\mathbb{G}_m$ and identify its character group $X^*(T_M)$ with the module $\mathbb{Z}[\Sigma\sqcup\Sigma c]$ of formal linear combinations of embeddings of M into $\overline{\mathbb{Q}}$. By the identity: $(X(\xi\mathcal{A}),\Lambda(\xi\mathcal{A}),\eta_{ord}(\xi\mathcal{A})=\xi\eta_{ord}(\mathcal{A}))_{/\mathcal{W}}\cong(X(\mathcal{A}),\xi\xi^c\Lambda(\mathcal{A}),\eta_{ord}(\mathcal{A})\circ\rho_{\mathcal{A}}(\xi))_{/\mathcal{W}}$, we may assume that for $k,\kappa\in\mathbb{Z}[\Sigma]$,

(G3$'$) $f(x(\xi\mathcal{A}))=f(\rho_R(\xi^{(\mathfrak{l})})(x(\mathcal{A})))=\xi^{-k-\kappa(1-c)}f(x(\mathcal{A}))$ for $\xi\in T_M(\mathbb{Z}_{(\ell)})$.

It is known that for the p–adic differential operator d_σ of Dwork-Katz ([K78] 2.5-6) corresponding to $\frac{1}{2\pi i}\frac{\partial}{\partial z_\sigma}$ for $\sigma\in\Sigma$, $\theta^\kappa f$ ($\theta^\kappa=\prod_\sigma d_\sigma^{\kappa_\sigma}$) satisfies (G3$'$) if $f\in G_k(B)$.

3.3. **Hecke operators.** Suppose that the $\mathfrak{l}$–component $K_{\mathfrak{l}}$ of the level subgroup is equal to $\Gamma_0(\mathfrak{l}^r)$ ($r\geq 0$). Let $e_1={}^t(1,0),e_2={}^t(0,1)$ be the standard basis of $F^2\otimes\mathbb{A}^{(p\infty)}$. Then, under (3.4), for each triple $(X,\overline{\Lambda},\eta_{ord})_{/A}$ with $\eta_{ord}=\eta_p^{ord}\times\eta^{(p)}$,

$$C=\eta_{\mathfrak{l}}(\mathfrak{l}^{-r}O_{\mathfrak{l}}e_1+O_{\mathfrak{l}}e_2)/\eta_{\mathfrak{l}}(O_{\mathfrak{l}}^2)$$

gives rise to an A–rational cyclic subgroup of X of order $\mathfrak{l}^r$, that is, a finite group subscheme defined over A of $X_{/A}$ isomorphic to $O/\mathfrak{l}^r$ étale locally. Since $\Gamma_0(\mathfrak{l}^r)$ fixes $(\mathfrak{l}^{-r}O_{\mathfrak{l}}e_1+O_{\mathfrak{l}}e_2)/O_{\mathfrak{l}}^2$, the level $\Gamma_0(\mathfrak{l}^r)$ moduli problem is equivalent to the classification of quadruples $(X,\overline{\Lambda},C,\overline{\eta}_{ord}^{(\mathfrak{l})})_{/A}$ for a subgroup C of order $\mathfrak{l}^r$ in X, where $\eta_{ord}^{(\mathfrak{l})}$ is the (p–ordinary) level structure outside $\mathfrak{l}$. Thus we may define for $f\in G_k(\Gamma_0(\mathfrak{l}^r\mathfrak{N});B)$ the value of f at $(X,\overline{\Lambda},C,\eta^{(p\mathfrak{l})},\omega)$ by $f(X,\overline{\Lambda},C,\eta^{(p\mathfrak{l})},\omega):=f(X,\overline{\Lambda},\eta^{(p)},\omega)$. When f is a p–adic modular form, we replace the ingredient ω by the ordinary level structure η_p^{ord} in order to define the value $f(X,\overline{\Lambda},C,\eta^{(p\mathfrak{l})},\eta_p^{ord})$.

We shall define Hecke operators $T(1,\mathfrak{l}^n)$ and $U(\mathfrak{l}^n)$ over (p–adic) modular forms of level K (with $K_{\mathfrak{l}}=\Gamma_0(\mathfrak{l}^r)$). The operator $U(\mathfrak{l}^n)$ is defined when $r>0$, and $T(1,\mathfrak{l}^n)$ is defined when $r=0$. Since $\mathfrak{l}$ is prime to p (and B is a $\mathcal{W}$–algebra), any cyclic subgroup C' of X of order $\mathfrak{l}^n$ is isomorphic to $O/\mathfrak{l}^n$ étale locally. We make the quotient $\pi:X\twoheadrightarrow X/C'$, and Λ, η_p^{ord} and ω induce canonically a polarization $\pi_*\Lambda$ a canonical level structure $\pi_*\eta_p^{ord}=\pi\circ\eta_p^{ord}$, $\pi_*\eta^{(p\mathfrak{l})}=\pi\circ\eta^{(p\mathfrak{l})}$ and a differential $(\pi^*)^{-1}\omega$ on X/C'. If $C'\cap C=\{0\}$ for the $\Gamma_0(\mathfrak{l}^r)$–structure C (in this case, we call that C' and C are disjoint), $\pi(C)=C+C'/C'$ gives rise to the level $\Gamma_0(\mathfrak{l}^r)$–structure on X/C'. We write $\underline{X/C'}$ for the new test object of the same level as the test object $\underline{X}=(X,\overline{\Lambda},C,\eta_{ord}^{(\mathfrak{l})},\omega)$ we started with. When f is p–adic, we suppose not to have ω in $\underline{X}$, and when f is classical, we ignore the ingredient η_p^{ord} in $\underline{X}$. Then we define (for $r>0$)

$$f|U(\mathfrak{l}^n)(\underline{X})=\frac{1}{N(\mathfrak{l}^n)}\sum_{C'}f(\underline{X/C'}),\tag{3.7}$$

where C' runs over all étale cyclic subgroups of order $\mathfrak{l}^n$ disjoint from C. The newly defined $f|U(\mathfrak{l}^n)$ is a modular form of the same level as f and $U(\mathfrak{l}^n)=U(\mathfrak{l})^n$. Since the polarization ideal class of X/C' is given by $\mathfrak{c}\mathfrak{l}^n$ for the polarization ideal class $\mathfrak{c}$ of X, the operators $U(\mathfrak{l}^n)$ permute the components $f_{\mathfrak{c}}$.

We recall some other isogeny actions on modular forms. For ideals $\mathfrak{A}$ in F, we can think of the association $X\mapsto X\otimes_O\mathfrak{A}$ for each AVRM X. This operation will be made explicit in terms of the lattice $L=\pi_1(X)$ in $Lie(X)$. There are a natural polarization and a level structure on $X\otimes\mathfrak{A}$ induced by those of X. Writing $(X,\Lambda,\eta)\otimes\mathfrak{z}$ for the triple made out of (X,Λ,η) after tensoring $\mathfrak{z}$, we define $f|\langle\mathfrak{z}\rangle(X,\Lambda,\eta) = f((X,\Lambda,\eta)\otimes\mathfrak{z})$ (see [PAF, §4.1.9] for more details of this definition, though $\langle\mathfrak{z}\rangle$ here is $\langle\mathfrak{z}^{-1}\rangle$ in [PAF, §4.1.9]). For $X(\mathcal{A})$, we have $\langle\mathfrak{z}\rangle(\underline{X}(\mathcal{A})) = \underline{X}(\mathfrak{z}\mathcal{A})$.

(3.8) *The effect of* $\langle\mathfrak{z}\rangle$ *on the Fourier expansion at* $(\mathfrak{a},\mathfrak{b})$ *is given by that at* $(\mathfrak{z}\mathfrak{a},\mathfrak{z}^{-1}\mathfrak{b})$

(e.g., by [PAF, §4.2.9], noting $\langle\mathfrak{z}\rangle$ here is $\langle\mathfrak{z}^{-1}\rangle$ in [PAF]).

Let $\mathfrak{q}$ be a prime ideal of F outside $p\mathfrak{l}$. For a test object $(X,\overline{\Lambda},C,\eta_{ord}^{(\mathfrak{q})},\omega)$ of level $\Gamma_0(\mathfrak{q})$, we can construct canonically its image under $\mathfrak{q}$–isogeny:

$$[\mathfrak{q}](X,\overline{\Lambda},C,\eta_{ord}^{(p\mathfrak{q})},\omega) = (X',\overline{\Lambda},\pi_*\eta_{ord}^{(p\mathfrak{q})},\overline{\eta}_{\mathfrak{q}},(\pi^*)^{-1}\omega)$$

for the projection $\pi : X \twoheadrightarrow X' = X/C$, where $\overline{\eta}_{\mathfrak{q}} = \eta_{\mathfrak{q}}\cdot GL_2(O_{\mathfrak{q}})$ for any level $\mathfrak{q}$–structure $\eta_{\mathfrak{q}}$ identifying $\mathcal{T}_{\mathfrak{q}}(X')$ with $O_{\mathfrak{q}}^2$. Then we have a linear operator $[\mathfrak{q}] : V(\Gamma_1(\mathfrak{l}^r\mathfrak{N});B) \to V(\Gamma_0(\mathfrak{q}\mathfrak{l}^r\mathfrak{N});B)$ given by $f|[\mathfrak{q}](\underline{X}) = f([\mathfrak{q}](\underline{X}))$. See [H04, (4.14)] for the description of this operator in terms of the lattice of X.

If $\mathfrak{q}$ splits into $\mathfrak{Q}\overline{\mathfrak{Q}}$ in M/F, choosing $\eta_{\mathfrak{q}}$ induced by

$$X(\mathcal{A})[\mathfrak{q}^\infty] \cong M_{\mathfrak{Q}}/R_{\mathfrak{Q}} \times M_{\overline{\mathfrak{Q}}}/R_{\overline{\mathfrak{Q}}} \cong F_{\mathfrak{q}}/O_{\mathfrak{q}} \times F_{\mathfrak{q}}/O_{\mathfrak{q}},$$

we always have a canonical level $\mathfrak{q}$–structure on $X(\mathcal{A})$ dependent on the choice of the factor $\mathfrak{Q}$. Then $[\mathfrak{q}](X(\mathcal{A})) = X(\mathcal{A}[\mathfrak{Q}]^{-1})$ for $[\mathfrak{Q}] \in Cl_\infty$. When $\mathfrak{q}$ ramifies in M/F as $\mathfrak{q} = \mathfrak{Q}^2$, $X(\mathcal{A})$ has a subgroup $C = X(\mathcal{A})[\mathfrak{Q}_n]$ isomorphic to $O/\mathfrak{q}$ for $\mathfrak{Q}_n = \mathfrak{Q}\cap R_n$; so, we can still define $[\mathfrak{q}](X(\mathcal{A})) = X(\mathcal{A}\mathfrak{Q}_n^{-1}) = X(\mathcal{A}[\mathfrak{Q}]^{-1})$.

The effect on the q–expansion of the operator $[\mathfrak{q}]$ can be computed similarly to $\langle\mathfrak{z}\rangle$ (e.g. [DR] 5.8; see also [PAF] 4.2.9), and the q–expansion of $f|[\mathfrak{q}]$ at the cusp $(\mathfrak{a},\mathfrak{b})$ is given by the q–expansion of f at the cusp $(\mathfrak{q}\mathfrak{a},\mathfrak{b})$.

These operators $[\mathfrak{q}]$ and $\langle\mathfrak{z}\rangle$ change polarization ideals (as we will see later in [H04, §4.2]); so, they permute components $f_{\mathfrak{c}}$. By the q–expansion principle, $f \mapsto f|[\mathfrak{q}]$ and $f \mapsto f|\langle\mathfrak{z}\rangle$ are injective.

3.4. **Anti-cyclotomic measure.** Choose a $U(\mathfrak{l})$-eigenform $f \in V(\Gamma_1(\mathfrak{l}^r\mathfrak{N});A)$ with a central character for a p-adic ring A in which ℓ is invertible. We suppose that $f|U(\mathfrak{l}) = (a/\lambda(\mathfrak{l})N(\mathfrak{l}))f$ for either a unit $a \in A$ or $a = 0$. This f is an element of $V(\Gamma_1(\mathfrak{l}^r\mathfrak{N});A)$ defined over the non-connected Hilbert modular Shimura variety whose geometrically connected components are indexed by the strict class group Cl_F^+ of F. Our geometrically irreducible component V carries $x(\mathcal{A})$ for $\mathcal{A} \in Cl^{alg}\cap K_0$ for $K_0 := \mathrm{Ker}(Cl_\infty \twoheadrightarrow Cl_0)$. Anyway $f(x(\mathcal{A}))$ is well defined for all $\mathcal{A} \in Cl^{alg}$ possibly $x(\mathcal{A})$ sitting in another geometrically connected component.

Choose a Hecke character λ such that

(f1) λ has infinity type $k+\kappa(1-c)$ of conductor $\mathfrak{C}$ which is a product of split primes over F,
(f2) Decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c$ for integral ideals $\mathfrak{F}$ and $\mathfrak{F}_c$ such that $\mathfrak{F}+\mathfrak{F}_c = R$, $\mathfrak{F} \subset \mathfrak{F}_c^c$, the Neben character of f as in [H07, (S1-3)] is given by $(\lambda_{\mathfrak{F}_c},\lambda_{\mathfrak{F}},(\lambda|_{F_{\mathbb{A}}^\times})|\cdot|_{F_{\mathbb{A}}}^2)$.

The condition (f1) imposes the condition $k_\sigma = k_\tau$ for all $\sigma,\tau \in \Sigma$. It might appear strange to have the module character in the description of the central character $(\lambda|_{F_{\mathbb{A}}^\times})|\cdot|_{F_{\mathbb{A}}}^2$ of f, but when we extend a geometric modular form to an automorphic form on $G(\mathbb{A})$, we multiply the factor $|\det(g)|_{\mathbb{A}}$ as the adelic Fourier expansion has the factor $|\det(g)|_{\mathbb{A}}$ in front of the Fourier expansion sum in [HMI, (2.3.15)]; so, the central action on a geometric modular form and the adelic one has this discrepancy. See [HMI, §2.3.2, §4.3.7] for more details on the relation of geometric Hilbert modular forms and adelic ones. Then by (f2), $f([\mathcal{A}]) = \lambda(\mathcal{A})^{-1}f(x(\mathcal{A}))$ for $\mathcal{A}$ prime to p depends only on the class of $\mathcal{A}$ in $Cl_n^- = Cl_n/Cl_F$. The existence of the character satisfying (f2) implies $k_\sigma = k_\tau$ for any two embeddings $\sigma,\tau : F \hookrightarrow \mathbb{R}$; so, hereafter, often we identify k with the integer k_σ.

For the p–adic avatar $\widehat{\lambda}(x) = \lambda(xR)x_p^{k+\kappa(1-c)}$, we also have $f([\mathcal{A}]) = \widehat{\lambda}(\mathcal{A})^{-1}f(x(\mathcal{A}))$. This new definition is valid even for $\mathcal{A}$ with non-trivial common factor with p. Then often we regard f as a function of $C^{(\infty)} = \bigsqcup_n C_n$ (embedded into $Sh_{/\mathcal{W}}^{(p)}$ or $Ig_{/\mathbb{F}}$ by $\mathcal{A} \mapsto x(\mathcal{A})$).

Writing $X(\mathcal{A})/C = X(\mathfrak{A})$ for $C \neq X(\mathcal{A})[\mathfrak{l}_n]$ for R_n-proper ideal $\mathcal{A}$ prime to $\mathfrak{l}$, $\mathfrak{A}$ is a proper R_{n+1}-ideal such that $\mathfrak{l}(R_n\mathfrak{A}) = \mathcal{A}$. Since there are $N(\mathfrak{l})$ proper R_{n+1}-ideal such that $\mathfrak{l}(R_n\mathfrak{A}) = \mathcal{A}$ if

$n > 0$, we have

$$(a/\lambda N(\mathfrak{l}))\lambda(\mathcal{A})f([\mathcal{A}]) = (a/\lambda N(\mathfrak{l}))f(x(\mathcal{A})) = f|U(\mathfrak{l})(x(\mathcal{A})) = N(\mathfrak{l})^{-1} \sum_{\mathfrak{A}:\mathfrak{l}(R_n\mathfrak{A})=\mathcal{A}} f(x(\mathfrak{A}))$$
$$= N(\mathfrak{l})^{-1}\lambda(\mathfrak{A}) \sum_{\mathfrak{A}:\mathfrak{l}(R_n\mathfrak{A})=\mathcal{A}} f([\mathfrak{A}]) \overset{\lambda(\mathfrak{A})=\lambda(\mathfrak{l})^{-1}\lambda(\mathcal{A})}{=} \lambda N(\mathfrak{l})^{-1}\lambda(\mathcal{A}) \sum_{\mathfrak{A}:\mathfrak{l}(R_n\mathfrak{A})=\mathcal{A}} f([\mathfrak{A}]) \text{ if } n > 0.$$

Since $f([\mathfrak{A}])$ only depends on the class of Cl^-_{n+1}, this implies

(1) $a \cdot f([\mathcal{A}]_n) = \sum_{[\mathcal{B}]_{n+1}:Cl^-_{n+1}\ni[\mathcal{B}]_{n+1}\mapsto[\mathcal{A}]_n} f([\mathcal{B}]_{n+1})$,
(2) $f|U(\mathfrak{l})([\mathcal{A}]_n) = \lambda N(\mathfrak{l})^{-1} \sum_{[\mathcal{B}]_{n+1}\in Cl^-_{n+1}:[\mathcal{B}]_{n+1}\mapsto[\mathcal{A}]_n} f([\mathcal{B}]_{n+1})$.

We can rewrite the above relation (1) as

$$a \cdot f([\mathcal{A}]_n) = \sum_{[\mathcal{B}]_{n+1}\in Cl^-_{n+1}:[\mathcal{B}]_{n+1}\mapsto\mathcal{A}} f([\mathcal{B}]_{n+1}) \text{ if } n > 0. \tag{3.9}$$

More generally as seen in [H04, (3.8)], we get, for integers $n > m \geq 1$,

$$\sum_{[\mathcal{B}]\in Cl^-_n,\ [\mathcal{B}]_n\mapsto[\mathcal{A}]_m\in Cl^-_m} f([\mathcal{B}]_n) = a^{n-m}f([\mathcal{A}]_m), \tag{3.10}$$

where $\mathcal{A}$ runs over all elements in Cl^-_n which project down to $\mathfrak{l}^{m-n}\mathfrak{A} \in Cl^-_m$. The second relation (2) can be written

(C1) $f|U(\mathfrak{l})([\mathcal{A}]_n) = \lambda(\mathfrak{l})^{-1}f([\mathcal{B}]_{n+1})$ if $f([\mathcal{A}]_{n+1}) = f([\mathcal{B}]_{n+1})$ for any $[\mathcal{B}]_{n+1}$ with $[R_n\mathcal{B}]_n = [\mathcal{A}]_n$,
(C2) $f|U(\mathfrak{l}^m)([\mathcal{A}]_n) = \lambda N(\mathfrak{l})^{-m} \sum_{[\mathcal{B}]_{n+m}\in Cl^-_{n+m}:[\mathcal{B}]_{n+m}\mapsto[\mathcal{A}]_n} f([\mathcal{B}]_{n+m})$.

For each function $\phi : Cl^-_\infty \to A$ factoring through Cl^-_n, assuming $a \in A^\times$, we define

$$\int_{Cl^-_\infty} \phi d\varphi_f = a^{-n} \sum_{\mathcal{A}\in Cl^-_n} \phi(\mathcal{A}^{-1})f([\mathcal{A}]). \tag{3.11}$$

Then for $n > m \geq 1$, assuming $a \in A^\times$, we find

$$a^{-n} \sum_{\mathcal{A}\in Cl^-_n} \phi(\mathcal{A}^{-1})f([\mathcal{A}]) = a^{-m} \sum_{\mathfrak{A}\in Cl^-_m} \phi(\mathfrak{A}^{-1})a^{m-n} \sum_{\mathcal{A}\in Cl^-_n,\ \mathcal{A}\mapsto\mathfrak{A}} f([\mathcal{A}])$$
$$\overset{(3.10)}{=} a^{-m} \sum_{\mathfrak{A}\in Cl^-_m} \phi(\mathfrak{A}^{-1})a^{m-n}f|U((\mathfrak{l})^{n-m})([\mathfrak{A}]) = \int_{Cl^-_\infty} \phi(x)d\varphi_f(x).$$

Thus φ_f gives an A–valued distribution on Cl^-_∞ well defined independently of the choice of m for which ϕ factors through Cl^-_m, because $U(\mathfrak{l}^m) = U(\mathfrak{l})^m$.

Remark 3.1. *The assumption that $a \in A^\times$ is not essential. If $a = 0$, we just define for each finite n and a function $\phi : Cl^-_n \to A$*

$$\int_{Cl^-_n} \phi d\varphi_f = \sum_{\mathcal{A}\in Cl^-_n} \phi(\mathcal{A}^{-1})f([\mathcal{A}]) \tag{3.12}$$

without dividing by a. Though we lose the distribution relation above, we have well defined value $\int_{Cl^-_n} \phi d\varphi_f$ dependent on n. Changing ∞ by n, all the formulas independent of the distribution relation holds even when $a = 0$. So hereafter we allow the case where $a = 0$, and as a convention, we use n in place of ∞. If $a \in A^\times$, we can replace n by ∞ since $\int_{Cl^-_\infty} = \int_{Cl^-_n}$ as long as the integral factors through Cl^-_n. Thus if $a = 0$, by (3.9), $\int_{Cl^-_n} \phi d\varphi_f \neq 0$ happens for a unique $n > 0$. This n is a minimal n for which ϕ factors through Cl^-_n. To write formulas uniform, we define $\mathbf{a} = 1$ if $a = 0$ and $\mathbf{a} = a$ if $a \neq 0$ in $\mathbb{F}$.

Classical modular forms are actually defined over a number field; so, we assume that f is defined over the localization $\mathcal{V}$ of the integer ring in a number field E. We assume that E is the smallest field containing M' for the reflex (M', Σ') of (M, Σ) and the values $\lambda(\mathfrak{A})$ for all M-fractional ideals $\mathfrak{A}$. We write $\mathcal{P}|p$ for the prime ideal of the p–integral closure $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ in $\overline{\mathbb{Q}}$ corresponding to $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

More generally, if $f = \theta^\kappa g$ for a classical modular form g integral over $\mathcal{V}$, the value $f([\mathcal{A}])$ is algebraic, abelian over M' and $\mathcal{P}$–integral over $\mathcal{V}$ by a result of Shimura and Katz (see [EAI, §8.1.1] and [K78]).

Let $f = \theta^\kappa g$ for $g \in G_k(\Gamma_0(\mathfrak{l}); \mathcal{V})$. Suppose $f|U(\mathfrak{l}) = (a/\lambda(\mathfrak{l})N(\mathfrak{l}))f$ for a giving a unit of $\widetilde{\mathcal{V}}/\mathcal{P}$. For the moment, let φ be the measure associated to f with values in $A = \widetilde{\mathcal{V}}$. We have a well defined measure $\varphi \mod \mathcal{P}$. Let E_f be the field of rationality of $x(\mathcal{A})$ for all $[\mathcal{A}] \in Cl^{alg}$ over $E[\mu_{\ell^\infty}]$. Then E_f/E is an abelian extension unramified outside ℓ, and we have the Frobenius element $\sigma_{\mathfrak{b}} \in \mathrm{Gal}(E_f/E)$ (that is, the image of $\mathfrak{b}$ under the Artin reciprocity map) for each ideal $\mathfrak{b}$ of E prime to ℓ. By Shimura's reciprocity law ([ACM] 26.8), writing (M', Σ') for the reflex CM type of (M, Σ), we find for $\sigma = \sigma_{\mathfrak{b}}$, $x(\mathcal{A})^\sigma = x(N(\mathfrak{b})^{-\Sigma'}\mathcal{A})$ for the norm $N : E \to M'$. As for $\eta_p^{ord}(\mathcal{A})$, we find $\sigma \circ \eta_p^{ord}(\mathcal{A}) = u\eta_p^{ord}$ for $u \in R_{\Sigma_p}^\times$. Since $\mathcal{A}_p \cong R_p$, we have $X(R)[p^\infty] \cong X(\mathcal{A})[p^\infty]$ as a Galois module. Thus we conclude $u = \psi_E(\mathfrak{b})$ for the Hecke character ψ_E of $E_{\mathbb{A}}^\times/E^\times$ giving rise to the zeta function of $X(R)$. From this, we see $f([\mathcal{A}])^\sigma = f([N(\mathfrak{b})^{-\Sigma'}\mathcal{A}])$ for any ideal $\mathfrak{b}$, since $\psi_E(\mathfrak{b}) \in M$ generates the ideal $N(\mathfrak{b})^{\Sigma'} \subset M$ ([ACM] Sections 13 and 19) and hence $\psi_E(\mathfrak{b})^{k\Sigma+\kappa(1-c)} = \lambda(N(\mathfrak{b})^{\Sigma'})$. We then have

$$\sigma \cdot \left(\int_{Cl_n^-} \phi(x) d\varphi_f(x) \right) = \int_{Cl_n^-} \sigma \circ \phi(N(\mathfrak{b})^{\Sigma'} x) d\varphi_f(x), \tag{3.13}$$

where $N(\mathfrak{b})$ is the norm of $\mathfrak{b}$ over M'. Writing $\mathbb{F}_q$ for $q := p^{r_0}$ for the residue field of $E \cap \mathcal{P}$, any modular form defined over $\mathbb{F}_q$ is a reduction modulo $\mathcal{P}$ of a classical modular form defined over $\mathcal{V}$ of sufficiently high weight. Since $\xi^\Sigma \in M'$ for $\xi \in M$ as the reflex of Σ' is a sub-CM-type of Σ, we have $\mathbb{F}_q \subset \mathbb{F}_{\mathbf{q}}$. Thus the above identity is valid for $\sigma = \Phi^s$ $(s \in \mathbb{Z})$ for the Frobenius element $\Phi \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. In this case, $N(\mathfrak{b})$ is a power of a prime ideal $\mathfrak{p}|p$ in M'.

We now assume that $A = \mathbb{F} = \widetilde{\mathcal{V}}/\mathcal{P}$ and regard the measure φ_f as having values in $\mathbb{F}$. Then (3.13) shows that if ϕ is a character χ of Cl_n^- with arbitrary $n > 0$, for $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$,

$$\int_{Cl_n^-} \chi(x) d\varphi_f(x) = 0 \iff \int_{Cl_n^-} \sigma \circ \chi(x) d\varphi_f(x) = 0. \tag{3.14}$$

Let $\mathbb{F}_q[\mu_\ell]$ be the finite subfield of $\mathbb{F}$ generated by all ℓ–th roots of unity over $\mathbb{F}_q$; so, it is the field of rationality of λ, f and μ_ℓ over the residue field of $M' \cap \mathcal{P}$.

3.5. **Measure projected to Γ and Γ_n.** Since each fractional R–ideal $\mathfrak{A}$ prime to $\mathfrak{l}$ defines a class $[\mathfrak{A}]$ in Cl_∞^-, we can embed the ideal group of fractional ideals prime to $\mathfrak{l}$ into Cl_∞^-. We write C^{alg} for its image. Thus the projection of $[\mathfrak{Q}]$ in Cl_n^- is $[\mathfrak{Q}]_n$ as specified for the integral ideal $\mathfrak{Q}$ above. Then $\Delta^{alg} = \Delta^- \cap C^{alg}$ is generated by prime ideals of M ramified over F. We choose a complete representative set for Δ^{alg} made of product of prime ideals in M ramified over F prime to $p\mathfrak{l}$. We may choose this set as $\{\mathfrak{R}^{-1}|\mathfrak{r} \in \mathcal{R}\}$, where $\mathcal{R}$ is made of square-free product of primes non-principal outside $\mathfrak{l}$ in F ramifying in M/F, and $\mathfrak{R}$ is a unique ideal in M with $\mathfrak{R}^2 = \mathfrak{r}$. Note that $\{\mathfrak{R}|\mathfrak{r} \in \mathcal{R}\}$ is a complete representative set for 2–torsion elements in the quotient Cl_0^-.

In [H04] and [H07], we used Cl_n in place of Cl_n^-; so, we had to choose a complete representative set $\mathcal{S}$ of the image $\overline{Cl}_F$ of Cl_F in Cl_n, which is not necessary. Indeed, since $f([\mathcal{A}]) = f([\mathfrak{s}\mathcal{A}])$ for an O-ideal $\mathfrak{s}$ by our choice of λ, we have $\overline{h}f([\mathcal{A}]) = \sum_{\mathfrak{s}\in\mathcal{S}} f([\mathfrak{s}\mathcal{A}])$ for $\overline{h} := |\overline{Cl}_F|$, and if we make our choice of λ, this implies the triviality of the measure if $\ell|\overline{h}$. To avoid this, we do not sum over $\mathcal{S}$. We fix a character $\psi : \Delta^- \to \mathbb{F}^\times$, and define

$$f_\psi = \sum_{\mathfrak{r}\in\mathcal{R}} \lambda\psi^{-1}(\mathfrak{R}) f|[\mathfrak{r}]. \tag{3.15}$$

In [H04] and [H07], f_ψ is defined by

$$\sum_{\mathfrak{r}\in\mathcal{R}} \lambda\psi^{-1}(\mathfrak{R}) \left(\sum_{\mathfrak{s}\in\mathcal{S}} \psi\lambda^{-1}(\mathfrak{s}) f|\langle\mathfrak{s}\rangle \right) |[\mathfrak{r}],$$

and we do not follow this definition.

Choose a complete representative set $\mathcal{Q}$ for $Cl_\infty^-/\Gamma\Delta^{alg}$ made of primes $\mathfrak{Q}$ of M split over F outside $p\mathfrak{l}$ except for the trivial element R representing $1 \in Cl_\infty^-/\Gamma\Delta^{alg}$. Thus $\mathfrak{q} := N_{M/F}(\mathfrak{Q})$ is a prime ideal of O if $\mathfrak{Q} \neq R$ (and $\mathfrak{q} = O$ if $\mathfrak{Q} = R$). We choose $\eta_n^{(p)}$ out of the base (w_1, w_2) of $\widehat{R}_n$ so that at $\mathfrak{q} = \mathfrak{Q} \cap F$, $w_{1,\mathfrak{q}} = (1, 0) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$ and $w_{2,\mathfrak{q}} = (0, 1) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$. Since all operators $\langle\mathfrak{s}\rangle$, $[\mathfrak{q}]$ and $[\mathfrak{r}]$ commute with $U(\mathfrak{l})$, $f_\psi|[\mathfrak{q}]$ is still an eigenform of $U(\mathfrak{l})$ with the same eigenvalue as f. Thus in particular, we have a measure $\varphi_{f_\psi|[\mathfrak{q}]}$. We then define another measure φ_f^ψ on Γ by

$$\int_{\Gamma_n} \phi d\varphi_f^\psi = \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) \int_{\Gamma_n} \phi|\mathfrak{Q} d\varphi_{f_\psi|[\mathfrak{q}]}, \tag{3.16}$$

where $\phi|\mathfrak{Q}(y) = \phi(y[\mathfrak{Q}]_\Gamma)$ for the projection $[\mathfrak{Q}]_\Gamma$ in Γ of the class $[\mathfrak{Q}] \in Cl_\infty^-$. As already remarked, $\phi \mapsto \phi|\mathfrak{Q}$ is a transcendental action unless $\mathfrak{Q} = R$. If $\mathfrak{Q} = R$, $\phi|\mathfrak{Q} = \phi$ and $f|[\mathfrak{q}] = f$.

Lemma 3.2. *If $\chi : \Gamma_n \to \mathbb{F}^\times$ and $\psi : \Delta^- \to \mathbb{F}^\times$ are characters, we have*

$$\int_{\Gamma_n} \chi d\varphi_f^\psi = \int_{Cl_n^-} \chi\psi d\varphi_f.$$

Here recall the image Γ_n of Γ in Cl_n^-.

Proof. For a proper R_n–ideal $\mathcal{A}$, by the above definition of these operators,

$$f|[\mathfrak{r}]|[\mathfrak{q}]([\mathcal{A}]) = \lambda(\mathcal{A})^{-1} f(x(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathcal{A})).$$

Since $\chi = \psi$ on Δ^-, we have

$$\begin{aligned}\int_{\Gamma_n} \chi d\varphi_f^\psi &= \sum_{\mathfrak{Q}\in\mathcal{Q}} \sum_{\mathfrak{r}\in\mathcal{R}} (\lambda\chi^{-1}\psi^{-1})(\mathfrak{Q}\mathfrak{R}) \sum_{\mathcal{A}\in\Gamma_n} \chi(\mathcal{A}) f|[\mathfrak{r}]|[\mathfrak{q}]([\mathcal{A}]) \\ &= \sum_{\mathcal{A},\mathfrak{Q},\mathfrak{r}} \chi\psi(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathcal{A}) f([\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathcal{A}]) = \int_{Cl_n^-} \chi\psi d\varphi_f,\end{aligned}$$

because $Cl_n^- = \bigsqcup_{\mathfrak{Q}\mathfrak{R}} [\mathfrak{Q}^{-1}\mathfrak{R}^{-1}]\Gamma_n$. □

We write $\mathbb{F}_{\mathbf{q}} := \mathbb{F}_q[\psi] \subset \mathbb{F}$ for the field of rationality of ψ over $\mathbb{F}_q$. Then $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{\mathbf{q}})$ preserves $d\varphi_f^\psi$. Then (3.14) shows that if χ is a character of Γ, for $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{\mathbf{q}})$,

$$\int_{\Gamma_n} \chi(x) d\varphi_f^\psi(x) = 0 \iff \int_{\Gamma_n} \sigma\circ\chi(x) d\varphi_f^\psi(x) = 0. \tag{3.17}$$

3.6. **Trace relation.** For any finite extensions $\kappa/\kappa'/\mathbb{F}_{\mathbf{q}}[\mu_\ell]$, we consider the trace map: $\mathrm{Tr}_{\kappa/\kappa'}(\xi) = \sum_{\sigma\in\mathrm{Gal}(\kappa/\kappa')} \sigma(\xi)$ for $\xi \in \kappa$. Recall the image Γ_n of Γ in Cl_n. Define

$$f_\psi^{\mathcal{Q}}([\mathcal{A}]) = \sum_{\mathfrak{Q}\in\mathcal{Q}} \psi(\mathfrak{Q})^{-1} f_\psi([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma) \text{ for the projection } [\mathfrak{Q}]_\Gamma \in \Gamma \text{ of } [\mathfrak{Q}]. \tag{3.18}$$

Let $\chi : \Gamma_n \to \mathbb{F}^\times$ be a character. Suppose that $\mathrm{Im}(\chi_\ell) \cap \mathbb{F}_{\mathbf{q}}[\mu_\ell]^\times$ has order r and that χ_ℓ has order ℓ^ν. Fix $j \geq r$, and write $\Phi = \Phi_n := \Gamma_n \cap \chi^{-1}(\mathbb{F}_{\mathbf{q}}[\mu_{\ell^j}]^\times)$ and $[\mathcal{A}_y] = [\mathcal{A}_y]_n$ for the image of y in Γ_n. We suppose that $[R_{n-1}\mathcal{A}_y]_n = [\mathcal{A}_y]_{n-1}$. If $\nu \geq j$, for

$$d = [\mathbb{F}_{\mathbf{q}}[\chi] : \mathbb{F}_{\mathbf{q}}[\mu_{\ell^j}]] = [\mathrm{Im}(\chi_\ell) : \mathrm{Im}(\chi_\ell) \cap \mathbb{F}_{\mathbf{q}}[\mu_\ell]^\times] = \ell^{\nu-j},$$

$$\begin{aligned}&\int_{\Gamma_n} \mathrm{Tr}_{\mathbb{F}_{\mathbf{q}}[\chi]/\mathbb{F}_{\mathbf{q}}[\chi^{(\ell)},\mu_\ell]} \circ \chi(y^{-1}x) d\varphi_f^\psi(x) \\ &\qquad = \frac{d}{\mathbf{a}^n} \sum_{\mathcal{A}\in\Gamma_n:\mathcal{A}y^{-1}\in\Phi_n} \chi(y^{-1}\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}]) = \frac{d}{\mathbf{a}^n} \sum_{[\mathcal{A}]\in\Phi_n} \chi(\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}][\mathcal{A}_y]),\end{aligned} \tag{3.19}$$

because for an ℓ–power root of unity and a finite extension $\kappa/\mathbb{F}_{\mathbf{q}}[\mu_{\ell^j}]$, $\zeta \in \mu_{\ell^\nu} - \mu_{\ell^j}$,

$$\mathrm{Tr}_{\kappa[\mu_{\ell^\nu}]/\kappa[\mu_{\ell^j}]}(\zeta^s) = \begin{cases} \ell^{\nu-j}\zeta^s & \text{if } \zeta^s \in \kappa[\mu_{\ell^j}] \text{ and } \kappa[\mu_\ell] \cap \mu_{\ell^\infty} = \mu_{\ell^j} \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

Thus by (3.17), we have

$$\sum_{[\mathcal{A}]\in\Phi_n} \chi(\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}][\mathcal{A}_y]) = 0 \text{ if } \int_{Cl_n^-} \chi\psi d\varphi_f = 0. \tag{3.21}$$

Let $\mathcal{F}(\Phi_n[\mathcal{A}_y],\mathbb{F})$ be the space of functions $\phi : \Phi_n[\mathcal{A}_y] \to \mathbb{F}$. Consider the linear form $\ell_\chi : \mathcal{F}(\Phi_n[\mathcal{A}_y],\mathbb{F}) \to \mathbb{F}$ given by $\ell_\chi(\phi) = \sum_{[\mathcal{A}]\in\Phi_n} \chi([\mathcal{A}])\phi([\mathcal{A}])$. Since the orthogonal complement of the space spanned by $\{\ell_{\chi^\sigma}\}_{\sigma\in\mathrm{Gal}(\mathbb{Q}[\mu_{\ell^\varepsilon}]/\mathbb{Q})}$ in $\mathcal{F}(\Phi_n,\mathbb{F})$ under the pairing

$$\langle\phi,\phi'\rangle = \sum_{[\mathcal{A}]\in\Phi_n} \phi([\mathcal{A}][\mathcal{A}_y])\phi'([\mathcal{A}])$$

is spanned by characters of order $\leq \ell^{\varepsilon-1}$. If $\varepsilon = 1$, the orthogonal complement is made of constant functions on Φ_n. Thus assuming that the integral (3.19) vanishes with $\Phi_n \cong \mu_\ell$ and that $\mathrm{Gal}(\mathbb{F}_{\mathbf{q}}[\mu_\ell]/\mathbb{F}_{\mathbf{q}}) = \mathrm{Gal}(\mathbb{Q}[\mu_\ell]/\mathbb{Q})$, $[\mathcal{A}] \mapsto f_\psi^{\mathcal{Q}}([\mathcal{A}][\mathcal{A}_y])$ is a constant function of $[\mathcal{A}]$ whose value is $f([\mathcal{A}_y])$, for $\alpha_1 = \mathrm{diag}[1,\varpi_{\mathfrak{l}}]$, we have

$$\ell f_\psi^{\mathcal{Q}}([\mathcal{A}_y]_n) = \sum_{[\mathcal{B}]\in\Phi_n y} f_\psi^{\mathcal{Q}}([\mathcal{B}]_n) = \sum_{u \mod \mathfrak{l}} f_\psi^{\mathcal{Q}}|(\varrho(u/\varpi_{\mathfrak{l}})\alpha_1)([\mathcal{A}_y]_{n-1}) = a f_\psi^{\mathcal{Q}}([\mathcal{A}_y]_{n-1}).$$

This is easy to see if we choose $\mathcal{A}_y$ prime to $\mathfrak{l}$ (i.e., $\mathcal{A}_{y,\mathfrak{l}} = R_{n,\mathfrak{l}}$). Hereafter exclusively

$$r \text{ is defined by } \mu_{\ell^\infty} \cap \mathbb{F}_{\mathbf{q}}[\mu_\ell]^\times = \mu_{\ell^r}. \tag{3.22}$$

4. Proof of Theorem 0.1

Write $\widehat{\mathbb{G}}_{m/\mathbb{Z}_\ell}$ for the formal completion over $\mathbb{Z}_\ell$ at the origin of $\mathbb{G}_m(\mathbb{F}_\ell)$. Let $\mathrm{Hom}(\Gamma,\mu_{\ell^\infty})$ embed into $\mathbb{G}^d_{m/\mathbb{Z}_\ell}$ and $\widehat{\mathbb{G}}^d_{m/\mathbb{Z}_\ell}$ by choosing a basis $(\gamma_1,\dots,\gamma_d)$ of Γ over $\mathbb{Z}_\ell$ and sending $\chi \in \mathrm{Hom}(\Gamma,\mu_{\ell^\infty})$ to $(\chi(\gamma_1),\dots,\chi(\gamma_d))$. A subset S of $\mathrm{Hom}(\Gamma,\mu_{\ell^\infty})$ therefore has its Zariski closure $\overline{S}$ (resp. $\widehat{S}$) in $\mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell)$ (resp. $\widehat{\mathbb{G}}^d_m(\overline{\mathbb{Q}}_\ell)$). Since $\mathrm{Aut}(\widehat{\mathbb{G}}^d_m) = \mathrm{GL}_d(\mathbb{Z}_\ell)$, the isomorphism class of $\widehat{S}$ is independent of the choice of the basis. As we will see later for our choice of S that $\dim\widehat{S} = \dim\overline{S}$, and hence $\overline{S}$ being a proper Zariski closed set is independent of the choice of basis.

Fix a character $\psi : \Delta \to \mathbb{F}^\times$. Let

$$\mathcal{X} = \mathcal{X}_\psi := \{\chi \in \mathrm{Hom}(\Gamma,\mu_{\ell^\infty}) | \int_{Cl_n^-} \chi\psi d\varphi_f \neq 0 \text{ for some } n\}.$$

If $a = 0$, as seen in Remark 3.1, $\int_{Cl_n^-} \chi\psi d\varphi_f \neq 0$ for one value n; in other words, for n given by $\mathrm{cond}(\chi) = \mathfrak{l}^n$, the integral is not defined over $Cl_{n'}$ for $n' < n$ and the integral vanishes for $n' > n$. On the contrary, if $a \neq 0$, the vanishing (and non-vanishing) of the integral is independent of n as long as it is well defined.

Assume the following condition:

(4.1) *The Zariski closure $\overline{\mathcal{X}}_\psi$ in $\mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell)$ of the set $\mathcal{X}_\psi$ has dimension $< d$,*

and we are going to deduce absurdity.

4.1. **Proof.** We prepare a lemma. Let $\mathbb{C}_\ell$ be the ℓ-adic completion of $\overline{\mathbb{Q}}_\ell$. Let $\mathbf{W}$ be a discrete valuation ring finite over the Witt vector ring $W(\overline{\mathbb{F}}_\ell)$ inside $\mathbb{C}_\ell$ for an algebraic closure $\overline{\mathbb{F}}_\ell$, and write $\mathcal{K}$ for its quotient field. For a formal subscheme X of $\widehat{\mathbb{G}}_{m/\mathbf{W}}$, we write $X(\mathbb{C}_\ell) := X(\overline{\mathbf{W}})$ for the integral closure $\overline{\mathbf{W}}$ of $\mathbf{W}$ in $\mathbb{C}_\ell$. The map $t \mapsto t^{\overline{z}_n}$ is an automorphism of μ_{ℓ^n} for $\overline{z}_n \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. Take a sequence of $z_n \in \mathbb{Z}$ lifting $\overline{z}_n$ and assuming $\{z_n\}$ converges to $z \in \mathbb{Z}_\ell^\times$. Then $\zeta \mapsto t^{\overline{z}_n}$ gives rise to an automorphism $z \in \mathbb{Z}_\ell^\times$ of μ_{ℓ^∞}. In this way, ℓ-adic unit z acts on $\mu^d_{\ell^\infty}$. If $z \in \mathbb{Q}\cap\mathbb{Z}_\ell^\times$ prime to ℓ, this automorphism of $\mu^d_{\ell^\infty}$ extends to an isogeny $t \mapsto t^z$ of $\mathbb{G}^d_m$. If we identify $\mu_{\ell^\infty} = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, $t \mapsto t^z$ turns into a multiplication $\tau \mapsto z\tau$ by z on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. In the following lemma, we take $z = p^m$ for $m \in \mathbb{Z}$ and a prime $p \neq \ell$.

Lemma 4.1. *Let p and ℓ be distinct primes and $r > 0$ be an integer. Let $\mathcal{X}$ be a subset of $\mu^d_{\ell^\infty}$ and $\overline{\mathcal{X}}$ be the Zariski closure of $\mathcal{X}$ in $\mathbb{G}^d_m(\overline{\mathbb{Q}}_\ell)$ for $d \geq 1$. Suppose that $\overline{\mathcal{X}}$ is a subscheme stable under $t \mapsto t^{p^{rn}}$ for all $n \in \mathbb{Z}$ and a fixed $r > 0$ (this means $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$). Assume $\dim\overline{\mathcal{X}} < d$. Identify $\mu^d_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$ with $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^d$ as ℓ–divisible groups. Then, for a given d-tuple $(a_1,\dots,a_d)$ of positive integers, we can find a sufficiently large p^r–power $P = p^j$ with an r-muliple $j = rj'$ and a positive integer N such that there exists a sequence of subsets $\{\Upsilon_n\}_{n=N}^\infty$ outside $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell)$ such that*

$$\Upsilon_n = \left\{ \left(\frac{P^{k_1}e_1}{\ell^{n+a_1}} + \cdots + \frac{P^{k_d}e_d}{\ell^{n+a_d}}\right) \mod \mathbb{Z}_\ell^d \middle| (k_i) \in \mathbb{Z}^d \right\}$$

if we choose a base $\{e_i\}$ *of* $\mathbb{Z}_\ell^d$ *suitably.*

Proof. We choose the p^r–power P so that $P \equiv 1 \mod \ell$. Let $\Gamma_P = P^{\mathbb{Z}_\ell} \subset \mathbb{Z}_\ell^\times$, which is an open subgroup of $1 + \ell\mathbb{Z}_\ell$. Let $\overline{\mathcal{X}}[\ell^\infty] := \overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell)$. Since $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$, we have $\overline{\mathcal{X}}[\ell^\infty]^{p^r} \subset \overline{\mathcal{X}}[\ell^\infty]$; so, the Zariski closure of $\overline{\mathcal{X}}[\ell^\infty]$ is stable under $t \mapsto t^{p^r}$. We may replace $\overline{\mathcal{X}}$ by the Zariski closure of $\overline{\mathcal{X}}[\ell^\infty]$ as the lemma only concerns about $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell)$, and after the replacement, the stability $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$ is intact. If $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell)$ is a finite set, the assertion plainly follows; so, we may assume that $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell)$ is infinite. Since $\overline{\mathcal{X}}$ is noetherian, we may also assume that $\dim \overline{\mathcal{X}} > 0$ as otherwise, $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell)$ is finite.

The variety $\overline{\mathcal{X}}$ is defined over a finite extension $\mathcal{K}$ of $\mathrm{Frac}(W(\overline{\mathbb{F}}_\ell))$. Write $\mathbf{W}$ be the ℓ-adic integer ring of $\mathcal{K}$. Let $\overline{\mathcal{X}}_{/\mathbf{W}}$ be the schematic closure of $\overline{\mathcal{X}}_{/\mathcal{K}}$ in $\mathbb{G}_{m/\mathbf{W}}^d$. Writing $\mathbb{G}_{m/\mathbf{W}}^d = \mathrm{Spec}(R)$ for $R = \mathbf{W}[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ and $\overline{\mathcal{X}} = \mathrm{Spec}(R \otimes_{\mathbf{W}} \mathcal{K}/\mathfrak{a})$, $\overline{\mathcal{X}}_{/\mathbf{W}} = \mathrm{Spec}(R/\mathfrak{A})$ for $\mathfrak{A} := \mathfrak{a} \cap R$. Thus $\overline{\mathcal{X}}_{/\mathbf{W}}$ is flat over $\mathbf{W}$. Let

$$\mathfrak{m} = \mathfrak{m}_{\mathbf{W}} + (t_1 - 1, \ldots, t_d - 1) \subset \mathbf{W}[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$$

and $\widehat{\mathfrak{A}}$ be the $\mathfrak{m}$-adic closure of $\mathfrak{a}$ in $\widehat{R} = \mathbf{W}[[T_1, \ldots, T_d]] = \varprojlim_n R/\mathfrak{m}^n$ for $T_i = t_i - 1$. We write $\widehat{M}$ for $\mathfrak{m}$-adic completion of an R-module M and $\mathrm{gr}(M) = \bigoplus_{j=0}^\infty \mathfrak{m}^n M/\mathfrak{m}^{n+1} M$ (the graded module over $\mathrm{gr}(R)$). Note that $\widehat{\mathbb{G}}_{m/\mathbf{W}}^d = \mathrm{Spf}(\widehat{R})$ is the formal completion of $\mathbb{G}_{m/\mathbf{W}}^d$ along the identity of $\widehat{\mathbb{G}}_{m/\overline{\mathbb{F}}_\ell}$. Define $\widehat{\mathcal{X}} := \mathrm{Spf}(A)$ for $A := W[[T_1, \ldots, T_d]]/\widehat{\mathfrak{A}}$, which is a formal subscheme of $\widehat{\mathbb{G}}_{m/\mathbf{W}}^d$ over $\mathbf{W}$. Since $A = R/\mathfrak{A} \otimes_R W[[T_1, \ldots, T_n]] = \widehat{R/\mathfrak{A}}$, $\widehat{\mathcal{X}}$ is a flat over $\mathbf{W}$. Since $\dim(A) = \dim \mathrm{gr}(A) = \dim \mathrm{gr}(R/\mathfrak{A}) = \dim(R/\mathfrak{A})$ (e.g., [CRT, Theorem 15.7]), we find $\dim \widehat{\mathcal{X}} = \dim \overline{\mathcal{X}} < d$.

Since $\overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell) \neq \emptyset$ and $\mu_{\ell^\infty}^d \times_{\mathbf{W}} \overline{\mathbb{F}}_\ell$ has only one geometric point, we see $\widehat{\mathcal{X}}(\mathbf{W}[\mu_{\ell^\infty}]) \supset \overline{\mathcal{X}}(\overline{\mathbb{Q}}_\ell) \cap \mu_{\ell^\infty}^d$. Since $\overline{\mathcal{X}}^{p^r} \subset \overline{\mathcal{X}}$, we still have $\widehat{\mathcal{X}}^{p^r} \subset \widehat{\mathcal{X}}$ inside $\widehat{\mathbb{G}}_{m/\mathbf{W}}^d$. Thus $\widehat{\mathcal{X}}$ is stable under an open subgroup U of $1 + \ell\mathbb{Z}_\ell$. Here an element $s \in 1 + \ell\mathbb{Z}_\ell$ acts on $\widehat{\mathbb{G}}_m^d$ by $t \mapsto t^s$. Since $\widehat{\mathcal{X}}$ is noetherian, it has finitely many geometrically irreducible components, and U permutes them. Thus replacing U by its open subgroup, we may assume that U fix each geometrically irreducible component. By extending scalar, we may assume that each geometrically irreducible component is defined over $\mathbf{W}$. Then by Lemma 4.4 below, $\widehat{\mathcal{X}} = \bigcup_{\zeta \in Z, i} \zeta \mathcal{T}_{\zeta,i}$, where $\mathcal{T}_{\zeta,i}$ is a formal subtorus of $\widehat{\mathbb{G}}_{m/\mathbf{W}}^d$ and Z is a finite subset of $\mu_{\ell^\infty}^d(\mathbf{W}[\mu_{\ell^\infty}])$.

We first assume that $Z = \{1\}$. By this assumption, $\widehat{\mathcal{X}}$ is a union of subtori $\{\mathcal{T}_i\}_{i \in J}$ with $|J| < \infty$ and $\dim \mathcal{T}_i < d$. Thus we have its ℓ-adic Tate module $T\mathcal{T}_i = \varprojlim_n \mathcal{T}[\ell^n] \subset T := T\mu_{\ell^\infty}^d$. In particular, $\overline{\mathcal{X}}[\ell^n] = \overline{\mathcal{X}} \cap \mu_{\ell^n}^d$ is the image of $\bigcup_i \mathbb{Q}_\ell T\mathcal{T}_i$ in $\mu_{\ell^n}^d = T/\ell^n T$. Then we can choose a base $\{e_1, \ldots, e_n\}$ of T over $\mathbb{Z}_\ell$ outside $T\widehat{\mathcal{X}}$ so that $\mathbb{Z}_\ell e \cap T\widehat{\mathcal{X}} = \{0\}$ for $e = e_1 + e_2 + \cdots + e_d$. Then the ℓ–adic distance from the $\mathbb{Q}_\ell$-span $\mathbb{Q}_\ell T\widehat{\mathcal{X}}$ to the point $\frac{e}{\ell^n}$ is larger than or equal to $c\ell^n$ for a positive constant c independent of n. Thus we can find sufficiently large power P of p^r (ℓ–adically very close to 1) so that $U_n = \Gamma_P \frac{e_1}{\ell^{a_1+n}} + \cdots + \Gamma_P \frac{e_d}{\ell^{a_d+n}}$ for $\Gamma_P = P^{\mathbb{Z}_\ell}$ gives rise to an open neighborhood of $\frac{e}{\ell^n}$ disjoint from $\mathbb{Q}_\ell T\widehat{\mathcal{X}}$. Then the image Υ_n of U_n in $\mu_{\ell^\infty}^d$ is disjoint from $\widehat{\mathcal{X}}[\ell^\infty]$ and hence from $\overline{\mathcal{X}}$ for all $n \geq 1$.

When $Z \neq \{1\}$, we consider the subgroup $\langle Z \rangle$ of $\mu_{\ell^\infty}^d$ generated by Z. The group $\langle Z \rangle$ is finite. Consider the projection $\pi : \widehat{G}_m^d \to \widehat{G}_m^d/\langle Z \rangle$. The image of $\pi(\widehat{\mathcal{X}})$ under π is a formal subtorus and hence stable under scalar multiplication. Using the result proven under the condition $Z = \{1\}$ applied to $\pi(\widehat{\mathcal{X}})$, we write Υ'_n for the sets constructed for $\pi(\mu_{\ell^\infty}^d) = \mu_{\ell^\infty}/\langle Z \rangle$. Then we find that for $n > N$ any Γ_P^d–orbit of an element in the pull-back image $\Upsilon_n := \pi^{-1}(\Upsilon'_n)$ gives a desired set $\Upsilon_n \subset \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell)$. This finishes the proof. □

Choose a $\mathbb{Z}_\ell$-basis $\gamma_1, \ldots, \gamma_d$ for $d = \mathrm{rank}_{\mathbb{Z}_\ell} \Gamma$. Then identify $\mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ with $\mu_{\ell^\infty}^d$ by $\chi \mapsto (\chi(\gamma_1), \ldots, \chi(\gamma_d)) \in \mu_{\ell^\infty}^d \subset \mathbb{G}_m^d$. Here is a more accurate version of Theorem 0.1.

Theorem 4.2. *Suppose that for a given class* $v \in (O_{\mathfrak{l}}/\mathfrak{l}^j)^\times$ *with a sufficiently large* $j \geq r > 0$ *for* r *as in* Theorem 0.1 *and a cusp* $(\mathfrak{a}, \mathfrak{b})$, *there exists* $\xi \in \mathfrak{ab} \cap -v$ *such that* $a(\xi, f_\psi) \neq 0$ *in* $\mathbb{F}$. *Then the set of characters* $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ *with non-vanishing* $v(\chi) = v$ *and* $\int_{Cl_n} \psi\chi d\varphi_f \neq 0$ *for* $n > 0$ *given by* $\mathrm{cond}(\chi) = \mathfrak{l}^n$ *is Zariski dense in* $\mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$. *If* $\mathrm{rank}_{\mathbb{Z}_\ell} \Gamma = 1$, *we can take* $j = r$.

Though the minimal possible r depends on $\mathfrak{l}$, the assumption in the theorem is in appearance weaker than

(h) There exists a strict ideal class $\mathfrak{c}$ of F such that $\mathfrak{c}(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s})$ is in $\mathfrak{c}$ for some $(\mathfrak{Q},\mathfrak{R},\mathfrak{s}) \in \mathcal{Q} \times \mathcal{S} \times \mathcal{R}$ and for any given integer $j \geq r > 0$, the $N(\mathfrak{l})^j$ modular forms $f_{\psi,\mathfrak{c}}|\left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right)$ for $u \in \mathfrak{l}^{-j}/O$ are linearly independent,

which is assumed in [H04, Theorem 3.3].

Proof. Let

$$\mathcal{X} = \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) \hookrightarrow \mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell} | \int_\Gamma \chi d\varphi^\psi_f \neq 0 \text{ and } v(\chi) = v\}$$

and $\widehat{\mathcal{X}}$ (resp. $\overline{\mathcal{X}}$) be the formal Zariski (resp. Zariski) closure of $\mathcal{X}$ in $\widehat{\mathbb{G}}^d_{m/\mathbf{W}}$ (resp. $\mathbb{G}^d_m$). Note that $\overline{\mathcal{X}}^P \subset \overline{\mathcal{X}}$ and $\widehat{\mathcal{X}}^P \subset \widehat{\mathcal{X}}$. Suppose $\overline{\mathcal{X}}$ is a proper Zariski closed subset of $\mathbb{G}^d_m$ and get a contradiction.

First suppose $d = 1$. Then $\overline{\mathcal{X}}$ is a finite set. Take $j := r$. So there exists $B > 0$ such that if the conductor of a character χ is $\mathfrak{l}^n$ with $n > B$, by (3.21), identifying $\Phi_n = O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^r$ by $x(\mathcal{A}_u) = \varrho(u/\ell^r)x(R_n) \leftrightarrow O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^r$,

$$\int_\Gamma \chi d\varphi^{\mathcal{Q}}_f = \sum_{[\mathcal{A}]\in\Phi_n} \chi(\mathcal{A}) f^{\mathcal{Q}}_\psi|\varrho(u/\varpi^r_\mathfrak{l})([\mathcal{A}\mathcal{A}_y]) = \sum_{u \mod \mathfrak{l}^r} \zeta^{vu}_r f^{\mathcal{Q}}_\psi|\varrho(u/\ell^r)([\mathcal{A}\mathcal{A}_y]) = 0.$$

Here $[\mathcal{A}_y]$ is any element in Γ_n. Let $\Xi_n = \{x([\mathcal{A}])|[\mathcal{A}] \in \Gamma_n\}$ and $\Xi := \bigsqcup_{n>B} \Xi_n \cap V$. Then Ξ is associated to an infinite arithmetic progression of difference m (for minimal exponent $\mathfrak{l}^m$ is generated by $N_{M/F}(R)$).

Since $\chi|_{\Phi_n} : O/\mathfrak{l}^r \to \mathbb{F}^\times$ is an arbitrary character of order ℓ^r, we may fix a character $\chi_v(u) = \zeta^{vu}_r$ for $v \in (O/\mathfrak{l}^r)^\times$ independent of $n > B$ as an additive character of $O/\mathfrak{l}^r$. Writing $g_\mathfrak{Q} := \sum_{u\in O/\mathfrak{l}^r} \chi_v(u) f_\psi|\varrho(u/\varpi^r_\mathfrak{l})$, we find $\sum_{\mathfrak{Q}\in\mathcal{Q}} \psi^{-1}(\mathfrak{Q}) g_\mathfrak{Q}|[\mathfrak{q}]([\mathcal{A}][\mathfrak{Q}]_\Gamma) = 0$ for all $[\mathcal{A}] \in \Xi$. By Corollary 2.11, Ξ is Zariski dense in $\mathcal{V} = V^{\mathcal{Q}}$, and hence we conclude $g_\mathfrak{Q}|[\mathfrak{q}] = 0$. Since the q-expansion of a modular form $h|[\mathfrak{q}]$ at $(\mathfrak{a}, \mathfrak{b})$ is given by the q-expansion of h at $(\mathfrak{q}\mathfrak{a}, \mathfrak{b})$; so, by q-expansion principle, $g_\mathfrak{Q}|[\mathfrak{q}] = 0 \Leftrightarrow g_\mathfrak{Q} = 0$ (e.g., [H10, (5.10)]). Note $a(\xi, g_\mathfrak{Q}) = N(\mathfrak{l})^r a(\xi, f_\psi)$ as long as $\xi \equiv -v \mod \mathfrak{l}^r$. Since v is arbitrary, we can choose v so that ξ as in the theorem satisfies $\xi \equiv -v \mod \mathfrak{l}^r$; so, $g_\mathfrak{Q} \neq 0$, a contradiction.

We now assume $d \geq 2$. Applying Lemma 4.1 to $\overline{\mathcal{X}} \subset \mathbb{G}^d_m$, for the base $e_1, \ldots, e_d$ as in Lemma 4.1 of the Tate module $T\mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) = \varprojlim_n \mathrm{Hom}(\Gamma, \mu_{\ell^n})$, we choose the corresponding basis of Γ as $\gamma_1, \ldots, \gamma_d$; so, the $\mathbb{Z}_\ell$-module $\gamma_i^{\mathbb{Z}_\ell}$ is sent isomorphically onto $\mathbb{Z}_\ell e_i$ for each i. Recall $Cl^-_\infty = \varprojlim_n Cl^-_n$ and $Cl_\infty = \Gamma \times \Delta$ for a finite group Δ. Pick the smallest integer $0 < a \in \mathbb{Z}$ so that $\mathrm{Ker}(Cl_\infty \to Cl_a) \subset \Gamma$. Choose $a_1, \cdots, a_d$ so that $\prod_i \gamma_i^{\ell^{a_i+n}\mathbb{Z}_\ell} = \mathrm{Ker}(Cl_\infty \to Cl_{a+n})$ for $n \geq 0$. Let $P = p^j$ with $j \geq r$ as in Lemma 4.1.

Suppose $\mathfrak{l}^m$ is principal generated by $\varpi = \varphi\varphi^c$ for $\varphi \in R$. Then $\Upsilon = \bigcup_{i\geq N} \Upsilon_i$ is disjoint from $\overline{\mathcal{X}}$ by Lemma 4.1 for some positive integer N. Put $\Xi_{a+im} = \{x(\mathcal{A})|\mathcal{A} \in \mathrm{Ker}(Cl_{a+im} \to Cl_a)\}$, replacing m by a positive multiple so that $m \geq N - a$. Define an infinite arithmetic progression $\underline{n} := \{a + im|i = 1, 2, \ldots\}$. Then $\mathbf{T}_{a,m}$ acts transitively on Ξ, and by Theorem 2.6 and the proof of [H04, Proposition 2.8], Ξ embedded in $V^{\mathcal{Q}}$ by $\mathcal{A} \mapsto \mathbf{x}([\mathcal{A}]) := (x([\mathcal{A}][\mathfrak{Q}]_\Gamma))_{\mathfrak{Q}\in\mathcal{Q}}$ is Zariski dense.

For each $\chi \in \Upsilon$,

$$\sum_{\mathfrak{Q}} \psi(\mathfrak{Q})^{-1} \sum_{\mathcal{A}\in y\widetilde{\chi}^{-1}(\mu^d_{\ell^j})} \chi(\mathcal{A}) f_{\psi,\mathfrak{c}}([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma) = 0$$

holds by (3.21) (see also [H04, page 770]) for $\mathcal{A}$ with $\mathbf{x}([\mathcal{A}]) \in \Xi_{\underline{n}}$.

Identify again $\Phi_n = O_\mathfrak{l}/(\mathfrak{l}O_\mathfrak{l})^j$ Let $g_{\mathcal{Q}} := \sum_{u\in O/\mathfrak{l}^j} \chi_v(u) f_\psi|\varrho(u/\varpi^j_\mathfrak{l})$ for $\chi_v(u) = \zeta_j^{\mathrm{Tr}(vu)}$ for $\mathrm{Tr} := \mathrm{Tr}_{O_\mathfrak{l}/\mathbb{Z}_\ell}$. Then

$$\sum_{\mathfrak{Q}} \psi(\mathfrak{Q})^{-1} \sum_{\mathcal{A}\in\Phi_n} g_{\mathcal{Q}}|[\mathfrak{q}]([\mathcal{A}][\mathfrak{Q}]_\Gamma) = 0 \text{ for } \mathbf{x}([\mathcal{A}]) \in \Xi. \tag{4.2}$$

By Zariski density of Ξ in $V^{\mathcal{Q}}$, we conclude $g_\mathfrak{Q}|[\mathfrak{q}] = 0$. Since $[\mathfrak{q}] \in \mathrm{Aut}(Sh^{(p)})$, we conclude $f_\psi = 0$.

For a chosen class $v \in (O/\mathfrak{l}^j)$, we find ξ such that $\xi \in -v$ and $a(\xi, f_\psi) \neq 0 \Leftrightarrow a(\xi, g_\mathfrak{Q}) \neq 0$, and from this we conclude contradiction against $f_\psi = 0$. □

Here is an obvious corollary of the proof of Theorem 4.2:

Corollary 4.3. *Let the notation be as in* Theorem 4.2. *Suppose* $d = 1$ *and* $a(\xi, f_\psi) \neq 0$ *for some* $\xi \in -v$ *for a given* $v \in (O/\mathfrak{l}^r)^\times$. *For a character* $\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}(\mathbb{F}))$, *define* $n(\chi)$ *by* $\mathrm{Ker}(\chi) = \mathrm{Ker}(\Gamma \to Cl_{n(\chi)})$. *Define a subset of* $\mathbb{Z}$ *by*

$$\underline{n}_v := \{n(\chi) | v(\chi) = v \ \ \textit{and} \ \int_{Cl^-_{n(\chi)}} \chi\psi d\varphi_\Gamma = 0\}.$$

Then $\underline{n}_v$ *cannot contain any infinite arithmetic progression.*

4.2. **A rigidity lemma.** We study formal subschemes of $\widehat{G} := \widehat{\mathbb{G}}^n_m$ stable under the action of $t \mapsto t^z$ for all z in an open subgroup U of $\mathbb{Z}_\ell^\times$. We recall with a proof the following result used in the proof of Theorem 0.1 from [H14, Lemma 4.1]:

Lemma 4.4. *Let* $\mathcal{K}$ *be a finite extension of* $\mathrm{Frac}(W(\overline{\mathbb{F}}_\ell))$ *and* $\mathbf{W}$ *be the integral closure of* $W(\overline{\mathbb{F}}_\ell)$ *in* $\mathcal{K}$. *Let* $T = \mathrm{Spf}(\mathcal{T})$ *be a closed formal subscheme of* $\widehat{G} = \widehat{\mathbb{G}}^n_{m/\mathbf{W}}$ *flat geometrically irreducible over* $\mathbf{W}$ *(i.e.,* $\mathcal{T} \cap \overline{\mathbb{Q}}_\ell = \mathbf{W}$*). Suppose there exists an open subgroup* U *of* $\mathbb{Z}_\ell^\times$ *such that* T *is stable under the action* $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$ *for all* $u \in U$. *If* T *contains a Zariski dense subset* $\Omega \subset T(\mathbb{C}_\ell) \cap \mu^n_{\ell^\infty}(\mathbb{C}_\ell)$, *then we have* $\omega \in \Omega$ *and a formal subtorus* $\mathbf{T}$ *such that* $T = \mathbf{T}\omega$.

A similar assertion is not valid for a formal group $\widehat{\mathbb{G}}^2_{m/K} = \mathrm{Spec}(K[[T, T']])$ over a characteristic 0 field K. Writing $t = 1 + T$ and $t' = 1 + T'$ for multiplicative variables, the formal subscheme Z defined by $t^{\log(t')} = 1$ is not a formal torus, but it is stable under $(t, t') \mapsto (t^m, t'^m)$ for any $m \in \mathbb{Z}$. See [C, Remark 6.6.1 (iv)] for an optimal expected form of the assertion similar to the above lemma.

Proof. Let T_s be the singular locus of the associated scheme $T^{sh} = \mathrm{Spec}(\mathcal{T})$ over $\mathbf{W}$, and put $T^\circ = T^{sh} \backslash T_s$. The scheme T_s is a closed formal subscheme of T. To see this, we note, by the structure theorem of complete neotherian ring, that $\mathcal{T}$ is finite over a power series ring $\mathbf{W}[[X_1, \dots, X_d]] \subset \mathcal{T}$ for $d = \dim_{\mathbf{W}} T$ (cf. [CRT, §29]). The sheaf of continuous differentials $\Omega_{\mathcal{T}/\mathrm{Spf}(\mathbf{W}[[X_1,\dots,X_d]])}$ is a torsion $\mathcal{T}$-module, and T_s is the support of the formal sheaf of $\Omega_{\mathcal{T}/\mathrm{Spf}(\mathbf{W}[[X_1,\dots,X_d]])}$ (which is a closed formal subscheme of T). The regular locus T° of T is open dense in the generic fiber $T^{sh}_{/\mathcal{K}} := T^{sh} \times_{\mathbf{W}} \mathcal{K}$ of T^{sh}. Then $\Omega^\circ := T^\circ \cap \Omega$ is Zariski dense in $T^{sh}_{/\mathcal{K}}$.

In this proof, by extending scalars, we always assume that $\mathbf{W}$ is sufficiently large so that for $\zeta \in \Omega$ we focus on, we have $\zeta \in \widehat{G}(\mathbf{W})$ and that we have a plenty of elements of infinite order in $T(\mathbf{W})$ and in $T^\circ(\mathcal{K}) \cap T(\mathbf{W})$, which we simply write as $T^\circ(\mathbf{W}) := T^\circ(\mathcal{K}) \cap T(\mathbf{W})$.

Note that the stabilizer U_ζ of $\zeta \in \Omega$ in U is an open subgroup of U. Indeed, if the order of ζ is equal to ℓ^a, then $U_\zeta = U \cap (1 + p^a\mathbb{Z}_\ell)$. Thus making a variable change $t \mapsto t\zeta^{-1}$ (which commutes with the action of U_ζ), we may assume that the identity $\mathbf{1}$ of $\widehat{G}$ is in Ω°.

Let $\widehat{G}^{an}$, T_{an} and T^s_{an} be the rigid analytic spaces associated to T and T^s (in Berthelot's sense in [J95, §7]). We put $T^\circ_{an} = T_{an} \setminus T^s_{an}$, which is an open rigid analytic subspace of T_{an}. Then we apply the logarithm $\log : \widehat{G}^{an}(\mathbb{C}_\ell) \to \mathbb{C}^n_\ell = Lie(\widehat{G}^{an}_{/\mathbb{C}_\ell})$ sending $(t_i)_i \in \widehat{G}^{an}(\mathbb{C}_\ell)$ (the ℓ-adic open unit ball centered at $\mathbf{1} = (1, 1, \dots, 1)$) to $(\log_\ell(t_i))_i) \in \mathbb{C}^n_\ell$ for the ℓ-adic Iwasawa logarithm map $\log_\ell : \mathbb{C}^\times_\ell \to \mathbb{C}_\ell$. Then for each smooth point $x \in T^\circ(\mathbf{W})$, taking a small analytic open neighborhood V_x of x (isomorphic to an open ball in $\mathbf{W}^d$ for $d = \dim_{\mathbf{W}} T$) in $T^\circ(\mathbf{W})$, we may assume that $V_x = G_x \cap T^\circ(\mathbf{W})$ for an n-dimensional open ball G_x in $\widehat{G}(\mathbf{W})$ centered at $x \in \widehat{G}(\mathbf{W})$. Since $\Omega^\circ \neq \emptyset$, $\log(T^\circ(\mathbf{W}))$ contains the origin $0 \in \mathbb{C}^n_\ell$. Take $\zeta \in \Omega^\circ$. Write T_ζ for the Tangent space at ζ of T. Then $T_\zeta \cong \mathbf{W}^d$ for $d = \dim_{\mathbf{W}} T$. The space $T_\zeta \otimes_{\mathbf{W}} \mathbb{C}_\ell$ is canonically isomorphic to the tangent space T_0 of $\log(V_\zeta)$ at 0.

If $\dim_{\mathbf{W}} T = 1$, there exists an infinite order element $t_1 \in T(\mathbf{W})$. We may (and will) assume that $U = (1 + \ell^m\mathbb{Z}_\ell)$ for $0 < m \in \mathbb{Z}$. Then T is the (formal) Zariski closure $\overline{t_1^U}$ of

$$t_1^U = \{t_1^{1+\ell^m z} | z \in \mathbb{Z}_\ell\} = t_1\{t_1^{\ell^m z} | z \in \mathbb{Z}_\ell\},$$

which is a coset of a formal subgroup Z. The group Z is the Zariski closure of $\{t_1^{\ell^m z} | z \in \mathbb{Z}_\ell\}$; in other words, regarding t_1^u as a $\mathbf{W}$-algebra homomorphism $t_1^u : \mathcal{T} \to \mathbb{C}_\ell$, we have $t_1 Z = \mathrm{Spf}(\mathcal{Z})$ for $\mathcal{Z} = \mathcal{T}/\bigcap_{u \in U} \mathrm{Ker}(t_1^u)$. Since t_1^U is an infinite set, we have $\dim_{\mathbf{W}} Z > 0$. From geometric irreducibility and $\dim_{\mathbf{W}} T = 1$, we conclude $T = t_1 Z$ and $Z \cong \widehat{\mathbb{G}}_m$. Since T contains roots of unity $\zeta \in \Omega \subset \mu^n_{\ell^\infty}(\mathbf{W})$, we confirm that $T = \zeta Z$ for $\zeta \in \Omega \cap \mu^n_{\ell^{m'}}$ for $m' \gg 0$. Replacing t_1 by $t_1^{\ell^m}$ for m as above if necessary, we have the translation $\mathbb{Z}_\ell \ni s \mapsto \zeta t_1^s \in Z$ of one parameter subgroup

$\mathbb{Z}_\ell \ni s \mapsto t_1^s$. Thus we have $\log(t_1) = \frac{dt_1^s}{ds}|_{s=0} \in T_\zeta$, which is sent by "$\log : \widehat{G} \to \mathbb{C}_\ell^n$" to $\log(t_1) \in T_0$. This implies that $\log(t_1) \in T_0$ and hence $\log(t_1) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ (under the identification of the tangent space at any $x \in \widehat{G}$ with $Lie(\widehat{G})$). Therefore T_ζ's over $\zeta \in \Omega^\circ$ can be identified canonically. This is natural as Z is a formal torus, and the tangent bundle on Z is constant, giving $Lie(Z)$.

Suppose that $d = \dim_{\mathbf{W}} T > 1$. Consider the Zariski closure Y of t^U for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^\circ$). Since U permutes finitely many geometrically irreducible components, each component of Y is stable under an open subgroup of U. Therefore $Y = \bigcup \zeta' \mathcal{T}_{\zeta'}$ is a union of formal subtori $\mathcal{T}_{\zeta'}$ of dimension ≤ 1, where ζ' runs over a finite set inside $\mu_{\ell^\infty}^n(\mathbb{C}_\ell) \cap T(\mathbb{C}_\ell)$. Since $\dim_{\mathbf{W}} Y = 1$, we can pick $\mathcal{T}_{\zeta'}$ of dimension 1 which we denote simply by $\mathcal{T}$. Then $\mathcal{T}$ contains t^u for some $u \in U$. Applying the argument in the case of $\dim_{\mathbf{W}} T = 1$ to $\mathcal{T}$, we find $u \log(t) = \log(t^u) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ and $t \in V_\zeta$. Summarizing our argument, we have found

- (T) The Zariski closure of t^U in T for an element $t \in V_\zeta$ of infinite order contains a coset $\xi\mathcal{T}$ of one dimensional subtorus $\mathcal{T}$, $\xi^{\ell^{m'}} = 1$ and $t^{\ell^{m'}} \in \mathcal{T}$ for some $m' > 0$;
- (D) Under the notation as above, we have $\log(t) \in T_\zeta$ for all $\zeta \in \Omega^\circ$.

Moreover, the image $\overline{V}_\zeta$ of V_ζ in $\widehat{G}/\mathcal{T}$ is isomorphic to $(d-1)$-dimensional open ball. If $d > 1$, therefore, we can find $\overline{t}' \in \overline{V}_\zeta$ of infinite order. Pulling back $\overline{t}'$ to $t' \in V_\zeta$, we find $\log(t), \log(t') \in T_\zeta$, and $\log(t)$ and $\log(t')$ are linearly independent in T_ζ. Inductively arguing this way, we find infinite order elements $t_1, \ldots, t_d$ in V_ζ such that $\log(t_i)$ span over the quotient field K of $\mathbf{W}$ the tangent space $T_{\zeta/K} = T_\zeta \otimes_{\mathbf{W}} \mathcal{K} \hookrightarrow T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{\mathbf{1}/\mathcal{K}} \subset T_0$ with $T_{\zeta/\mathcal{K}} \subset T_0$. Thus the tangent bundle over $T^\circ_{/\mathcal{K}}$ is constant as it is constant over the Zariski dense subset Ω°. Therefore T° is close to an open dense subscheme of a coset of a formal subgroup. We pin-down this fact.

Take $t_i \in V_\zeta$ as above $(i = 1, 2, \ldots, d)$ which give rise to a basis $\{\partial_i = \log(t_i)\}_i$ of the tangent space of $T_{\zeta/K} = T_{\mathbf{1}/K}$. Note that $t_i^u \in T$ and $u\partial_i = \log(t_i^u) = u \log(t_i) \in T_{\mathbf{1}/K}$ for $u \in U$. The embedding $\log : V_\zeta \hookrightarrow T_{\mathbf{1}} \subset Lie(\widehat{G}_{/\mathbf{W}})$ is surjective onto a open neighborhood of $0 \in T_{\mathbf{1}}$ (by extending scalars if necessary). For $t \in V_\zeta$, if we choose t closer to ζ, $\log(t)$ getting closer to 0. Thus by replacing $t_1, \ldots, t_d$ inside V_ζ to elements in V_ζ closer to ζ, we may assume that $\log(t_i) \pm \log(t_i)$ for all $i \neq j$ is in $\log(V_\zeta)$.

So, for each pair $i \neq j$, we can find $t_{i \pm j} \in V_\zeta$ such that $\log(t_i t_j^{\pm 1}) = \log(t_i) \pm \log(t_j) = \log(t_{i \pm j})$. The element $\log(t_{i \pm j})$ is uniquely determined in $\log(\widehat{G}_{an}(\mathbb{C}_\ell)) \cong \widehat{G}_{an}(\mathbb{C}_\ell)/\mu^n_{\ell^\infty}(\mathbb{C}_\ell)$. Thus we conclude $\zeta'_{i \pm j} t_i t_j^{\pm 1} = t_{i \pm j}$ for some $\zeta'_{i \pm j} \in \mu^n_{\ell^N}$ for sufficiently large N. Replacing T by its image under the ℓ-power isogeny $\widehat{G} \ni t \mapsto t^{\ell^N} \in \widehat{G}$ and t_i by $t_i^{\ell^N}$, we may assume that $t_i t_j^{\pm 1} = t_{i \pm j}$ all in T. Since $t_i^U \subset T$, by (T), for a sufficiently large $m' \in \mathbb{Z}$, we find a one dimensional subtorus $\widehat{H}_i$ containing $t_i^{\ell^{m'}}$ such that $\zeta_i \widehat{H}_i \subset T$ with some $\zeta_i \in \mu^n_{\ell^{m'}}$ for all i. Thus again replacing T by the image of the ℓ-power isogeny $\widehat{G} \ni t \mapsto t^{\ell^{m'}} \in \widehat{G}$, we may assume that the subgroup $\widehat{H}$ (Zariski) topologically generated by $t_1, \ldots, t_d$ is contained in T. Since $\{\log(t_i)\}_i$ is linearly independent, we conclude $\dim_{\mathbf{W}} \widehat{H} \geq d = \dim_{\mathbf{W}} T$, and hence T must be the formal subgroup $\widehat{H}$ of $\widehat{G}$. Since T is geometrically irreducible, $\widehat{H} = T$ is a formal subtorus. Pulling it back by the ℓ-power isogenies we have used, we conclude $T = \zeta\widehat{H}$ for the original T and $\zeta \in \mu^n_{\ell^{m'}N}(\mathbf{W})$. Since Ω is Zariski dense in T, we may assume that $\zeta \in \Omega$. This finishes the proof. □

4.3. **Semi-group action.** Though we do not need it, we add here an explicit determination of the action of α_m and α_m^{-1} on the point $x(\mathcal{A})$ defined in [H04, §2.1]. More generally we consider a pair $(L, \eta : \widehat{O}^2 \cong \widehat{L})$ of an O-lattice L of M and an $\widehat{O}$-linear isomorphism $\eta : (F_{\mathbb{A}}^{(p\infty)})^2 \cong \widehat{L} \otimes_O F_{\mathbb{A}}^{(p\infty)}$ with $\eta((\widehat{O}^{(p)})^2) = \widehat{L}^{(p)}$. We suppose that $L_p = R_p$. We define $Lg = \mathrm{Im}(\eta \circ g(\widehat{O}^2)) \cap M$ and $(L, \eta)g := (Lg, \eta \circ g)$. The pair gives rise to a point $x(L) \in Sh^{(p)}_{/W}$.

Choose a prime element $\varpi_{\mathfrak{l}}$ of $O_{\mathfrak{l}}$ and if $\mathfrak{l}$ ramifies in R, we suppose that $R_{\mathfrak{l}} = O_{\mathfrak{l}} + \sqrt{\varpi_{\mathfrak{l}}} O_{\mathfrak{l}}$. Recall $R_n = O + \mathfrak{l}^n R$. If ℓ is odd or ℓ dose not split in R, we write $R_{\mathfrak{l}} = O_{\mathfrak{l}} + \delta O_{\mathfrak{l}}$ so that $\delta = \sqrt{\varpi_{\mathfrak{l}}}$ if $\mathfrak{l}$ ramifies in R and $\delta = \sqrt{d}$ for $d \in O_{\mathfrak{l}}^\times$ if $\mathfrak{l}$ is unramified ($d = \delta^2$ is square if $\mathfrak{l} = \mathfrak{L}\mathfrak{L}^c$ splits and $d = (\delta, -\delta) \in R_{\mathfrak{L}} \times R_{\mathfrak{L}^c} = R_{\mathfrak{l}}$). If $\ell = 2$ and ℓ splits in R, we define $R'_{\mathfrak{l}} = \{x \in R_{\mathfrak{l}} | x \equiv x^c \mod 2\}$ and we start with this order, which has basis 1 and $(1, -1) \in O_{\mathfrak{l}} \times O_{\mathfrak{l}} = R_{\mathfrak{l}}$. We note in this case $R_1 = R'_{\mathfrak{l}} \cap \bigcap_{\mathfrak{q} \neq \mathfrak{l}} R_{\mathfrak{q}}$ for primes $\mathfrak{q}$ in O, and we put $\delta = (1, -1) \in R_{1,\mathfrak{L}}$ (so,we start with non-maximal order R_1). Then we put $\alpha_{\mathfrak{l}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}} \end{smallmatrix}\right) \in \mathrm{GL}_2(O_{\mathfrak{l}})$. We often regard $\alpha_{\mathfrak{l}} \in G(\mathbb{A})$ so that its component at a prime $\mathfrak{q} \neq \mathfrak{l}$ is equal to 1. We simply write R_n for the pair (R_n, η_n) with $\eta_n(a,b) = a + \varpi_{\mathfrak{l}}^n b$ at $\mathfrak{l}$ and outside $\mathfrak{l}$, we choose the basis given in [H04, page 741] and define η accordingly.

Then we put $\alpha_{\mathfrak{l}}^{\pm 1}(x(R_n)) = x(R_n \alpha_{\mathfrak{l}}^{\pm 1})$ under the action defined above. This action depends only on local component at $\mathfrak{l}$. As seen in [H04, page 760], we have

$$\alpha_{\mathfrak{l}}(x(R_n)) = x(R_{n+1}) \text{ and } \alpha_{\mathfrak{l}}^{-1}(x(R_n)) = x(R_{n-1}) \text{ if } n > 0. \tag{4.3}$$

Note

$$\alpha_{\mathfrak{l}}^{-n} \left[\begin{smallmatrix} 1 \\ \delta \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ \varpi_{\mathfrak{l}}^{-n}\delta \end{smallmatrix}\right] = \varpi_{\mathfrak{l}}^{-n}\delta \left[\begin{smallmatrix} \varpi_{\mathfrak{l}}^{n}\delta^{-1} \\ 1 \end{smallmatrix}\right]$$

at $\mathfrak{l}$, we need to change the original η_n to η'_n given by $\eta'_n(a,b) = \varpi_{\mathfrak{l}}^{-n}\delta^{-1}(a\varpi_{\mathfrak{l}}^{n}\delta^{-1} + b)$ at $\mathfrak{l}$ and outside $\mathfrak{l}$, the choice is the same as η_n. The lattice will change as follows

(unr) $R_{\mathfrak{l}} \mapsto \mathfrak{l}^{-n} R_{n,\mathfrak{l}}$ with $R_0 = R$ if $\mathfrak{l}$ remains prime or $\mathfrak{l}$ is odd and split in R;
(ram) $R_{\mathfrak{l}} \mapsto \mathfrak{l}^{-n}\mathfrak{L} R_{n,\mathfrak{l}}$ with $R_0 = R$ if $\mathfrak{l} = \mathfrak{L}^2$ in R;
(sp2) $R_{0,\mathfrak{l}} \mapsto \mathfrak{l}^{-n} R_{n,\mathfrak{l}}$ with $R_0 = R_1$ if $\mathfrak{l}|2$ and $\mathfrak{l}$ splits in R.

Denote $x'(\mathcal{A}) = (\mathcal{A}, \eta'_n)$ with $\mathcal{A}_{\mathfrak{l}}$ equal to the ideal as in (unr), (ram) and (sp2). Since

$$Cl_n = \frac{\{\text{fractional projective } R_n\text{-ideals}\}}{\{\text{principal } R_n\text{-ideals}\}},$$

we may allow R_n-ideals not prime to $\mathfrak{l}$. For an R_n-fractional ideal $\mathcal{A}$ prime to $\mathfrak{l}$, we denote $\mathcal{A}_n$ (resp. $\mathcal{A}'_n$) by the R_n-fractional ideal $\mathcal{A}_n$ (resp. $\mathcal{A}'_n$) with $\mathcal{A}_{n,\mathfrak{l}} = R_{n,\mathfrak{l}}$ (resp. $\mathcal{A}'_{n,\mathfrak{l}}$ given as in (unr), (ram) and (sp2)), and outside $\mathfrak{l}$, it is equal to the given $\mathcal{A}$. We have the following effect of $\alpha_{\mathfrak{l}}^m$ on the points $x(\mathcal{A}_n)$ and $x'(\mathcal{A}'_n)$;

(+) $\alpha_{\mathfrak{l}}^m(x(\mathcal{A}_n)) = x(\mathcal{A}_{n+m})$ and $\alpha_{\mathfrak{l}}^{-m}(x'(\mathcal{A}'_n)) = x'(\mathcal{A}'_{n+m})$ if $n > 0$ and $m \geq 0$;
(0) $\alpha_{\mathfrak{l}}^m(x'(\mathcal{A}'_n)) = x(\mathcal{A}_{m-n})$ and $\alpha_{\mathfrak{l}}^{-m}(x(\mathcal{A}_n)) = x'(\mathcal{A}'_{m-n})$ if $m \geq n$;
(−) $\alpha_{\mathfrak{l}}^m(x'(\mathcal{A}'_n)) = x'(\mathcal{A}'_{n-m})$ and $\alpha_{\mathfrak{l}}^{-m}(x(\mathcal{A}_n)) = x(\mathcal{A}_{n-m})$ if $n > m$.

5. A KEY STEP IN THE PROOF OF ANTICYCLOTOMIC MAIN CONJECTURE IN [H06]

A key step towards the proof of the anticyclotomic main conjecture is the following divisibility in the introduction of [H06]:

$$(h(M)/h(F))L_p^-(\psi^-)\big|H(\psi^-) \text{ in } \overline{W}[[\Gamma_M^-]]. \tag{L}$$

Here $h(M)$ (resp. $h(F)$) is the class number of M (resp. F), and Γ_M is the Galois group over M of the composite of all $\mathbb{Z}_p$-extensions of M and Γ_M^- is its anti-cylotomic projection. In [H06], $H(\psi^-)$ is written as $H(\psi)$, though it depends essentially only on ψ^- (strictly speaking, its is defined for ψ with minimal conductor giving a fixed ψ^-). All the ingredients in the above formula are described in the introduction of [H06]. In particular, $H(\psi^-)$ is a congruence power series associated to the p-adic analytic family $\boldsymbol{\theta}(\psi)$ of modular form containing the theta series of ψ (see [H06] for precise definition). In [H06], this is attributed to [H07, Corollary 5.6], whose proof relies on the stronger version [H07, Theorem 4.7] of Corollary 4.3 asserting finiteness of $\underline{n}_v$. In [H06], this corollary was quoted as Corollary 5.5, but it became Corollary 5.6 after publication of [H07] one year after the publication of [H06]. This stronger version is still an open question. However the proof of (L) can be given in the following two different ways.

Here is the first on relying on the vanishing of the μ-invariant of Katz p-adic L-functions [H11]. Indeed, in [HT93, Theorem I], (L) is proven under the vanishing of the μ-invariant of the Katz p-adic L functions, which was proven in [H11] 19 years later.

The second argument is a modification of the argument in [H07]. For a Hecke character φ of M of type A_0, regarding it as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ by class field theory, we write $\varphi_c(\sigma) = \varphi(c\sigma c)$ for complex conjugation c and $\varphi^- = \varphi/\varphi_c$. Following the technique of [HT93], the following formula was proven in [H07, Theorem 5.5]:

$$\frac{\mathcal{L}}{H(\psi^-)} = \frac{\mathcal{L}_p(\psi^{-1}\varphi)\mathcal{L}_p(\psi^{-1}\varphi_c)}{(h(M)/h(F))L_p^-(\psi^-)}, \tag{K0}$$

Here ψ is a given finite order branch character of M with conductor made of split primes of M/F for which we want to prove (L) and φ is a character of order ℓ-power of conductor $\mathfrak{l}$-power. The numerator $\mathcal{L} \in \overline{W}[[\Gamma_M]]$ interpolating Rankin product of the two CM families $\boldsymbol{\theta}(\psi)$ and $\boldsymbol{\theta}(\varphi)$. The numerator of the right-hand-side of the product of the two Katz p-adic L-functions $\mathcal{L}_p(\psi^{-1}\varphi)$

and $\mathcal{L}_p(\psi^{-1}\varphi_c)$ with branch characters $\psi^{-1}\varphi$ and $\psi^{-1}\varphi_c$, respectively. If the stronger version of Corollary 4.3 is valid, choosing $\mathfrak{l}$ so that $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma = 1$, we can arrange the two Katz p-adic L-functions $\mathcal{L}_p(\psi^{-1}\varphi)$ and $\mathcal{L}_p(\psi^{-1}\varphi_c)$ to be units in $\overline{W}[[\Gamma_M]]$ and (L) follows. Whether the actual Corollary 4.3 is sufficient for this argument is not clear. However there is a way-out. We choose one more CM quadratic extension M_1/F disjoint from M. For a Hecke character φ of $X = M, M_1$ and the composite $K = MM_1$, write $\widehat{\varphi} := \varphi \circ N_{K/X}$ as a Hecke character of K. Adjusting the notation to the formula (K0), in [H09, Theorem 3.5] (where slightly different notation was used), the following formula generalizing (K0) is proven:

$$\frac{\mathcal{L}}{H(\psi^-)} = \frac{\mathcal{L}_p(\widehat{\psi^{-1}}\widehat{\xi})}{(h(M)/h(F))L^-_M(\psi^-)}, \tag{K1}$$

where ξ is a branch Hecke character of M_1 whose conductor is an $\mathfrak{l}$-power for a split prime $\mathfrak{l}$ of M_1/F, $\mathcal{L} \in \overline{W}[[\Gamma_M \times \Gamma_{M_1}]]$ and $\mathcal{L}_p(\widehat{\psi^{-1}}\widehat{\xi})$ is the Katz p-adic L-function in $\overline{W}[[\Gamma_K]]$ projected to $\overline{W}[[\Gamma_M \times \Gamma_{M_1}]]$ by $N_{K/M} \times N_{K/M_1}$. The formula in [H09] is more general including the case where the conductor of ψ can have inert or ramified primes, and in such a case, there is an extra simple factor in the denominator of the right-hand-side. By Theorem 0.1 applied to Hilbert modular Eisenstein series for the maximal real subfield of K, one can find ξ for which $\mathcal{L}_p(\widehat{\psi^{-1}}\widehat{\xi})$ is a unit in $\overline{W}[[\Gamma_K]]$ and is a unit in $\overline{W}[[\Gamma_M \times \Gamma_{M_1}]]$ after the projection. This shows (L) also.

REFERENCES

Books

[ACM] G. Shimura, *Abelian Varieties with Complex Multiplication and Modular Functions*, Princeton University Press, 1998

[CAG] D. Eisenbud, *Commutative algebra, with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995.

[CRT] H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics **8**, Cambridge Univ. Press, 1986

[EAI] *Elliptic Curves and Arithmetic Invariants*, Springer Monographs in Mathematics, 2013.

[GME] H. Hida, *Geometric Modular Forms and Elliptic Curves*, Second edition, World Scientific, Singapore, 2012.

[HMI] H. Hida, *Hilbert modular forms and Iwasawa theory*, Oxford University Press, 2006

[PAF] H. Hida, *p-Adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics, Springer, 2004.

Articles

[C] C.-L. Chai, Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, Tate-linear subvarieties and Hecke orbits, preprint 2003, posted on the web: `https://www2.math.upenn.edu/~chai/papers_pdf/fam_ord_av.pdf`

[DR] P. Deligne and K. A. Ribet, Values of abelian L–functions at negative integers over totally real fields, Inventiones Math. **59** (1980), 227–286

[H04] H. Hida, Non-vanishing modulo p of Hecke L-values, In "Geometric Aspects of Dwork Theory" (A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz and F. Loeser, eds.), pp.735-784, Walter de Gruyter, Berlin, 2004.

[H06] H. Hida, Anticyclotomic main conjectures, Documenta Math. Extra volume Coates (2006), 465–532.

[H07] H. Hida, Non-vanishing modulo p of Hecke L–values and application, In "L-functions and Galois representations", London Mathematical Society Lecture Note Series **320** (2007), 207–269.

[H09] H. Hida, Quadratic exercises in Iwasawa theory, International Mathematics Research Notices, Vol. **2009**, Article ID rnn151, 41 pages.

[H10] H. Hida, The Iwasawa μ–invariant of p–adic Hecke L–functions, Annals of Mathematics, **172** (2010), 41–137.

[H11] H. Hida, Vanishing of the μ–invariant of p–adic Hecke L–functions, Compositio Math. **147** (2011), 1151–1178.

[H14] H. Hida, Hecke fields of Hilbert modular analytic families, Contemporary Mathematics, Vol. **614** (2014) 97–137.

[Hs14] M.-L. Hsieh, Special Values of Anticyclotomic Rankin-Selberg L-Functions, Documenta Math. **19** (2014) 709–767.

[HT93] H. Hida and J. Tilouine, Anticyclotomic Katz p–adic L–functions and congruence modules, Ann. Sci. Ec. Norm. Sup. 4-th series **26** (1993), 189–259.

[J95] A. J. De Jong, Crystalline Dieudonné module theory via formal and rigid geometry. Inst. Hautes Études Sci. Publ. Math. No. **82** (1995), 5–96

[K78] N. M. Katz, p-adic L-functions for CM fields, Inventiones Math. **49** (1978), 199–297