Non-vanishing modulo p of Hecke L–values

Haruzo Hida *

Contents

1. Introduction

It is a classical problem to study the divisibility of the Hecke L–values by a given prime p when one varies Hecke characters of ℓ–power conductor for another prime ℓ. Washington studied such a problem for Dirichlet L–values, and as its application, he proved that the exponent of a prime p in the class number is bounded

*The author is partially supported by the NSF grant: DMS 9988043 and DMS 0244401.

independently of ℓ–abelian extensions of $\mathbb{Q}$ unramified outside ℓ if $\ell \neq p$ ([W]). When $\ell = p$, this problem can be formulated as the determination of the Iwasawa μ–invariant of a given $\mathbb{Z}_p^d$–extension. In the cyclotomic case, a solution has been given as vanishing of the μ–invariant by Ferrero and Washington ([FW]). We studied in [H03b] the case $\ell = p$ for p–adic Hecke L–functions of p–ordinary CM fields, and the subject of the author's lecture at the Dwork memorial conference is on the μ–invariant of the p–adic Hecke L–functions.

In this paper, we assume $\ell \neq p$ and study critical values of the L–function of a Hecke character twisted by anticyclotomic finite order characters of $\mathfrak{l}$–power conductor. We limit our study to Hecke L–values of a p–ordinary CM field M and fix a prime factor $\mathfrak{l}$ of ℓ of the maximal totally real subfield F of M. Consider the torus $T_M = \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$. Then $T_M(\mathbb{A})$ is the idele group $M_{\mathbb{A}}^{\times}$. A continuous idele character $\lambda : T_M(\mathbb{A})/T_M(\mathbb{Q}) \to \mathbb{C}^{\times}$ is called an arithmetic Hecke character if its restriction to $T_M(\mathbb{R})$ is induced by algebraic character $\lambda_\infty \in X^*(T_M) = \mathrm{Hom}_{\text{alg-gp}}(T_M, \mathbb{G}_m)$. The $\mathbb{Z}$–module $\mathbb{Z}[I]$ of formal linear combinations of elements in $I = \mathrm{Hom}_{\text{field}}(M, \overline{\mathbb{Q}})$ can be identified with $X^*(T_M)$ regarding $\kappa = \sum_\sigma \kappa_\sigma \sigma \in \mathbb{Z}[I]$ as a character $x \mapsto x^\kappa = \prod_\sigma (\sigma(x))^{\kappa_\sigma}$. Thus $\lambda_\infty = \lambda|_{T_M(\mathbb{R})}$ is induced by $\kappa \in \mathbb{Z}[I]$, and the element $\kappa \in \mathbb{Z}[I]$ is called the infinity type of λ. Let R be the integer ring of M. Since λ is continuous, there exists an R–ideal $\mathfrak{f} \neq 0$ such that $\lambda(x) = 1$ for $x \in \widehat{R}^\times$ if $x - 1 \in \mathfrak{f}\widehat{R}$ for $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ ($\widehat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$). Among such $\mathfrak{f}$, there is a unique maximal one, which is called the conductor of λ. We write $\mathfrak{C} = \mathfrak{C}(\lambda)$ for the conductor of λ. For each finite idele a with trivial $\mathfrak{C}$-component, the value $\lambda(a)$ only depends on the ideal $\mathfrak{A} = a\widehat{R} \cap M$ prime to $\mathfrak{C}$. Thus by defining $\lambda(\mathfrak{A}) := \lambda(a)$, we get Hecke's grössen character λ of conductor $\mathfrak{C}$. The value of ideal character λ at a principal ideal (α) is given by $\lambda(\alpha^{(\infty)})$ if $\alpha \in T_M(\mathbb{Q}) = M^\times$ satisfies $\alpha \equiv 1 \mod \mathfrak{C}$, we find $\lambda((\alpha))\alpha^\kappa = \lambda(\alpha) = 1$ and hence $\lambda((\alpha)) = \alpha^{-\kappa}$ (and because of this fact, some authors call "$-\kappa$" the infinity type of λ). The character we deal with is a Hecke character of arithmetic type associated to a p–ordinary CM-type (M, Σ), which is the type of a CM abelian variety having ordinary good reduction at p, in the following sense: the infinity type of λ is given by $k\Sigma + \kappa(1-c)$ with $\kappa_\sigma \geq 0$ for all $\sigma \in \Sigma$ and $0 < k \in \mathbb{Z}$, where $k\Sigma = k\sum_{\sigma\in\Sigma}\sigma$. Take algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, and choose two embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

We fix a Hecke character λ of conductor prime to p and look at the critical values $L(0, \chi\lambda)$ for χ running over anticyclotomic characters of $\mathfrak{l}$–power conductor. The anticyclotomy means that $\chi(\mathfrak{A}^c) = \chi^{-1}(\mathfrak{A})$ for the non-trivial automorphism c of M/F. We have a pair of the periods $(\Omega, \Omega_p) \in (\mathbb{C}^\times)^\Sigma \times (W^\times)^\Sigma$ of a Néron differential on the abelian scheme $X_{/W}$ of CM type (M, Σ), where W is the Witt ring (regarded as a subring of the p–adic completion $\widehat{\overline{\mathbb{Q}}}_p$ of $\overline{\mathbb{Q}}_p$) with coefficients in an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. The abelian variety X has a polarization ideal $\mathfrak{c} \subset F$ such that $X \otimes \mathfrak{c}$ is isomorphic to the dual abelian variety of X. The strict ideal class of $\mathfrak{c}$ is uniquely determined by X. For a multi-index $\kappa = \sum_\sigma \kappa_\sigma \sigma$ for $\sigma \in \Sigma$, we write $\pi^{-\kappa} = \pi^{-\sum_\sigma \kappa_\sigma}$ and $\Omega^\kappa = \prod_\sigma \Omega_\sigma^{\kappa_\sigma}$. We often identify Σ with $\sum_{\sigma\in\Sigma}\sigma$; so, $\pi^{k\Sigma} = \pi^{k[F:\mathbb{Q}]}$ for an integer k. Let $\mathcal{W}$ be the pull-back of W

in $\overline{\mathbb{Q}}$; so, it is a discrete valuation ring with maximal ideal $\mathfrak{m} = \mathfrak{m}_{\mathcal{W}}$ unramified over the localization $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$. It is known by Shimura and Katz that $\frac{\Gamma_\Sigma(k\Sigma+\kappa)L^{(\mathfrak{l})}(0,\chi\lambda)}{\pi^{-\kappa}\Omega^{k\Sigma+2\kappa}} \in \mathcal{W}$ if 0 is critical for $\chi\lambda$, where $\Gamma_\Sigma(k\Sigma+\kappa) = \prod_{\sigma\in\Sigma}\Gamma(k+\kappa_\sigma)$ for the gamma function $\Gamma(s)$. This normalization (in particular, the gamma-factor: $\Gamma_\Sigma(k\Sigma+\kappa)$) gives optimal p-integrality since it appears as a special value of the Katz p-adic L-function (see [K1] 5.3.5 and [HT] Theorem II). We write $L^{(\mathfrak{l})}(s,\chi\lambda)$ for the possibly imprimitive L–function obtained from $L(s,\chi\lambda)$ removing its $\mathfrak{l}$–Euler factor. We shall prove

Theorem 1.1. *Suppose that $p > 2$ is unramified in $M/\mathbb{Q}$ and that (M,Σ) is ordinary for p. Fix a character λ of conductor* 1 *with infinity type $k\Sigma + \kappa(1-c)$, where k is a positive integer and $\kappa = \sum_{\sigma\in\Sigma}\kappa_\sigma\sigma$ with integers $\kappa_\sigma \geq 0$. Then $\frac{\Gamma_\Sigma(k\Sigma+\kappa)L^{(\mathfrak{l})}(0,\lambda\chi)}{\pi^{-\kappa}\Omega^{k\Sigma+2\kappa}} \not\equiv 0 \mod \mathfrak{m}$ for almost all anticyclotomic characters χ of $\mathfrak{l}$–power conductor, unless the following three conditions are satisfied simultaneously:*

(M1) *M/F is unramified everywhere;*

(M2) *The Artin symbol $\left(\frac{M/F}{\mathfrak{c}}\right)$ has the value -1;*

(M3) *For all ideal $\mathfrak{a}$ of F prime to p, $\lambda N(\mathfrak{a}) \equiv \left(\frac{M/F}{\mathfrak{a}}\right) \mod \mathfrak{m}$.*

Here the word "almost all" means "Zariski density of anticyclotomic characters modulo $\mathfrak{l}$–power" if $\dim_{\mathbb{Q}_\ell} F_\mathfrak{l} > 1$ and "except for finitely many anticyclotomic characters modulo $\mathfrak{l}$–power" if $\dim_{\mathbb{Q}_\ell} F_\mathfrak{l} = 1$. If the three conditions (M1-3) *are satisfied, the L–value as above vanishes modulo $\mathfrak{m}$ for all anti-cyclotomic characters χ modulo $\mathfrak{l}$-power.*

Since $\mathrm{Hom}(\mathbb{Z}_\ell^d, \mu_{\ell^\infty})$ can be considered as a subset of $\mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$ (by sending characters χ to $(\chi(e_i))_i \in \mathbb{G}_m^d$ for the standard basis e_i of $\mathbb{Z}_\ell^d$), the Zariski density of characters has well-defined meaning that a subset of characters is dense if its closure in $\mathbb{G}_m^d$ is $\mathbb{G}_m^d$ itself. In particular, except for the case where $\dim_{\mathbb{Q}_\ell} F_\mathfrak{l} = 1$, our notion of "almost all" is weaker than "outside a proper Zariski closed set", and this theorem when $\dim F_\mathfrak{l} > 1$ has still wide room for improvement.

Our idea could be applied to Hecke characters λ with non-trivial conductor. However the non-vanishing of the L–values is even more subtle for λ with non-trivial conductor because exceptional cases governed by the conditions similar to (M1-3) occur more frequently. To present our idea in a simpler form, we have chosen to assume λ to have conductor 1 leaving its generalization to a future work [H03c]. The CM-fields satisfying (M1-3) are rare: (M1) implies that the strict class number of F and $[F:\mathbb{Q}]$ are both even, but they exist (an example is given in [H03b] Section 5.4). This example shows that the polarization ideal $\mathfrak{c}$ depends on the choice of Σ, and $\mathfrak{c}$ satisfies (M2) for one such choice and not for the other choices.

Until the above theorem, all known results of this kind dealt with either cyclotomic or elliptic cases and $\mathbb{Z}_\ell$–extensions. The imaginary quadratic case has been studied by T. Finis [F] Corollary 4.7, reducing the assertion to the non-vanishing of the theta lift from $GL(2)$ to $U(2,1)$ and a rather deep Diophantine statement ([B] Théorème 1 and [F] Lemma 4.5) about a product of elliptic curves with complex multiplication by M. Actually Finis covers the case where Σ is non-ordinary at p although he needs to suppose the ordinarity for ℓ. A similar assertion for Coates-Wiles $\mathbb{Z}_\ell$–extensions had been studied much earlier by Gillard [G] who reduced the problem to a similar (but slightly stronger) Diophantine statement. Recently Vatsal has given another result of similar kind for elliptic modular L–functions over imaginary quadratic fields ([V] and [V1]).

When M is an imaginary quadratic field, we can interpret the above theorem in terms of the boundedness of the order of certain Selmer groups over the anti-cyclotomic $\mathbb{Z}_\ell$–tower. We hope to come back to this question for general CM fields in future.

Our proof is based on two ideas. One is the philosophy that the knowledge of reciprocity laws (studied by Shimura, e.g., [ACM] and [AAF]) at each place of an arithmetic automorphic function field is almost equivalent to the knowledge of the value of automorphic forms at the point, and the other is Sinnott's idea ([Si]) of relating the non-vanishing modulo p of such L–values to Zariski-density (modulo p) of special points of the algebraic variety underlying the L–values. In the cyclotomic case, the variety is just $\mathbb{G}_m$ and in the elliptic case treated by Gillard and Finis, it is a product of elliptic curves with complex multiplication by M, but in our case, they are more sophisticated Hilbert modular Shimura varieties. Non-density of characters with non-vanishing L–values modulo p leads to the vanishing of an Eisenstein series at densely many CM points with complex multiplication by M. The Zariski density of these CM points then yields the vanishing of the Eisenstein series itself. This vanishing is against the non-vanishing modulo p of the q–expansion of the Eisenstein series, and hence we obtain the desired assertion. The density of CM points of Siegel and Hilbert modular varieties modulo p has been studied by C.-L. Chai (for example [C1]). We quote from [H03b] in Section 2 a version of his result (in the Hilbert modular case) in a fashion suitable to our use. Over the complex field, not just Zariski density but stronger equidistribution of such CM points is known for many Shimura varieties by the works of a handful of mathematicians (e.g. [COU] and references therein).

A key to relating vanishing of L–values mod p to the vanishing of the Eisenstein series is the existence of a measure interpolating the special values of a Hecke eigenform at Hecke and Heegner points. The construction of the measure is a reminiscence of the modular symbol method of Mazur, and we propose to apply the method to global sections of automorphic vector bundles.

In earlier versions of this paper, the proof of Zariski density of CM points (modulo p) relied on a lifting lemma of the Zariski closure to a characteristic 0 formal scheme. Although lifting works well over the ordinary locus, C.-L. Chai pointed out a flaw in the proof that it may not fit well with a characteristic 0 formal scheme at super-singular points and suggested the author the use of his techniques in his four papers from [C1] to [C4] to recover the result. In particular, the proof of a crucial lemma (Lemma 2.4) is due to him, and also the use of Zarhin's theorem

in the proof of Proposition 2.8 is suggested by him. The author would like to thank Ching-Li Chai for his remarks and assistance. The author wishes to thank the organizers of the Dwork memorial conferences for their invitation. Finally but not least, the author would like to thank the referee of this paper for his careful reading and comments.

2. Density of CM points

Our principal tool for proving Theorem 1.1 is Proposition 2.8 which asserts the Zariski density of a small infinite set of CM points in the product of copies of the Hilbert modular variety. We study this density problem in this section.

We shall keep the notation in the introduction. In particular, M/F is a totally imaginary quadratic extension of a totally real field F with a CM type Σ. Thus $\Sigma \sqcup \Sigma c$ gives the set of all embeddings of M into $\overline{\mathbb{Q}}$. We write R (resp. O) for the integer ring of M (resp. F). We write $|\ |_p$ for the p–adic absolute value of $\overline{\mathbb{Q}}_p$ and define $\widehat{\mathbb{Q}}_p$ by the p–adic completion of $\overline{\mathbb{Q}}_p$ under $|\ |_p$. For an idele x and an ideal $\mathfrak{f}$, we write $x_{\mathfrak{f}}$ the components at places dividing $\mathfrak{f}$ and define $x^{(\mathfrak{f})} = x x_{\mathfrak{f}}^{-1}$.

An abelian variety of CM type (M, Σ) over $\mathbb{C}$ is a quotient of $\mathbb{C}^{\Sigma}$ by an O-lattice of M. As we will see below, an O–lattice $\mathcal{A}$ is a proper ideal of an order (possibly non-maximal) of M. Although an ideal of an order is traditionally (from the time of Kronecker) denoted by a lower case Gothic letter, we will use script letters $\mathcal{A}$ and $\mathcal{B}$ for them in order to use lower case Gothic letters exclusively for ideals of rings other than orders of M (particularly, ideals of F are denoted by lower case Gothic letters). We often use capital Gothic letters for ideals of R (with some exceptions).

2.1. CM abelian varieties.

For each $\sigma \in (\Sigma \cup \Sigma c)$, $i_p\sigma$ induces a p–adic place $\mathfrak{p}_\sigma$ whose p-adic absolute value is $|x|_{\mathfrak{p}_\sigma} = |i_p(\sigma(x))|_p$. Set $\Sigma_p = \{\mathfrak{p}_\sigma | \sigma \in \Sigma\}$ and $\Sigma_p c = \{\mathfrak{p}_{\sigma c} | \sigma \in \Sigma\}$.

(ord) We assume that Σ is p–ordinary: $\Sigma_p \cap \Sigma_p c = \emptyset$.

Such a CM type Σ is called a *p–ordinary* CM type. The existence of a p–ordinary CM type is equivalent to the fact that all prime factors of p in F split into a product of two distinct primes in M. We suppose

(unr) *p is unramified in $F/\mathbb{Q}$.*

This condition combined with (ord) implies that p is unramified in $M/\mathbb{Q}$.

We first recall the construction of CM abelian varieties from [ACM]. Recall the complex field embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For each O–lattice $\mathcal{A} \subset M$ with $\mathcal{A}_p = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong R \otimes_{\mathbb{Z}} \mathbb{Z}_p = R_p$, we consider the complex torus $X(\mathcal{A})(\mathbb{C}) = \mathbb{C}^{\Sigma}/\Sigma(\mathcal{A})$, where $\Sigma(\mathcal{A}) = \{(i_\infty(\sigma(a)))_{\sigma\in\Sigma} | a \in \mathcal{A}\}$. By a theorem of Shimura-Taniyama-Weil (cf. [ACM] 12.4), this complex torus is algebraizable to an abelian variety $X(\mathcal{A})$ of CM type (M, Σ) defined over a number field.

The main theorem of complex multiplication ([ACM] 18.6) combined with the criterion of good reduction over W in [ST] tells us that $X(\mathcal{A})$ is actually defined over the field of fractions $\mathcal{K}$ of $\mathcal{W}$ and extends to an abelian scheme (still written as $X(\mathcal{A})_{/\mathcal{W}}$) over $\mathcal{W}$. The special fiber of the abelian scheme $X(\mathcal{A})_{/\mathcal{W}}$ is ordinary under (ord). All endomorphisms of $X(\mathcal{A})$ are defined over $\mathcal{W}$. In particular, we have an embedding $M \hookrightarrow \mathrm{End}(X(\mathcal{A})_{/\mathcal{W}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ taking $\alpha \in M$ to the complex multiplication by $\Sigma(\alpha)$ on $X(\mathcal{A})(\mathbb{C}) = \mathbb{C}^\Sigma/\Sigma(\mathcal{A})$.

Let $R(\mathcal{A}) = \{\alpha \in R | \alpha\mathcal{A} \subset \mathcal{A}\}$. Then $R(\mathcal{A})$ is an order of M over O. An O–order $\mathcal{R}$ of M is determined by its conductor ideal $\mathfrak{f} \subset O$ (which satisfies $\mathfrak{f}R = \{\alpha \in R | \alpha R \subset \mathcal{R}\}$). We have $\mathcal{R} = O + \mathfrak{f}R$. The conductor $\mathfrak{f}(\mathcal{A})$ of $R(\mathcal{A})$ will be called the *conductor* of $\mathcal{A}$. We thus have $R(\mathcal{A}) = O + \mathfrak{f}(\mathcal{A})R$.

For any order $\mathcal{R}$ of M over O and any fractional $\mathcal{R}$-ideal $\mathcal{A}$, the following three conditions are equivalent (cf. [IAT] Proposition 4.11 and (5.4.2) and [CRT] Theorem 11.3):

(I1) $\mathcal{A}$ is $\mathcal{R}$–projective;

(I2) $\mathcal{A}$ is locally principal (that is, localization at each prime ideal is principal);

(I3) $\mathcal{A}$ is a proper $\mathcal{R}$–ideal (that is, $\mathcal{R} = R(\mathcal{A})$).

Define the class group $Cl(\mathcal{R})$ by the group of $\mathcal{R}$–projective fractional ideals modulo globally principal ideals; so, $Cl(\mathcal{R}) = \mathrm{Pic}(\mathcal{R})$. The group $Cl(\mathcal{R})$ is finite and called the ring class group of conductor $\mathfrak{f}$ if $\mathcal{R}$ has conductor $\mathfrak{f}$. Since $\mathcal{A}_p \cong R_p$, the order $R(\mathcal{A})$ has conductor prime to p.

For an abelian scheme X over a subring of an algebraically closed field k, we define the Tate module $\mathcal{T}(X) = \varprojlim_N X[N](k)$ for the kernel $X[N]$ of the multiplication by a positive integer N. In this subsection, we take $k = \overline{\mathbb{Q}}$. We choose a base $w = (w_1, w_2)$ of $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ over $\widehat{O}$ so that the p–component w_p is the standard basis $((1,0),(0,1))$ of $R_p = R_{\Sigma^c} \oplus R_\Sigma$, where $R_\Sigma = \prod_{\mathfrak{p}\in\Sigma_p} R_{\mathfrak{p}} \cong O_p \cong \prod_{\mathfrak{p}\in\Sigma_p^c} R_{\mathfrak{p}} = R_{\Sigma^c}$. The base w gives rise to a level N–structure $\eta_N : (O/NO)^2 \cong X(R)[N]$ given by $\eta_N(a,b) = \frac{aw_1+bw_2}{N} \in X(R)[N]$. Taking their limit and tensoring with $\mathbb{A}^{(\infty)}$, we get

$$\eta = \eta(R) = \varprojlim_N \eta_N : F^2 \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)} \cong V(X(R)) = \mathcal{T}(X(R)) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(\infty)}.$$

We remove the p–part of η and define

$$\eta^{(p)} = \eta^{(p)}(R) : F^2 \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(X(R)) = \mathcal{T}(X(R)) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(p\infty)}. \tag{2.1}$$

The level structure $\eta^{(p)}(R)$ is defined over $\mathcal{W}$ because $X[N]$ for $p \nmid N$ is étale hence constant over $\mathcal{W}$. Since we have

$$\mathcal{T}(X(\mathcal{A})) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(\infty)} = M \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)} = \mathcal{T}(X(R)) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(\infty)},$$

the level structure $\eta(R)$ induces the level structure $\eta_1(\mathcal{A})$ for all $X(\mathcal{A})$. As we will see later, this level structure $\eta_1(\mathcal{A})$ is not good for our purpose, since they do not distinguish two abelian varieties $X(\mathcal{A})$ and $X(R)$ as distinct points of the Shimura variety because of the commutativity of

$$\begin{array}{ccc} V^{(p)}(X(\mathcal{A})) & \longrightarrow & V^{(p)}(X(R)) \\ {\scriptstyle \eta_1(\mathcal{A})}\uparrow & & \uparrow{\scriptstyle \eta(R)} \\ M_{\mathbb{A}}^{(\infty)} & = & M_{\mathbb{A}}^{(\infty)}, \end{array}$$

where the top arrow is the isogeny induced by the bottom equality.

We choose a "good" level structure $\eta(\mathcal{A})$ of $X(\mathcal{A})$ so that $\eta(\mathcal{A})(\widehat{O}^2) = \widehat{\mathcal{A}}$ in the following way. First we choose a representative set $\{\mathfrak{A}_j\}$ of ideal classes of M (prime to $p\mathfrak{f}$). Then we can write $\widehat{\mathfrak{A}}_j = a_j\widehat{R}$ for an idele a_j with $a_j = a_j^{(\mathfrak{f}p\infty)}$ and choose $\alpha \in M$ so that $\mathcal{A}R = \alpha\mathfrak{A}_j$. If $\mathfrak{f}(\mathcal{A}) = O$ (so, $\mathcal{A}$ is an R–ideal), we define the level structure $\eta(\mathcal{A})$ by $(F_{\mathbb{A}}^{(\infty)})^2 \ni (a, b) \mapsto a\alpha a_j w_1 + b\alpha a_j w_2 \in M_{\mathbb{A}}^{(\infty)} = V(X(\mathcal{A}))$. When $\mathfrak{f}(\mathcal{A}) \neq O$, we first suppose that $\mathfrak{f} = (\varphi\varphi^c)$ for $\varphi \in M$. Take $\alpha \in M$ such that $\mathcal{A}R = \alpha\mathfrak{A}_j$, and choose a base $w(\mathcal{A})$ of $\widehat{\mathcal{A}}$ so that $w(\mathcal{A})^{(\mathfrak{f})} = (\varphi\alpha a_j w)^{(\mathfrak{f})}$ and $w(\mathcal{A})_{\mathfrak{f}} = \alpha w_{\mathfrak{f}} \cdot g$ for $g \in GL_2(F_{\mathfrak{f}})$ with $\det(g_{\mathfrak{f}}) = \varphi\varphi^c$. Then we define $\eta(\mathcal{A})(a, b) = a \cdot w_1(\mathcal{A}) + b \cdot w_2(\mathcal{A}) \in M_{\mathbb{A}}^{(\infty)}$. There is an ambiguity of the choice of α and φ up to units in R, but this does not cause any trouble later.

Suppose that $\mathfrak{f}(\mathcal{A})$ is not generated by a norm from M. Let $G = \mathrm{Res}_{F/\mathbb{Q}}GL(2)$ (so, $G(A) = GL_2(A \otimes_{\mathbb{Q}} F)$). We choose $g \in G(\mathbb{A}^{(\infty)})$ with $g^{(\mathfrak{f})} = 1$ so that $w(\mathcal{A}) = \alpha a_j w \cdot g$ gives a base over $\widehat{O}$ of $\widehat{\mathcal{A}}$, and define $\eta(\mathcal{A})$ by using $w(\mathcal{A})$. In the above two cases, we choose g independent of the ideals in the proper ideal class of $\mathcal{A}$; in other words, we choose $w(\beta\mathcal{A}) = \beta\alpha a_j w \cdot g$. We then define $g(\mathcal{A}) \in G(\mathbb{A}^{(\infty)})$ by $\eta(\mathcal{A}) = \eta(\mathfrak{A}_j) \cdot g(\mathcal{A})$. We will later specify the choice of g precisely.

We introduce a representation $\rho_{\mathcal{A}} : M_{\mathbb{A}}^{\times} \to G(\mathbb{A}^{(\infty)})$ by $\alpha\eta(\mathcal{A}) = \eta(\mathcal{A}) \cdot \rho_{\mathcal{A}}(\alpha)$. By our choice, we have $\rho_{\mathcal{A}} = \rho_R$ on $M_{\mathbb{A}}^{(\mathfrak{f}(\mathcal{A}))\times}$, and

$$\det(g(\mathcal{A})) \in F_+^{\times} \ \text{ if } \mathfrak{f}(\mathcal{A}) \text{ is generated by a norm from } M. \tag{2.2}$$

We choose a totally imaginary $\delta \in M$ with $\mathrm{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$. Then the alternating form $(a, b) \mapsto (c(a)b - ac(b))/2\delta$ gives an identity $R \wedge_O R = \mathfrak{c}^*$ for a fractional ideal $\mathfrak{c}$ of F. Here $\mathfrak{c}^* = \{x \in F | \mathrm{Tr}_{F/\mathbb{Q}}(x\mathfrak{c}) \subset \mathbb{Z}\} = \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ for the different $\mathfrak{d}$ of $F/\mathbb{Q}$. Identifying $M \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^{\Sigma}$ by $m \otimes r \mapsto (\sigma(m)r)_{\sigma\in\Sigma}$, we find that $(a, ia) = \frac{\sqrt{-1}}{\delta}a\overline{a} \gg 0$ for $a \in M^{\times}$. Thus $\mathrm{Tr}_{F/\mathbb{Q}} \circ (\cdot, \cdot)$ gives a Riemann form for the lattice $\Sigma(R)$, and therefore, a projective embedding of $\mathbb{C}^{\Sigma}/\Sigma(R)$ onto a projective abelian variety $X(R)_{/\mathbb{C}}$ (cf. [ABV] Chapter I). As we already remarked, $X(R)$ extends to an abelian scheme over $\mathcal{W}$ (unique up to isomorphisms). In this way, we get a $\mathfrak{c}$–polarization $\Lambda(R) : X(R)(\mathbb{C}) \otimes \mathfrak{c} \cong {}^tX(R)(\mathbb{C})$ for the dual abelian scheme ${}^tX(R) = \mathrm{Pic}^0_{X(R)/\mathcal{W}}$ ([ABV] Section 13). The same δ induces

$$\mathcal{R} \wedge \mathcal{R} = \mathfrak{f}(O \wedge R) + \mathfrak{f}^2(R \wedge R) = (\mathfrak{f}^{-1}\mathfrak{c})^* \ \text{ and } \ \mathcal{A} \wedge \mathcal{A} = (N_{M/F}(\mathcal{A})^{-1}\mathfrak{f}(\mathcal{A})^{-1}\mathfrak{c})^*,$$

where the exterior product is taken over O. Hereafter we fix δ so that $\mathfrak{c}$ is prime to $p\mathfrak{f}(\mathcal{A})\mathfrak{d}$, and write $\mathfrak{c}(\mathcal{A})$ for $N_{M/F}(\mathcal{A})^{-1}\mathfrak{f}(\mathcal{A})^{-1}\mathfrak{c}$ (so, $\mathfrak{c} = \mathfrak{c}(R)$). We can always choose such a δ, since in this paper we only treat $\mathcal{A}$ with $\mathfrak{l}$–power conductor.

Since a generically defined isogeny between abelian schemes over $\mathcal{W}$ extends to the entire abelian scheme (e.g. [GME] Lemma 4.1.16), we have a well defined $\mathfrak{c}(\mathcal{A})$–polarization $\Lambda(\mathcal{A}) : X(\mathcal{A}) \otimes \mathfrak{c}(\mathcal{A}) \cong {}^tX(\mathcal{A})$. Replacing $X(\mathcal{A})$ by an isomorphic $X(\alpha\mathcal{A})$ for $\alpha \in M$, we may assume that $\mathcal{A}_p = R_p$. Then

$$X(\mathcal{A})[\mathfrak{p}_F] = X(\mathcal{A})[\mathfrak{p}] \oplus X(\mathcal{A})[\mathfrak{p}^c]$$

for $\mathfrak{p}_F = \mathfrak{p} \cap F$ is isomorphic by $\Lambda(\mathcal{A})$ to its Cartier dual. Since the Rosati-involution $a \mapsto a^* = \Lambda(\mathcal{A}) \circ {}^ta \circ \Lambda(\mathcal{A})^{-1}$ is the complex conjugation c, $X(\mathcal{A})[\mathfrak{p}]_{/\mathcal{W}}$ is multiplicative (étale locally) if and only if $X(\mathcal{A})[\mathfrak{p}^c]$ is étale over $\mathcal{W}$.

Since $X(\mathcal{A})$ has ordinary reduction over $\mathcal{W}$, the connected component $X(\mathcal{A})[p]^\circ$ is isomorphic to $\mu_p^{[F:\mathbb{Q}]}$ and shares with $X(\mathcal{A})_{/\mathbb{F}}$ the tangent space $Lie(X(\mathcal{A}))$ at the origin. We confirm, from $Lie(X(\mathcal{A}))\otimes_{\mathcal{W}}\mathbb{C} = \mathbb{C}^\Sigma$ as R–modules, that R acts on $Lie(X(\mathcal{A}))$ by a representation isomorphic to $\Sigma = \bigoplus_{\sigma\in\Sigma}\sigma$. By (unr), $X(\mathcal{A})[\mathfrak{p}]_{/\mathbb{F}}$ is multiplicative if and only if $\mathfrak{p} \in \Sigma_p$. Let $\mathcal{A}_\Sigma = \prod_{\mathfrak{p}\in\Sigma_p}\mathcal{A}_\mathfrak{p}$, $\mathcal{A}_{\Sigma^c} = \prod_{\mathfrak{p}\in\Sigma_p^c}\mathcal{A}_\mathfrak{p}$, $M_{\Sigma^c} = \prod_{\mathfrak{p}\in\Sigma_p^c} M_\mathfrak{p}$ and $\mathbf{p} = \prod_{\mathfrak{p}\in\Sigma_p}\mathfrak{p}$. Then we may define an étale level p–structure η_p^{et} over $\mathcal{W}$ by $\eta_p^{et} : M_{\Sigma^c}/\mathcal{A}_{\Sigma^c} \cong X(\mathcal{A})[(\mathbf{p}^c)^\infty]$. By the duality under $\Lambda(\mathcal{A})$, this is equivalent to having $\eta_p^{ord}(\mathcal{A}) : \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{A}_\Sigma\mathfrak{d}^{-1} \cong X(\mathcal{A})[\mathbf{p}^\infty]$, which in turn induces an isomorphism $\widehat{\eta}_p^{ord}(\mathcal{A}) : \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{A}_\Sigma\mathfrak{d}^{-1} \cong \widehat{X}(\mathcal{A})$ of formal groups, where $\widehat{X}(\mathcal{A})$ is the formal completion of $X(\mathcal{A})_{/\mathcal{W}}$ at the origin of its mod p fiber. By $\mathcal{A}_p \cong R_p$, we have an identity (induced by our choice of $\eta(\mathcal{A})$): $X(R)[\mathfrak{p}^\infty](\overline{\mathbb{Q}}) = M_\mathfrak{p}/R_\mathfrak{p} \cong X(\mathcal{A})[\mathfrak{p}^\infty](\overline{\mathbb{Q}})$.

We choose and fix a generator $\omega = \omega(R)$ of $H^0(X(R), \Omega_{X(R)/\mathcal{W}})$ over $\mathcal{W}\otimes_\mathbb{Z} O$. If $\mathcal{A}_\Sigma = R_\Sigma$, $X(R\cap\mathcal{A})$ is an étale covering of the both $X(\mathcal{A})$ and $X(R)$; so, $\omega(R)$ induces a differential $\omega(\mathcal{A})$ so that the pull back of $\omega(\mathcal{A})$ and $\omega(R)$ to $X(R\cap\mathcal{A})$ coincide. We then have

$$H^0(X(\mathcal{A}), \Omega_{X(R)/\mathcal{W}}) = (\mathcal{W}\otimes_\mathbb{Z} O)\omega(\mathcal{A}).$$

In this way, we get many quadruples: $(X(\mathcal{A}), \Lambda(\mathcal{A}), \eta^{(p)}(\mathcal{A}) \times \eta_p^{ord}(\mathcal{A}), \omega(\mathcal{A}))_{/\mathcal{W}}$ as long as $\mathfrak{f}(\mathcal{A})$ is prime to p. We call this quadruple $x(\mathcal{A})$.

2.2. Hilbert modular Shimura varieties.

We shall create a lot of CM points on the Hilbert modular Shimura variety. Define an affine group scheme $G_{/\mathbb{Z}}$ by $G = \mathrm{Res}_{O/\mathbb{Z}}GL(2)$. We identify the symmetric space $\mathfrak{Z}$ for $G(\mathbb{R})$ with the collection of embeddings $h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \hookrightarrow G_{/\mathbb{R}}$ which are conjugates of $h_\mathbf{i}(a + b\sqrt{-1}) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ ($\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$). Then the pair $(G_{/\mathbb{Q}}, \mathfrak{Z})$ satisfies Deligne's axioms for having its Shimura variety ([D2] and [D3] 2.1.1). We write $\mathfrak{Z}^+$ for the identity connected component of $\mathfrak{Z}$ containing $h_\mathbf{i}$.

An abelian scheme X over a scheme S is called an abelian variety with real multiplication (AVRM) if O acts on X as S–endomorphisms with an O–linear isomorphism $Lie(X) \cong O \otimes_{\mathbb{Z}} \mathcal{O}_S$ Zariski locally (an AVRM is called a Hilbert-Blumenthal abelian variety in [K1]).

A polarization $\Lambda : X \to {}^tX = \mathrm{Pic}^0_{X/S}$ is an O–linear isogeny induced étale locally by an ample line bundle over X (see [DAV] I.1.6). Then Λ induces $X \otimes \mathfrak{c} \cong {}^tX$ for a fractional ideal $\mathfrak{c}$ of F. We always assume that the polarization ideal $\mathfrak{c}$ is prime to $p\mathfrak{d}$. Let $O_{(p)} = O \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \subset F$ (which is a semi-local Dedekind domain). A polarization class $\overline{\Lambda}$ is the set $\{\Lambda \circ \xi\}_{\xi \in O^\times_{(p)+}}$ for the multiplicative group $O^\times_{(p)+}$ of totally positive units in $O_{(p)}$.

Suppose now that the base S is a $\mathbb{Z}_{(p)}$–scheme. Thus $\Lambda \in \overline{\Lambda}$ is an étale isogeny; so, for each geometric point $s \in S$, $\pi_1(S, s)$ acts on $\overline{\Lambda}$. We say that $\overline{\Lambda}$ is defined over S if it is stable under the action of $\pi_1(S, s)$ for all geometric points $s \in S$. In this definition, by a standard argument, we only need to require the stability taking one geometric point on each connected component of S.

Suppose that $\overline{\Lambda}$ is defined over S. Then we find an irreducible étale Galois covering V/S so that we have a member $\Lambda : X \times_S V \to {}^tX \times_S V$ in $\overline{\Lambda}$. The map $\mathrm{Gal}(V/S) \ni \sigma \mapsto \Lambda^{1-\sigma} \in O^\times_{(p)+}$ is a homomorphism of the finite group $\mathrm{Gal}(V/S)$ into the torsion-free module $O^\times_{(p)+}$ (on which $\mathrm{Gal}(V/S)$ acts trivially). Thus $\Lambda^{1-\sigma} = 1$, and Λ is defined over S. By this descent argument, we can always find a member $\Lambda \in \overline{\Lambda}$ which is defined globally over S if $\overline{\Lambda}$ is defined over S. Thus our definition of S-integrality is equivalent to having a member Λ defined over S in the class $\overline{\Lambda}$ (which is the definition of integrality Kottwitz used in [Ko] Section 5). As shown in [GIT] Proposition 6.10, for the universal Poincaré bundle $\mathcal{P}$ over $X \times_S {}^tX$, the isogeny 2Λ is induced globally by $L = (1 \times \Lambda)^*\mathcal{P}$; so, if $p \neq 2$, we have Λ in $\overline{\Lambda}$ globally associated to an ample line bundle L.

An F–linear isomorphism $\eta^{(p)} : (F^{(p\infty)}_{\mathbb{A}})^2 \cong V^{(p)}(X) = \mathcal{T}(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)}$ is called a *level structure* of X. An adele $g \in G(\mathbb{A})$ acts on $\eta^{(p)}$ by $\eta^{(p)} \mapsto \eta^{(p)} \circ g^{(p\infty)}$. A coset $\overline{\eta}^{(p)} = \eta^{(p)}K$ for a closed subgroup $K \subset G(\mathbb{A}^{(\infty)})$ is called a *level K–structure*. A level K–structure $\overline{\eta}^{(p)}$ is defined over S if for each geometric point s and each $\sigma \in \pi_1(S, s)$, $\sigma \circ \eta^{(p)} = \eta^{(p)} \circ k$ for $k \in K$.

Two triples $\underline{X} = (X, \overline{\Lambda}, \overline{\eta}^{(p)})_{/S}$ and $\underline{X}' = (X', \overline{\Lambda}', \overline{\eta}'^{(p)})_{/S}$ are *isomorphic up to prime-to–p isogeny* if we have an isogeny $\phi : X_{/S} \to X'_{/S}$ of degree prime to p such that $\phi^{-1} \circ \overline{\Lambda}' \circ \phi = \overline{\Lambda}$ and $\phi \circ \overline{\eta}^{(p)} = \overline{\eta}'^{(p)}$. We write $\underline{X} \approx \underline{X}'$ if this is the case. If we insist ϕ above is an isomorphism of abelian scheme (not just an isogeny), we write $\underline{X} \cong \underline{X}'$. Let $Sh^{(p)} = Sh^{(p)}(G, X)_{/\mathcal{W}}$ be the p–integral model over $\mathcal{W}$ of the Shimura variety for (G, X) of prime-to–p level (e.g. [H03a] Lectures 6 and 9 and [PAF] Chapter 4), and the construction of such a model for general Shimura varieties of PEL type has been given by Kottwitz [Ko] (see [PAF] Chapter 7 for an exposition of the construction). By its construction, $Sh^{(p)}$ represents the functor $\mathcal{F}^{(p)} : \mathbb{Z}_{(p)}$–$SCH \to SETS$ given by

$$\mathcal{F}^{(p)}(S) = \Big\{(X, \overline{\Lambda}, \eta^{(p)})_{/S}\Big\} / \approx .$$

The scheme $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$. Each adele $g \in G(\mathbb{A})$ acts on $Sh^{(p)}$ by $\eta^{(p)} \mapsto \eta^{(p)} \circ g^{(p\infty)}$. For each open compact subgroup $K = K_p \times K^{(p)}$ of $G(\mathbb{A}^{(\infty)})$ with $K_p = G(\mathbb{Z}_p)$ (we call such an open compact subgroup "*maximal at* p"), the quotient $Sh^{(p)}_K = Sh^{(p)}/K$ represents the following quotient functor

$$\mathcal{F}^{(p)}_K(S) = \left\{(X, \overline{\Lambda}, \eta^{(p)} \circ K)_{/S}\right\} / \approx .$$

The scheme $Sh^{(p)}_K$ is quasi-projective of finite type over $\mathbb{Z}_{(p)}$, and we have $Sh^{(p)} = \varprojlim_K Sh^{(p)}_K$, where $K = G(\mathbb{Z}_p) \times K^{(p)}$ and $K^{(p)}$ runs over open-compact subgroups of $G(\mathbb{A}^{(p\infty)})$. If $K^{(p)}$ is sufficiently small, $Sh^{(p)}_{K/\mathbb{Z}_{(p)}}$ is smooth.

A triple $(X(\mathcal{A}), \overline{\Lambda}, \eta^{(p)})_{/\mathcal{W}}$ for an O–lattice $\mathcal{A}$ with $\mathcal{A}_p = R_p$ gives rise to a $\mathcal{W}$–point of the p–integral model of the Hilbert modular Shimura variety $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$. Here Λ may not be proportional to $\Lambda(\mathcal{A})$ and $\eta^{(p)}$ may not be equal to $\eta^{(p)}(\mathcal{A})$ we have chosen. A point of $Sh^{(p)}$ obtained in this way from an O–lattice $\mathcal{A}$ of conductor prime to p is called a *CM point* of $Sh^{(p)}$.

We fix an O–lattice $\mathcal{A} \subset M$ with $\mathfrak{f}(\mathcal{A})$ generated by a norm from M and $\mathcal{A}_p = R_p$. Let V_K be the geometrically irreducible component of $Sh^{(p)}_{K/\mathbb{Q}}$ containing the geometric point $x = x(\mathcal{A})_{/\overline{\mathbb{Q}}}$. In other words, $V_{K/\overline{\mathbb{Q}}}$ is a subscheme of $Sh^{(p)}_K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ which is irreducible over $\overline{\mathbb{Q}}$ and contains $x = x(\mathcal{A}) \otimes_{\mathcal{W}} \overline{\mathbb{Q}}$. Since the Galois action permutes geometrically irreducible components, we can think of the stabilizer $\mathfrak{G}_K$ of V_K in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The fixed field k_K of $\mathfrak{G}_K$ in $\overline{\mathbb{Q}}$ is the field of definition of V_K in Weil's sense that $V_{K/\overline{\mathbb{Q}}}$ descends to an irreducible closed subscheme V_{K/k_K} of $Sh^{(p)} \otimes_{\mathbb{Z}_{(p)}} k_K$ and that the algebraic closure of $\mathbb{Q}$ in the function field $k_K(V_K)$ coincides with k_K. As shown in [Sh1] II and [MS] Theorem 4.13, $k_K \subset \mathbb{Q}^{(p)} = \bigcup_{p \nmid N} \mathbb{Q}[\mu_N]$, and it is unramified over $\mathbb{Q}$ at p. Thus k_K is in the field of fractions $\mathcal{K}$ of $\mathcal{W}$, and the schematic closure of $V_K \otimes_{k_K} \mathcal{K}$ in $Sh^{(p)}_{K/\mathcal{W}}$ is smooth over $\mathcal{W}$ if $K^{(p)}$ is sufficiently small. We write again $V_{K/\mathcal{W}}$ for the schematic closure. By a result of Rapoport (see [R], [C] and also [DT] in this volume), we can compactify V over $\mathcal{W}$ so that the projective compactification $f : V^* \to Spec(\mathcal{W})$ (either smooth toroidal or minimal) is projective normal over $\mathcal{W}$ with $f_*(\mathcal{O}_{V^*}) = \mathcal{W}$. Then by Zariski's connectedness theorem, $V^*_K \otimes_{\mathcal{W}} \mathbb{F}$ and hence $V_K \otimes_{\mathcal{W}} \mathbb{F}$ remain irreducible. We take $V^{(p)} = \varprojlim_K V_{K/\mathcal{W}}$, where K runs over all open compact subgroups of $G(\mathbb{A}^{(\infty)})$ maximal at p. The scheme $V^{(p)}$ is therefore smooth over $\mathcal{W}$, and all its geometric fibers are irreducible. In this sense, we call $V^{(p)}_{/\mathcal{W}}$ a geometrically connected component of $Sh^{(p)}$ over $\mathcal{W}$ containing the $\mathcal{W}$–point $x = x(\mathcal{A})$ (so its generic and special fibers are both geometrically irreducible).

We can think of the following functor $\mathcal{F}^{\mathbb{Q}} : \mathbb{Q}\text{–}SCH \to SETS$ in place of $\mathcal{F}^{(p)}$:

$$\mathcal{F}^{\mathbb{Q}}(S) = \left\{(X, \overline{\Lambda}_{\mathbb{Q}}, \eta)_{/S}\right\} / \sim,$$

where $\eta : (F_{\mathbb{A}}^{(\infty)})^2 \cong V(X) = \mathcal{T}(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(\infty)}$ is an O–linear isomorphism and $\overline{\Lambda}_{\mathbb{Q}} = \{\Lambda \circ \xi | \xi \in F_+^\times\}$. The equivalence relation $(X, \overline{\Lambda}_{\mathbb{Q}}, \eta) \sim (X', \overline{\Lambda}'_{\mathbb{Q}}, \eta')$ is given by an isogeny $\phi : X \to X'$ similarly to the case of $\mathcal{F}^{(p)}$ but we do not require that $\deg(\phi)$ be prime to p. This functor is represented by a quasi-projective smooth $\mathbb{Q}$–scheme $Sh = Sh(G, X)$ ([Sh] and [D2] 4.16-21), and $Sh^{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = Sh/G(\mathbb{Z}_p)$. We consider $x_{\mathbb{Q}}(\mathcal{A}) = (X(\mathcal{A}), \overline{\Lambda}(\mathcal{A})_{\mathbb{Q}}, \eta(\mathcal{A}))_{/\mathcal{K}}$ as a closed point $x_{\mathbb{Q}}(\mathcal{A}) \in Sh(\mathcal{K})$. We then take the geometrically irreducible component $V \subset Sh \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ containing $x_{\mathbb{Q}}(\mathcal{A})$. Then V is actually defined over $\mathbb{Q}^{ab} = \bigcup_N \mathbb{Q}[\mu_N]$ in the sense of Weil, and we have $V^{(p)} \otimes_{\mathbb{Q}^{(p)} \cap \mathcal{W}} \mathbb{Q}^{(ab)} = V/G_1(\mathbb{Z}_p)$, where G_1 is the derived group of G. The complex point of the Shimura variety Sh has the following expression ([D3] Proposition 2.1.10 and [M] page 324 and Lemma 10.1)

$$Sh(\mathbb{C}) = G(\mathbb{Q}) \backslash \left(\mathfrak{Z} \times G(\mathbb{A}^{(\infty)})\right) / \overline{Z(\mathbb{Q})},$$

where $\overline{Z(\mathbb{Q})}$ is the topological closure in $G(\mathbb{A}^{(\infty)})$ for the center Z of G and the action is given by $\gamma(z, g)u = (\gamma(z), \gamma g u)$ for $\gamma \in G(\mathbb{Q})$ and $u \in \overline{Z(\mathbb{Q})}$. Thus we have another geometrically irreducible component $V_{\mathbf{i}} \subset Sh$ defined over $\mathbb{Q}^{ab}$ which contains the image of $\mathfrak{Z}^+ \times 1$. If we choose the base $w = (w_1, w_2)$ of $\widehat{R}$ rationally so that $R = Ow_1 + \mathfrak{c}^* w_2$ with $\mathrm{Im}(w_2/w_1) \gg 0$, we actually have $V = V_{\mathbf{i}}$. The right action $(z, h) \mapsto (z, gh)$ is induced by the action of $G(\mathbb{A}^{(\infty)})$ on the level structure η. Thus $G(\mathbb{A}^{(\infty)})$ acts on the Shimura variety by $\eta \mapsto \eta \circ g$ as scheme automorphisms over $\mathbb{Q}$, and $G(\mathbb{A}^{(\infty)})$ acts transitively on the set $\pi_0(Sh_{/\overline{\mathbb{Q}}})$ of all geometrically irreducible components of $Sh_{/\overline{\mathbb{Q}}}$. Each $\xi \in F^\times$ gives an endomorphism $\mathrm{End}^{\mathbb{Q}}(X) = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ taking η to $\eta \circ \xi$. Thus the center $Z(\mathbb{Q})$ acts trivially on Sh. By the above expression of $Sh(\mathbb{C})$, the action of $G(\mathbb{A}^{(\infty)})/Z(\mathbb{Q})$ on Sh factors through $G(\mathbb{A}^{(\infty)})/\overline{Z(\mathbb{Q})}$ for the topological closure $\overline{Z(\mathbb{Q})}$ of $Z(\mathbb{Q})$.

Let

$$\mathcal{E} = \left\{ x \in G(\mathbb{A}) \middle| \det(x) \in \overline{\mathbb{A}^\times F^\times F_{\mathbb{R}+}^\times} \subset F_{\mathbb{A}}^\times \right\} / \overline{Z(\mathbb{Q}) G(\mathbb{R})^+} \hookrightarrow \frac{G(\mathbb{A}^{(\infty)})}{\overline{Z(\mathbb{Q})}},$$

where $G(\mathbb{R})^+$ is the identity connected component of $G(\mathbb{R})$, and $F_{\mathbb{R}+}^\times$ (for $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R}$) is the identity connected component of the multiplicative group $F_{\mathbb{R}}^\times$.

The stabilizer of the geometrically connected component $V_{\mathbf{i}}$ is given by $\mathcal{E}$ (cf. [Sh1] II, [MS] and [PAF] 4.2.2). Actually $\mathrm{Aut}(V_{\mathbf{i}/\mathbb{Q}}) = \mathcal{E} \rtimes \mathrm{Aut}(F)$ (cf. [IAT] Theorem 6.23, [Sh1] II 6.5, [D3] 2.4-6, [MS] 4.13 and [PAF] Theorem 4.14), where $\sigma \in \mathrm{Aut}(F)$ acts on Sh through its Galois action on $G(\mathbb{A})$. Since V is a conjugate of $V_{\mathbf{i}}$ under an element of $G(\mathbb{A}^{(\infty)})$ and $\mathcal{E}$ is normal in $G(\mathbb{A}^{(\infty)})/\overline{Z(\mathbb{Q})}$, we also have $\mathrm{Aut}(V_{/\mathbb{Q}}) = \mathcal{E} \rtimes \mathrm{Aut}(F)$ in the same way. We write $\tau(g)$ ($g \in \mathcal{E}$) for the automorphism of V induced by the action of g (if it is necessary to indicate that we regard $g \in G(\mathbb{A})$ as an automorphism of the Shimura variety V). We define

$$\mathcal{E}^{(p)} = \frac{\left\{ x \in G(\mathbb{A}^{(p)} \times \mathbb{Z}_p) \middle| \det(x) \in \overline{(\mathbb{A}^{(p)})^\times O_{(p)}^\times F_{\mathbb{R}+}^\times} \right\}}{\overline{G(\mathbb{Z}_p) G(\mathbb{R})^+ Z(\mathbb{Z}_{(p)})}}. \tag{2.3}$$

Then $g \in \mathcal{E}^{(p)}$ acts on $V^{(p)}_{/\mathbb{Q}}$ by $\eta^{(p)} \mapsto \eta^{(p)} \circ g^{(p)}$ as scheme automorphisms, and if $x \in \mathcal{E}^{(p)}$ has $\det(x) \in \overline{O^\times_{(p)} F^\times_{\mathbb{R}+}}$, the action of x is an automorphism of the scheme $V^{(p)}$ over $\mathcal{W}$. Thus putting

$$\mathcal{D}^{(p)} = \frac{\left\{x \in G(\mathbb{A}^{(p)} \times \mathbb{Z}_p) \middle| \det(x) \in \overline{O^\times_{(p)} F^\times_{\mathbb{R}+}}\right\}}{\overline{G(\mathbb{Z}_p)G(\mathbb{R})^+ Z(\mathbb{Z}_{(p)})}}, \tag{2.4}$$

one can show that $\mathrm{Aut}(V^{(p)}_{/\overline{\mathbb{F}}_p}) = \mathcal{D}^{(p)} \rtimes \mathrm{Aut}(F)$ (see the proof of Proposition 2.8 in the text, [PAF] Theorem 4.17 and [H03d]).

If $\mathcal{B} \cdot R$ is in the ideal class of $\mathcal{A} \cdot R$ and $\mathfrak{f}(\mathcal{B})$ is generated by a norm from M, $x(\mathcal{A})$ and $x(\mathcal{B})$ are on the same geometrically irreducible component V, because $\eta(\mathcal{B}) = \eta(\mathcal{A}) \circ g$ for $g \in \mathcal{D}^{(p)}$ by (2.2). We have infinitely many CM points in the component V. Recall that $\mathfrak{l}$ is a prime factor in F of the prime $\ell \neq p$ fixed in the introduction. Let $\mathcal{L}$ be the closed subgroup of $\mathcal{D}^{(p)}$ made up of elements represented by $g \in G(\mathbb{A})$ whose prime-to–$\mathfrak{l}$ component $g^{(\mathfrak{l})}$ is 1. We would like to state

Conjecture 2.1. *If $X(R)_{/\mathbb{F}}$ is an ordinary abelian variety with $\mathrm{End}(X(R)_{/\mathbb{F}}) = R$, then any infinite subset of $\{g(x(R))|g \in \mathcal{L}\}$ is Zariski dense in $V^{(p)}(\mathbb{F})$.*

This type of questions has been studied by C.-L. Chai (e.g. [C1] and [C4]) in the Hilbert and Siegel modular cases. This conjecture is related to the following folkloric version of a conjecture of André and Oort (cf. [COU]):

Conjecture 2.2. *For a geometrically irreducible component $V_{/\mathbb{C}}$ of $Sh(G,X)_{/\mathbb{C}}$, if a subvariety $\mathcal{V}_{/\mathbb{C}} \subset V$ contains infinitely many special points, then $\mathcal{V}$ contains an irreducible component of a Shimura subvariety of $Sh(G,X)_{/\mathbb{C}}$.*

When we drop the assumption: $\mathrm{End}(X(R))_{/\mathbb{F}} = R$, the assertion of Conjecture 2.1 is not true for an arbitrary infinite subset of $\{g(x(R))|g \in \mathcal{L}\}$. One of the ways to remedy this is to take an infinite sequence of orders $\mathcal{R}_j$ with $x(\mathcal{B}) \in V^{(p)}$ for a proper ideal $\mathcal{B}$ of $\mathcal{R}_j$ and to assert the density in $V^{(p)}(\mathbb{F})$ of $\{x(\mathcal{B}) \in V^{(p)}(\mathbb{F})\}_{j,\mathcal{B}}$. We will study this version in Proposition 2.7 via the method of Chai. In Subsection 2.4, we generalize Proposition 2.7 to the case of a product of copies of the Hilbert modular variety.

2.3. Density modulo p.

Let K be an open compact subgroup of $G(\mathbb{A}^{(\infty)})$ maximal at p. We suppose that K is sufficiently small so that $V^{(p)}_{K/\mathcal{W}}$ is smooth. In this setting, fixing a lattice $L \subset F^2$ such that $K\widehat{L} = \widehat{L}$ and the alternating form $\langle \cdot, \cdot \rangle$ induces $\mathrm{Hom}(L_p, O_p) \cong L_p$, we have an isomorphism of $\mathcal{F}^{(p)}_K$ onto the following functor $\mathcal{F}_K$:

$$\mathcal{F}_K(S) = \left\{\underline{X}_{/S} \middle| \underline{X} \approx \exists \underline{X}'_{/S} \in \mathcal{F}^{(p)}(S) \ \text{ and } \ \eta^{(p)}(\widehat{L}^{(p)}) = \mathcal{T}^{(p)}(X)\right\} / \cong$$

where $\mathcal{T}^{(p)}(X) = \mathcal{T}(X) \otimes \widehat{\mathbb{Z}}^{(p)}$, $\underline{X} = (X, \Lambda, \eta^{(p)})$ and "$\cong$" is induced by the isomorphisms of abelian schemes (not just an isogeny). The point here is that we imposed extra condition: $\eta^{(p)}(\widehat{L}^{(p)}) = \mathcal{T}^{(p)}(X)$ but also tighten the equivalence from "prime-to–p isogenies" to "isomorphisms". Thus $\mathcal{F}_K^{(p)} \cong \mathcal{F}_K$ can be shown by finding a unique triple $(X, \Lambda, \eta^{(p)})$ (up to isomorphisms) in a given prime-to–p S-isogeny class of a given $\underline{X}$, which can be done in a standard way of the construction of Shimura varieties (see [PAF] 4.2.1 for the argument).

If we choose K suitably (see [PAF] 4.2.1), V_K is isomorphic to the (fine or coarse) moduli $\mathfrak{M}(\Gamma(N))_{/\mathbb{Z}_{(p)}[\mu_N]}$ representing the following functor $\mathcal{F}_N : \mathbb{Z}_{(p)}$–$SCH \to SETS$ (cf. [K1] and [PAF] 4.1.2):

$$\mathcal{F}_N(S) = \left\{(X, \Lambda, \phi_N)_{/S}\right\} / \cong,$$

where Λ is a O–polarization and ϕ_N is an isomorphism: $(NO)^*/O^* \times (O/NO) \cong A[N]$ of locally free group schemes with $e_N(\phi(a,b), \phi(c,d)) = \exp(2\pi i \mathrm{Tr}(ad - bc))$ (see [ABV] Section 20). Thus $V^{(p)}_{/\mathcal{W}}$ is isomorphic to $V_{\mathbf{i}}^{(p)} = \varprojlim_{p \nmid N} \mathfrak{M}(\Gamma(N))_{/\mathcal{W}}$.

Recall that $V_K(\mathcal{W})$ has the CM point $x = x(\mathcal{A})$. By the Serre-Tate deformation theory (cf. [H03a] Lecture 8, [H03b] Section 2 and [PAF] 8.2), we have an identification depending on $\eta_p^{ord}(\mathcal{A})$:

$$\widehat{\mathcal{O}}_{V,x/\mathbb{F}} \cong \widehat{\mathcal{O}}_{\mathbb{G}_m \otimes_{\mathbb{Z}} O, 1} = \varprojlim_n \mathbb{F}[t^\xi]_{\xi \in O}/(t^{\xi_1} - 1, \cdots, t^{\xi_d} - 1)^n$$

for a $\mathbb{Z}$–base $\xi_1, \xi_2, \cdots, \xi_d$ of O. We write the ring at the right-hand-side as $\mathbb{F}[[t^\xi]]_{\xi \in O}$ symbolically (though it is not exactly a power series ring). The ring O_p acts on $\widehat{G}_m \otimes_{\mathbb{Z}} O$ through the right factor which is induced by the variable change $t \mapsto t^a$ in the affine ring $\mathbb{F}[[t^\xi]]_\xi$. The coordinate t therefore depends on $\eta_p^{ord}(\mathcal{A})$, since by the Serre-Tate theory, the deformation space is identified via $\eta_p^{ord}(\mathcal{A})$ with

$$\mathrm{Hom}(\mathcal{T}_p(X(\mathcal{A})_{/\mathbb{F}}) \otimes_O \mathcal{T}_p(X(\mathcal{A})_{/\mathbb{F}}), \widehat{\mathbb{G}}_m) \quad \text{for } \mathcal{T}_p(X(\mathcal{A})_{/\mathbb{F}}) = \varprojlim_n X(\mathcal{A})[p^n](\mathbb{F}),$$

and the identification of $\mathcal{T}_p(X(\mathcal{A}))$ with O_p depends on the level structure $\eta_p^{ord}(\mathcal{A})$.

We consider a torus $T_{/\mathbb{Z}}$ defined by the following exact sequence:

$$1 \to \mathrm{Res}_{O/\mathbb{Z}} \mathbb{G}_m \to \mathrm{Res}_{R/\mathbb{Z}} \mathbb{G}_m \to T \to 1.$$

By Hilbert's theorem 90, the homomorphism $\alpha \mapsto \alpha^{1-c}$ induces an isomorphism of group schemes from $T_{/\mathbb{Z}_{(p)}}$ to the kernel of the norm map: $\mathrm{Res}_{R_{(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_m \to \mathrm{Res}_{O_{(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_m$. We identify the two tori by this isomorphism. The embedding $\rho_{\mathcal{A}} : (M_{\mathbb{A}}^{(\infty)})^\times \to G(\mathbb{A}^{(\infty)})$ induces an embedding of $T(\mathbb{Z}_{(p)})$ into $\mathcal{D}^{(p)} \subset \mathrm{Aut}(V^{(p)}_{/\mathbb{Z}_{(p)}})$ by composing $T(\mathbb{Z}_{(p)}) \hookrightarrow G(\mathbb{Z}_{(p)})/Z(\mathbb{Z}_{(p)})$ with a diagonal inclusion as principal ideles. Then $T(\mathbb{Z}_{(p)})$ fixes the point $x = x(\mathcal{A}) \in V(\mathbb{F})$ because $\alpha \in R_{(p)}$ induces a prime-to–p isogeny

$$\alpha : (X(\mathcal{A}), \overline{\Lambda}(\mathcal{A}), \eta^{(p)}(\mathcal{A})) \to (X(\mathcal{A}), \overline{\Lambda}(\mathcal{A}), \alpha\eta^{(p)} = \eta^{(p)}(\mathcal{A}) \circ \rho_{\mathcal{A}}(\alpha)).$$

Let $\underline{\mathbb{X}} = (\mathbb{X}, \overline{\boldsymbol{\Lambda}}, \boldsymbol{\eta}^{(p)})$ be the universal triple over $Sh^{(p)}$. Then the Igusa tower over $Sh^{(p)}_{/\mathbb{F}}$ is defined by $\varprojlim_n \mathrm{Isom}_{\text{gp-sch}}(\mu_{p^n} \otimes \mathfrak{d}^{-1}_{/\mathbb{F}}, \mathbb{X}[p^n]^\circ_{/\mathbb{F}})$ for the connected component $\mathbb{X}[p^n]^\circ_{/\mathbb{F}}$ of the locally free group scheme $\mathbb{X}[p^n]_{/\mathbb{F}}$. We write $Ig_{/V^{(p)}}$ for the pull back of this Igusa tower to $V^{(p)}_{/\mathbb{F}}$. The scheme Ig is étale faithfully flat over the ordinary locus of $V^{(p)}_{/\mathbb{F}}$. The p–ordinary level structure $\eta_p^{ord}(\mathcal{A})$ defines a point, again denoted by $x = x(\mathcal{A})$, of the Igusa tower $Ig/V^{(p)}$ (see [H03a] Lectures 7 and 10 for the Igusa tower). Since $R_\Sigma \cong O_p$ canonically, the multiplicative group $R_{(p)}^\times$ acts on ordinary level structures and hence on the stalk $\mathcal{O}_{Ig,x/\mathbb{F}}$ through the quotient $T(\mathbb{Z}_{(p)})$. Thus $T(\mathbb{Z}_{(p)})$ is identified with a p–adically dense subgroup of $\mathrm{Aut}_O(\widehat{S}) = O_p^\times$ for $\widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O$ which is isomorphic via $\eta_p^{ord}(\mathcal{A})$ to the formal completion of Ig at x (see [H03a] Lecture 8 and [H03b] Section 2). By the Serre-Tate theory, this identification is equivariant under the action of $\alpha \in T(\mathbb{Z}_{(p)})$ through the automorphism $\rho_{\mathcal{A}}(\alpha)$ of Ig and multiplication by α^{1-c} on the right factor $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O$ (that is, sending the Serre-Tate coordinate t to $t^{\alpha^{1-c}}$). We can guess this fact heuristically from the identity: $R_{(p)}^\times/O_{(p)}^\times \subset R_p^\times/O_p^\times$ and $\alpha \equiv \alpha^{1-c}$ mod $O_p^\times$, although the real proof is more complicated (via a full use of the Serre-Tate theory), because a non-trivial element in $O_p^\times/O_{(p)}^\times$ (which is the center of $G(\mathbb{Z}_p)/Z(\mathbb{Z}_{(p)})$) really moves x to another point on Ig.

We state here formally the fact as a lemma and give a brief outline of the proof (see [H03b] Proposition 3.3 for more details):

Lemma 2.3. *We have a canonical identification:*

$$\widehat{Ig}_{/\mathbb{F}} \cong \widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O = \prod_{\mathfrak{p} \in \Sigma_p} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}},$$

where $\widehat{Ig}$ is the formal completion of Ig along $x = x(\mathcal{A})$. Under this identification, the action on Ig of $\rho_{\mathcal{A}}(\alpha) \in \mathcal{D}^{(p)}$ for an element $\alpha \in R_{(p)}^\times$ induces on $\widehat{S}$ an action given by $t \mapsto t^{\alpha^{1-c}}$ for the Serre-Tate coordinate t of $\widehat{S}$, where c is the generator of $\mathrm{Gal}(M/F)$ *(that is, "complex conjugation").*

Here is an outline of the proof: In the proof, we use Katz's notation on the Serre-Tate moduli in [K2] without much explanation to make our exposition short. In particular, we write q for the Serre-Tate coordinate (keeping t exclusively for the corresponding coordinate on Ig centered at x). The first assertion follows from the universality of the Serre-Tate local moduli and the global universality of Ig. To show the second assertion, let $X_0 = X(\mathcal{A}) \otimes_{\mathcal{W}} \mathbb{F}$. By the Serre-Tate theory, the abelian variety $(X_0, \overline{\Lambda}(\mathcal{A}), \eta^{ord}(\mathcal{A}))$ is sitting on the origin $\mathbf{1}$ of $\widehat{\mathbb{G}}_m \otimes O$.

We now compute the effect of the isogeny $\alpha : X_0 \to X_0$ (for $\alpha \in R$) on the deformation space $\widehat{S}$. Pick a deformation $X_{/A}$ of $X_0 = A_{x/\mathbb{F}}$ for an artinian W-algebra A with residue field $\mathbb{F}$, and we look into the following diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{T}X_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(A)) & \hookrightarrow & X[p^n](A) & \twoheadrightarrow & X_0[p^n]^{et}(A) \\ {\scriptstyle \alpha^{-c}}\uparrow & & & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}(\mathcal{T}X_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(A)) & \hookrightarrow & X[p^n](A) & \twoheadrightarrow & X_0[p^n]^{et}(A). \end{array} \tag{2.5}$$

Take $u = \varprojlim_n u_n \in \mathcal{T}X_0[p^\infty]^{et}$, and lift it to $v = \varprojlim_n v_n$ for $v_n \in X[p^n]$. Then $q(u) = \varprojlim_n q_n(u_n) \in \mathrm{Hom}(\mathcal{T}X_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m)$ for $q_n(u_n) =$"p^n"v_n, where "p^n" is the Drinfeld lift of multiplication by p^n in [K2] Lemma 1.1.2. Note that the identification of $\mathrm{Hom}(\mathcal{T}X_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m)$ with the formal group $\widehat{X}$ of X is given by the Cartier duality composed with the polarization; so, given by α^{-c} because the complex conjugation is the Rosati involution. Thus the effect of α on q is given by $q \mapsto q^{\alpha^{1-c}}$. Once the identification of $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$ is given (via $\eta_p^{ord}(\mathcal{A})$), $\alpha \in R$ prime to p acts on the coordinate t by $t \mapsto t^{\alpha^{1-c}}$. $\square$

Let L be a $\mathbb{Z}_p$–free module of finite rank on which $T(\mathbb{Z}_p)$ acts by a $\mathbb{Q}_p$–rational linear representation. Then $\widehat{S}_{L/\mathbb{F}} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L$ inherits the action of T from L. Here we quote a lemma whose proof has been supplied to me by Ching-Li Chai:

Lemma 2.4. *Suppose that the trivial representation of $T(\mathbb{Z}_p)$ is not a subquotient of $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $Z_{/\mathbb{F}}$ is a reduced irreducible formal subscheme of $\widehat{S}_L$ stable under the action of an open subgroup of $T(\mathbb{Z}_p)$, then there exists a $\mathbb{Z}_p$–direct summand $L_Z \subset L$ stable under $T(\mathbb{Z}_p)$ such that $Z = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L_Z$, and Z is a smooth formal subtorus of $\widehat{T}$.*

This is proven in [C1] Proposition 4 assuming further the smoothness of Z at the origin of $\widehat{S}_L$. Recently Chai has removed the smoothness assumption from [C1] Proposition 4 and has actually generalized his result in much more general situations (see [C3]). His proof in the above setting can be found in [H03b] Lemma 3.5. We actually need the lemma only under the smoothness assumption.

Let $\mathfrak{b}_K$ be a prime ideal of $\mathcal{O}_{V_K,x/\mathbb{F}}$ for $x = x(\mathcal{A})$. We consider the Zariski closure $\mathcal{X}_K$ of $Spec(\mathcal{O}_{V_K,x}/\mathfrak{b}_K)$ in $V_{K/\mathbb{F}}$. Since $\mathcal{O}_{Ig,x}$ is étale over $\mathcal{O}_{V_K,x}$, we have a unique prime ideal $\mathfrak{b} \subset \mathcal{O}_{Ig,x}$ with $\mathfrak{b} \cap \mathcal{O}_{V_K,x} = \mathfrak{b}_K$. We write $\mathcal{X}$ for the Zariski closure of $Spec(\mathcal{O}_{Ig,x}/\mathfrak{b})$ in Ig. Then $\mathcal{X}_K$ is the Zariski closure of the image of $\mathcal{X}$ in V_K (supplemented with the non-ordinary loci). Since $T(\mathbb{Z}_{(p)})$ (strictly speaking $\tau(\rho_{\mathcal{A}}(T(\mathbb{Z}_{(p)}))))$ fixes x, it acts on the stalk $\mathcal{O}_{Ig,x}$ as endomorphisms of the ring $\mathcal{O}_{Ig,x}$, and in particular, $T(\mathbb{Z}_{(p)})$ acts by automorphisms.

Proposition 2.5. *Let $\mathcal{T}$ be a subgroup of $T(\mathbb{Z}_{(p)})$ whose p–adic closure contains an open subgroup of $T(\mathbb{Z}_p)$. Suppose that $\dim \mathcal{X}_{/\mathbb{F}} > 0$ and $\mathcal{X}$ is stable under $\mathcal{T}$. Then we have $\mathcal{X}_{/\mathbb{F}} = Ig_{/\mathbb{F}}$ and hence $\mathcal{X}_{K/\mathbb{F}} = V_{K/\mathbb{F}}$.*

We shall repeat the proof given in [H03b] 3.4 because this is crucial in the sequel.

Proof. We follow the argument in [C1] Section 4 and 5. The proof is separated into the following three steps:

1. To show the existence of a subset $\Sigma' \subset \Sigma_p$ such that the formal completion $\widehat{\mathcal{X}}_y$ of $\mathcal{X}$ along any ordinary point $y \in \mathcal{X}(\mathbb{F})$ is given by $\prod_{\mathfrak{p}\in\Sigma'} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$ in the Serre-Tate deformation space giving an infinitesimal neighborhood of y. In particular, for the pull-back A to $\mathcal{X}_K$ of the universal abelian scheme, we

show that $A[\mathfrak{p}^\infty]$ for $\mathfrak{p} \notin \Sigma_p - \Sigma'$ gives an étale constant Barsotti-Tate group over the ordinary locus of $\mathcal{X}_K$;

2. To show that $\mathcal{X}_K$ contains a super-singular point z;
3. Draw a curve C from an ordinary closed point to the super-singular z inside $\mathcal{X}_K$. Then by Step 1, $A[\mathfrak{p}^\infty]$ for $\mathfrak{p} \notin \Sigma_p - \Sigma'$ is an étale constant deformation of $A_z[\mathfrak{p}^\infty]$ (which is impossible because A_z is super-singular); so, we conclude $\Sigma' = \Sigma_p$ and hence $\mathcal{X} = Ig$.

Step 1: By the stability of $\mathcal{X}$ under $\mathcal{T}$, the formal completion $\widehat{\mathcal{X}}$ along the point x is stable under its closure $\overline{\mathcal{T}} \subset T(\mathbb{Z}_p)$. Then, applying Lemma 2.4 to $\widehat{S}$ ($L = O_p$), we find that

$$\widehat{\mathcal{X}} = \prod_{\mathfrak{p}\in\Sigma'} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}} \tag{2.6}$$

for a subset $\Sigma' \subset \Sigma_p$. In particular, $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}_K$ is a smooth formal subgroup of $\widehat{S}$, and $\mathcal{X}$ is stable under $T(\mathbb{Z}_{(p)})$.

On Ig, by the argument in [C1] page 473, the tangent bundle T_{Ig} is decomposed into the direct sum of eigenspaces under the O–action:

$$T_{Ig} \cong \mathcal{O}_{Ig} \otimes_{\mathbb{Z}} O \text{ locally, and } T_{Ig} = \bigoplus_{\mathfrak{p}\in\Sigma_p} T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is a locally free $O_{Ig} \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$-module of rank 1. By the above expression, the tangent space $T_{\widehat{\mathcal{X}}}$ of $\widehat{\mathcal{X}}$ coincides with $\oplus_{\mathfrak{p}\in\Sigma'} T_{\mathfrak{p}} \otimes_{\mathcal{O}_{Ig,x}} \widehat{\mathcal{O}}_{Ig,x}$. Through faithfully flat descent, we have $T_{\mathcal{X}} \otimes_{\mathcal{O}_{Ig}} \mathcal{O}_{Ig,x} = \oplus_{\mathfrak{p}\in\Sigma'} T_{\mathfrak{p}} \otimes_{\mathcal{O}_{Ig}} \mathcal{O}_{Ig,x}$. Thus on an open dense subscheme $\mathcal{X}^\circ \subset \mathcal{X}$ inside the smooth locus $\mathcal{X}^{sm} \subset \mathcal{X}$, we have

$$T_{\mathcal{X}^\circ} = \oplus_{\mathfrak{p}\in\Sigma'} T_{\mathfrak{p}} \otimes_{\mathcal{O}_{Ig}} \mathcal{O}_{\mathcal{X}^\circ}.$$

Since the tangent space is locally defined, if $T_{\mathcal{X}^\circ,y} = \oplus_{\mathfrak{p}\in\Sigma'} T_{\mathfrak{p}} \otimes_{\mathcal{O}_{Ig}} \mathcal{O}_{\mathcal{X}^\circ,y}$ at $y \in \mathcal{X}$, the same identity hold at the image of y under the toric action of $T(\mathbb{Z}_{(p)})$. Thus we may suppose that $\mathcal{X}^\circ$ is stable under $T(\mathbb{Z}_{(p)})$.

Suppose that $\Sigma' \neq \Sigma_p$ and let $\mathfrak{p} \in \Sigma_p - \Sigma'$. We only need to prove that $V_K = \mathcal{X}_K$ for a choice of an open subgroup K maximal at p. Choosing K sufficiently small, we may assume that V_K is smooth over $\mathbb{F} = \overline{\mathbb{F}}_p$. Let $\mathbb{X}_{/V_K}$ be the universal abelian scheme, and define $A = A_K = \mathbb{X} \times_{V_K} \mathcal{X}_K$. Write $\mathfrak{p}_F = F \cap \mathfrak{p}$, and consider the $\mathfrak{p}_F$–divisible group $A[\mathfrak{p}_F^\infty]$. We choose the connected component $\mathcal{Y}_{\eta,r} = \mathcal{Y}_{K,\eta,r}$ of the generic fiber $A[\mathfrak{p}_F^r]_\eta$ (for the generic point η of $\mathcal{X}_K$) whose projection to the maximal étale quotient $A[\mathfrak{p}_F^r]_\eta^{et}$ gives the generator over $O/\mathfrak{p}_F^r$. We then take the schematic closure $\mathcal{Y}_r = \mathcal{Y}_{K,r}$ of $\mathcal{Y}_{\eta,r}$ in $A[\mathfrak{p}_F^r]$. The subscheme $\mathcal{Y}_{K,r} \subset A[\mathfrak{p}_F^r]$ projects down to the generator of $A[\mathfrak{p}_F^r]^{et}$ in the sense of [AME] Sections 1.9-10. We arrange $\{\mathcal{Y}_{K,r}\}_r$ coherently so that it gives a tower of irreducible varieties over $\mathcal{X}_K$. The reduced scheme $\mathcal{Y}_{K,\eta,r}^{red}$ is étale over $\mathcal{X}_{K,\eta}$; they give rise to the tower of function fields $\mathbb{F}(\mathcal{X}_K^{red}) \subset \mathbb{F}(\mathcal{Y}_{K,1}^{red}) \subset \cdots \subset \mathbb{F}(\mathcal{Y}_{K,r}^{red}) \subset \cdots$. Since $\mathcal{Y}_{K,r}^{red}$ is the generator of $A[\mathfrak{p}_F^r]^{et}$, the fiber $\mathcal{Y}_{K,r}$ of $A[\mathfrak{p}_F^r]$ over $\mathcal{Y}_{K,r}^{red} \subset A[\mathfrak{p}_F^r]^{et}$ can be identified with the connected component $A[\mathfrak{p}_F^r]^\circ$. Thus the scheme $A[\mathfrak{p}_F^r]^\circ$ is a connected finite flat group scheme over $\mathcal{Y}_{K,r}^{red}$.

Pick any function $\phi \in \mathbb{F}(\mathcal{Y}_{K,r}^{red})$ finite at x, and expand ϕ into a power series $\phi(t_{\mathfrak{p}'})$ around x using the variables $t_{\mathfrak{p}'}$ of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}'}$ for $\mathfrak{p}' \in \Sigma_p$. Since $\mathfrak{p} \notin \Sigma'$, $\phi(t_{\mathfrak{p}'})$ is constant with respect to $t_{\mathfrak{p}}$. Since $t_{\mathfrak{p}}$ is the variable of the universal deformation space over $\mathbb{F}$ of $X(\mathcal{A})[\mathfrak{p}_F^\infty] = A[\mathfrak{p}_F^\infty] \times_{\mathcal{X}_K,x} \mathbb{F}$, $A[\mathfrak{p}_F^r]$ as a deformation of $X(\mathcal{A})[\mathfrak{p}_F^r]$ has to be constant on $\widehat{\mathcal{X}}_K$. Since $X(\mathcal{A})$ is ordinary, we have $X(\mathcal{A})[\mathfrak{p}^\infty] \cong (\mu_{p^\infty} \otimes O_{\mathfrak{p}}) \times F_{\mathfrak{p}}/O_{\mathfrak{p}}$. Thus over $\widehat{\mathcal{X}}$, $A[\mathfrak{p}_F^\infty]_{/\widehat{\mathcal{X}}}$ is again constant, and $A[\mathfrak{p}_F^\infty]_{/\widehat{\mathcal{X}}} \cong (\mu_{p^\infty} \otimes O_{\mathfrak{p}}) \times F_{\mathfrak{p}}/O_{\mathfrak{p}}$. In other words, the morphism of $\widehat{\mathcal{X}}_K$ into the universal deformation space $\widehat{S}_{\mathfrak{p}} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$ of $X(\mathcal{A})[\mathfrak{p}_F^\infty]_{/\mathbb{F}}$ induced by $A[\mathfrak{p}_F^\infty]_{/\widehat{\mathcal{X}}}$ is the constant morphism factoring through $Spec(\mathbb{F}) \hookrightarrow \widehat{S}_{\mathfrak{p}}$. In particular, $A[\mathfrak{p}_F^r]_{/\widehat{\mathcal{X}}}$ is defined by equations with coefficients in $\mathbb{F}$; so, it is defined over a finite field $\mathbb{F}_q$ $(q = p^f)$. Therefore the q–th power relative Frobenius map $\widehat{\Phi}$ of $A[\mathfrak{p}_F^r]_{/\widehat{\mathcal{X}}}^{et}$ is constant. Since the relative Frobenius map $\widehat{\Phi}$ is the base-change (from $\mathcal{O}_{\mathcal{X}_K,x}$ to $\widehat{\mathcal{X}}_K$) of the relative Frobenius map Φ of $A[\mathfrak{p}_F^r]_{/\mathcal{O}_{\mathcal{X},x}}^{et}$, Φ has to be constant, since $\widehat{\mathcal{O}}_{\mathcal{X},x}$ is faithfully flat over $\mathcal{O}_{\mathcal{X},x}$. Thus we conclude that $A[\mathfrak{p}_F^r]_{/\mathcal{O}_{\mathcal{X},x}}^{et}$ is a constant étale group (before completion) defined over $\mathbb{F}_q$. Hence it is constant over an open dense subscheme $\mathcal{X}_K^\circ$ of $\mathcal{X}_K$ (see Remark in [C1] page 473). We hereafter rewrite $\mathcal{X}^\circ$ for $\varprojlim_K \mathcal{X}_K^\circ$ (for K maximal at p).

By the constancy of $A[\mathfrak{p}_F^r]^{et}$ over $\mathcal{X}_K^\circ$, we find that $\mathbb{F}(\mathcal{Y}_{K,r}^{red}) = \mathbb{F}(\mathcal{X}_K)$ for all $r > 0$, because $\mathbb{F}(\mathcal{Y}_{K,r}^{red})$ is the function field of an irreducible component of $A[\mathfrak{p}_F^r]^{et}$. Since $\mathcal{Y}_{K,r}^{red}$ is étale over the ordinary locus of $\mathcal{X}_K$, the two ordinary loci coincide, and $\mathcal{X}_K^\circ$ is the ordinary locus of $\mathcal{X}_K$, which is stable under $T(\mathbb{Z}_{(p)})$ and is open dense in $\mathcal{X}_K$. The étale group scheme $A[\mathfrak{p}_F^r]^{et}$ is constant over $\mathcal{X}^\circ$ (independent of r). Thus for any geometric point $y \in \mathcal{X}^\circ$, the formal completion $\widehat{\mathcal{X}}_y$ along y is isomorphic to $\prod_{\mathfrak{p}' \in \Sigma'} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}'}$ as formal O–modules.

Step 2: We proceed in a manner similar to the proof of Proposition 7 and Theorem 1 of [C1], though in our case, we take a slightly different approach, because $T(\mathbb{Z}_{(p)})$ is much smaller than the full group of prime-to–p Hecke operators treated in [C1].

For simplicity, hereafter we write V for $V^{(p)} = \varprojlim_{K:\text{maximal at } p} V_K$ if no confusion is likely. Let V^* be the Satake compactification of V. The automorphism group $\mathcal{D}^{(p)}$ still acts on the compactification V^* since V_K^* is given by the "Proj" of the graded algebra of parallel weight modular forms (see [C]). The Hasse-Witt matrix extends to the compactification V^* and is invertible at the cusp. To see this, we first make the smooth toroidal compactification $\overline{V}_K$ which carries a universal semi-abelian scheme (depending on the toroidal compactification data: see [C]). Thus the Hasse-Witt matrix extends to the toroidal compactification $\overline{V}_K$. It is constant on the boundary divisors, since in the Hilbert modular case, the abelian variety fully degenerates towards the cusp. Thus the matrix actually factors through the Satake compactification. Then V^* can be stratified so that at each stratum the rank of the Hasse-Witt matrix is constant $0 \le r \le [F:\mathbb{Q}]$, that is, one of $r \times r$ minors of the matrix does not vanish on the stratum but $(r+1) \times (r+1)$ minors all vanish. Each stratum is known to be quasi affine (see [O] and the discussion in [C1] Section 5). Each stratum is stable under the action of $\mathcal{D}^{(p)}$ because r gives the p–rank of the abelian varieties over the stratum. Let $\mathcal{X}^*$ be the closure of the image of $\mathcal{X}$ in V^*. Since $\mathcal{X}^*$ is proper and the cusp is contained in the generic stratum, $\mathcal{X}^*$ cannot be contained in the generic stratum (because of affineness of the ordinary locus). Let $\mathcal{S}$ be one of the non-generic stratum of V^* such that $\mathcal{X}^* \cap \mathcal{S}$ is non-empty. Note that $\mathcal{S} = \varprojlim_K \mathcal{S}_K$ for $\mathcal{S}_K \subset V_K$. Then $\mathcal{X}^* \cap \mathcal{S}$ is stable under $T(\mathbb{Z}_{(p)})$, and $\mathcal{S}$ does not contain any cusp. For each $y \in \mathcal{S}$, the abelian variety A_y sitting over y is non-ordinary. If $\overline{\alpha} = \tau(\rho_{\mathcal{A}}(\alpha))$ for $\alpha \in T(\mathbb{Z}_{(p)})$ fixes the point y, it is induced by a prime-to–p isogeny: $\mathbb{X} \to \overline{\alpha}^*\mathbb{X} = \mathbb{X} \times_{V,\overline{\alpha}} V$ by the universality of the Hilbert modular Shimura variety, which induces an isogeny $\alpha : A_y \to A_y$. This implies $M = F[\alpha] \hookrightarrow \mathrm{End}(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}$. By [C1] Lemma 6, if the p–rank of A_y is positive, the commutant of F in $\mathrm{End}(A_y) \otimes \mathbb{Q}$ is equal to M, and the relative Frobenius q–th power endomorphism $\phi \in \mathrm{End}(A_y)$ (for a sufficient large p–power q) generates M over F. Thus, the multiplication by M determines (through the level structure $\eta_y^{(p)}$ of A_y) a unique subtorus $T_y \cong \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$ of G, which is the centralizer of ϕ and is different from T_x since the p–rank of A_y is less than $[F:\mathbb{Q}]$. The quotient torus T_y/Z for the center Z of G gives the stabilizer of y; so, $T_x = T_y$. This is a contradiction, since the p–rank is less than $[F:\mathbb{Q}]$. So, the stabilizer in $T(\mathbb{Z}_{(p)})$ of A_y is trivial. Thus $\mathcal{S}_K \cap \mathcal{X}_K$ is an infinite set, and $\mathcal{X}^*$ meets some of the lower strata. This shows that $\mathcal{X}^*$ contains a point z which carries a super-singular AVRM (a p–rank 0 abelian scheme).

Write $(A_z, i_z : O \hookrightarrow \mathrm{End}(A_z), \phi)$ for the abelian variety sitting over the super-singular point $z \in V_K$, where ϕ is a level K–structure. Now we denote by $\widehat{\mathcal{X}}_K$ (resp. $\widehat{V}_K$) the formal completion of $\mathcal{X}_K$ (resp. V_K) along $z \in V_K$. We have a splitting $\widehat{V}_K = \prod_{\mathfrak{p} \in \Sigma_p} \widehat{V}_{\mathfrak{p}}$, where $\widehat{V}_{\mathfrak{p}}$ is the deformation space of the p–divisible group of $A_z[\mathfrak{p}_F^\infty]$ for $\mathfrak{p}_F = \mathfrak{p} \cap F$. Choose $z \in \mathcal{X}^*$ projecting down to $z \in \mathcal{X}_K$. Then $z \in V$ carries the triple $(A_z, \overline{\lambda}, \eta_z^{(p)})$.

Step 3: Suppose (contrary to the desired conclusion) that $\mathcal{X}$ is a proper subscheme of Ig; so, $\mathcal{X}_K$ is a proper subscheme of V_K. Since z is in the Zariski closure of $\mathcal{X}_K^\circ$ in V_K, we have a formal curve $\xi : Spf(\mathbb{F}[[t]]) \to \widehat{\mathcal{X}}_K$ such that

1. The abelian scheme $A_\xi = A \times_{\widehat{\mathcal{X}}_K, \xi} Spf(\mathbb{F}[[t]])$ over ξ is generically ordinary. In particular, the projection $\xi_{\mathfrak{p}} : Spf(\mathbb{F}[[t]]) \to \widehat{\mathcal{X}} \to \widehat{V}_{\mathfrak{p}}$ is determined by $A_\xi[\mathfrak{p}_F^\infty]$ which is a generically ordinary p–divisible group;
2. Over $\mathbb{F}((t))$, the image of ξ is contained in $\prod_{\mathfrak{p} \in \Sigma'} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$, and the tangent space T_ξ° over the generic point is given by $\oplus_{\mathfrak{p} \in \Sigma'} T_{\mathfrak{p}} \otimes_{\mathcal{O}_V} \mathbb{F}((t))$ for $\Sigma' \subsetneq \Sigma_p$;
3. ξ is the formal completion around z of an irreducible smooth algebraic curve $C \to \mathcal{X}_K$ with open dense subset $C^\circ = C \times_{\mathcal{X}_K} \mathcal{X}_K^\circ$.

Choose $\mathfrak{p} \notin \Sigma'$. Since $A_\xi[\mathfrak{p}_F^\infty]$ is generically ordinary but deforms into a super-singular p–divisible group at special fiber, $\xi_{\mathfrak{p}}$ is non-constant. The formal group

$A_C[\mathfrak{p}_F^\infty] = A[\mathfrak{p}_F^\infty] \times_{\mathcal{X}_K} C$ is constant on the dense open subscheme C° in C. We choose a minimal non-trivial p–divisible subgroup $\Gamma \subset A_C[\mathfrak{p}_F^\infty]$ and consider its generator $C_r \in \Gamma[p^r]$ (similarly defined as $\mathcal{Y}_{K,r}$ in the sense of [AME] Sections 1.9-10) as a covering of C. Then the relative dimension of the formal group $\widehat{\Gamma}$ of Γ is equal to 1, and we have a tower of algebraic curves $\{C_r^{red}\}_r$ which carries the connected finite flat group scheme $\Gamma[p^r]^\circ$. We consider the function field $\mathfrak{K} = \mathbb{F}(C_\infty^{red}) = \cup_r \mathbb{F}(C_r^{red})$, its Galois closure $\mathfrak{K}^{gal}$ over $\mathbb{F}(C)$ and the Galois group $\mathfrak{G} = \mathrm{Gal}(\mathfrak{K}^{gal}/\mathbb{F}(C))$. We may assume that the formal group $\widehat{\Gamma}$ is of height 2 over z and is of height one at the generic point, because it is constant and ordinary generically. By a classical result of Igusa, the formal deformation space of a height 2 formal group over $\mathbb{F}$ is a smooth formal curve $Spf(\mathbb{F}[[T]])$, and the Galois representation on the p–adic Tate module $\mathcal{T}_p(\mathcal{G}) \cong \mathbb{Z}_p$ of the universal p–divisible group $\mathcal{G}$ has full image $\mathbb{Z}_p^\times$ (cf. [I] and [K] Theorem 4.3). There is a generalization of this fact to height $h > 0$ formal groups over $Spf(\mathbb{F}[[t]])$ in [C2]. This Galois action of course factors through the inertia group $I \subset \mathfrak{G}$ of z (by the very definition of C_r^{red}/C). By the universality, we have a morphism $Spf(\mathbb{F}[[t]]) \to Spf(\mathbb{F}[[T]])$, which is non-constant since the special fiber of Γ is non-ordinary. Thus the Galois representation of I on $\mathcal{T}_p(\Gamma)$ has open image in $\mathbb{Z}_p^\times$, and hence $\mathfrak{G}$ is non-trivial. Since $A_C[\mathfrak{p}_F^\infty]^{et}$ is constant over C°, the image of the representation of $\mathfrak{G}$ on $\mathcal{T}_p(\Gamma)$ has to be trivial, hence a contradiction. Thus we conclude that $\Sigma' = \Sigma_p$, which finishes the proof. $\square$

Recall that $\mathcal{O}_{x/\mathbb{F}} = \mathcal{O}_{x/\mathcal{W}} \otimes_{\mathcal{W}} \mathbb{F}$ is the stalk of $x = x(\mathcal{A})$ of $\mathcal{O}_{Ig/\mathbb{F}}$. The following result is a key to prove the linear independence.

Proposition 2.6. *Let $\mathbb{F} = \overline{\mathbb{F}}_p$ and the notation be as in Proposition 2.5 and its proof. We consider $\mathcal{O}_{x/\mathbb{F}} = \mathcal{O}_{Ig,x/\mathbb{F}}$ and write $S = Spec(\mathcal{O}_x)$. For a positive integer m, let $\mathcal{X}$ be a reduced irreducible closed subscheme of S^m containing $x^m = (x, x, \dots, x)$ for the closed point $x \in S$. Suppose that $\mathcal{X}$ is stable under the diagonal action of $\mathcal{T}$. If the projection to the first $(m-1)$–factor S^{m-1} induces a surjective morphism $\pi_{\mathcal{X}} : \mathcal{X} \to S^{m-1}_{/\mathbb{F}}$, then either $\mathcal{X} = S^m$ or $\mathcal{X}$ is integral over S^{m-1} under the projection to the first $(m-1)$–factor. If further $\pi_{\mathcal{X}}$ induces a surjection of the tangent space at x^m onto that of S^{m-1} at x^{m-1} and $\mathcal{X}$ is a proper subscheme of S^m, $\pi_{\mathcal{X}} : \mathcal{X} \to S^{m-1}$ is étale, where the 0 fold product S^0 is supposed to be equal to $Spec(\mathbb{F})$.*

Proof. Let $\widehat{S} = Spf(\widehat{\mathcal{O}}_x)$ for the formal completion of S along x. We have $\widehat{S} \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O$. Since the case $m = 1$ has already been taken care of by Proposition 2.5, we may assume that $m \geq 2$. If $\dim \mathcal{X} = 0$, we see that $Supp(\mathcal{X})$ is made up of the point induced by x, which is impossible since $\mathcal{X}$ covers $Spec(\mathcal{O}_x)^{m-1}$. Thus we have $\dim \mathcal{X} > 0$. Since $\mathcal{X}$ is stable under the diagonal action of $\mathcal{T} \subset \mathrm{Aut}(\mathcal{O}_x)$ (which is p–adically dense in $\mathrm{Aut}_O(\widehat{S}) = O_p^\times$) and $\mathcal{X}$ is irreducible reduced, by Chai's lemma (Lemma 2.4), $\widehat{\mathcal{X}}$ is a formal O_p–submodule of $\widehat{S}^m$ of the form $\widehat{\mathbb{G}}_m \otimes L$ for an O_p–direct summand L of the cocharacter group $X_*(\widehat{S}^m) \cong O_p^m$. In particular, $\mathcal{X}$ is stable under $T(\mathbb{Z}_{(p)})$.

Since $\pi_{\mathcal{X}} : \widehat{\mathcal{X}} \to \widehat{S}^{m-1}$ is surjective, the projected image $L^{\mathcal{X}} = \pi_{\mathcal{X}}(L)$ of L in the first $(m-1)$–factor $O_p^{m-1} \cong X_*(\widehat{S}^{m-1})$ is an O_p–submodule of finite index, and $L_{\mathcal{X}} = \mathrm{Ker}(\pi_{\mathcal{X}} : L \to O_p^{m-1})$ is an O_p–direct summand of $O_p = \mathrm{Ker}(\pi : X_*(\widehat{S}^m) \to X_*(\widehat{S}^{m-1}))$ given by $\prod_{\mathfrak{p}\in\Sigma'} O_{\mathfrak{p}}$ for a subset Σ' of Σ_p.

We consider the diagonal formal torus $\widehat{\Delta} \subset \widehat{S}^m$ ($\widehat{\Delta} \cong \widehat{S}$) and its image $\alpha(\widehat{\Delta})$ under the component-wise action of $\alpha \in T(\mathbb{Z}_{(p)})^m$. Since a formal subtorus $\widehat{S}' \subset \widehat{S}^m$ is determined by its cocharacter group $X_*(\widehat{S}') = \mathrm{Hom}(\widehat{\mathbb{G}}_m, \widehat{S}')$, the dimension of $\widehat{S}'$ is given by $\mathrm{rank}_{\mathbb{Z}_p} X_*(\widehat{S}')$. The dimension of the intersection of two tori is determined by the rank of the intersection of the cocharacter groups of the two tori in $X_*(\widehat{S}^m)$. As we have already seen, the cocharacter group of $\widehat{\mathcal{X}}$ at x^m contains the subspace $\prod_{\mathfrak{p}\in\Sigma'} O_{\mathfrak{p}}^m$ in $X_*(\widehat{S}^m) = O_p^m$. Thus $\mathrm{rank}(X_*(\alpha(\widehat{\Delta})) \cap X_*(\widehat{\mathcal{X}})) \geq |\Sigma'| = \sum_{\mathfrak{p}\in\Sigma'} [F_{\mathfrak{p}}, \mathbb{Q}_p]$, and by the p–adic density of $T(\mathbb{Z}_{(p)})$ in $\mathrm{Aut}_O(\widehat{S})$, the equality is attained for some $\alpha \in T(\mathbb{Z}_{(p)})^m$. Thus the intersection $\alpha(\widehat{\Delta}) \cap \widehat{\mathcal{X}}$ gives rise to an irreducible subscheme in $Spec(\mathcal{O}_x)$ of dimension $|\Sigma'|$ stable under the action of $T(\mathbb{Z}_{(p)})$. Then by Proposition 2.5, we conclude that $|\Sigma'| = 0$ or $\Sigma' = \Sigma_p$. If $\Sigma' = \Sigma_p$, we conclude $\mathcal{X} = S^m$.

Now suppose $|\Sigma'| = 0$. Thus $\mathcal{X} = Spec(\mathcal{R})$ is integral over S^{m-1}. If further $\pi_{\mathcal{X}} : L \to X_*(S^{m-1})$ is surjective ($\Leftrightarrow \pi_{\mathcal{X}}$ is surjective at the level of tangent spaces), $\pi_{\mathcal{X}} : L \to X_*(S^{m-1})$ is an isomorphism; so, $\widehat{\mathcal{X}} \cong \widehat{S}^{m-1}$ by $\pi_{\mathcal{X}}$. Thus we conclude that $\mathcal{X}$ is étale over S^{m-1}. □

Remark 2.1. Suppose that $\mathcal{X} \subset V^m$ is the Zariski closure of an infinite set of distinct points of the form $x = (x(\mathcal{A}_1), \ldots, x(\mathcal{A}_m))$ for O–lattices $\mathcal{A}_j \subset M$ of conductor prime to p and is stable under a subgroup U_0 of $T(\mathbb{Z}_{(p)})$ whose p–adic closure is an open subgroup of $T(\mathbb{Z}_p)$. Here T acts diagonally on V^m. Then we can apply Proposition 2.6, because we can use any point x as above on an irreducible component of $\mathcal{X}$ which is fixed by a subgroup of finite index in U_0. In particular, by changing x, we may assume that the formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along x is smooth; so, we conclude $\mathcal{X} = V^m$ (if $\dim \mathcal{X} > (m-1)\dim V$ and $\mathcal{X}$ covers any $(m-1)$–factor V^{m-1}) from Proposition 4 of [C1] (by the same proof of Proposition 2.5 and Proposition 2.6) without using the refined version: Lemma 2.4.

Recall the prime factor $\mathfrak{l}$ in O of $\ell \neq p$ fixed in the introduction. Let $R_n = O + \mathfrak{l}^n R$, and write $Cl_n = \mathrm{Pic}(R_n)$. Let $\mathcal{A}$ be a proper ideal of R_n. Each $\alpha \in R_{(p)}^\times$ induces an isomorphism:

$$(X(\mathcal{A}), \overline{\Lambda}(\mathcal{A}), \eta^{(p)}(\mathcal{A})) \cong (X(\alpha\mathcal{A}), \overline{\Lambda}(\alpha\mathcal{A}), \eta^{(p)}(\alpha\mathcal{A})).$$

Thus the isomorphism class of the triple $x(\mathcal{A})_{/\mathbb{F}} = x(\mathcal{A}) \times_{\mathcal{W}} \mathbb{F}$ for a proper R_n–ideal $\mathcal{A} \subset M$ only depends on the class $[\mathcal{A}] \in Cl_n$. Let $C^{alg} \subset Cl_\infty$ be the subgroup made up of the image $[x] = \varprojlim_n [x\widehat{R}_n \cap M]$ of $x \in M_{\mathbb{A}}^\times$ with $x_{\mathfrak{l}} = 1$. Since the image of $F^\times \subset M_{\mathbb{A}}^\times$ is trivial in Cl_∞, we may embed the image $\mathcal{T}_{\mathfrak{l}}$ of $M \cap R_{\mathfrak{l}}^\times$ in $T(\mathbb{Q})$ into Cl_∞ by $\alpha \mapsto [\alpha^{(\mathfrak{l})}]$. Then we have an exact sequence: $1 \to \mathcal{T}_{\mathfrak{l}} \to C^{alg} \to Cl_M \to 1$ for the class group Cl_M of M.

Remark 2.2. As remarked by the referee of this paper, this group C^{alg} is obviously close to the rational points $S_1^M(\mathbb{Q})$ of the Serre group S_1^M relative to the CM field M (of the empty modulus of support; see [ALR] II.2.2) and is isomorphic to $S_1^M(\mathbb{Q})/T_1^F(\mathbb{Q})$ when $\mathfrak{l} = (\ell)$ and $\mathfrak{l}$ remains prime in M, where T_1^F is the quotient of $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ by the Zariski closure of $O^\times$ (regarded as a subgroup of S_1^M). In any case, C^{alg} is a countable subgroup in the compact profinite group Cl_∞, in a way similar to the inclusion of the Serre groups: $S_{\mathfrak{m}}(\mathbb{Q}) \subset S_{\mathfrak{m}}(\mathbb{Q}_\ell)$.

Proposition 2.7. *Suppose* (unr) *and* (ord) *for* p. *Let* $\underline{n} = 0 < n_1 < n_2 < \cdots$ *be an infinite sequence of integers with* $\mathfrak{l}^{n_j}$ *generated by an element in* $N_{M/F}(R_{(p)}^\times)$, *and let* $\Xi = \{x(\mathcal{A}) \in V(\mathbb{F}) | \mathcal{A} \in Cl_{n_j}$ *with* $[\mathcal{A}] \in \mathrm{Ker}(Cl_{n_j} \twoheadrightarrow Cl_r)$ $(j = 1, 2, \ldots)\}$ *for an integer* r *with* $0 \le r \le n_1$. *Then* Ξ *is Zariski dense in* $V(\mathbb{F})$.

Proof. Let $K = G_1(\widehat{\mathbb{Z}})$. Then $V_{1/\mathbb{F}} = V/K$ is the coarse moduli scheme of abelian variety with real multiplication by O with a given polarization ideal (cf. [H03a] Lecture 9 or [PAF] Chapter 4). The image Ξ_1 of Ξ in $V_1(\mathbb{F})$ has infinitely many points (see the remark after the proposition); so, the dimension of an irreducible component of the Zariski closure X of Ξ is positive. We pick such a component X_2, and suppose that $x(\mathcal{A}) \in \Xi$ is in X_2. For a sufficiently small open compact subgroup S of K (maximal at p), $V_{/\mathbb{F}}$ is étale over $V_{S/\mathbb{F}}$. Thus if X has a singularity at $y \in X$, it has singularity at the image of y in $V_S(\mathbb{F})$, which is a smooth irreducible variety over $\mathbb{F}$ of finite type. Thus the singular locus in the image X_S of X in $V_{S/\mathbb{F}}$ is a proper closed subscheme, and we may assume that $x = x(\mathcal{A})_{/\mathbb{F}}$ is a smooth point of X (by changing $\mathcal{A}$ if necessary). We suppose that $\mathcal{A}$ is a proper R_n–ideal (for $n = n_j$ with an index j).

Since $\tau(\rho_{\mathcal{B}}(\alpha))(x(\mathcal{B})) = x(\mathcal{B})$ for $\alpha \in R_{(p)}^\times$, we find that

$$\tau(\rho_{\mathcal{B}}(\alpha_{\mathfrak{l}})^{-1})(x(\mathcal{B})) = \tau(\rho_{\mathcal{B}}(\alpha^{(\mathfrak{l})}))(x(\mathcal{B})).$$

By our construction, we have $\rho_{\mathcal{B}}(\alpha^{(\mathfrak{l})}) = \rho_{\mathcal{A}}(\alpha^{(\mathfrak{l})}) = \rho_R(\alpha^{(\mathfrak{l})})$. Since we have a commutative diagram for $a \in M_{\mathbb{A}}^\times$ with $a_p = a_{\mathfrak{l}} = 1$:

$$\begin{tikzcd}
(F_{\mathbb{A}}^{(p\infty)})^2 \arrow[r, "\text{multiplication by } \rho_R(a)"'] \arrow[d, "\eta^{(p)}(\mathcal{B})"'] & (F_{\mathbb{A}}^{(p\infty)})^2 \arrow[d, "\eta^{(p)}(a\mathcal{B})"] \\
V^{(p)}(X(\mathcal{B})) \arrow[r, "\phi_a"] & V^{(p)}(X(a\mathcal{B}))
\end{tikzcd}$$

for a prime-to–p isogeny $\phi_a : X(\mathcal{B}) \to X(a\mathcal{B})$, the action of $\tau(\rho_R(\alpha^{(\mathfrak{l})}))$ brings the point $x(\mathcal{B})$ to $x([\alpha^{(\mathfrak{l})}]\mathcal{B})$, and hence it induces a permutation of Ξ as long as $(\alpha^{(\mathfrak{l})}R_r)$ is in the identity class in Cl_r ($\Leftrightarrow \alpha \equiv 1 \mod \mathfrak{l}^r$). If $\alpha \equiv 1 \mod \mathfrak{l}^n$, the class $[\alpha R_n]$ in Cl_n is trivial and $[\alpha\mathcal{A}] = [\alpha^{(\mathfrak{l})}][\mathcal{A}] = [\mathcal{A}]$, which implies $\tau(\rho_{\mathcal{A}}(\alpha^{(\mathfrak{l})}))(x(\mathcal{A})) = x(\mathcal{A})$. The image $\mathcal{T}$ of $\{\alpha \in R_{(p)}^\times \cap R_{\mathfrak{l}} | \alpha \equiv 1 \mod \mathfrak{l}^{\max(r,n)}\}$ in $T(\mathbb{Z}_p)$ is p–adically dense and preserves X and fixes $x(\mathcal{A})$. Since $t \in \mathcal{T}$ fixes $x = x(\mathcal{A})$, $t \in \mathcal{T}$ permutes irreducible components of the Zariski closure X of Ξ passing through x. In particular, if t really moves the irreducible component X_2 of X, $x \in X_2 \cap t(X_2)$ is not a smooth point of X. Thus we conclude that X_2 is stable under $\mathcal{T}$. By our choice, we have $\dim X_2 > 0$. Thus we can apply Proposition 2.5 to the ideal $\mathfrak{b}_K$ of X_K in $\mathcal{O}_{V_K,x/\mathbb{F}}$, which is stable under $\mathcal{T}$. By Proposition 2.5, we have $X_2 = X = V$. □

Remark 2.3. By a result of Deligne and Tate ([D1] and [T]), each isomorphism class of an abelian variety over $\mathbb{F}$ is determined by the action of the relative Frobenius map on its $\mathfrak{l}$–adic Tate module (over the field of definition), all abelian varieties sitting over the points in Ξ are non-isomorphic.

2.4. Linear independence.

Keeping the notation and the assumption of Proposition 2.7, let $\mathcal{C} = \mathcal{C}_\Xi$ denote the space of functions defined over Ξ with values in $\mathbf{P}^1(\mathbb{F}) = \mathbb{F} \sqcup \{\infty\}$. The class group Cl_∞ acts on $\mathcal{C}$ by translation: $f(x) \mapsto f(xy)$ ($y \in Cl_\infty$). By Zariski density of Ξ in $V = V^{(p)}_{/\mathbb{F}}$, we can embed into $\mathcal{C}$ the function field $\mathbb{F}(V)$ of V.

Proposition 2.8. *For a finite set $\Delta = \{\gamma_1, \ldots, \gamma_m\} \subset Cl_\infty$ independent modulo C^{alg}, the fields $\gamma_1(\mathbb{F}(V)), \ldots, \gamma_m(\mathbb{F}(V))$ are linearly disjoint over $\mathbb{F}$ in $\mathcal{C}_\Xi$. In other words, the subset $\{(x(\delta(\mathcal{A}))_{\delta\in\Delta} | x(\mathcal{A}) \in \Xi\}$ is Zariski dense in the product $V^\Delta_{/\mathbb{F}}$ of Δ copies of $V_{/\mathbb{F}}$.*

The following proof is modeled after an argument of C.-L. Chai ([C4] Section 8) proving a special subvariety (called a Tate linear subvariety) of a product of copies of Hilbert modular varieties is actually a Shimura subvariety. Particularly the use of Zarhin's theorem is his idea, which shortened substantially the original argument in the older versions of this paper.

Proof. We write $x_m(\mathcal{A})$ for $(x(\gamma_j\mathcal{A})) \in V^m$. Let X be the Zariski closure of

$$\{x_m(\mathcal{A}) \in V^m_{/\mathbb{F}} | x(\mathcal{A}) \in \Xi\}$$

in V^m. The assertion follows from the density of the above set: $X = V^m$. We shall prove this by induction on m. The case $m = 1$ is already treated (by Proposition 2.7); so, we assume that $m \geq 2$.

The projection of X to any of the factor V^{m-1} is surjective by the induction hypothesis. Thus we can choose an irreducible component X_1 of X so that the projection to each factor V^{m-1} is onto. By the same argument as in the proof of Proposition 2.7, we may assume that $x_m(\mathcal{A})$ is a smooth point on X. As we have seen, the action of $\tau(\rho_R(\alpha^{(\mathfrak{l})}))$ sends $x(\mathcal{B})$ to $x([\alpha^{(\mathfrak{l})}]\mathcal{B})$. Recall the image $\mathcal{T} \subset C^{alg}$ of $\alpha^{(\mathfrak{l})}$ for $\alpha \in M^\times$ prime to $\mathfrak{l}$ with (α) trivial in Cl_r and Cl_n, where n is the index such that $\mathcal{A}$ is a proper R_n–ideal. Then the action of $\alpha^{(\mathfrak{l})} \in \mathcal{T}$ as described above coincides with the action of $\tau(\rho_R(\alpha^{(\mathfrak{l})}))$. Thus $(X, x_m(\mathcal{A}))$ is stable under the diagonal action $\mathcal{T}$. Since there are only finitely many irreducible components of X crossing at $x_m(\mathcal{A})$, the smoothness of X at $x_m(\mathcal{A})$ tells us that X_1 is stable under $\mathcal{T}$. The p–adic closure of $\mathcal{T}$ coincides with $T(\mathbb{Z}_p)$. Let $\mathcal{O}_{Ig^m,x^m/\mathbb{F}}$ for $x = x(\mathcal{A})$

be the stalk at x^m. We apply Proposition 2.6 to X_1, which claims that either $X_1 = Ig^m$ or X_1 is algebraic over each factor $Ig^{m-1}_{/\mathbb{F}}$.

By changing γ_j in the coset $\gamma_j C^{alg}$, we may assume that $\gamma_j(x(\mathcal{A})) = x(\mathcal{A})$. We first suppose that $m = 2$. If $X_1 = Ig^2$, there is nothing to prove. Suppose X_1 is a proper subscheme. Let $i : X_1 \hookrightarrow Ig^2$ and $\mathbb{Y} = i^*\mathbb{X}^2 = \mathbb{X}^2 \times_{Ig^2} X_1$. Since the two projections $\pi_j : X_1 \to Ig$ are dominant, $\mathrm{End}(\pi_j^*\mathbb{X}) \otimes \mathbb{Q} = F$ for $\pi_j^*\mathbb{X} = \mathbb{X} \times_{Ig} X_1$. Thus there are only two possibilities of $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = \mathrm{End}(\mathbb{Y}) \otimes \mathbb{Q}$: Either $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = F \times F$ or $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$.

We suppose $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = F \times F$ and try to get a contradiction (in order to prove that $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$). We pick a sufficiently small $K^{(p)}$ so that V_K is smooth, and write $\mathbb{Y}_K$ for the abelian scheme $i^*\mathbb{X}^2_K$ over the image $X_{1,K}$ of X_1 in Ig^2_K for the Igusa tower Ig_K over V_K. Then $X_{1,K}$ and $\mathbb{Y}_K$ are varieties over a finite field $\mathbb{F}_q$ for a p–power q. Let η be the generic point of $X_{1,K/\mathbb{F}_q}$, and write $\overline{\eta}$ for the geometric point over η so that $\mathbb{F}_q(\overline{\eta})$ is a separable algebraic closure of $\mathbb{F}_q(\eta)$. Take an odd prime ℓ, and consider the ℓ–adic Tate module $\mathcal{T}_\ell\mathbb{Y}_{\overline{\eta}}$ for the generic fiber $\mathbb{Y}_{\overline{\eta}}$ of $\mathbb{Y}$. We consider the image of the Galois action $\mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})/\mathbb{F}_q(\eta)))$ in $GL_{O_\ell \times O_\ell}(\mathcal{T}_\ell\mathbb{Y}_{\overline{\eta}})$. Then by a result of Zarhin ([Z] and [DAV] Theorem V.4.7) combined with a standard argument, $\mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})/\mathbb{F}_q(\eta)))$ is an ℓ–adically open subgroup of $\mathbb{Q}_\ell$–points of a reductive group $\mathcal{G}$ defined over $\mathbb{Q}$. By Zarhin's theorem, the centralizer of $\mathcal{G}$ is $\mathrm{End}(\mathbb{Y}) \otimes \mathbb{Q}_\ell$, and hence the derived group $\mathcal{G}_1(\mathbb{Q}_\ell)$ of $\mathcal{G}(\mathbb{Q}_\ell)$ has to be $SL_2(F_\ell \times F_\ell)$. Then by Chebotarev's density theorem, we can find a set of closed points $x \in X_{1,K}(\mathbb{F})$ with positive density such that the Zariski closure in $\mathcal{G}$ of the subgroup generated by the Frobenius element $Frob_x \in \mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})/\mathbb{F}_q(\eta)))$ at x is a torus containing a maximal torus T_x of the derived group $\mathcal{G}_1$ of $\mathcal{G}$. In particular the centralizer of T_x in $\mathcal{G}_1$ is itself. Thus $\mathbb{Y}_x$ is isogenous to a product of two non-isogenous absolutely simple abelian varieties Y_1 and Y_2 with multiplication by F defined over a finite field. The endomorphism algebra $M_j = \mathrm{End}^{\mathbb{Q}}(Y_j)$ is a CM quadratic extension of F generated over $\mathbb{Q}$ by the relative Frobenius map ϕ_j induced by $Frob_x$. The relative Frobenius map $Frob_x$ acting on $X_*(\widehat{Ig}_{x_1}) \cong O_p$ has $[F : \mathbb{Q}]$ distinct eigenvalues $\{\phi_1^{(1-c)\sigma} | \sigma \in \Sigma_1\}$ for the CM type Σ_1 of Y_1, which differ from the eigenvalues of $Frob_x \in \mathrm{End}(Y_2)$ on $X_*(\widehat{Ig}_{x_2}) \cong O_p$. Since it has been proven in [H03b] Proposition 3.11 (using many results of Chai in [C4]) that over an open dense subscheme U of X_1, the formal completion of U at $(x_1, x_2) \in X_1 \subset Ig^2$ is canonically isomorphic to a formal subtorus $\widehat{Z} \subset \widehat{Ig}_{x_1} \times \widehat{Ig}_{x_2}$ with co-character group $X_*(\widehat{Z}) \cong O_p$, we may assume that our point $x = (x_1, x_2)$ as above is in U (because such x has positive density). Projecting $X_*(\widehat{Z})$ down to the left and the right factor Ig, the projection map $X_*(\widehat{Z}) \to X_*(\widehat{Ig}_{x_j})$ is actually an injection commuting with the action of $Frob_x$. Thus $Frob_x$ has more than $[F : \mathbb{Q}]$ distinct eigenvalues on $X_*(\widehat{Z})$, which is a contradiction. Thus we conclude that $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$. This implies that we have a prime-to-p isogeny $\alpha : \pi_1^*\mathbb{X}_K \to \pi_2^*\mathbb{X}_K$. Writing $\boldsymbol{\eta}_j$ for the level structure (including the ordinary level structure at p) of $\pi_j^*\mathbb{X}$, we find that $\alpha \circ \boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 \circ g_K^{-1}$ for $g_K \in O_p^\times \times G(\mathbb{A}^{(p\infty)})$. Thus $X_{1,K}$ is a graph of the action of g_K in $Ig_K \times Ig_K$.

Shrinking K and taking the limit, we have $g = \lim_{K \to \{1\}} g_K$ in $O_p^\times \times G(\mathbb{A}^{(p\infty)})$, and X_1 is the graph of the action of g. Thus $g\gamma_1 = \gamma_2$. Since g fixes $x(\mathcal{A})$, we find that it is induced by $\alpha \in \mathrm{End}^{\mathbb{Q}}(X(\mathcal{A})) = M$ prime to p. In other words, $\gamma_1^{-1}\gamma_2 \in C^{alg}$, which is a contradiction. Thus $X_1 = Ig^2$, and we get the desired assertion when $m = 2$.

We now deal with the case where $m > 2$. By the induction hypothesis (and Proposition 2.6), we know that $\dim_{\mathbb{F}} X_1 = d(m-1)$ or dm for $d = [F : \mathbb{Q}]$. We need to prove $\dim_{\mathbb{F}} X_1 = dm$. Suppose now that $\dim_{\mathbb{F}} X_1 = d(m-1)$ to get a contradiction. We consider the partial product for $S = Spec(\mathcal{O}_x)$:

$$T_{i,j} = \left(\{x_{i-1}(\mathcal{A})\} \times \overset{i}{S} \times \{x_{j-i-1}(\mathcal{A})\} \times \overset{j}{S} \times \{x_{m-j}(\mathcal{A})\} \right) \subset S^m$$

for $1 \le i < j \le m$. We see that $\dim(\widehat{X}_1 \cap \widehat{T}_{i,j}) \ge d$ by Chai's lemma (Lemma 2.4), taking intersection of the formal cocharacter groups of $\widehat{T}_{i,j}$ and $\widehat{X}_1$. If the strict inequality holds for all (i,j), by Proposition 2.6 applied to $(X_1 \cap T_{i,j}) \subset T_{i,j} \cong S^2$, we find that $\dim(\widehat{X}_1 \cap \widehat{T}_{i,j}) = 2d$ and $X_1 \supset T_{i,j}$. Again by Chai's lemma, this implies $X_1 = S^m$, because the cocharacter group of $\widehat{X}_1$ contains the cocharacter group of $\widehat{T}_{i,j}$ for all (i,j). This contradicts our assumption: $\dim_{\mathbb{F}} X_1 = d(m-1)$. Thus we find that $\dim(T_{i,j} \cap X_1) = d$ for at least one pair (i,j). The argument in the case of $m = 2$ tells us that $\gamma_i^{-1}\gamma_j \in C^{alg}$, which is a contradiction. □

The linear independence applied to the global sections of a line bundle (regarded as sitting inside the function field) yields the following result:

Corollary 2.9. *Let the notation and the assumption be as in Proposition* 2.8. *Let* $\mathcal{L}$ *be a line bundle over* $Ig_{/\mathbb{F}}$. *Then for a finite set* $\Delta \subset Cl_\infty$ *independent modulo* C^{alg} *and a set* $\{s_\delta \in \mathcal{L}\}_{\delta \in \Delta}$ *of non-constant global sections* s_δ *of* $\mathcal{L}$ *finite at* Ξ, *the functions* $s_\delta \circ \delta$ $(\delta \in \Delta)$ *are linearly independent in* $\mathcal{C}_\Xi$.

3. Measure associated to a Hecke eigenform

To each Hecke eigenform f, we associate a measure supported on Cl_∞.

3.1. Hecke relation among CM points.

We write the left action: $G(\mathbb{A}^{(\infty)}) \times Sh^{(p)} \to Sh^{(p)}$ simply as $(g, x) \mapsto g(x) := \tau(g)^{-1}(x)$. Here the action of $\tau(g)$ is a right action induced by $\eta \mapsto \eta \circ g$. For each point $x = (X, \Lambda, \eta) \in Sh$, we can associate a lattice $\widehat{L} = \eta^{-1}(\mathcal{T}(X)) \subset (F_{\mathbb{A}}^{(\infty)})^2$. Then the level structure η is determined by the choice of a base $w = (w_1, w_2)$ of $\widehat{L}$ over $\widehat{O}$. In view of the base w, the inverted action $x \mapsto g(x)$ is matrix multiplication: ${}^t w \mapsto g\,{}^t w$, because $(\eta \circ g^{-1})^{-1}(\mathcal{T}(X)) = g\eta^{-1}(\mathcal{T}(X)) = g\widehat{L}$.

Recall the order $R_n = O + \mathfrak{l}^n R \subset M$ and the class groups $Cl_n = \mathrm{Pic}(R_n)$ and $Cl_\infty = \varprojlim_n Cl_n$. By class field theory, Cl_n gives the Galois group $\mathrm{Gal}(H_n/M)$ of the ring class field H_n of conductor $\mathfrak{l}^n$. The ideal $\mathfrak{l}_n = \mathfrak{l} + \mathfrak{l}^n R = \mathfrak{l}R_{n-1}$ is a prime ideal of R_n but is not proper (it is a proper ideal of R_{n-1}). Since $X(R_n)[\mathfrak{l}_n] \cong R_n/\mathfrak{l}_n = O/\mathfrak{l}$ and $\mathfrak{l}_n R_{n-1} \subset R_n$, we find that $X(R_n)[\mathfrak{l}_n] = R_{n-1}/R_n$. In other words, we have $X(R_n)/X(R_n)[\mathfrak{l}_n] \cong X(R_{n-1})$. We pick a subgroup $C \subset X(R_n)[\mathfrak{l}]$ isomorphic to $O/\mathfrak{l}$ but different from $X(R_n)[\mathfrak{l}_n]$. We look into $X(R_n)/C$. Take a lattice $\mathfrak{A}$ so that $X(R_n)/C = X(\mathfrak{A}) \Leftrightarrow \mathfrak{A}/R_n = C$. Since C is an O–submodule, $\mathfrak{A}$ is an O–lattice of M. Since $\mathfrak{l}C = 0$, we find $\mathfrak{l}R_n\mathfrak{A} \subset \mathfrak{A}$. Thus $\mathfrak{A}$ is R_{n+1}–ideal, because $R_{n+1} = O + \mathfrak{l}R_n$. Since C is not an R_n–submodule, the ideal $\mathfrak{A}$ is not R_n–ideal; so, it is a proper R_{n+1}–ideal. Since C generates over R_n all $\mathfrak{l}$–torsion points of $X(R_n)$, we find $R_n\mathfrak{A} = \mathfrak{l}^{-1}R_n$. In this way, we have created ℓ proper R_{n+1}–ideals $\mathfrak{A}$ with $\mathfrak{A}R_n = \mathfrak{l}^{-1}R_n$.

We have chosen a base $w = (w_1, w_2)$ of $\widehat{R}$ over $\widehat{O}$ in 2.1. We also specified the base of $\widehat{R}_n^{(\mathfrak{l})}$ to be $w^{(\mathfrak{l})}$ there, because $\widehat{R}_n^{(\mathfrak{l})} = \widehat{R}^{(\mathfrak{l})}$. To specify the base $w_\mathfrak{l}$ of $R_\mathfrak{l}$, we take $d \in O_\mathfrak{l}$ so that $R_\mathfrak{l} = O_\mathfrak{l}[\sqrt{d}] \subset M_\mathfrak{l}$. We assume that d is a $\mathfrak{l}$–adic unit if $\mathfrak{l}$ is unramified in M/F and d generates $\mathfrak{l}$ if $\mathfrak{l}$ ramifies in M/F. Then we choose $w_\mathfrak{l} = (1, \sqrt{d})$.

Since the base of $R_{n,\mathfrak{l}}$ is given by $\alpha_n{}^t(1, \sqrt{d})$ for $\alpha_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{l}^n \end{smallmatrix}\right)$ with a prime element $\varpi_\mathfrak{l}$ of $O_\mathfrak{l}$, we find that $\alpha_n(x(R)) = x(R_n)$ and $\alpha_1(x(R_{n-1})) = x(R_n)$. Moreover, for a suitable $u \in O$

$$\varpi_\mathfrak{l}(x(\mathcal{A})) = \begin{pmatrix} 1 & \frac{u}{\varpi_\mathfrak{l}} \\ 0 & 1 \end{pmatrix} (x(R_{n+1})) \quad \text{if } x(\mathcal{A}) = x(R_n)/C \text{ for } C \text{ as above,} \tag{3.1}$$

because the base of $\varpi_\mathfrak{l}\mathcal{A}_\mathfrak{l}$ is given by $\begin{pmatrix} 1+\varpi_\mathfrak{l}^n u\sqrt{d} \\ \varpi_\mathfrak{l}^{n+1}\sqrt{d} \end{pmatrix} = \begin{pmatrix} 1 & \frac{u}{\varpi_\mathfrak{l}} \\ 0 & 1 \end{pmatrix} \alpha_{n+1} \begin{pmatrix} 1 \\ \sqrt{d} \end{pmatrix}$. Here the action of $\varpi_\mathfrak{l} : x(\mathcal{A}) \mapsto \varpi_\mathfrak{l}(x(\mathcal{A}))$ may bring $x(\mathcal{A})$ on a geometrically irreducible component of $Sh^{(p)}$ to a different one.

Now we consider $x(\mathcal{A})$ in V_K for an open subgroup $K \subset G(\mathbb{A}^{(\infty)})$ containing $Z(\widehat{\mathbb{Z}})$. By repeating (3.1), if $x(\mathcal{A}) = x(R_n)/C$ for $C \cong O/\mathfrak{l}^m$ with $C \cap X(R_n)[\mathfrak{l}_n] = \{0\}$, then $\mathcal{A}$ is a proper R_{n+m}–ideal. If further $\mathfrak{l}^m$ is generated by an element $\varpi \in F$, we get $x(\mathcal{A}) = x(\varpi\mathcal{A}) = \varpi_\mathfrak{l}^m(x(\mathcal{A}))$ in V_K (because $\varpi/\varpi_\mathfrak{l}^m \in K$) and

$$x(\mathcal{A}) = \begin{pmatrix} 1 & \frac{u}{\varpi_\mathfrak{l}^m} \\ 0 & 1 \end{pmatrix} (x(R_{n+m})) = \left(\begin{smallmatrix} 1 & \frac{u}{\varpi} \\ 0 & 1 \end{smallmatrix}\right) (x(R_{n+m})) \quad \text{for a suitable } u \in O. \tag{3.2}$$

The set $\{x(\mathcal{A}) | [\mathcal{A}R_n] = [\mathfrak{A}]\}/\cong$ for $\mathcal{A} \in Cl_{n+m}$ running through ideal classes $\mathcal{A}$ projecting down to a given ideal class $[\mathfrak{A}] \in Cl_n$ is in bijection with $O/\mathfrak{l}^m$ by associating u to $\mathcal{A}$ in (3.2) (see Proposition 4.2).

3.2. Geometric modular forms.

Let k be a weight of $T = \mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$, that is, $k : T(A) = (A \otimes_\mathbb{Z} O)^\times \to A^\times$ is a homomorphism given by $(a \otimes \xi)^k = \prod (a\xi^\sigma)^{k_\sigma}$ for integers k_σ indexed by field embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$. Let B be a base ring, which is a $\mathcal{W}$–algebra. We consider quadruples $(X, \overline{\Lambda}, \eta^{(p)}, \omega)_{/A}$ for a B–algebra A with a differential ω generating

$H^0(X, \Omega_{X/A})$ over $A \otimes_{\mathbb{Z}} O$. We impose the following condition:

$$\eta^{(p)}(\widehat{L}_{\mathfrak{c}}^{(p)}) = \mathcal{T}(X) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)} \quad \text{for } L_{\mathfrak{c}} = O \oplus \mathfrak{c}^* \text{ with a fixed } \mathfrak{c}. \tag{3.3}$$

Under this condition, as seen in 2.3, the classification up to prime-to–p isogenies of the quadruples is equivalent to the classification up to isomorphisms. A modular form f (integral over B) of weight k is a functorial rule of assigning a value $f(X, \overline{\Lambda}, \eta^{(p)}, \omega) \in A$ to (the A–isomorphism class of) each quadruple $(X, \overline{\Lambda}, \eta^{(p)}, \omega)_{/A}$ (called a *test object*) defined over a B–algebra A. Here Λ is a $\mathfrak{c}$–polarization which (combined with $\eta^{(p)}$) induces $L_{\mathfrak{c}} \wedge L_{\mathfrak{c}} \cong \mathfrak{c}^*$ given by $((a \oplus b), (a' \oplus b')) \mapsto ab' - a'b$. The Tate test object at the cusp $(\mathfrak{a}, \mathfrak{b})$ for two factional ideals with $\mathfrak{a}^*\mathfrak{b} = \mathfrak{c}^*$ is an example of such test objects. The Tate semi-AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is defined over $\mathbb{Z}[[q^\xi]]_{\xi \in (\mathfrak{ab})_+}$ and is given by the algebraization of the formal quotient $(\widehat{\mathbb{G}}_m \otimes \mathfrak{a}^*)/q^{\mathfrak{b}}$ (see [K1] Chapter I and [HT] 1.7 for details of this construction). The rule f is supposed to satisfy the following three axioms:

(G1) For a B–algebra homomorphism $\phi : A \to C$, we have

$$f((X, \overline{\Lambda}, \eta^{(p)}, \omega) \times_{A,\phi} C) = \phi(f(X, \overline{\Lambda}, \eta^{(p)}, \omega)).$$

(G2) f is finite at all cusps, that is, the q–expansion of f at every Tate test object does not have a pole at $q = 0$ (see [K1] Chapter I and [HT] 1.7).

(G3) $f(X, \overline{\Lambda}, \eta^{(p)}, \alpha\omega) = \alpha^{-k} f(X, \overline{\Lambda}, \eta^{(p)}, \omega)$ for $\alpha \in T(A)$.

We write $G_k(\mathfrak{c}; B)$ for the space of all modular forms f satisfying (G1-3) for B–algebras A. We put

$$G_k(B) = \bigoplus_{\mathfrak{c}} G_k(\mathfrak{c}; B), \tag{3.4}$$

where $\mathfrak{c}$ prime to ℓp runs over a representative set of strict ideal classes of F.

An element $g \in G(\mathbb{A}^{(\infty)})$ fixing $\widehat{L}_{\mathfrak{c}}$ acts on $f \in G_k(\mathfrak{c}; B)$ by

$$f|g(X, \overline{\Lambda}, \eta^{(p)}, \omega) = f(X, \overline{\Lambda}, \eta^{(p)} \circ g, \omega).$$

For a closed subgroup $K \subset K_{\mathfrak{c}} = GL(\widehat{L}_{\mathfrak{c}}) \cap G_1(\mathbb{A}^{(\infty)})$, we write $G_k(\mathfrak{c}; K; B)$ for the space of all K–invariant modular forms; thus,

$$G_k(\mathfrak{c}; K; B) = H^0(K, G_k(\mathfrak{c}; B)).$$

Take an O–ideal $\mathfrak{N}$ prime to $p\mathfrak{c}$. Then the $\mathfrak{N}$–component of $K_{\mathfrak{c}}$ is $SL_2(O_{\mathfrak{N}})$. Let

$$\Gamma_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(O_{\mathfrak{l}}) \middle| c \in \mathfrak{N}O_{\mathfrak{N}} \right\},$$

and define for an open subgroup $\Gamma \subset SL_2(O_{\mathfrak{N}})$

$$G_k(\Gamma; B) = \bigoplus_{\mathfrak{c}} G_k(\mathfrak{c}; \Gamma \times K_{\mathfrak{c}}^{(\mathfrak{N})}; B).$$

A W–algebra B is called a p–adic algebra if $B = \varprojlim_n B/p^n B$. We write η_{ord} for the pair of level structures $(\eta_p^{ord} : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \hookrightarrow X[p^\infty], \eta^{(p)})$. A p–adic modular form f over a p–adic W–algebra B is a functorial rule of assigning a value in A to triples $(X, \overline{\Lambda}, \eta_{ord})_{/A}$ with $\mathfrak{c}$–polarization class $\overline{\Lambda}$ satisfying an obvious version of (G1-2) for p–adic B–algebras A (not just B–algebras). In general, we do not impose (G3) on p–adic modular forms. We write $V(\mathfrak{c}; B)$ for the space of p–adic modular forms defined over B. We again define

$$V(B) = \bigoplus_{\mathfrak{c}} V(\mathfrak{c}; B) \quad \text{and} \quad V(\Gamma; B) = \bigoplus_{\mathfrak{c}} V(\mathfrak{c}; \Gamma \times K_{\mathfrak{c}}^{(p\mathfrak{N})}; B), \tag{3.5}$$

where $V(\mathfrak{c}; K; B) = H^0(K, V(\mathfrak{c}; B))$. For $f \in V(B)$, we write $f_{\mathfrak{c}} \in V(\mathfrak{c}; B)$ for the $\mathfrak{c}$–component of f, and we say that f is of *level* $\mathfrak{N}$ if f in either in $G_k(\Gamma; B)$ or in $V(\Gamma; B)$ for $\Gamma \subset SL_2(O_{\mathfrak{N}})$.

Since η_p^{ord} induces the identification $\widehat{\eta}_p^{ord} : \widehat{\mathbb{G}}_m \otimes O^* \cong \widehat{X}$ for the formal completion of X along the origin, by pushing forward the differential $\frac{dt}{t}$, we can associate $(X, \overline{\Lambda}, \eta^{(p)}, \widehat{\eta}_{p,*}^{ord} \frac{dt}{t})$ to a quadruple $(X, \overline{\Lambda}, \eta_p^{ord}, \eta^{(p)})$. In this way, any modular form f satisfying (G1-3) can be regarded as a p–adic modular form by

$$f(X, \overline{\Lambda}, \eta_{ord}) = f(X, \overline{\Lambda}, \eta^{(p)}, \widehat{\eta}_{p,*}^{ord} \frac{dt}{t}).$$

By the q–expansion principle (cf. [K1] (1.2.15-16) and [PAF] Corollary 4.23), we thus have a canonical embedding of $G_k(B)$ into $V(B)$ which keeps the q–expansion. A p–adic modular form associated to a modular form in $G_k(B)$ satisfies the following replacement of (G3):

(g3) $f(X, \overline{\Lambda}, \alpha \cdot \eta_p^{ord}, \eta^{(p)}) = \alpha^{-k} f(X, \overline{\Lambda}, \eta_p^{ord}, \eta^{(p)})$ for $\alpha \in O_p^{\times}$.

Although we do not impose the condition (G3) on p–adic modular forms f, we limit ourselves to the study of forms satisfying the following condition (G3$'$) in order to define the modified value $f([\mathcal{A}])$ later at CM points $x(\mathcal{A})$ truly independent of the choice of $\mathcal{A}$ in its proper ideal class. Here abusing our notation, $x(\mathcal{A})$ is the quadruple $(X(\mathcal{A}), \Lambda(\mathcal{A}), \eta_{ord}(\mathcal{A}), \omega(\mathcal{A}))_{/\mathcal{W}}$ introduced in 2.1. We consider the torus $T_M = \mathrm{Res}_{R/\mathbb{Z}} \mathbb{G}_m$ and identify its character group $X^*(T_M)$ with the module $\mathbb{Z}[\Sigma \sqcup \Sigma c]$ of formal linear combinations of embeddings of M into $\overline{\mathbb{Q}}$. By the identity: $(X(\alpha\mathcal{A}), \Lambda(\alpha\mathcal{A}), \alpha\eta_{ord}(\alpha\mathcal{A}))_{/\mathcal{W}} \cong (X(\mathcal{A}), \alpha\alpha^c\Lambda(\mathcal{A}), \eta_{ord}(\mathcal{A}) \circ \rho_{\mathcal{A}}(\alpha))_{/\mathcal{W}}$, we may assume that for $k, \kappa \in \mathbb{Z}[\Sigma]$,

(G3$'$) $f(x(\alpha\mathcal{A})) = f(\rho_R(\alpha^{(\mathfrak{l})})(x(\mathcal{A}))) = \alpha^{-k-\kappa(1-c)} f(x(\mathcal{A}))$ for $\alpha \in T_M(\mathbb{Z}_{(\ell)})$.

It is known that for the p–adic differential operator d_σ of Dwork-Katz ([K1] 2.5-6) corresponding to $\frac{1}{2\pi i}\frac{\partial}{\partial z_\sigma}$ for $\sigma \in \Sigma$, $d^\kappa f$ satisfies (G3$'$) if $f \in G_k(B)$.

3.3. Hecke operators.

Suppose that the $\mathfrak{l}$–component $\Gamma_{\mathfrak{l}}$ of the level subgroup Γ is equal to $\Gamma_0(\mathfrak{l}^r)$ $(r \geq 0)$. Let $e_1 = {}^t(1, 0), e_2 = {}^t(0, 1)$ be the standard basis of $F^2 \otimes \mathbb{A}^{(p\infty)}$. Then, under

(3.3), for each triple $(X, \overline{\Lambda}, \eta_{ord})_{/A}$ with $\eta_{ord} = \eta_p^{ord} \times \eta^{(p)}$,

$$C = \eta_{\mathfrak{l}}(\mathfrak{l}^{-r} O_{\mathfrak{l}} e_1 + O_{\mathfrak{l}} e_2)/\eta_{\mathfrak{l}}(O_{\mathfrak{l}}^2)$$

gives rise to an A–rational cyclic subgroup of X of order $\mathfrak{l}^r$, that is, a finite group subscheme defined over A of $X_{/A}$ isomorphic to $O/\mathfrak{l}^r$ étale locally. Since $\Gamma_0(\mathfrak{l}^r)$ fixes $(\mathfrak{l}^{-r} O_{\mathfrak{l}} e_1 + O_{\mathfrak{l}} e_2)/O_{\mathfrak{l}}^2$, the level $\Gamma_0(\mathfrak{l}^r)$ moduli problem is equivalent to the classification of quadruples $(X, \overline{\Lambda}, C, \overline{\eta}_{ord}^{(\mathfrak{l})})_{/A}$ for a subgroup C of order $\mathfrak{l}^r$ in X, where $\eta_{ord}^{(\mathfrak{l})}$ is the (p–ordinary) level structure outside $\mathfrak{l}$. Thus we may define for $f \in G_k(\Gamma; B)$ with Γ as above the value of f at $(X, \overline{\Lambda}, C, \eta^{(p\mathfrak{l})}, \omega)$ by $f(X, \overline{\Lambda}, C, \eta^{(p\mathfrak{l})}, \omega) := f(X, \overline{\Lambda}, \eta^{(p)}, \omega)$. When f is a p–adic modular form of level Γ, we replace the ingredient ω by the ordinary level structure η_p^{ord} in order to define the value $f(X, \overline{\Lambda}, C, \eta^{(p\mathfrak{l})}, \eta_p^{ord})$.

We shall define Hecke operators $T(1, \mathfrak{l}^n)$ and $U(\mathfrak{l}^n)$ over (p–adic) modular forms of level Γ (with $\Gamma_{\mathfrak{l}} = \Gamma_0(\mathfrak{l}^r)$). The operator $U(\mathfrak{l}^n)$ is defined when $r > 0$, and $T(1, \mathfrak{l}^n)$ is defined when $r = 0$. Since $\mathfrak{l}$ is prime to p (and B is a $\mathcal{W}$–algebra), any cyclic subgroup C' of X of order $\mathfrak{l}^n$ is isomorphic to $O/\mathfrak{l}^n$ étale locally. We make the quotient $\pi : X \twoheadrightarrow X/C'$, and Λ, η_p^{ord} and ω induce canonically a polarization $\pi_*\Lambda$ a canonical level structure $\pi_*\eta_p^{ord} = \pi \circ \eta_p^{ord}$, $\pi_*\eta^{(p\mathfrak{l})} = \pi \circ \eta^{(p\mathfrak{l})}$ and a differential $(\pi^*)^{-1}\omega$ on X/C'. If $C' \cap C = \{0\}$ for the $\Gamma_0(\mathfrak{l}^r)$–structure C (in this case, we call that C' and C are disjoint), $\pi(C) = C + C'/C'$ gives rise to the level $\Gamma_0(\mathfrak{l}^r)$–structure on X/C'. We write $\underline{X/C'}$ for the new test object of the same level as the test object $\underline{X} = (X, \overline{\Lambda}, C, \eta_{ord}^{(\mathfrak{l})}, \omega)$ we started with. When f is p–adic, we suppose not to have ω in $\underline{X}$, and when f is classical, we ignore the ingredient η_p^{ord} in $\underline{X}$. Then we define (for $r > 0$)

$$f|U(\mathfrak{l}^n)(\underline{X}) = \frac{1}{N(\mathfrak{l}^n)} \sum_{C'} f(\underline{X/C'}), \tag{3.6}$$

where C' runs over all étale cyclic subgroups of order $\mathfrak{l}^n$ disjoint from C. We also define (for $r = 0$)

$$f|T(1, \mathfrak{l}^n)(\underline{X}) = \frac{1}{N(\mathfrak{l}^n)} \sum_{C} f(\underline{X/C}), \tag{3.7}$$

where C runs over all étale cyclic subgroups of order $\mathfrak{l}^n$ and $\underline{X}$ and $\underline{X/C}$ do not contain any datum of cyclic $\mathfrak{l}$–subgroups. The newly defined $f|\overline{U(\mathfrak{l}^n)}$ and $f|T(1, \mathfrak{l}^n)$ are modular forms of the same level as f and $U(\mathfrak{l}^n) = U(\mathfrak{l})^n$. Since the polarization ideal class of X/C' is given by $\mathfrak{c}\mathfrak{l}^n$ for the polarization ideal class $\mathfrak{c}$ of X, the operators $U(\mathfrak{l}^n)$ and $T(1, \mathfrak{l}^n)$ permute the components $f_{\mathfrak{c}}$.

We recall some other isogeny actions on modular forms. For ideals $\mathfrak{A}$ in F, we can think of the association $X \mapsto X \otimes_O \mathfrak{A}$ for each AVRM X. This operation will be made explicit in terms of the lattice $L = \pi_1(X)$ in $Lie(X)$ in (4.12). There are a natural polarization and a level structure on $X \otimes \mathfrak{A}$ induced by those of X (as specified later below (4.12)). Writing $(X, \Lambda, \eta) \otimes \mathfrak{A}$ for the triple made out of (X, Λ, η) after tensoring $\mathfrak{A}$, we define $f|\langle\mathfrak{A}\rangle(X, \Lambda, \eta) = f((X, \Lambda, \eta) \otimes \mathfrak{A})$ (see [PAF]

4.1.9 for more details of this definition). For $X(\mathcal{A})$, we have $\langle\mathfrak{A}\rangle(\underline{X}(\mathcal{A})) = \underline{X}(\mathfrak{A}\mathcal{A})$. The effect of the operator $\langle\mathfrak{A}\rangle$ on the Fourier expansion at $(\mathfrak{a},\mathfrak{b})$ is given by that at $(\mathfrak{A}^{-1}\mathfrak{a},\mathfrak{A}\mathfrak{b})$ (by [DR] 5.8; see also [PAF] 4.2.9).

Let $\mathfrak{q}$ be a prime ideal of F outside $p\mathfrak{l}$. For a test object $(X,\overline{\Lambda},C,\eta^{(\mathfrak{q})}_{ord},\omega)$ of level $\Gamma_0(\mathfrak{q})$, we can construct canonically its image under $\mathfrak{q}$–isogeny:

$$[\mathfrak{q}](X,\overline{\Lambda},C,\eta^{(p\mathfrak{q})}_{ord},\omega) = (X',\overline{\Lambda},\pi_*\eta^{(p\mathfrak{q})}_{ord},\overline{\eta}_{\mathfrak{q}},(\pi^*)^{-1}\omega)$$

for the projection $\pi : X \twoheadrightarrow X' = X/C$, where $\overline{\eta}_{\mathfrak{q}} = \eta_{\mathfrak{q}}\cdot GL_2(O_{\mathfrak{q}})$ for any level $\mathfrak{q}$–structure $\eta_{\mathfrak{q}}$ identifying $\mathcal{T}_{\mathfrak{q}}(X')$ with $O^2_{\mathfrak{q}}$. Then we have a linear operator $[\mathfrak{q}] : V(\Gamma;B) \to V(\Gamma_0(\mathfrak{q})\times\Gamma;B)$ given by $f|[\mathfrak{q}](\underline{X}) = f([\mathfrak{q}](\underline{X}))$. See (4.14) for the description of this operator in terms of the lattice of X.

If $\mathfrak{q}$ splits into $\mathfrak{Q}\overline{\mathfrak{Q}}$ in M/F, choosing $\eta_{\mathfrak{q}}$ induced by

$$X(\mathcal{A})[\mathfrak{q}^\infty] \cong M_{\mathfrak{Q}}/R_{\mathfrak{Q}} \times M_{\overline{\mathfrak{Q}}}/R_{\overline{\mathfrak{Q}}} \cong F_{\mathfrak{q}}/O_{\mathfrak{q}} \times F_{\mathfrak{q}}/O_{\mathfrak{q}},$$

we always have a canonical level $\mathfrak{q}$–structure on $X(\mathcal{A})$ dependent on the choice of the factor $\mathfrak{Q}$. Then $[\mathfrak{q}](X(\mathcal{A})) = X(\mathcal{A}[\mathfrak{Q}]^{-1})$ for $[\mathfrak{Q}] \in Cl_\infty$. When $\mathfrak{q}$ ramifies in M/F as $\mathfrak{q} = \mathfrak{Q}^2$, $X(\mathcal{A})$ has a subgroup $C = X(\mathcal{A})[\mathfrak{Q}_n]$ isomorphic to $O/\mathfrak{q}$ for $\mathfrak{Q}_n = \mathfrak{Q}\cap R_n$; so, we can still define $[\mathfrak{q}](X(\mathcal{A})) = X(\mathcal{A}\mathfrak{Q}_n^{-1}) = X(\mathcal{A}[\mathfrak{Q}]^{-1})$.

The effect on the q–expansion of the operator $[\mathfrak{q}]$ can be computed similarly to $\langle\mathfrak{A}\rangle$ (e.g. [DR] 5.8; see also [PAF] 4.2.9), and the q–expansion of $f|[\mathfrak{q}]$ at the cusp $(\mathfrak{a},\mathfrak{b})$ is given by the q–expansion of f at the cusp $(\mathfrak{q}\mathfrak{a},\mathfrak{b})$.

These operators $[\mathfrak{q}]$ and $\langle\mathfrak{A}\rangle$ change polarization ideals (as we will see later in 4.2); so, they permute components $f_{\mathfrak{c}}$. By the q–expansion principle ([K1] (1.2.15)), $f \mapsto f|[\mathfrak{q}]$ and $f \mapsto f|\langle\mathfrak{A}\rangle$ are injective.

3.4. Anti-cyclotomic measure.

Choose a Hecke character λ with infinity type $k+\kappa(1-c)$ of conductor 1. Then by (G3′), $f([\mathcal{A}]) = \lambda(\mathcal{A})^{-1}f(x(\mathcal{A}))$ for $\mathcal{A}$ prime to p depends only on the class of $\mathcal{A}$ in Cl_n if $n \geq m$ for the conductor $\mathfrak{l}^m$ of λ. For the p–adic avatar $\widehat{\lambda}(x) = \lambda(xR)x_p^{k+\kappa(1-c)}$, we also have $f([\mathcal{A}]) = \widehat{\lambda}(\mathcal{A})^{-1}f(x(\mathcal{A}))$. This new definition is valid even for $\mathcal{A}$ with non-trivial common factor with p. Hereafter we regard f as a function of $Cl^{(\infty)} = \bigsqcup_n Cl_n$ (embedded into $Sh^{(p)}_{/\mathcal{W}}$ or $Ig_{/\mathbb{F}}$ by $\mathcal{A} \mapsto x(\mathcal{A})$).

Suppose that f is defined over a $\mathcal{W}$–algebra A in which ℓ is invertible. By the result in 3.1 combined with (3.6), we have, for an integer $n > m$,

$$\sum_{\mathcal{A}\in Cl_n,\ \mathcal{A}\mapsto\mathfrak{l}^{m-n}\mathfrak{A}\in Cl_m} f([\mathcal{A}]) = (\lambda N(\mathfrak{l}))^{n-m} f|U(\mathfrak{l}^{n-m})([\mathfrak{A}]), \tag{3.8}$$

where $\mathcal{A}$ runs over all elements in Cl_n which project down to $\mathfrak{l}^{m-n}\mathfrak{A} \in Cl_m$. Let h be the strict class number of F. If $h|m-n$, the sum on the left-hand-side of (3.8) is over $\mathcal{A} \in Cl_n$ projecting down to $\mathfrak{A} \in Cl_m$, because $[\mathfrak{l}^{m-n}\mathfrak{A}] = [\mathfrak{A}]$ in Cl_m.

We suppose that $f|U(\mathfrak{l}) = (a/\lambda N(\mathfrak{l}))f$ with a unit $a \in A$. For each function $\phi : Cl_\infty \to A$ factoring through Cl_{mh}, we define

$$\int_{Cl_\infty} \phi d\varphi_f = a^{-mh} \sum_{\mathcal{A}\in Cl_{mh}} \phi(\mathcal{A}^{-1}) f([\mathcal{A}]). \tag{3.9}$$

Then for $n > m$, we find

$$\begin{aligned} a^{-nh} \sum_{\mathcal{A}\in Cl_{nh}} \phi(\mathcal{A}^{-1}) f([\mathcal{A}]) &= a^{-mh} \sum_{\mathfrak{A}\in Cl_{mh}} \phi(\mathfrak{A}^{-1}) a^{(m-n)h} \sum_{\mathcal{A}\in Cl_{nh},\ \mathcal{A}\mapsto\mathfrak{A}} f([\mathcal{A}]) \\ &\overset{(3.8)}{=} a^{-mh} \sum_{\mathfrak{A}\in Cl_{mh}} \phi(\mathfrak{A}^{-1}) a^{(m-n)h} f|U((\pi)^{(n-m)h})([\mathfrak{A}]) = \int_{Cl_\infty} \phi(x) d\varphi_f(x). \end{aligned}$$

Thus φ_f gives an A–valued distribution on Cl_∞ well defined independently of the choice of m for which ϕ factors through Cl_m, because $U(\mathfrak{l}^\alpha) = U(\mathfrak{l})^\alpha$.

Classical modular forms are actually defined over a number field; so, we assume that f is defined over the localization $\mathcal{V}$ of the integer ring in a number field E. We assume that E contains M' for the reflex (M', Σ') of (M, Σ). We write $\mathcal{P}|p$ for the prime ideal of the p–integral closure $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ in $\overline{\mathbb{Q}}$ corresponding to $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. More generally, if $f = d^\kappa g$ for a classical modular form g integral over $\mathcal{V}$, the value $f([\mathcal{A}])$ is algebraic, abelian over M' and $\mathcal{P}$–integral over $\mathcal{V}$ by a result of Shimura and Katz (see [Sh2] and [K1]).

Let $f = d^\kappa g$ for $g \in G_k(\Gamma_0(\mathfrak{l}); \mathcal{V})$. Suppose $f|U(\mathfrak{l}) = (a/N(\mathfrak{l}))f$ for a giving a unit of $\widetilde{\mathcal{V}}/\mathcal{P}$. For the moment, let φ be the measure associated to f with values in $A = \widetilde{\mathcal{V}}$. We have a well defined measure $\varphi \mod \mathcal{P}$. Let E_f be the field generated by $f([\mathcal{A}])$ over $E[\mu_{\ell^\infty}]$. Then E_f/E is an abelian extension unramified outside ℓ, and we have the Frobenius element $\sigma_\mathfrak{b} \in \mathrm{Gal}(E_f/E)$ (that is, the image of $\mathfrak{b}$ under the Artin reciprocity map) for each ideal $\mathfrak{b}$ of E prime to ℓ. By Shimura's reciprocity law ([ACM] 26.8), we find for $\sigma = \sigma_\mathfrak{b}$, $x(\mathcal{A})^\sigma = x(N(\mathfrak{b})^{-\Sigma'}\mathcal{A})$ for the norm $N : E \to M'$. As for $\eta_p^{ord}(\mathcal{A})$, we find $\sigma \circ \eta_p^{ord}(\mathcal{A}) = u\eta_p^{ord}$ for $u \in R_{\Sigma_p}^\times$. Since $\mathcal{A}_p \cong R_p$, we have $X(R)[p^\infty] \cong X(\mathcal{A})[p^\infty]$ as a Galois module. Thus we conclude $u = \psi(\mathfrak{b})$ for the Hecke character ψ of $E_\mathbb{A}^\times/E^\times$ giving rise to the zeta function of $X(R)$. From this, if we extend E further if necessary, we see $f([\mathcal{A}])^\sigma = f([N(\mathfrak{b})^{-\Sigma'}\mathcal{A}])$ for any ideal $\mathfrak{b}$, since $\psi(\mathfrak{b}) \in M$ generates the ideal $N(\mathfrak{b})^{\Sigma'} \subset M$ ([ACM] Sections 13 and 19) and hence $\psi(\mathfrak{b})^{k\Sigma+\kappa(1-c)} = \lambda(N(\mathfrak{b})^{\Sigma'})$. We then have

$$\sigma \cdot \left(\int_{Cl_\infty} \phi(x) d\varphi_f(x)\right) = \int_{Cl_\infty} \sigma \circ \phi(N(\mathfrak{b})^{\Sigma'} x) d\varphi_f(x), \tag{3.10}$$

where $N(\mathfrak{b})$ is the norm of $\mathfrak{b}$ over M'. Writing $\mathbb{F}_{p^r}$ for $\mathcal{V}/\mathcal{P}\cap\mathcal{V}$, any modular form defined over $\mathbb{F}_{p^r}$ is a reduction modulo $\mathcal{P}$ of a classical modular form defined over $\mathcal{V}$ of sufficiently high weight. Thus the above identity is valid for $\sigma = \Phi^s$ ($s \in \mathbb{Z}$) for the Frobenius element $\Phi \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^r})$. In this case, $N(\mathfrak{b})$ is a power of a prime ideal $\mathfrak{p}|p$ in M'.

We now assume that $A = \mathbb{F} = \widetilde{\mathcal{V}}/\mathcal{P}$ and regard the measure φ_f as having values in $\mathbb{F}$. Then (3.10) shows that if ϕ is a character χ of Cl_∞, for $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^r})$,

$$\int_{Cl_\infty} \chi(x) d\varphi_f(x) = 0 \iff \int_{Cl_\infty} \sigma \circ \chi(x) d\varphi_f(x) = 0. \tag{3.11}$$

Decompose Cl_∞ into a product of the maximal torsion-free ℓ–profinite subgroup Γ_f and a finite group Δ. Let $\mathbb{F}[f,\lambda]$ be the finite subfield of $\mathbb{F}$ generated by all $\ell|\Delta|$–th roots of unity over the field $\mathbb{F}_{p^r}$ of rationality of f and λ. For any finite extension $\kappa/\mathbb{F}[f,\lambda]$, we consider the trace map: $\mathrm{Tr}_{\kappa/\mathbb{F}[f,\lambda]}(\xi) = \sum_{\sigma \in \mathrm{Gal}(\kappa/\mathbb{F}[f,\lambda])} \sigma(\xi)$ for $\xi \in \kappa$. If $\chi : Cl_n \to \mathbb{F}^\times$ is a character, we find, for $d = [\mathrm{Im}(\chi) : \mathrm{Im}(\chi) \cap \mathbb{F}[f,\lambda]^\times]$,

$$\int_{Cl_\infty} \mathrm{Tr}_{\mathbb{F}[f,\lambda](\chi)/\mathbb{F}[f,\lambda]} \circ \chi(y^{-1}x) d\varphi_f(x) = \frac{d}{a^n} \sum_{\mathcal{A} \in Cl_n : \chi(\mathcal{A}y^{-1}) \in \mathbb{F}[f,\lambda]} \chi(y^{-1}\mathcal{A}) f([\mathcal{A}]), \tag{3.12}$$

because for an ℓ–power root of unity $\zeta \in \mu_{\ell^n} - \mu_\ell$,

$$\mathrm{Tr}_{\mathbb{F}[f,\lambda](\zeta)/\mathbb{F}[f,\lambda]}(\zeta^s) = \begin{cases} \ell^{n-m}\zeta^s & \text{if } \zeta^s \in \mathbb{F}[f,\lambda] \text{ and } \mathbb{F}[f,\lambda] \cap \mu_{\ell^\infty} = \mu_{\ell^m} \\ 0 & \text{otherwise.} \end{cases}$$

For the moment, suppose that $\Gamma_f \cong \mathbb{Z}_\ell$. Suppose also $\int_{Cl_\infty} \chi(x) d\varphi_f(x) = 0$ for an infinite set Ξ of characters χ. For sufficiently large m, we always find a character $\chi \in \Xi$ such that $\mathrm{Ker}(\chi) \subset \Gamma_f^{\ell^m}$. Then writing $\mathrm{Ker}(\chi) = \Gamma_f^{\ell^n}$ for $n \geq m$, we have the vanishing from (3.11)

$$\int_{Cl_\infty} \sigma \circ \chi d\varphi_f = 0 \;\text{ for all } \sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}[f,\lambda]).$$

This combined with (3.12), we find $\sum_{y \in \chi^{-1}(\mathbb{F}[f,\lambda]^\times)} \chi(y\mathcal{A}) f(y[\mathcal{A}])) = 0$ for all $\mathcal{A} \in \Gamma_n$, where Γ_n is the image of Γ_f in Cl_n.

Since each fractional R–ideal $\mathfrak{A}$ prime to $\mathfrak{l}$ defines a class $[\mathfrak{A}]$ in Cl_∞, we can embed the ideal group of fractional ideals prime to $\mathfrak{l}$ into Cl_∞. We write C^{alg} for its image. Thus the projection of $[\mathfrak{Q}]$ in Cl_n is $[\mathfrak{Q}_n]$ as specified for the integral ideal $\mathfrak{Q}$ above. Then $\Delta^{alg} = \Delta \cap C^{alg}$ is generated by prime ideals of M non-split over F. We choose a complete representative set for Δ^{alg} made of product of prime ideals in M non-split over F prime to $p\mathfrak{l}$. We may choose this set as $\{\mathfrak{s}\mathfrak{R}^{-1} | \mathfrak{s} \in \mathcal{S},\ \mathfrak{r} \in \mathcal{R}\}$, where $\mathcal{S}$ is made of prime ideals of F, $\mathcal{R}$ is made of square-free product of primes outside $\mathfrak{l}$ in F ramifying in M/F, and $\mathfrak{R}$ is a unique ideal in M with $\mathfrak{R}^2 = \mathfrak{r}$. The set $\mathcal{S}$ is a complete representative set for the image Cl_F^0 of the class group of F in Cl_0 and $\{\mathfrak{R} | \mathfrak{r} \in \mathcal{R}\}$ is a complete representative set for 2–torsion elements in the quotient Cl_0/Cl_F^0. We fix a character $\psi : \Delta \to \mathbb{F}^\times$, and define

$$f_\psi = \sum_{\mathfrak{r} \in \mathcal{R}} \lambda\psi^{-1}(\mathfrak{R}) \left(\sum_{\mathfrak{s} \in \mathcal{S}} \psi\lambda^{-1}(\mathfrak{s}) f | \langle \mathfrak{s} \rangle \right) | [\mathfrak{r}]. \tag{3.13}$$

Choose a complete representative set $\mathcal{Q}$ for $Cl_\infty/\Gamma_f\Delta^{alg}$ made of primes $\mathfrak{Q}$ of M split over F outside $p\mathfrak{l}$. We choose $\eta_n^{(p)}$ out of the base (w_1, w_2) of $\widehat{R}_n$ so that at $\mathfrak{q} = \mathfrak{Q} \cap F$, $w_{1,\mathfrak{q}} = (1,0) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$ and $w_{2,\mathfrak{q}} = (0,1) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$. Since all operators $\langle\mathfrak{s}\rangle$, $[\mathfrak{q}]$ and $[\mathfrak{r}]$ involved in this definition commutes with $U(\mathfrak{l})$, $f_\psi|[\mathfrak{q}]$ is still an eigenform of $U(\mathfrak{l})$ with the same eigenvalue as f. Thus in particular, we have a measure $\varphi_{f_\psi|[\mathfrak{q}]}$. We then define another measure φ_f^ψ on Γ_f by

$$\int_{\Gamma_f} \phi d\varphi_f^\psi = \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) \int_{\Gamma_f} \phi|\mathfrak{Q} d\varphi_{f_\psi|[\mathfrak{q}]}, \tag{3.14}$$

where $\phi|\mathfrak{Q}(y) = \phi(y[\mathfrak{Q}]_f^{-1})$ for the projection $[\mathfrak{Q}]_f$ in Γ_f of the class $[\mathfrak{Q}] \in Cl_\infty$.

Lemma 3.1. *If $\chi : Cl_\infty \to \mathbb{F}^\times$ is a character inducing ψ on Δ, we have*

$$\int_{\Gamma_f} \chi d\varphi_f^\psi = \int_{Cl_\infty} \chi d\varphi_f.$$

Proof. Write $\Gamma_{f,n}$ for the image of Γ_f in Cl_n. For a proper R_n–ideal $\mathcal{A}$, by the above definition of these operators,

$$f|\langle\mathfrak{s}\rangle|[\mathfrak{r}]|[\mathfrak{q}]([\mathcal{A}]) = \lambda(\mathcal{A})^{-1} f(x(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s}\mathcal{A})).$$

For sufficiently large n, χ factors through Cl_n. Since $\chi = \psi$ on Δ, we have

$$\begin{aligned}\int_{\Gamma_f} \chi d\varphi_f^\psi &= \sum_{\mathfrak{Q}\in\mathcal{Q}} \sum_{\mathfrak{s}\in\mathcal{S}} \sum_{\mathfrak{r}\in\mathcal{R}} \sum_{\mathcal{A}\in\Gamma_{f,n}} \lambda\chi^{-1}(\mathfrak{Q}\mathfrak{R}\mathfrak{s}^{-1}\mathcal{A}) f|\langle\mathfrak{s}\rangle|[\mathfrak{r}]|[\mathfrak{q}]([\mathcal{A}]) \\ &= \sum_{\mathcal{A},\mathfrak{Q},\mathfrak{s},\mathfrak{r}} \chi(\mathfrak{Q}\mathfrak{R}\mathfrak{s}^{-1}\mathcal{A}) f([\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s}\mathcal{A}]) = \int_{Cl_\infty} \chi d\varphi_f,\end{aligned}$$

because $Cl_\infty = \bigsqcup_{\mathfrak{Q},\mathfrak{s},\mathfrak{R}} [\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s}]\Gamma_f$. □

We write $\mathbb{F}_p[f]$ for the minimal field of definition of $f \in V(\mathbb{F})$. Similarly $\mathbb{F}_p[\lambda]$ (resp. $\mathbb{F}_p[\psi]$) is the subfield of $\mathbb{F}$ generated by the values $\lambda([\mathcal{A}]) \mod \mathcal{P}$ (resp. $\psi([\mathcal{A}])$) for all $[\mathcal{A}] \in C^{alg}$. Define $\mathbb{F}_p[f,\lambda,\psi]$ by the composite of these fields. Note that $\mathbb{F}_p[f,\lambda,\psi]$ is a finite extension of $\mathbb{F}_p$ if $f = d^\kappa g$ for a classical modular form $g \in G_k(\Gamma;\mathbb{F})$ of finite level. For general $h \in V(\mathbb{F})$, we write the q–expansion at the cusp $(O, \mathfrak{c}^{-1})$ of the $\mathfrak{c}$–component $h_\mathfrak{c}$ as $\sum_{\xi\in\mathfrak{c}^{-1}} a(\xi, h_\mathfrak{c}) q^\xi$.

Theorem 3.2. *Let $f \in V(\mathbb{F})$ of level $\Gamma_0(\mathfrak{l})$ at $\mathfrak{l}$ be an eigenform for $U(\mathfrak{l})$ with non-zero eigenvalue $a \in \mathbb{F}$. Suppose that $f = d^\kappa g$ for a classical modular form $g \in G_k(\mathbb{F})$ rational over a finite field $\mathbb{F}_p[f]$. We fix a character $\psi : \Delta \to \mathbb{F}^\times$. Suppose the following conditions in addition to* (unr) *and* (ord)*:*

(H) *Write the order of the Sylow ℓ–subgroup of $\mathbb{F}_p[f,\psi,\lambda]^\times$ as $\ell^{r(\psi)}$. Then there exists a strict ideal class $\mathfrak{c} \in Cl_F$ such that the polarization ideal of $x(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s})$ is in $\mathfrak{c}$ for some $(\mathfrak{Q},\mathfrak{R},\mathfrak{s}) \in \mathcal{Q}\times\mathcal{S}\times\mathcal{R}$ and for every $u \in O$ prime to $\mathfrak{l}$, we can find $\xi \equiv u \mod \mathfrak{l}^{r(\psi)}$ with $a(\xi, f_{\psi,\mathfrak{c}}) \neq 0$.*

Suppose further that the torsion-free part Γ_f of Cl_∞ has $\mathbb{Z}_\ell$ rank 1. Then the integral $\int_{Cl_\infty}\chi d\varphi_f$ vanishes for only finitely many characters χ with $\chi|_\Delta = \psi$.

Proof. By definition, the projection $\{[\mathfrak{Q}]_f\}_{\mathfrak{Q}\in\mathcal{Q}}$ of $[\mathfrak{Q}]$ in Γ_f are all distinct in Cl_∞/C^{alg}. By Lemma 3.1, we need to prove that the integral $\int_{\Gamma_f}\chi d\varphi_f^\psi$ vanishes only for finitely many characters χ of Γ_f. Suppose towards contradiction that the integral vanishes for characters χ in an infinite set $\mathcal{X}$.

Let $\Gamma(n) = \Gamma_f^{\ell^{n-r}}/\Gamma_f^{\ell^n}$ for $r = r(\psi)$. By applying (3.12) to a character in $\mathcal{X}$ with $\mathrm{Ker}(\chi) = \Gamma_f^{\ell^n}$, we find

$$\sum_{\mathfrak{Q}\in\mathcal{Q}}\psi(\mathfrak{Q})^{-1}\sum_{\mathcal{A}\in y\chi^{-1}(\mu_{\ell^r})}\chi(\mathcal{A})f_\psi([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_f) = 0. \tag{3.15}$$

Note here $f_\psi([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_f) = f_{\psi,\mathfrak{c}}([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_f)$ for the strict class $\mathfrak{c}$ of $\mathfrak{c}(\mathcal{A}\mathfrak{Q}^{-1})$.

Fix $\mathfrak{Q}\in\mathcal{Q}$. By (3.2), $\{x(\mathcal{A})|[\mathcal{A}]\in y\chi^{-1}(\mu_{\ell^r})\}$ is given by $\alpha(\frac{u}{\varpi_{\mathfrak{l}}^r})(x(\mathcal{A}_0))$ for any member $\mathcal{A}_0 \in y\chi^{-1}(\mu_{p^r})$, where $\alpha(t) = \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$. Actually $\mathcal{A}\mapsto u \mod \mathfrak{l}^r$ gives a bijection of $y\chi^{-1}(\mu_{\ell^r})$ onto $O/\mathfrak{l}^r$. We write the element $\mathcal{A}$ corresponding to u as $\alpha(\frac{u}{\varpi_{\mathfrak{l}}^r})\mathcal{A}_0$. This shows, choosing a primitive ℓ^r–th root of unity $\zeta = \exp(2\pi i/\ell^r)$ and $\mathcal{A}_y \in y\chi^{-1}(\mu_{\ell^r})$ so that $\chi(\alpha(\frac{u}{\varpi_{\mathfrak{l}}^r})\mathcal{A}_y) = \zeta^{uv}$ for an integer $0 < v < \ell^r$ prime to ℓ (independent of y), the inner sum of (3.15) is equal to, for $\mathfrak{c}$ in (H),

$$\sum_{u \mod \mathfrak{l}^r}\zeta^{uv}(f_{\psi,\mathfrak{c}}|\alpha(\frac{u}{\varpi_{\mathfrak{l}}^r}))([\mathcal{A}_y\mathfrak{Q}^{-1}][\mathfrak{Q}]_f).$$

The choice of v depends on χ. Since $\mathcal{X}$ is infinite, we can choose an infinite subset $\mathcal{X}'$ of $\mathcal{X}$ for which v is independent of the element in $\mathcal{X}'$. Then write n_j for the integers given by $\Gamma_f^{\ell^{n_j}} = \mathrm{Ker}(\chi)$ for $\chi\in\mathcal{X}'$ (in increasing order), and define Ξ to be the set of points $x(\mathcal{A})$ for $\mathcal{A}\in Cl_{n_j}$ with $[\mathcal{A}R_{n_1}] = [R_{n_1}]$ in Cl_{n_1}. Define also $g_{\mathfrak{Q}} = \sum_{u \mod \mathfrak{l}^r}\zeta^{uv}f_{\psi,\mathfrak{c}}|\alpha(\frac{u}{\varpi_{\mathfrak{l}}^r})$, because (3.15) is now the sum:

$$\sum_{\mathfrak{Q}\in\mathcal{Q}}\psi(\mathfrak{Q})^{-1}g_{\mathfrak{Q}}|[\mathfrak{q}]([\mathcal{A}][\mathfrak{Q}]_f) = 0,$$

where $\mathfrak{q} = \mathfrak{Q}\cap F$. By Corollary 2.9, this implies that $g_{\mathfrak{Q}} = 0$.

The q–expansion coefficient $a(\xi, g_{\mathfrak{Q}})$ of $g_{\mathfrak{Q}}$ is given by $\ell^r a(\xi, f_{\psi,\mathfrak{c}})$ if $\xi \equiv -v \mod \mathfrak{l}^r$ and vanishes otherwise. This contradicts to the assumption (H). □

We now treat the general case. A naive question when $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma_f > 1$ is:

Question: *Let $f \neq 0$ be an eigenform of $U(\mathfrak{l})$ which is rational over a finite extension $\mathbb{F}_{p^r}$ of $\mathbb{F}_{\mathfrak{p}}$). Suppose (unr) and (ord). For each $\mathbb{Z}_\ell$–rank 1 quotient $\pi:\Gamma_f \twoheadrightarrow \mathbb{Z}_\ell$, does $\int_{\Gamma_f}\chi\circ\pi d\varphi_f$ vanish only for finitely many characters of $\mathbb{Z}_\ell$?*

Under the condition (H), we are tempted to believe that the above question is affirmative. However our result in the general case is weaker than this expectation.

Since $\mathrm{Hom}(\Gamma_f, \mathbb{F}^\times) \cong \mathrm{Hom}(\Gamma_f, \mu_{\ell^\infty})$, we may regard $\mathrm{Hom}(\Gamma_f, \mu_{\ell^\infty})$ as a subset of $\mathbb{G}_m(\overline{\mathbb{Q}})^d$. We call a subset $\mathcal{X}$ of characters of Γ_f *Zariski-dense* if it is Zariski-dense as a subset of the algebraic group $\mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$. What we can prove is

Theorem 3.3. *Suppose* (unr) *and* (ord) *for* p. *Let the notation be as in Theorem 3.2. In addition to the finiteness of* $\mathbb{F}_p[f]$, *if* f_ψ *as in* (3.13) *satisfies the following condition:*

(h) *There exists a strict ideal class* $\mathfrak{c}$ *of* F *such that* $\mathfrak{c}(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s})$ *is in* $\mathfrak{c}$ *for some* $(\mathfrak{Q}, \mathfrak{R}, \mathfrak{s}) \in \mathcal{Q} \times \mathcal{S} \times \mathcal{R}$ *and for any given integer* $r > 0$, *the* $N(\mathfrak{l})^r$ *modular forms* $f_{\psi,\mathfrak{c}}|\left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right)$ *for* $u \in \mathfrak{l}^{-r}/O$ *are linearly independent,*

then the set of characters $\chi : \Gamma_f \to \mathbb{F}^\times$ *with non-vanishing* $\int_{Cl_\infty} \psi\chi d\varphi_f \neq 0$ *is Zariski dense. Here* $\psi\chi$ *is the character of* $Cl_\infty = \Gamma_f \times \Delta$ *given by* $\psi\chi(\gamma, \delta) = \psi(\delta)\chi(\gamma)$ *for* $\gamma \in \Gamma_f$ *and* $\delta \in \Delta$.

Proof. By the same argument in the proof of Theorem 3.2, we create f_ψ and work with the measure φ_f^ψ on Γ_f. Identify $\mathrm{Hom}(\Gamma_f, \mu_{\ell^\infty})$ with $\mu_{\ell^\infty}^d \subset \mathbb{G}_m^d$ by sending a character to its value at d independent generators of Γ_f. Write $\varphi = \varphi_f^\psi$. Suppose that the Zariski closure X in $\mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$ of the set of all characters χ with non-vanishing integral $\int_{\Gamma_f} \chi d\varphi \neq 0$ for characters χ is a proper subset of $\mathbb{G}_m^d$. Since the case $d = 1$ is already proven, we may assume that $d \geq 2$.

For a sufficiently large p–power P, $\mathbb{F}_p[f_\psi, \lambda] = \mathbb{F}_P$. Since $\chi^P = \sigma \circ \chi \in X \Leftrightarrow \chi \in X$ for the Frobenius automorphism $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_P)$, we find $X^P \subset X$. Thus X is stable under any power of the P–th power homomorphism: $t \mapsto t^P$ of $\mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$.

Let $t \in X$ be a point of infinite order. We consider the ℓ–adic logarithm map $\log : \widehat{\mathbb{G}}^d_{m/\mathbb{Q}_\ell} \to \widehat{\mathbb{G}}^d_{a/\mathbb{Q}_\ell}$. The subvariety X contains the Zariski closure of the subset $\{t^{P^n}\}$, whose image under log is the one-parameter subgroup generated by $\log(t)$. Thus X is contained in a proper closed subset which is a finite union of translations by elements in $\mu^d_{\ell^\infty}$ of proper irreducible closed subschemes stable under scalar multiplication.

By the lemma following this theorem, we can find an infinite sequence of d–tuple of characters $\Xi = \{(\chi_{1,j}, \cdots, \chi_{d,j})\}_{j=1,2,\ldots}$ such that $K_j = \bigcap_{i=1}^d \mathrm{Ker}(\chi_{i,j})$ gives rise to a system of open neighborhoods of the identity of Γ_f, and

$$\int_{\Gamma_f} \prod_{i=1}^d \chi_{i,j}^{P^{k_i}} d\varphi_f^\psi = 0$$

for all $(k_1, \ldots, k_d) \in \mathbb{Z}^d$. We define for each fixed j a finite subset of Ξ by

$$\Xi_j = \{(\chi_{1,j}^{P^{k_1}}, \ldots, \chi_{d,j}^{P^{k_d}}) | (k_1, \ldots, k_d) \in \mathbb{Z}^d\}.$$

From this, we find

$$\begin{aligned} 0 &= \sum_{\mathfrak{Q} \in \mathcal{Q}} \psi(\mathfrak{Q})^{-1} \sum_{(\chi_1, \ldots, \chi_d) \in \Xi_j} \int_{\Gamma_f} \chi_1\chi_2 \cdots \chi_d d\varphi_{f_\psi} \\ &= \sum_{\mathfrak{Q} \in \mathcal{Q}} \psi(\mathfrak{Q})^{-1} \sum_{\mathcal{A}} \prod_i \left(\sum_{k_i=1}^{r_i} \chi_{i,j}^{P^{k_i}}(\mathcal{A}) \right) f_\psi([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_f), \end{aligned}$$

where $\{\chi_{i,j}^{P^{k_i}} | k_i = 1, \cdots, r_i\}$ is the set of all distinct conjugates of $\chi_{i,j}$ under the power of the P–power Frobenius element, and $\mathcal{A}$ runs over all classes in $\Gamma_f/\Gamma_f^{P^N}$ for N sufficiently large depending on j. For an ℓ–power root of unity $\zeta \in \mu_{\ell^n} - \mu_\ell$, we write $r = |\{\zeta^{P^j} | j \in \mathbb{Z}\}|$. Again we use the following fact: we have $\sum_{k=1}^{r} \zeta^{sP^k} = 0$ unless $\zeta^s \in \mu_\ell$. From this, we have $\sum_{k_i=1}^{r_i} \chi_{i,j}^{P^{k_i}}(\mathcal{A}) = 0$ unless $\mathcal{A} \in \chi_{i,j}^{-1}(\mathbb{F}_P)$. Writing $\mu_{\ell^r} = \mu_{\ell^\infty} \cap \mathbb{F}_P$, we then find for any $y \in \Gamma_f/\Gamma_f^{P^N}$ and $j = 1, 2, \ldots$

$$\sum_{\mathfrak{Q}} \psi(\mathfrak{Q})^{-1} \sum_{\mathcal{A} \in y\widetilde{\chi}_j^{-1}(\mu_{\ell^r}^d)} \chi_{1,j}(\mathcal{A})\chi_{2,j}(\mathcal{A}) \cdots \chi_{d,j}(\mathcal{A}) f_{\psi,\mathfrak{c}}([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_f) = 0.$$

where $\widetilde{\chi}_j(x) = (\chi_{1,j}(x), \ldots, \chi_{d,j}(x)) \in \mathbb{F}^d$. From this by the same argument which proves the previous theorem, we deduce a contradiction against (h), which shows that X is equal to $\mathbb{G}_m^d$ as desired. $\square$

Lemma 3.4. *Let p and ℓ be distinct primes and $r > 0$ be an integer. Let $X \subset \mathbb{G}^d_{m/\overline{\mathbb{Q}}_\ell}$ for $d \geq 2$ be a proper Zariski closed subset which is a finite union of translations by $\zeta \in \mu_{\ell^\infty}^d$ of closed subschemes stable under $t \mapsto t^{p^{rn}}$ for all $n \in \mathbb{Z}$. Identify $\mu_{\ell^\infty}^d$ with $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^d$ as ℓ–divisible groups. Then we can find a sufficiently large p^r–power P and an infinite sequence $\underline{n} = \{n_1 < n_2 < \cdots\}$ of positive integers such that there exists a sequence of subsets Ξ_j outside X such that*

$$\Xi_j = \left\{ \left(\frac{P^{k_1}e_1}{\ell^{n_j}}, \ldots, \frac{P^{k_d}e_n}{\ell^{n_j}}\right) \mod \mathbb{Z}_\ell^d \middle| (k_j) \in \mathbb{Z}^d \right\}$$

if we choose a base $\{e_j\}$ of $\mathbb{Z}_\ell^d$ suitably.

Proof. We choose a p^r–power P so that $P \equiv 1 \mod \ell$. Let $\Gamma_P = P^{\mathbb{Z}_\ell}$, which is an open neighborhood of 1 in $1 + \ell\mathbb{Z}_\ell$.

Let $V = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^d$, and write $V[\ell^n]$ for the kernel inside V of the multiplication by ℓ^n. By the argument using the ℓ–adic logarithm in the theorem, we may assume that $X = \bigcup_{\zeta \in V[\ell^N]} (\zeta + C_\zeta)$ for finitely many proper irreducible closed subschemes C_ζ. We first assume that $N = 0$. By this assumption, X is stable under $t \mapsto t^s$ for $s \in \mathbb{Z}$; so, it is something like a projective cone centered at the origin. In other words, X is a union of one parameter subgroups of $\mathbb{G}_m$. In particular, if we put $X_n = V[\ell^n] \cap X$, the scalar multiplication leaves stable X (because raising the power on μ_{ℓ^∞} is scalar multiplication on V), and hence multiplication by $\ell^{m-n} : X_m \to X_n$ for $m > n$ is surjective and induces a projective system $\{X_n\}_n$. We consider $\widehat{X} = \varprojlim_n X_n$, which is a projective cone in $\widehat{V} = \varprojlim_n V[\ell^n]$. Then by definition, X_n is the image of $\widehat{X}$ in $V[\ell^n]$. In other words, the image of the cone $C(X) = \{x \in \widehat{V} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell | tx \in \widehat{X}\}$ in $\widehat{V} \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell) = V$ is the subset X. Since $\widehat{X}$ is a proper closed analytic subset in a compact space $\widehat{V}$, we can find a base $e_1, \ldots, e_n$ of $\widehat{V}$ over $\mathbb{Z}_\ell$ outside X so that $\mathbb{Q}_\ell e \cap C(X) = \{0\}$ for $e = e_1 + e_2 + \cdots + e_d$. Then the ℓ–adic distance from $C(X)$ to the point $\frac{e}{\ell^n}$ is larger than or equal to $c\ell^n$ for a positive constant c independent of n. Thus we can find sufficiently large power P of p^r (ℓ–adically very close to 1) so that $\widehat{\Xi}_n = \Gamma_P \frac{e_1}{\ell^n} + \cdots + \Gamma_P \frac{e_d}{\ell^n}$ gives rise to an open neighborhood of $\frac{e}{\ell^n}$ disjoint from $C(X)$. Then the image Ξ_n of $\widehat{\Xi}_n$ in V satisfies the desired property.

When $N > 0$, we consider the projection $\pi : V \to V/V[\ell^N]$. The image of X under π is stable under scalar multiplication. Using the result proven under the condition $N = 0$ applied to $V/V[\ell^N]$, we write Ξ'_n for the sets constructed for $V/V[\ell^N]$. Then we find that for $n > N$ any Γ_P^d–orbit of an element in the pull-back image $\pi^{-1}(\Xi_n)$ gives a desired set $\Xi_n \subset V$. This finishes the proof. $\square$

4. Hecke L–values

We compute integrals of characters for the measure associated to Eisenstein series in terms of L–values. Then Theorem 1.1 follows from Theorem 3.3.

4.1. Anti-cyclotomic L–functions.

We determine the type of Hecke L–function obtained by values of Eisenstein series at CM points. A result equivalent to the one presented here has also been obtained independently by H. Yoshida [LAP] V.3.2; so, our exposition will be brief. We leave the proof of the following two propositions to the reader.

Proposition 4.1. *Recall $R_n = O + \mathfrak{l}^n R$. Then we have the identity:*

$$\{\text{non-proper } R_{n+1}\text{–ideals}\} = \{\mathfrak{l}\mathcal{A} | \mathcal{A} \text{ is an } R_n\text{–ideal}\}.$$

Proposition 4.2. *Let I_n be the group of all proper fractional R_n–ideals. Associating to each R_{n+1}–ideal $\mathcal{A}$ the R_n–ideal $R_n\mathcal{A}$, we get the following homomorphism of groups $\pi_n : I_{n+1} \to I_n$. The homomorphism π is surjective, and the kernel of π is isomorphic to $R^\times_{n,\mathfrak{l}}/R^\times_{n+1,\mathfrak{l}}$. After taking the quotient by principal ideals, we still have the following exact sequence:*

$$1 \to R^\times_{n,\mathfrak{l}}/R^\times_{n+1,\mathfrak{l}}R^\times_n \to Cl_{n+1} \to Cl_n \to 1.$$

Writing a prime element of $\mathfrak{L}$ and $\mathfrak{l}$ as ϖ and ϖ_F, respectively, we find that

$$R_{n+1,\mathfrak{l}}/R^\times_{n+1,\mathfrak{l}} = \{1\} \sqcup \varpi_F R_{n,\mathfrak{l}}/R^\times_{n+1,\mathfrak{l}}. \tag{4.1}$$

When $\mathfrak{l}$ splits into a product of two primes $\mathfrak{L}\overline{\mathfrak{L}}$ in M, we have

$$R_{n,\mathfrak{l}}/R^\times_{n,\mathfrak{l}} \cong \left(\sqcup_{j=0}^{n-1}(\varpi\overline{\varpi})^j R^\times_{n-j,\mathfrak{l}}/R^\times_{n,\mathfrak{l}}\right) \bigsqcup \left(\sqcup_{i,j=0}^{\infty}\varpi^{n+i}\overline{\varpi}^{n+j} R^\times_{\mathfrak{l}}/R^\times_{n,\mathfrak{l}}\right). \tag{4.2}$$

When $\mathfrak{l}$ ramifies in M/F,

$$R_{n,\mathfrak{l}}/R^\times_{n,\mathfrak{l}} \cong \left(\sqcup_{j=0}^{n-1}\varpi^{2j} R^\times_{n-j,\mathfrak{l}}/R^\times_{n,\mathfrak{l}}\right) \bigsqcup \left(\sqcup_{j=0}^{\infty}\varpi^{j} R^\times_{\mathfrak{l}}/R^\times_{n,\mathfrak{l}}\right). \tag{4.3}$$

When $\mathfrak{l}$ is inert in M/F, we have

$$R_{n,\mathfrak{l}}/R_{n,\mathfrak{l}}^{\times} \cong \left(\sqcup_{j=0}^{n-1}\varpi^j R_{n-j,\mathfrak{l}}^{\times}/R_{n,\mathfrak{l}}^{\times}\right)\bigsqcup\left(\sqcup_{j=0}^{\infty}\varpi^j R_{\mathfrak{l}}^{\times}/R_{n,\mathfrak{l}}^{\times}\right). \tag{4.4}$$

Let χ be a character of the group of fractional proper ideals of R_n. By the above proposition, χ gives rise to a unique character of the full group of fractional ideals of M. In particular, taking an idele generator $a \in M_{\mathbb{A}}^{\times}$ with $a_\infty = 1$ of a proper R_n–ideal $\mathcal{A}$, we define $N(\mathcal{A}) = |a|_{\mathbb{A}}^{-1}$. We note that $N(\mathcal{A}) = [R_n : \mathcal{A}] = [R : R\mathcal{A}]$. We then define a formal L–function:

$$L^n(s,\chi) = \sum_{\mathcal{A}\subset R_n} \chi(\mathcal{A})N(\mathcal{A})^{-s}, \tag{4.5}$$

where $\mathcal{A}$ runs over all proper R_n–ideals. We write $L(s,\chi)$ for $L^0(s,\chi)$, which is the classical abelian L–function. This L–function depends on n, because the set of proper R_n–ideals depends on n. Since each proper ideal $\mathcal{B} \subset R_n$ equivalent to $\mathcal{A}^{-1}$ can be written as $\mathcal{B} = \alpha\mathcal{A}^{-1}$ with $\alpha \in \mathcal{A}$, we have

$$\sum_{\alpha\in\mathcal{A}} \chi((\alpha))N(\alpha)^{-s} = \chi(\mathcal{A})N(\mathcal{A})^{-s}\sum_{\mathcal{B}\sim\mathcal{A}^{-1}}\chi(\mathcal{B})N(\mathcal{B})^{-s}.$$

We write $L^n_{\mathcal{A}^{-1}}(s,\chi) = \sum_{\mathcal{B}\sim\mathcal{A}^{-1}}\chi(\mathcal{B})N(\mathcal{B})^{-s}$. Then we have

$$L^n(s,\chi) = \sum_{\mathcal{A}\in Cl_n} L^n_{\mathcal{A}^{-1}}(s,\chi) = \sum_{\mathcal{A}\in Cl_n}\chi(\mathcal{A})^{-1}N(\mathcal{A})^s\sum_{\alpha\in\mathcal{A}}\chi((\alpha))N(\alpha)^{-s}.$$

For a primitive character χ of Cl_n, we shall compute $L^{n+k}(s,\chi)$ $(k > 0)$. The L–function has Euler product, and we only need to compute the $\mathfrak{l}$–factor $L^{n+k}_{\mathfrak{l}}(s,\chi)$ given by $\sum_{\mathcal{A}\in I_{n+k,\mathfrak{l}}}\chi(\mathcal{A})N(\mathcal{A})^{-s}$, where $I_{n+k,\mathfrak{l}} = (R_{n+k,\mathfrak{l}} - \{0\})/R_{n+k,\mathfrak{l}}^{\times}$. We first deal with the case where $\mathfrak{l}$ splits in M/F. Then by (4.2), we have

$$\begin{aligned} L^{n+k}_{\mathfrak{l}}(s,\chi) = &\sum_{j=0}^{n+k-1}\chi(\varpi\overline{\varpi})^j N(\mathfrak{l})^{-2sj}\sum_{u\in U_{n+k-j}}\chi(u) \\ &+ \chi(\varpi\overline{\varpi})^n N(\mathfrak{l})^{-2(n+k)s}\sum_{i,j}\chi(\varpi^i\overline{\varpi}^j)N(\mathfrak{l})^{-(i+j)s}\sum_{u\in U_{n+k}}\chi(u), \end{aligned}$$

where $U_j = R_{n+k-j,\mathfrak{l}}^{\times}/R_{n+k,\mathfrak{l}}^{\times}$. We have $\sum_{u\in U_{n+k-j}}\chi(u) = 0$ unless $j = 0, 1, \ldots, k$, and we get

$$L^{n+k}_{\mathfrak{l}}(s,\chi) = \sum_{j=0}^{k}\chi(\varpi\overline{\varpi})^j N_{F/\mathbb{Q}}(\mathfrak{l})^{j-2sj} \quad \text{if } \mathfrak{l} \text{ splits in } M/F \text{ and } n > 0. \tag{4.6}$$

Similarly, for ramified and inert primes, we have, for $n > 0$,

$$\begin{aligned} L_{\mathfrak{l}}^{n+k}(s,\chi) &= \sum_{j=0}^{k} \chi(\varpi)^{2j} N_{F/\mathbb{Q}}(\mathfrak{l})^{j-2sj} \quad \text{if } \mathfrak{l} \text{ ramifies in } M/F, \\ L_{\mathfrak{l}}^{n+k}(s,\chi) &= \sum_{j=0}^{k} \chi(\varpi)^{j} N_{F/\mathbb{Q}}(\mathfrak{l})^{j-2sj} \quad \text{if } \mathfrak{l} \text{ remains prime in } M/F. \end{aligned} \tag{4.7}$$

When χ is of conductor 1, we have

$$\begin{aligned} L_{\mathfrak{l}}^{k}(s,\chi) - \left(N_{F/\mathbb{Q}}(\mathfrak{l}) - \left(\frac{M/F}{\mathfrak{l}}\right)\right)\chi(\mathfrak{l})^k N_{F/\mathbb{Q}}(\mathfrak{l})^{k-1-2ks} L_{\mathfrak{l}}^{0}(s,\chi) \\ = 1 + \sum_{j=1}^{k-1} \chi(\mathfrak{l})^j N_{F/\mathbb{Q}}(\mathfrak{l})^{j-2sj} \quad (k > 0), \end{aligned} \tag{4.8}$$

$$L_{\mathfrak{l}}^{n+1}(s,\chi) - \chi(\mathfrak{l}) N_{F/\mathbb{Q}}(\mathfrak{l})^{1-2s} L_{\mathfrak{l}}^{n}(s,\chi) = 1 \quad \text{for } n > 0. \tag{4.9}$$

and

$$L_{\mathfrak{l}}^{1}(s,\chi) - \left(N_{F/\mathbb{Q}}(\mathfrak{l}) - \left(\frac{M/F}{\mathfrak{l}}\right)\right)\chi(\mathfrak{l}) N_{F/\mathbb{Q}}(\mathfrak{l})^{-2s} L_{\mathfrak{l}}^{0}(s,\chi) = 1. \tag{4.10}$$

4.2. A generalization of a result of Hurwitz.

In the late 19th century, Hurwitz proved an analogue of the von Staut theorem for Hurwitz numbers ([Hz]). We shall give a generalization of his result to Hecke L–values of CM fields. The definition of Hilbert modular Eisenstein series is classical going back to Hecke; so, we first reproduce from [K1] and [HT] relevant results.

Let $E_{\mathfrak{c}} = E_k(\phi, \mathfrak{c})$ be the Eisenstein series of weight k defined in [HT] (2.3). Here we take $\phi : O_p \times O_p \to \mathbb{C}$ to be the constant function with value 1, and $\mathfrak{c}$ is a fixed polarization ideal. Over $\mathbb{C}$, having a pair (X, Λ) with $\mathfrak{c}$–polarization Λ is equivalent to having a lattice $L \subset \mathbb{C}^{\Sigma}$ with $X(\mathbb{C}) = \mathbb{C}^{\Sigma}/L$ and an alternating pairing $L \wedge L \cong \mathfrak{c}^*$ with certain positivity (see [K1] I.1.4). Thus $E_{\mathfrak{c}}$ is a function of lattices $L \subset F_{\mathbb{C}}$ with $\Lambda : L \wedge L \cong \mathfrak{c}^*$ satisfying $E_{\alpha\mathfrak{c}}(L, \alpha^{-1}\Lambda) = E_{\mathfrak{c}}(L, \Lambda)$. We then have

$$E_{\mathfrak{c}}((L,\Lambda)) = \frac{(-1)^{k[F:\mathbb{Q}]}\Gamma_{\Sigma}((k+s)\Sigma)}{\sqrt{|D|}} \sum_{w \in L - \{0\}} N(w)^{-k} |N(w)|^{-s}\Big|_{s=0}, \tag{4.11}$$

where $N(w)$ is the norm of $F_{\mathbb{C}}$ to $\mathbb{C}$ induced by the norm map $N_{F/\mathbb{Q}} : F \to \mathbb{Q}$. By this definition, $E_{\mathfrak{c}}$ only depends on the strict ideal class of $\mathfrak{c}$ (and $E_{\mathfrak{c}}((L,\Lambda))$ depends only on $(L, \overline{\Lambda})$ for the polarization class $\overline{\Lambda}$). We write $E = (E_{\mathfrak{c}})_{\mathfrak{c} \in Cl_F} \in G_{kI}(\mathbb{C})$, where I is the sum of all embedding of F into $\overline{\mathbb{Q}}$ and k is a positive integer.

We may specify an O–lattice:

$$L = L_z^{\mathfrak{b}} = 2\pi i(\mathfrak{b}\mathfrak{c}^{-1}z + \mathfrak{b}^*) \subset \mathbb{C}^{\Sigma} = F \otimes_{\mathbb{Q}} \mathbb{C} \quad (z \in \mathbb{C}^{\Sigma},\ \mathrm{Im}(z_\sigma) > 0\ \forall \sigma \in \Sigma)$$

with $\Lambda_z : L \wedge_O L \cong \mathfrak{c}^*$ by the isomorphism induced by an alternating pairing

$$\langle 2\pi i(az+b), 2\pi i(cz+d)\rangle = -(ad-bc).$$

By this, we may regard $E_{\mathfrak{c}}$ as a holomorphic function defined on $\mathfrak{H}^\Sigma$ for the upper half complex plane $\mathfrak{H}$.

Since polarization is fixed as above, we often regard $f \in G_{kI}(\mathbb{C})$ as a function of O–lattices L with $L \wedge L = \mathfrak{c}^*$, omitting Λ_z from the notation. For a given lattice L, we write $\mathfrak{c}(L)$ for its polarization ideal. We define an operator for an ideal $\mathfrak{l}$

$$f|\langle\mathfrak{l}\rangle(L) = f(\mathfrak{l}L). \tag{4.12}$$

Then $f|\langle\mathfrak{l}\rangle$ is defined over O–lattices L with $\mathfrak{c}\mathfrak{l}^2$–polarization, because $\mathfrak{l}L \wedge \mathfrak{l}L = \mathfrak{l}^2\mathfrak{c}(L)^* \Leftrightarrow \mathfrak{c}(\mathfrak{l}L) = \mathfrak{l}^{-2}\mathfrak{c}(L)$. Since f is of weight kI, if $\mathfrak{l}$ is a principal ideal generated by ϖ, we have $f|\langle\mathfrak{l}\rangle(L) = f(\varpi L) = N(\varpi)^{-k}f(L)$. Thus, if $\mathfrak{l}$ is principal, we have

$$E_{\mathfrak{c}}|\langle\mathfrak{l}\rangle(L) = sgn(N(\mathfrak{l}))^k N(\mathfrak{l})^{-k}E_{\mathfrak{c}}(L).$$

Here $sgn(N(\mathfrak{l})) = N(\varpi)/|N(\varpi)|$ for the generator ϖ of $\mathfrak{l}$. This operator $\langle\mathfrak{l}\rangle$ is equal to the one introduced in 3.3 under the same symbol.

We shall compute the Hecke eigenvalue of E for $T(\mathfrak{l})$. We consider the sum

$$(*) = \sum_{L'} \sum_{w\in L'-\{0\}} N(w)^{-k}|N(w)|^{-s}\Big|_{s=0},$$

where L' runs over all O–lattices containing L with $L'/L \cong O/\mathfrak{l}$. Since there are $1+N(\mathfrak{l})$ such lattices L', in the sum, each element in L contributes $1+N(\mathfrak{l})$ times, and each element in $\mathfrak{l}^{-1}L$ outside L contribute once. Thus

$$(*) = N(\mathfrak{l}) \sum_{w\in L-\{0\}} N(w)^{-k}|N(w)|^{-s}\Big|_{s=0} + \sum_{w\in \mathfrak{l}^{-1}L-\{0\}} N(w)^{-k}|N(w)|^{-s}\Big|_{s=0}.$$

If $\mathfrak{l}$ is principal generated by ξ, we find the sum over $\mathfrak{l}^{-1}L$ is just equal to $sgn(N(\mathfrak{l}))^k N(\mathfrak{l})^k = N(\xi)^k$ times the sum over L. The Hecke operator $T(\mathfrak{l})$ is given by the sum divided by $N(\mathfrak{l})$; so, we get

$$\begin{aligned} E_{\mathfrak{c}}|T(\mathfrak{l}) =&(1+sgn(N(\mathfrak{l}))^k N(\mathfrak{l})^{k-1})E_{\mathfrak{c}} \text{ if } \mathfrak{l} \text{ is principal.} \\ E_{\mathfrak{c}\mathfrak{l}}|T(\mathfrak{l}) =&N(\mathfrak{l})^{-1}E_{\mathfrak{c}\mathfrak{l}^2}|\langle\mathfrak{l}^{-1}\rangle + E_{\mathfrak{c}} \text{ if } \mathfrak{l} \text{ is not principal.} \end{aligned} \tag{4.13}$$

Take an object (L,Λ,C) of level $\Gamma_0(\mathfrak{l})$; so, $C \cong O/\mathfrak{l}$ is a subgroup of $X(L)$. This is equivalent to choose $L_C \supset L$ with $L_C/L \cong O/\mathfrak{l}$. For $L = L_z^{\mathfrak{b}}$, we choose $C_z = (\mathfrak{l}\mathfrak{b})^*/\mathfrak{b}^* \subset X(L_z^{\mathfrak{b}})$. Then we see $X(L_z^{\mathfrak{b}})/C_z \cong X(L_z^{\mathfrak{l}\mathfrak{b}}))$, where $L_z^{\mathfrak{l}\mathfrak{b}}$ on the right-hand-side has polarization ideal $\mathfrak{c}\mathfrak{l}$; so, it is $\mathfrak{b}\mathfrak{c}^{-1}z + (\mathfrak{l}\mathfrak{b})^*$. Then we define another operator $[\mathfrak{l}]$ by

$$f|[\mathfrak{l}](L,C) = N(\mathfrak{l})^{-1}f(L_C). \tag{4.14}$$

Then $f|[\mathfrak{l}]$ is defined over O–lattices with $\mathfrak{c}\mathfrak{l}^{-1}$–polarization. This definition is compatible with the operator $[\mathfrak{l}]$ in 3.3. By definition, we have

$$f|T(\mathfrak{l})(L) = N(\mathfrak{l})^{-1} \sum_{L':L/L\cong O/\mathfrak{l}} f(L')$$
$$f|U(\mathfrak{l})(L,C) = N(\mathfrak{l})^{-1} \sum_{L':L'/L\cong O/\mathfrak{l},L'\neq L_C} f(L',C'),$$

where $C' = L_C + L'/L'$. Again $f|T(\mathfrak{l})$ and $f|U(\mathfrak{l})$ is defined over lattices with $\mathfrak{c}\mathfrak{l}^{-1}$-polarization. Then we find

$$E_{\mathfrak{c}}|U(\mathfrak{l}) = E_{\mathfrak{c}}|T(\mathfrak{l}) - E_{\mathfrak{c}}|[\mathfrak{l}]. \tag{4.15}$$

From (4.14), we have

$$N(\mathfrak{l})(E_{\mathfrak{c}}|[\mathfrak{l}])|U(\mathfrak{l})(L,C) = \sum_{L':L'/L\cong O/\mathfrak{l},L\neq L_C} E_{\mathfrak{c}}|[\mathfrak{l}](L',C') = E_{\mathfrak{c}}|\langle\mathfrak{l}^{-1}\rangle(L,C). \tag{4.16}$$

From (4.16), (4.15) and (4.13) combined, we get

$$\begin{aligned}(E_{\mathfrak{c}} - N(\mathfrak{l})E_{\mathfrak{c}\mathfrak{l}^{-1}}|(\langle\mathfrak{l}\rangle[\mathfrak{l}])|U(\mathfrak{l}) =&(E_{\mathfrak{c}\mathfrak{l}} - N(\mathfrak{l})E_{\mathfrak{c}}|\langle\mathfrak{l}\rangle[\mathfrak{l}])|\left(N(\mathfrak{l})^{-1}\langle\mathfrak{l}^{-1}\rangle\right)\\ (E_{\mathfrak{c}} - E_{\mathfrak{c}\mathfrak{l}}|[\mathfrak{l}])|U(\mathfrak{l}) =&(E_{\mathfrak{c}\mathfrak{l}^{-1}} - E_{\mathfrak{c}}|[\mathfrak{l}]).\end{aligned} \tag{4.17}$$

We write $E_{1,\mathfrak{c}} = E_{\mathfrak{c}} - E_{\mathfrak{c}\mathfrak{l}}|[\mathfrak{l}]$ and $E_{\mathfrak{l}^{k-1},\mathfrak{c}} = E_{\mathfrak{c}\mathfrak{l}} - N(\mathfrak{l})E_{\mathfrak{c}}|\langle\mathfrak{l}\rangle[\mathfrak{l}]$. Defining $E_1 = (E_{1,\mathfrak{c}})_{\mathfrak{c}} \in G_{kI}(\mathbb{C})$ and $E_{\mathfrak{l}^{k-1}} = (E_{\mathfrak{l}^{k-1},\mathfrak{c}})_{\mathfrak{c}} \in G_{kI}(\mathbb{C})$, we have

$$E_1|U(\mathfrak{l}) = E_1 \quad\text{and}\quad E_{\mathfrak{l}^{k-1}}|U(\mathfrak{l}) = N(\mathfrak{l})^{k-1}E_{\mathfrak{l}^{k-1}}. \tag{4.18}$$

We can compute Fourier expansion of the Eisenstein series (see [K1] III), and find

$$E_{\mathfrak{l}^{k-1}} \in G_{kI}(\mathbb{Z}_{(p)}),\ E_1 \in G_{kI}(\mathbb{Q}) \text{ removed constant term is } p\text{–integral.} \tag{4.19}$$

Let d^κ be the p–adic analytic differential operator of Dwork-Katz whose effect on q–expansions is given by $a(\xi, d^\kappa f) = \xi^\kappa a(\xi, f)$ with $\xi^\kappa = \prod_\sigma \sigma(\xi)^{\kappa_\sigma}$ ([K1] II). Under (unr), d^κ is an integral operator. Recall the Hecke character λ of conductor 1 and of infinity type $k\Sigma + \kappa(1-c)$. Then $L^n_{\mathcal{A}^{-1}}(0,\lambda)$ does not depend on the choice of $\mathcal{A}$ in its strict ideal class. We define $f = d^\kappa E_{\mathfrak{l}^{k-1}} \in V(W)$ and $f' = d^\kappa E_1 \in V(W) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The construction of the measures φ_f and $\varphi_{f'}$ makes sense for f and f'. Decompose $Cl_\infty = \Gamma_f \times \Delta$. We fix a character ψ of Δ and define again two measures $\mathcal{E}_\psi = \varphi^\psi_f$ and $\mathcal{E}'_\psi = \varphi^\psi_{f'}$ by (3.14) for the above f and f'. From the definition: $E_{1,\mathfrak{c}} = E_{\mathfrak{c}} - E_{\mathfrak{c}\mathfrak{l}}|[\mathfrak{l}]$ and $E_{\mathfrak{l}^{k-1},\mathfrak{c}} = E_{\mathfrak{c}\mathfrak{l}} - N(\mathfrak{l})E_{\mathfrak{c}}|\langle\mathfrak{l}\rangle[\mathfrak{l}]$, we find, for $n > 1$,

$$\begin{aligned}f(x(\mathcal{A})) &= c_0\left(\lambda(\mathcal{A})L^n_{\mathcal{A}^{-1}}(0,\lambda) - \lambda(\mathfrak{l}(\mathcal{A}R_{n-1}))L^{n-1}_{(\mathfrak{l}\mathcal{A}R_{n-1})^{-1}}(0,\lambda)\right)\\ f'(x(\mathcal{A})) &= c_0\left(\lambda(\mathcal{A})L^n_{\mathcal{A}^{-1}}(0,\lambda) - N(\mathfrak{l})^{-1}\lambda(\mathcal{A}R_{n-1})L^{n-1}_{(\mathcal{A}R_{n-1})^{-1}}(0,\lambda)\right),\end{aligned} \tag{4.20}$$

where $c_0 = \frac{(-1)^{k[F:\mathbb{Q}]}\Gamma_\Sigma(k\Sigma+\kappa)\pi^\kappa}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}}$ for the discriminant D of F. For a character $\psi\chi$ of Γ_f with conductor $\mathfrak{l}^n$ ($n > 0$), we have for λ of infinity type $k\Sigma + \kappa(1-c)$,

$$\int_{\Gamma_f} \chi d\mathcal{E}_\psi = N(\mathfrak{l})^{n(1-k)} \int_{\Gamma_f} \chi d\mathcal{E}'_\psi = \frac{\pi^\kappa \Gamma_\Sigma(k\Sigma+\kappa) L(0, \psi^{-1}\chi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa \Omega^{k\Sigma+2\kappa}} \tag{4.21}$$

where $L(s,\alpha)$ is the primitive L–function of α. If the conductor of $\chi\psi$ is 1, by (4.9), we have slightly different expressions:

$$\int_{\Gamma_f} \chi d\mathcal{E}_\psi = \begin{cases} \frac{(1-\psi^{-1}\lambda(\mathfrak{L}))(1-\psi^{-1}\lambda(\overline{\mathfrak{L}}))\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L(0,\psi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}} & \text{if } \mathfrak{l} = \mathfrak{L}\overline{\mathfrak{L}}, \\ (1-\psi^{-1}\lambda(\mathfrak{l}))\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L(0,\psi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}} & \text{if } \mathfrak{l} \text{ is inert}, \\ (1-\psi^{-1}\lambda_k(\mathfrak{L}))\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L(0,\psi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}} & \text{if } \mathfrak{l} = \mathfrak{L}^2, \end{cases} \tag{4.22}$$

and

$$\begin{aligned} &N(\mathfrak{l})^{1-k} \int_{\Gamma_f} \chi d\mathcal{E}'_\psi \\ &= \begin{cases} N(\mathfrak{l})^{-1}\frac{(1-N(\mathfrak{l})\psi^{-1}\lambda(\mathfrak{L}))(1-N(\mathfrak{l})\psi^{-1}\lambda(\overline{\mathfrak{L}}))\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L(0,\psi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}} & \text{if } \mathfrak{l} = \mathfrak{L}\overline{\mathfrak{L}}, \\ -N(\mathfrak{l})^{-1}(1-\psi^{-1}\lambda(\mathfrak{l})N(\mathfrak{l})^2)\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L(0,\psi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}} & \text{if } \mathfrak{l} \text{ is inert}, \\ -\psi^{-1}\lambda(\mathfrak{L})(1-\psi^{-1}\lambda(\mathfrak{L})N(\mathfrak{l}))\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L(0,\psi^{-1}\lambda)}{\sqrt{|D|}\,\mathrm{Im}(\delta)^\kappa\Omega^{k\Sigma+2\kappa}} & \text{if } \mathfrak{l} = \mathfrak{L}^2, \end{cases} \end{aligned} \tag{4.23}$$

Now we get the following generalization of Hurwitz's theorem, which has been proven in [H96] Theorem 8.4.1 by a different method.

Theorem 4.3. *Let the notation and the assumption be as in Theorem* 1.1. *Fix a Hecke character λ of conductor* 1 *with $\lambda((a)) = a^{-k\Sigma}$ for all principal ideal (a) of M with a positive integer k. Then for a finite order character χ of Cl_∞, $\frac{\Gamma_\Sigma(k\Sigma)L(0,\lambda\chi)}{\Omega^{k\Sigma}}$ is $\mathfrak{P}$–integral and belongs to $\mathcal{W}$ unless $p-1|k$ and the conductor of χ is equal to* 1. *If $(p-1)|k$ and $p^m|k$ exactly, writing $L_\mathfrak{l}(s,\lambda)$ for the Euler $\mathfrak{l}$–factor of $L(s,\lambda)$, $\frac{L(0,\lambda)}{\Omega^{k\Sigma}}$ belongs to the field of fractions of $\mathcal{W}$, and we have*

$$\begin{aligned} &p^m L_\mathfrak{l}(-1,\lambda)\frac{\Gamma_\Sigma(k\Sigma)L(0,\lambda)}{\Omega^{k\Sigma}} \\ &\qquad \equiv \varepsilon(1-N(\mathfrak{l}))\frac{(1-\left(\frac{M/F}{\mathfrak{l}}\right)N(\mathfrak{l}))hR_p}{w\sqrt{D}} \prod_{\mathfrak{p}\in\Sigma_p}\left(1-\frac{1}{N(\mathfrak{p})}\right) \mod \mathfrak{P}^m, \end{aligned}$$

where h is the class number of M, w is the number of roots of unity in M, D is the discriminant of $F/\mathbb{Q}$ and R_p is the p–adic regulator of F and $\varepsilon = 1$ if $\mathfrak{l}$ is unramified in M/F and if $\mathfrak{l}$ ramifies, $\varepsilon = \lambda(\mathfrak{L})^{-1}$ for the ramified prime $\mathfrak{L}|\ell$.

As in [Cz], the ratio $R_p/\sqrt{D}$ is canonically defined although the numerator and denominator depends on the choice of ordering of embedding of F into $\overline{\mathbb{Q}}_p$.

Proof. Since the rationality of the L–values follows from Shimura's rationality result [Sh2] combined with [K1] 5.3.0, we only need to prove the last congruence formula. We assume that $F \neq \mathbb{Q}$ for simplicity (because the case of $k = 2$ and $F = \mathbb{Q}$ needs an extra care). We need to look into the q–expansion at the cusp $(\mathfrak{b}, \mathfrak{b}\mathfrak{c}^{-1})$ given by, if $k \geq 2$ is even,

$$N(\mathfrak{b})^{-1}E_{\mathfrak{c}}\big|\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)(q) = 2^{-d}N(\mathfrak{b})^{k-1}\zeta_{\mathfrak{b}^{-1}}(1-k) + \sum_{0 \ll \xi \in \mathfrak{b}^2\mathfrak{c}^{-1}} \sum_{(\alpha,\beta)\in(\mathfrak{b}\times\mathfrak{b}\mathfrak{c}^{-1}),\ \alpha\beta=\xi} |N(\alpha)|^{k-1}q^{\xi}, \quad (4.24)$$

where $d = [F : \mathbb{Q}]$ and $\zeta_{\mathfrak{b}^{-1}}(s) = \sum_{\mathfrak{x}\sim\mathfrak{b}^{-1},\ \mathfrak{x}\subset O} N(\mathfrak{x})^{-s}$ is the partial zeta function of the ideal class $\mathfrak{b}^{-1}$. Here $b \in F_{\mathbb{A}}^{\times}$ is an adelic generator of the ideal $\mathfrak{b}$ with $b_\infty = 1$, and $f\big|\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)$ is as in [DR] 5.8 (see also [H88] Theorem 4.9). We can rewrite the expansion as

$$N(\mathfrak{b})^{-k}E_{\mathfrak{c}}\big|\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)(q) = 2^{-d}\zeta_{\mathfrak{b}^{-1}}(1-k) + \sum_{0 \ll \xi \in \mathfrak{b}^2\mathfrak{c}^{-1}} \sum_{\mathcal{A}|\xi\mathfrak{b}^{-2}\mathfrak{c},\ \mathcal{A}\sim\mathfrak{b}^{-1}} N(\mathcal{A})^{k-1}q^{\xi}, \quad (4.25)$$

where $\mathcal{A}$ runs over all integral factors of $\xi\mathfrak{b}^{-2}\mathfrak{c}$ in the ideal class of $\mathfrak{b}^{-1}$.

Taking $\mathfrak{b} = O$, we have $a(0, E_{1,\mathfrak{c}}) = (1 - N(\mathfrak{l})^{k-1})2^{-d}\zeta_O(1-k)$. Then we find from a result of Colmez [Cz] that

$$p^m 2^{-d}\zeta_O(1-k) \equiv \frac{R_p}{2\sqrt{D}} \prod_{\mathfrak{p}\in\Sigma_p}\left(1 - \frac{1}{N(\mathfrak{p})}\right) \mod \mathfrak{P}^m.$$

Then the desired formula follows from (4.23) by the q–expansion principle ([K1] (1.2.16)). □

4.3. Proof of Theorem 1.1.

By (4.25) and the rationality of the differential operator d (cf. [K1] II), we have $\mathbb{F}_p[f] = \mathbb{F}_p$ for $f = d^\kappa E_{\mathfrak{l}^{k-1}}$. By Theorems 3.2 and 3.3, we need to verify the condition (H) in Theorem 3.2 for f (which implies the condition (h) in Theorem 3.3). For a given q–expansion $g(q) = \sum_\xi a(\xi, g)q^\xi \in \mathbb{F}[[q^\xi]]_{\xi\in\mathfrak{c}^{-1}}$ at the infinity cusp $(O, \mathfrak{c}^{-1})$, we know that, for $u \in O_{\mathfrak{l}} \subset F_{\mathbb{A}}$,

$$a(\xi, g|\alpha_u) = \mathbf{e}_F(u\xi)a(\xi, g) \quad \text{for } \alpha_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1\end{smallmatrix}\right)\ (\mathbf{e}_F(x) = \exp(2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x)).$$

The condition (h) for g concerns about the linear independence of $g|\alpha_u$ for $u \in \mathfrak{l}^{-r}O_{\mathfrak{l}}/O_{\mathfrak{l}}$. For any function $\phi : \mathfrak{c}^{-1}/\mathfrak{l}^r\mathfrak{c}^{-1} = O/\mathfrak{l}^r \to \mathbb{F}$, put $g|\phi = \sum_\xi \phi(\xi)a(\xi, g)q^\xi$. By definition, we have $g|R_\phi = \sum_{u\in O/\mathfrak{l}^r} \phi(u)g|\alpha_u = g|\phi^*$ for the Fourier transform $\phi^*(v) = \sum_u \phi(u)\mathbf{e}_F(uv)$. For the characteristic function χ_v of a singleton $\{v\} \subset \mathfrak{c}^{-1}/\mathfrak{l}^r\mathfrak{c}$, its Fourier transform is given by $u \mapsto \mathbf{e}_F(vu)$. By the Fourier inversion formula (and the q–expansion principle: [K1] 1.9.17), the linear independence of $\{g|\alpha_u = g|\chi_u^*\}_u$ is equivalent to the linear independence of $\{g|\chi_u\}_u$.

We recall that f_ψ is a tuple $(f_{\psi,\mathfrak{c}})_{\mathfrak{c}} \in \bigoplus_{\mathfrak{c}} V(\mathfrak{c}; W)$. Thus we need to prove that there exists $\mathfrak{c}$ such that for a given congruence class $u \in \mathfrak{c}^{-1}/\mathfrak{l}^r\mathfrak{c}^{-1}$

$$a(\xi, f_{\psi,\mathfrak{c}}) \not\equiv 0 \mod \mathfrak{m}_W \quad \text{for at least one } \xi \in u. \tag{4.26}$$

Since $a(\xi, d^\kappa h) = \xi^\kappa a(\xi, h)$, (4.26) is achieved if

$$a(\xi, g_{\psi,\mathfrak{c}}) \not\equiv 0 \mod \mathfrak{m}_W \quad \text{for at least one } \xi \in u \text{ prime to } p \tag{4.27}$$

holds for $g = (E_{\mathfrak{c}} - N(\mathfrak{l})E_{\mathfrak{c}\mathfrak{l}^{-1}}|\langle\mathfrak{l}\rangle|[\mathfrak{l}])_{\mathfrak{c}}$, because $\mathfrak{l} \nmid p$. Recall (3.13):

$$g_\psi = \sum_{\mathfrak{r}\in\mathcal{R}} \lambda\psi^{-1}(\mathfrak{R}) \left(\sum_{\mathfrak{s}\in\mathcal{S}} \psi\lambda^{-1}(\mathfrak{s}) g|\langle\mathfrak{s}\rangle \right) |[\mathfrak{r}]. \tag{4.28}$$

We described the effect on the q–expansion of the operators $[\mathfrak{l}]$ and $\langle\mathfrak{l}\rangle$ in 3.3. This combined with the q–expansion of the Eisenstein series described in the proof of Theorem 4.3 gives us the following q–expansion $g_{\mathfrak{a},\mathfrak{b}}(q)$ of g at the cusp $(\mathfrak{a}, \mathfrak{b})$ (carrying $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) = (\mathbb{G}_m \otimes \mathfrak{a}^*)/q^{\mathfrak{b}}$):

$$\begin{aligned} a(\xi, g_{\mathfrak{a},\mathfrak{b}}) &= N(\mathfrak{a}) \sum_{\substack{(a,b)\in(\mathfrak{a}\times\mathfrak{b})/O^\times \\ ab=\xi}} \frac{N(a)^k}{|N(a)|} - N(\mathfrak{a}) \sum_{\substack{(a,b)\in(\mathfrak{a}\times\mathfrak{l}\mathfrak{b})/O^\times \\ ab=\xi}} \frac{N(a)^k}{|N(a)|} \\ &= N(\mathfrak{a}) \sum_{\substack{(a,b)\in(\mathfrak{a}\times(\mathfrak{b}-\mathfrak{l}\mathfrak{b}))/O^\times \\ ab=\xi}} \frac{N(a)^k}{|N(a)|} \end{aligned} \tag{4.29}$$

We now study the question (4.27) using this explicit formula (4.29). Our evaluation points $x(\mathcal{A})$ have polarization ideals $\mathfrak{c}(\mathcal{A})$ which cover all classes in the coset $N_{M/F}(Cl_M)\mathfrak{c}(R)$. Therefore, we only need to look into the contribution of all $g|\langle\mathfrak{s}\rangle|[\mathfrak{r}]|[\mathfrak{q}]$ belonging to a single class $\mathfrak{c}$ in $N_{M/F}(Cl_M)\mathfrak{c}(R)$. In the class $\mathfrak{c}^{-1}$, we pick a prime $\mathfrak{t}$ outside $p\mathfrak{l}$. We take an element $0 \ll \xi \in u$ for $u \in O/\mathfrak{l}^r$ so that $(\xi) = \mathfrak{t}\mathfrak{n}\mathfrak{l}^e$ for an integral ideal $\mathfrak{n} \nmid p\mathfrak{l}\mathfrak{C}$ prime to the relative discriminant D of M/F and $0 \le e \le r$. Since we have a freedom of choosing ξ modulo $\mathfrak{l}^r$, the ideal $\mathfrak{n}$ varies freely in a given ray class modulo $\mathfrak{l}^{r-e}$.

We pick a pair $(a, b) \in F^2$ with $ab = \xi$ with $a \in \mathfrak{s}^{-1}$ and $b \in \mathfrak{t}\mathfrak{s}$. Then $(a) = \mathfrak{s}^{-1}\mathfrak{l}^\alpha\mathfrak{x}$ for an integral ideal $\mathfrak{x}$ prime to $\mathfrak{l}$ and $(b) = \mathfrak{s}\mathfrak{t}\mathfrak{l}^{e-\alpha}\mathfrak{x}'$ for an integral ideal $\mathfrak{x}'$ prime to $\mathfrak{l}$. Since $(ab) = \mathfrak{t}\mathfrak{n}\mathfrak{l}^e$, we find that $\mathfrak{x}\mathfrak{x}' = \mathfrak{n}$. By (4.29), b has to be prime to $\mathfrak{l}$; so, we find $\alpha = e$. Since $\mathfrak{x}\mathfrak{x}' = \mathfrak{n}$ and hence $\mathfrak{r} = O$ because $\mathfrak{n}$ is prime to D. Thus for each factor $\mathfrak{x}$ of $\mathfrak{n}$, we could have two possible pairs $(a_{\mathfrak{x}}, b_{\mathfrak{x}})$ with contribution to the q–expansion such that

$$a_{\mathfrak{x}} b_{\mathfrak{x}} = \xi \ \text{ and } \ ((a_{\mathfrak{x}}) = \mathfrak{s}_{\mathfrak{x}}^{-1}\mathfrak{l}^e\mathfrak{x},\ (b_{\mathfrak{x}}) = (\xi a_{\mathfrak{x}}^{-1}) = \mathfrak{s}_{\mathfrak{x}}\mathfrak{t}\mathfrak{n}\mathfrak{x}^{-1})$$

for $\mathfrak{s}_{\mathfrak{x}} \in \mathcal{S}$ representing the ideal class of the ideal $\mathfrak{l}^e\mathfrak{x}$. We then write down the q–expansion coefficient of q^ξ at the cusp $(O, \mathfrak{t})$ of g_ψ:

$$\begin{aligned} a(\xi, g_\psi) &= \sum_{\mathfrak{x}|\mathfrak{n}} N(\mathfrak{s}_{\mathfrak{x}})^{-1}\psi^{-1}\lambda(\mathfrak{s}_{\mathfrak{x}})\frac{N(a_{\mathfrak{x}})^k}{|N(a_{\mathfrak{x}})|} \\ &= \psi^{-1}\lambda(\mathfrak{l})^{-e}N(\mathfrak{l})^{-e}\sum_{\mathfrak{x}|\mathfrak{n}}\frac{1}{\psi^{-1}\lambda(\mathfrak{x})N(\mathfrak{x})} \\ &= \psi^{-1}\lambda(\mathfrak{n}\mathfrak{l}^e)^{-1}N(\mathfrak{n}\mathfrak{l}^e)^{-1}\prod_{\mathfrak{y}|\mathfrak{n}}\frac{1-(\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y}))^{e(\mathfrak{y})+1}}{1-\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y})}, \end{aligned} \tag{4.30}$$

where $\mathfrak{n} = \prod_{\mathfrak{y}|\mathfrak{n}} \mathfrak{y}^{e(\mathfrak{y})}$ is the prime factorization of $\mathfrak{n}$.

We define, for the valuation v of W (normalized so that $v(p) = 1$)

$$\mu_C(\psi) = \mathrm{Inf}_{\mathfrak{n}} v\left(\prod_{\mathfrak{y}|\mathfrak{n}}\frac{1-(\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y}))^{e(\mathfrak{y})+1}}{1-\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y})}\right), \tag{4.31}$$

where $\mathfrak{n}$ runs over a ray class C modulo $\mathfrak{l}^{r-e}$ made of all integral ideals prime to $D\mathfrak{l}$ of the form $\mathfrak{t}^{-1}\xi\mathfrak{l}^{-e}$, $0 \ll \xi \in u$. If $\mu_C(\psi) = 0$ for one C, we get the desired non-vanishing. Since $\mu_C(\psi)$ only depends on the class C, we may assume (and will assume) that $e = 0$ without losing generality; thus ξ is prime to $\mathfrak{l}$, and C is the class of $u[\mathfrak{t}^{-1}]$.

Suppose that $\mathfrak{n}$ is a prime $\mathfrak{y}$. Then by (4.30), we have

$$a(\xi, g_\psi) = 1 + (\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y}))^{-1}.$$

If $\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y}) \equiv -1 \mod \mathfrak{m}_W$ for all prime ideals $\mathfrak{y}$ in the ray class C modulo $\mathfrak{l}^r$, the character $\mathcal{A} \mapsto (\psi^{-1}\lambda(\mathcal{A})N(\mathcal{A}) \mod \mathfrak{m}_W)$ is of conductor $\mathfrak{l}^r$. We write $\widetilde{\psi}$ for the character: $\mathcal{A} \mapsto (\psi^{-1}\lambda(\mathcal{A})N(\mathcal{A}) \mod \mathfrak{m}_W)$ of the ideal group of F with values in $\mathbb{F}^\times$. This character therefore has conductor $\widetilde{\mathfrak{C}}|\mathfrak{l}^r$. Since ψ is anticyclotomic, its restriction to $F_{\mathbb{A}}^\times$ has conductor 1. Since λ has conductor 1, the conductor of $\widetilde{\psi}$ is a factor of the conductor of $\lambda \mod \mathfrak{m}_W$, which is a factor of p. Thus $\widetilde{\mathfrak{C}}|p$. Since $\mathfrak{l} \nmid p$, we find that $\widetilde{\mathfrak{C}} = 1$.

We shall show that if $\mu_C(\psi) > 0$, M/F is unramified and $\widetilde{\psi} \equiv \left(\frac{M/F}{}\right)$ mod $\mathfrak{m}_W$. We now choose two prime ideals $\mathfrak{y}$ and $\mathfrak{y}'$ so that $\mathfrak{t}\mathfrak{y}\mathfrak{y}' = (\xi)$ with $\xi \in u$. Then by (4.30), we have

$$a(\xi, g_\psi) = \left(1 + \frac{1}{\psi^{-1}\lambda(\mathfrak{y})N(\mathfrak{y})}\right)\left(1 + \frac{1}{\psi^{-1}\lambda(\mathfrak{y}')N(\mathfrak{y}')}\right). \tag{4.32}$$

Since $\widetilde{\psi}(\mathfrak{y}\mathfrak{y}') = \widetilde{\psi}(u[\mathfrak{t}^{-1}]) = \widetilde{\psi}(C) = -1$, we find that if $a(\xi, g_\psi) \equiv 0 \mod \mathfrak{m}_W$,

$$-1 = \widetilde{\psi}(\mathfrak{y}/\mathfrak{y}') = \widetilde{\psi}(\mathfrak{l}^{-1})\widetilde{\psi}(\mathfrak{y}^2) = -\widetilde{\psi}(\mathfrak{y}^2).$$

Since we can choose $\mathfrak{y}$ arbitrary, we find that $\widetilde{\psi}$ is quadratic. Thus $\mu_C(\psi) > 0$ if and only if $\widetilde{\psi}(\mathfrak{c}) = -1$, which is independent of the choice of u. We now move the strict ideal class $\mathfrak{c}$ in $\mathfrak{c}(R)N_{M/F}(Cl_M)$. By class field theory, assuming that $\widetilde{\psi}$ has conductor 1, we have

$$\begin{aligned}\widetilde{\psi}(\mathfrak{c}) = -1 \text{ for all } \mathfrak{c} \in \mathfrak{c}(R)N_{M/F}(Cl_M) \\ \text{if and only if } \widetilde{\psi}(\mathfrak{c}(R)) = -1 \text{ and } \widetilde{\psi}(\mathcal{A}) = \left(\frac{M/F}{\mathcal{A}}\right) \text{ for all } \mathcal{A} \in Cl_F.\end{aligned} \quad (4.33)$$

If M/F is unramified, by definition, $2\delta\mathfrak{c}^* = 2\delta\mathfrak{d}^{-1}\mathfrak{c}^{-1} = R$. Taking square, we find that $(\mathfrak{d}\mathfrak{c})^2 = 4\delta^2 \ll 0$. Thus $1 = \widetilde{\psi}(\mathfrak{d}^{-2}\mathfrak{c}^{-2}) = (-1)^{[F:\mathbb{Q}]}$, and this never happens when $[F:\mathbb{Q}]$ is odd. Thus (4.33) is equivalent to the following three conditions:

(M1) M/F is unramified everywhere (so the strict class number of F and $[F:\mathbb{Q}]$ are even);

(M2) The strict ideal class of the polarization ideal $\mathfrak{c}$ of $X(R)$ in F is not a norm class of an ideal class of M ($\Leftrightarrow \left(\frac{M/F}{\mathfrak{c}}\right) = -1$);

(M3) $\widetilde{\psi}$ is the character of M/F.

Thus the non-vanishing result stated in the theorem is the case where ψ is the identity character. The condition (M1) and (M3) combined is equivalent to $\widetilde{\psi}^* \equiv \widetilde{\psi}$ mod $\mathfrak{m}_W$, where the dual character ϵ^* is defined by $\epsilon^*(x) = \epsilon(x^{-c})N(x)^{-1}$. The vanishing under (M1-3) of $L(0, \chi^{-1}\psi^{-1}\lambda) \equiv 0$ for all anti-cyclotomic $\chi\psi$ follows from the functional equation of the p-adic Katz measure interpolating the p–adic Hecke L–values (see [HT] Theorem II), because the constant term of the functional equation is given by $\widetilde{\psi}(\mathfrak{c}) = -1$. This finishes the proof. □

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