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Constancy of adjoint $\mathcal{L}$-invariant

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ABSTRACT

We prove that Greenberg's (adjoint) $\mathcal{L}$-invariant is constant over a slope 0 p-adic analytic family if and only if the family has complex multiplication.

Contents

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Fix a prime $p > 2$. For a (nonzero) Hecke eigenform $f \in S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ ($p \nmid N, r \geqslant 0$) and a subfield K of $\mathbb{C}$, the Hecke field $K(f)$ inside $\mathbb{C}$ is generated over K by the eigenvalues $a_n = a(n, f)$ of f for the Hecke operators $T(n)$ for all n. Then $\mathbb{Q}(f)$ is a finite extension of $\mathbb{Q}$ sitting inside the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$. Let Γ be the maximal torsion-free quotient of $\mathbb{Z}_p^\times$. We choose and fix a generator $\gamma \in \Gamma$ so that $\Gamma = \gamma^{\mathbb{Z}_p}$ and identify the Iwasawa algebra $\Lambda = W[[\Gamma]]$ with the power series ring $W[[T]]$ by $\Gamma \ni \gamma \mapsto (1+T)$ (for a discrete valuation ring W finite flat over $\mathbb{Z}_p$). A p-adic slope 0 analytic family of Hecke eigenforms $\mathcal{F} = \{f_P \mid P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)\}$ is indexed by points of $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$, where $\mathrm{Spec}(\mathbb{I})$ is a finite reduced irreducible covering of $\mathrm{Spec}(\Lambda)$ (giving an irreducible component of the 'big' ordinary Hecke algebra of prime-to-p level N; see the discussion after the following theorem for a definition of the family). For each $P \in \mathrm{Spec}(\mathbb{I})$, f_P is a p-adic modular form of slope 0 of level Np^{r+1} for a fixed prime to p-level N ($p \nmid N$) with a suitable exponent r. The family is called analytic because $a(n) : P \mapsto a(n, f_P)$ is a p-adic analytic function on $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$ belonging to the structure sheaf $\mathbb{I}$. Often we write the value at P of this function as $a_P(n)$; so, $a_P(n) = P(a(n)) \in \overline{\mathbb{Q}}_p$ if $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ (regarding P as a W-algebra homomorphism $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$). We call P *arithmetic* of weight $k = k(P) \in \mathbb{Z}$ with character $\varepsilon_P : \Gamma \to \mu_{p^\infty}(\mathbb{C}_p)$ if P kills $(1 + T - \varepsilon_P(\gamma)\gamma^k) \in \Lambda$ and $k(P) \geqslant 1$ (so the weight of the associated cusp form is $k(P) + 1 \geqslant 2$). We write $p^{r(P)}$ for the order of ε_P (so, f_P has level $Np^{r(P)+1}$). If P is arithmetic, f_P is known to be a p-stabilized classical Hecke eigenform and has Neben character ψ_P whose restriction to Γ is given by ε_P. After the work of Shimura and Deligne, it is proven in [13] (see also [29, Proposition 1 in Section 9]) that for each $P \in \mathrm{Spec}(\mathbb{I})$, we have a p-adic semi-simple Galois representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\kappa(P))$ with coefficients in the residue field $\kappa(P)$ of P unramified outside Np such that $\mathrm{Tr}(\rho_P(Frob_l)) = a(l, f_P)$ for all primes outside Np. Write $\rho_{\mathbb{I}}$ for ρ_P if $P = (0)$ and $\overline{\rho}$ for $\rho_{\mathfrak{m}}$ for the maximal ideal $\mathfrak{m}$ of $\mathbb{I}$. The representation $\overline{\rho}$ has values in $GL_2(\mathbb{F})$ for the finite field $\mathbb{F} = \kappa(\mathfrak{m})$. For each p-decomposition subgroup $D \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the restriction $\rho_P|_D$ is known to be isomorphic to a reducible upper triangular representation with unramified quotient character (e.g., [16, Theorem 4.2.6]). Consider the adjoint representation $Ad(\rho_P)$ realized on the trace zero subspace in $\mathfrak{sl}_2(\kappa(P)) \subset M_2(\kappa(P))$ by conjugation action: $x \mapsto \rho_P(\sigma)x\rho_P(\sigma)^{-1}$. Since ρ_P is motivic and $Ad(\rho_P)$ is critical at $s = 1$ if P is arithmetic, $Ad(\rho_P)$ has an $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho_P))$ defined by Greenberg [7] (see also [20, Section 1.5.2]). Thus we get a function $P \mapsto \mathcal{L}(Ad(\rho_P))$ defined on the set of arithmetic points of $\mathrm{Spec}(\mathbb{I})$. This function can be interpolated analytically on $\mathrm{Spec}(\mathbb{I})$, and we just write $P \mapsto \mathcal{L}(Ad(\rho_P))$ for this analytic function (as in Theorem 2.4 in Section 2.5). We prove in this paper

Theorem. *Let $p > 2$. Either suppose smoothness at an arithmetic point P_0 of the ordinary deformation space of ρ_{P_0} (an almost known conjecture; see Conjecture 2.2 below) or define the $\mathbb{I}$-adic $\mathcal{L}$-invariant as in Section 2.5. Then the analytic function $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p) \ni P \mapsto \mathcal{L}(Ad(\rho_P))$ is constant if and only if the family $\mathcal{F}$ has complex multiplication.*

After recalling the definition of the 'big' cuspidal Hecke algebra, we prove the theorem in Section 1.1 assuming a formula of $\mathcal{L}(Ad(\rho_P))$ in terms of a derivative of the analytic function $a(p)$ with respect to T. We recall from [18] the proof of the derivative formula in Section 2. By this theorem, if $\mathcal{F}$ is a non-CM family, $P \mapsto \mathcal{L}(Ad(\rho_P))$ is a non-constant analytic function; so, except for finitely many modular adjoint Galois representations in the family, the conjecture of Greenberg (see [7]) predicting the non-vanishing of $\mathcal{L}(Ad(V))$ is true. The importance of characterizing CM families out of general

p-adic analytic families of cusp forms will be discussed in the last section, where one can find also an overview of such characterizations (including a list of known/expected characterizations).

1. Big cuspidal Hecke algebra and Galois representation

Before proving the theorem, here is a concise definition of p-adic analytic families of slope 0. Fix a prime $p \geqslant 3$, field embeddings $\mathbb{C} \overset{i_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ and a positive integer N prime to p. We sometimes identify $\mathbb{C}_p$ and $\mathbb{C}$ by a fixed field isomorphism compatible with the above embeddings. Consider the space of cusp forms $S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ with $(p \nmid N, r \geqslant 0)$. Let the rings $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively. The Hecke algebra $h = h_{k+1}(\Gamma_0(Np^{r+1}), \psi; \mathbb{Z}[\psi])$ over $\mathbb{Z}[\psi]$ is

$$h = \mathbb{Z}[\psi]\big[T(n) \bigm| n = 1, 2, \ldots\big] \subset \operatorname{End}\big(S_{k+1}\big(\Gamma_0\big(Np^{r+1}\big), \psi\big)\big).$$

For any $\mathbb{Z}[\psi]$-algebra $A \subset \mathbb{C}$, $h_{k+1}(\Gamma_0(Np^{r+1}), \psi; A) = h \otimes_{\mathbb{Z}[\psi]} A$ is actually the subalgebra generated over A by $T(l)$'s in $\operatorname{End}(S_{k+1}(\Gamma_0(Np^{r+1}), \psi))$. Simply we write

$$h_{k+1,\psi} = h_{k+1,\psi/W} = h_{k+1}\big(\Gamma_0\big(Np^{r+1}\big), \psi; W\big) := h \otimes_{\mathbb{Z}[\psi]} W$$

for a p-adic discrete valuation ring $W \subset \overline{\mathbb{Q}}_p$ containing $\mathbb{Z}_p[\psi]$. Sometimes our $T(p)$ is written as $U(p)$ as the level is divisible by p. The ordinary part $\mathbf{h}_{k+1,\psi/W} \subset h_{k+1,\psi/W}$ is then the maximal ring direct summand on which $U(p)$ is invertible. We write e for the idempotent of $\mathbf{h}_{k+1,\psi/W}$; so, e is the p-adic limit in $h_{k+1,\psi/W}$ of $U(p)^{n!}$ as $n \to \infty$. By the fixed isomorphism $\mathbb{C}_p \cong \mathbb{C}$, the idempotent e not only acts on the space of modular forms with coefficients in W but also on the classical space $S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$. We write the image of the idempotent as S^{ord}_{k+1} for modular forms and $\mathcal{S}^{ord}_{k+1}$ for cusp forms. Let $\psi_1 = \psi_N \times$ the tame p-part of ψ. Then, as constructed in [12] and [13], we have a unique 'big' Hecke algebra $\mathbf{h} = \mathbf{h}_{\psi_1/W}$ such that

(1) $\mathbf{h}$ is free of finite rank over $\Lambda := W[[T]]$ equipped with $T(n) \in \mathbf{h}$ for all n,
(2) if $k \geqslant 1$ and $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a character,

$$\mathbf{h}/\big(1 + T - \psi(\gamma)\varepsilon(\gamma)\gamma^k\big)\mathbf{h} \cong \mathbf{h}_{k+1,\varepsilon\psi_k} \quad (\gamma = 1 + p) \text{ for } \psi_k := \psi_1\omega^{1-k},$$

sending $T(n)$ to $T(n)$, where ω is the Teichmüller character.

We often identify Λ with the completed group algebra $W[[\Gamma]]$ of $\Gamma = 1 + p\mathbb{Z}_p$ generated by $\gamma = (1 + p)$ by $(1 + T) \leftrightarrow \gamma$.

Let Q be the quotient field of Λ. Each (reduced) irreducible component $\operatorname{Spec}(\mathbb{I}) \subset \operatorname{Spec}(\mathbf{h})$ has a two-dimensional semi-simple Galois representation $\rho_{\mathbb{I}}$ (of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) with coefficients in the quotient field $Q_{\mathbb{I}}$ of $\mathbb{I}$ which is a finite extension of Q (see [13]). This representation preserves an $\mathbb{I}$-lattice $L \subset Q_{\mathbb{I}}^2$ (i.e., an $\mathbb{I}$-submodule of $Q_{\mathbb{I}}$ of finite type which spans $Q_{\mathbb{I}}^2$ over $Q_{\mathbb{I}}$), and as a map of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into the profinite group $\operatorname{Aut}_{\mathbb{I}}(L)$, it is continuous. Write $a(l)$ for the image of $T(l)$ $(l \nmid Np)$ in $\mathbb{I}$ and $a(p)$ for the image of $U(p)$. The representation $\rho_{\mathbb{I}}$ restricted to the p-decomposition group D is reducible with nontrivial unramified quotient. We write $\rho_{\mathbb{I}}^{ss}$ for its semi-simplification over D. As is well known now (e.g., [16, Section 4.2]), $\rho_{\mathbb{I}}$ satisfies

$$\operatorname{Tr}\big(\rho_{\mathbb{I}}(Frob_l)\big) = a(l) \quad (l \nmid Np),$$

$$\rho_{\mathbb{I}}^{ss}\big([\gamma^s, \mathbb{Q}_p]\big) \sim \left(\begin{pmatrix} (1+T)^s & 0 \\ 0 & 1 \end{pmatrix}\right) \quad \text{and} \quad \rho_{\mathbb{I}}^{ss}\big([p, \mathbb{Q}_p]\big) \sim \left(\begin{pmatrix} * & 0 \\ 0 & a(p) \end{pmatrix}\right), \tag{Gal}$$

where $\gamma^s = (1+p)^s \in \mathbb{Z}_p^\times$ for $s \in \mathbb{Z}_p$ and $[x, \mathbb{Q}_p]$ is the local Artin symbol. We call a prime ideal $P \subset \mathbb{I}$ a *prime divisor* if $\mathrm{Spec}(\mathbb{I}/P)$ has codimension 1 in $\mathrm{Spec}(\mathbb{I})$. If a prime divisor P of $\mathrm{Spec}(\mathbb{I})$ contains $(1+T-\varepsilon\psi_k(\gamma)\gamma^k)$ with $k \geqslant 1$, we therefore have a Hecke eigenform $f_P \in S_{k+1}(\Gamma_0(Np^{r(P)+1}), \varepsilon\psi_k)$ such that its eigenvalue for $T(n)$ is given by $a_P(n) := (T(n)|_{\mathbb{I}} \bmod P) \in \overline{\mathbb{Q}}_p$ for all n. A prime divisor P with $P \cap \Lambda = (1+T-\varepsilon\psi_k(\gamma)\gamma^k)$ with $k \geqslant 1$ is called an *arithmetic* point (or prime). We write $\varepsilon_P = \varepsilon$ and $k(P) = k \geqslant 1$ for an arithmetic P. Thus $\mathbb{I}$ gives rise to an analytic family $\mathcal{F}_{\mathbb{I}} = \{f_P \mid \text{arithmetic } P \in \mathrm{Spec}(\mathbb{I})\}$. A (cuspidal) component $\mathbb{I}$ is called a *CM component* if there exists a nontrivial character $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{I}^\times$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \xi$. Then ξ is necessarily an odd quadratic character; so, $M := \overline{\mathbb{Q}}^{\mathrm{Ker}(\xi)}$ is an imaginary quadratic field (see [23, (CM1–3) in Section 1]). If a cuspidal $\mathbb{I}$ is not a CM component, we call it a non-CM component.

For each prime $P \in \mathrm{Spec}(\mathbb{I})$, $\mathrm{Tr}(\rho_{\mathbb{I}}) \bmod P$ has values in $\mathbb{I}/P$. Let $\kappa(P)$ be the field of fractions of $\mathbb{I}/P$. As we already mentioned, we have a unique semi-simple Galois representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\kappa(P))$ such that

$$\mathrm{Tr}\big(\rho_P(Frob_l)\big) = \big(a(l) \mod P\big).$$

The easiest way of constructing ρ_P is by the technique of pseudo-representations (though the technique is not logically necessary; see [29, Proposition 1 in Section 9]).

A family has complex multiplication if one of the following equivalent conditions is satisfied (see [23, (CM1–3) in Section 1] and [22, Proposition 3.2] for the equivalence):

(1) there exist an arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$ and a nontrivial Galois character ξ such that $\rho_P \otimes \xi \cong \rho_P$,
(2) for all arithmetic points $P \in \mathrm{Spec}(\mathbb{I})$ and a nontrivial Galois character ξ, we have $\rho_P \otimes \xi \cong \rho_P$.

If the above equivalent conditions are satisfied, ξ cuts out an imaginary quadratic field M so that $\xi = (\frac{M/\mathbb{Q}}{})$, and the above conditions are equivalent to the each of the two following conditions:

(3) there exists an arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$ such that f_P is a theta series of the norm form of M, i.e., $f_P = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) q^{N(\mathfrak{a})}$ for a Hecke character λ of M, where $\mathfrak{a}$ runs over integral ideals of M,
(4) for all arithmetic points $P \in \mathrm{Spec}(\mathbb{I})$, f_P is a theta series of the norm form of M.

Let $F/\mathbb{Q}$ be a totally real finite extension field. Suppose $F \neq \mathbb{Q}$. For each prime factor $\mathfrak{p}|p$ in F, we write $\Gamma_{\mathfrak{p}}$ for the Galois group of cyclotomic $\mathbb{Z}_p$-extension of the local field $F_{\mathfrak{p}}$ (the $\mathfrak{p}$-adic completion of F). Let $\Gamma_F = \prod_{\mathfrak{p}|p} \Gamma_{\mathfrak{p}}$. In the Hilbert modular case for the totally real number field F, the locally cyclotomic nearly ordinary family (with fixed central character) is indexed by $\mathrm{Spec}(\mathbb{I})$ for an integral domain finite flat over the Iwasawa algebra $W[\![\Gamma_F]\!]$. For each point $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, we still have an associated Galois representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\kappa(P))$ (see [20, Sections 2.3.8 and 3.2.8]) and we have an analytic function $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p) \ni P \to \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_P))$ interpolating Greenberg's $\mathcal{L}$ invariant of $\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_P)$ (see [20, Theorem 3.73]). Similarly as above (e.g., [14]), we define CM families of Hilbert modular forms. We may conjecture

Conjecture 1.1. *The analytic function* $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p) \ni P \mapsto \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_P))$ *is constant if and only if the family $\mathcal{F}$ has complex multiplication.*

The conjecture implies that for a non-CM component $\mathbb{I}$, the function $P \mapsto \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_P))$ is non-constant; so, it vanishes only on a very thin subset of $\mathrm{Spec}(\mathbb{I})$.

1.1. Proof of the theorem

Before starting the proof, we prepare a lemma which is a characteristic 0 version of a lemma of Chai [2, Theorem 4.3]. Identify Λ with $\widehat{W[t,t^{-1}]} = \varprojlim_n W[t]/(t^{p^n}-1)$ by $t = 1+T$. Then the formal completion $\widehat{\mathbb{G}}_m$ of the multiplicative group $\mathbb{G}_m$ along the identity $1 \in \mathbb{G}_m(\mathbb{F})$ is given by $\mathrm{Spf}(\Lambda)$. Here note that our $\widehat{\mathbb{G}}_m$ is the formal completion along the characteristic p identity not along $1 \in \mathbb{G}_m(W)$ (i.e., p-adic completion, which could be somehow standard), though the completion of $W[t, t^{-1}]$ with respect to (p, T) and with respect to the filter $\{(t^{p^n} - 1)\}_n$ are actually equal. Therefore we may regard our $\widehat{\mathbb{G}}_m$ as a group functor from the category of pro-artinian local rings over W into the category of groups; so, $\widehat{\mathbb{G}}_m(R) = 1 + \mathfrak{m}_R$ for any pro-artinian local W-algebra R with maximal ideal $\mathfrak{m}_R$. Write $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m = \mathrm{Spf}(W[t, \widehat{t^{-1}, t'}, t'^{-1}])$ for the variable t of the left factor and t' of the right factor, and let $z \in \mathbb{Z}_p^\times$ act on $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ by $(t, t') \mapsto (t^z, t'^z)$ for $z \in \mathbb{Z}_p^\times$, where $t^z = (1+T)^z = \sum_{n=0}^\infty \binom{z}{n} T^n \in W[\![T]\!]$.

Lemma 1.2. *Let $\Phi(t, t') = t'^d + a_1(t)t'^{d-1} + \cdots + a_d(t) \in \Lambda[t']$ with $\Phi(1, 1) = 0$ be an irreducible polynomial over $Q = \mathrm{Frac}(\Lambda)$. Suppose that the integral closure of $\mathbb{Z}_p$ in $\mathbb{J} := W[t, \widehat{t^{-1}, t'}, t'^{-1}]/(\Phi(t, t'))$ is equal to W. If the integral formal closed subscheme $Z = \mathrm{Spf}(\mathbb{J})$ of $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_{m/W}$ of codimension 1 is stable under the action of an open subgroup U of $\mathbb{Z}_p^\times$, we have $\Phi(t, t') = t'^d - t^b$ for $b \in \mathbb{Z}_p$ with $d\mathbb{Z}_p + b\mathbb{Z}_p = \mathbb{Z}_p$ for a p-power d.*

Proof. Write $\mathbb{J} := \Lambda[t']/(\Phi(t, t'))$. Regard $\mathbb{J}$ as embedded in an algebraic closure $\overline{Q}$ of Q. Since $\overline{\mathbb{Q}}_p \cap \mathbb{J} = W$ in $\overline{Q}$, $\mathbb{J} \otimes_W \overline{\mathbb{Q}}_p$ is an integral domain by [30, Section 3.5]. Thus $\mathrm{Spec}(\mathbb{J})$ (and $\mathrm{Spec}(\mathbb{J} \otimes_W \overline{\mathbb{Q}}_p)$) is geometrically irreducible with dominant projection to the right factor $\widehat{\mathbb{G}}_m = \mathrm{Spf}(\Lambda)$. By assumption, $Z \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ is stable under the diagonal action $(t, t') \mapsto (t^s, t'^s)$ for $s \in U$. We may assume that $U = 1 + p^r\mathbb{Z}_p$ for $r > 0$. Since Z is flat of relative dimension 1 over W, replacing W by a finite extension if necessary, we find in Z a W-point $(t_0, t'_0) \in \widehat{\mathbb{G}}_m^2(W)$ of infinite order. Thus we have an infinite set $\Xi = \{(t_0^s, t_0'^s) \mid s \in U\} = \{(t_0 t_0^{p^r u}, t'_0 t_0'^{p^r u}) \mid u \in \mathbb{Z}_p\}$ inside Z. By translation $\tau : (t, t') \mapsto (tt_0^{-1}, t't_0'^{-1})$, we find that $\tau(Z)$ contains

$$\Xi_0 = \{(t_0^{p^r u}, t_0'^{p^r u}) \mid u \in \mathbb{Z}_p\}.$$

Since Z is finite flat over $\mathrm{Spf}(\Lambda)$ with dominant projection to the right factor $\widehat{\mathbb{G}}_m$, by flatness, $\mathbb{J}$ is a free module over Λ, and $\bigcap_{P \in \Xi_0}(P \cap \Lambda)\mathbb{J} = 0$. So $\tau(Z)$ is the Zariski closure of the infinite subgroup Ξ_0 of $\widehat{\mathbb{G}}_m^2$. Then $\tau(Z)$ must be a formal subgroup of $\widehat{\mathbb{G}}_m^2$ of codimension 1, and Z is a coset $(t_0, t'_0)\tau(Z)$ in $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$. Since Z contains the identity $\mathbf{1}$ of $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ as $\Phi(1, 1) = 0$, we must have $\tau(Z) = Z$. Since the left projection $\pi : Z \to \mathrm{Spf}(\Lambda) = \widehat{\mathbb{G}}_m$ is finite flat of degree d, the kernel $\mathrm{Ker}(\pi)$ is a finite flat group scheme over W, which is therefore a finite flat subgroup of $\mu_{p^n} \times \mu_{p^n}$ for sufficiently large n isomorphic to $\mu_{p^m} \times \mu_{p^{m'}}$ (for some $m \leqslant n$ and $m' \leqslant n$). Thus Z is an extension of $\widehat{\mathbb{G}}_m$ by $\mu_{p^m} \times \mu_{p^{m'}}$. By geometric irreducibility of $Z = \mathrm{Spec}(\mathbb{J})$, multiplication by p induces a dominant morphism on $Z \to Z$ and hence $Z[p^\infty]$ is p-divisible. Since $\mathrm{Ker}(\pi)$ is finite flat, $Z[p^\infty]$ is a Barsotti–Tate group fitting into an exact sequence $0 \to \mathrm{Ker}(\pi) \to Z[p^\infty] \to \mu_{p^\infty} \to 0$; so, we may assume that $m' = 0$. By Cartier duality applied to this sequence, the dual of $Z[p^\infty]$ is étale, and hence $Z[p^\infty]$ must be isomorphic to μ_{p^∞}. Therefore $Z \cong \varprojlim_n \mu_{p^n}$ is a formal multiplicative group, and Z is a formal subtorus of $\widehat{\mathbb{G}}_m^2$. Thus writing $X^*(\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^2$ for the formal character group of $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$, Z is the kernel of $\chi \in X^*(\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m)$. Writing $\chi(t, t') = t^a t'^{-b}$, we have $Z = \{(t, t') \mid t^{b'} = t'^a\}$ for $a, b' \in \mathbb{Z}_p$. Thus we may assume that $a = up^n$ for $u \in \mathbb{Z}_p^\times$ by the original form of Φ. Then $Z = \{(t, t') \mid t^b = t'^{d'}\}$ for $d' = p^n$ and $b = b'u^{-1}$. Thus $d = d'd''$ for an integer d'' prime to p. Since we can extract d''-th root inside $1 + \mathfrak{m}_\Lambda$ for the maximal ideal $\mathfrak{m}_\Lambda$ of Λ (as $1 + \mathfrak{m}_\Lambda$ is p-profinite), irreducibility of $\Phi(t, t')$ in $Q[t']$ implies $d = d'$, and $\Phi(t, t') = (t'^d - t^b)u(t, t')$ for a unit $u(t, t') \in W[t, \widehat{t^{-1}, t'}, t'^{-1}]$. Since $\Phi(t, t')$ is irreducible of degree $d = d'$ in $Q[t']$, $u(t, t')$ must be a unit in Λ. Since $\Phi(t, t')$ and $t'^d - t^b$ are monic with respect to t', we conclude $u(t, t') = 1$. This finishes the proof. $\square$

Remark 1.1. In the above proof, the two conditions $\Phi(1, 1) = 0$ and geometric irreducibility are both indispensable. If we do not suppose $\Phi(1, 1) = 0$, for a one-dimensional formal subtorus $G \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$, a coset $(\zeta, \zeta')G$ for $(1, 1) \neq (\zeta, \zeta') \in \mu_{p^m}(W) \times \mu_{p^m}(W)$ is stable under $U = 1 + p^m\mathbb{Z}_p$ but is not a formal subgroup. As a special instance of this example, we have the case when $\Phi(t,t')$ is an irreducible factor $\Psi(t')$ of $t'^{p^n}-1$ (and in this case, $Z=\widehat{\mathbb{G}}_m\times\zeta$ for a root ζ of $\Psi(t')=0$ in W). Similarly if we do not suppose geometric irreducibility of $\mathrm{Spec}(\mathbb{J})$, $Z\subset\widehat{\mathbb{G}}_m\times\widehat{\mathbb{G}}_m$ could be of the form $(\mu_{p^m}\times\mu_{p^r})\cdot G$ (with $m+r>0$) which is not a formal subtorus but a geometrically reducible formal subgroup. In the above example of $\Phi(t,t')=\Psi(t')$, geometric irreducibility of $\mathrm{Spec}(\mathbb{J})$ forces $\Phi(t')=t'-1$ (and hence $d=1$ and $b=0$ under the notation of the above lemma).

We now start the proof of the theorem. Hereafter we assume $p>2$ in the rest of the paper. We may assume that the integral closure of $\mathbb{Z}_p$ in $\mathbb{I}$ is equal to W (by extending scalars if necessary). By Theorem 2.1 (which assumes Conjecture 2.2), $\mathcal{L}(Ad(\rho_P))$ (for densely populated arithmetic points P in $\mathrm{Spec}(\mathbb{I})$) is a constant multiple of

$$\left(a(p)^{-1}\frac{da(p)}{dX}\right)\bigg|_{X=0},$$

if $P\cap\Lambda=(X)$ for $X=\gamma^{-k}\zeta^{-1}t-1$ and $t=1+T$. After proving the theorem assuming this formula, we recall the proof of the formula (though it was proven in [18]). By variable change, we get

$$\left(a(p)^{-1}\frac{da(p)}{dX}\right)\bigg|_{X=0}=\left(a(p)^{-1}t\frac{da(p)}{dt}\right)\bigg|_{t=\zeta\gamma^k}.$$

Thus the constancy of $\mathcal{L}(Ad(\rho_P))$ implies constancy of

$$a(p)^{-1}(1+T)\frac{da(p)}{dT}=a(p)^{-1}t\frac{da(p)}{dt}.$$

Thus $t\frac{da}{dt}=s\cdot a$ for $a(t)=a(p)(t)$ for $s\in W\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. In other words, putting $b(x)=\log_p\circ a(\exp_p(x))$ (for $x=\log_p(t)$), as $dx=\frac{dt}{t}$, we get from the chain rule,

$$\frac{db}{dx}=\frac{da}{dx}\frac{db}{da}=\frac{da}{dx}\frac{d\log_p(a)}{da}=s\cdot a\cdot\frac{1}{a}=s.$$

Thus b is a linear function of x with slope s:

$$\log_p(a)=sx+c\quad\Leftrightarrow\quad a=C\exp_p\big(s\cdot\log_p(t)\big)=Ct^s\quad\big(C=\exp_p(c)\big).$$

Then $a(p)=Ct^s\in F[\![T]\!]$ for $t^s=\exp_p(s\cdot\log_p(t))$, where F is the quotient field of W. Since $a(p)$ is a unit in $\mathbb{I}$, C is a p-adic unit in $W^\times$; so, $\Phi(t):=t^s=C^{-1}a(p)\in\mathbb{I}$. Consider the subalgebra $\mathbb{J}=\Lambda[\Phi(t)]\subset\mathbb{I}$ topologically generated by $C^{-1}a(p)$. We have a surjective W-algebra homomorphism $W[\widehat{t,t^{-1},t',t'^{-1}}]\twoheadrightarrow\mathbb{J}$ sending (t,t') to $(1+T,\Phi(t))$. This surjection gives rise to an integral formal closed subscheme $Z\cong\mathrm{Spf}(\mathbb{J})$ in $\widehat{\mathbb{G}}_m\times\widehat{\mathbb{G}}_m$. Since $C^{-1}a(p)$ is a power series $t^s=\sum_{n=0}^{\infty}\binom{s}{n}T^n\in F[\![T]\!]$ convergent over a closed disk $D_\epsilon=\{x\in F\mid |x|_p\leqslant\epsilon\}$ for some $\epsilon\in|F^\times|_p$, it is in an affinoid ring $F\{T\}$ which is the convergent power series ring over D_ϵ. Since the action $t\mapsto t^z$ for $z\in\mathbb{Z}_p^\times$ extends to $F\{T\}$, we have $\Phi(t^z)=\Phi(t)^z$ in $F\{T\}$ as $\Phi(t)=t^s$. Thus $\Phi(t^z)=\Phi(t)^z$ for all $z\in U=\mathbb{Z}_p^\times$ in $\mathbb{J}$ as we may regard $\mathbb{J}$ as a subring of $F\{T\}$. Then the formal subscheme Z is stable under the action $(t,t')\mapsto(t^z,t'^z)$ of all $z\in U:=\mathbb{Z}_p^\times$ on $\widehat{\mathbb{G}}_m\times\widehat{\mathbb{G}}_m$. Thus we conclude $s\in\mathbb{Q}_p$ from the above lemma. Let us write $f_{P,\zeta}=f_P$ for the eigenform in the family corresponding to a prime P above $(X)=(\gamma^{-k}\zeta^{-1}t-1)$ with $\zeta\in\mu_{p^r}$ to emphasize dependence on ζ. The cusp form $f_{P,\zeta}$ is a Hecke eigenform in $S_k(\Gamma_1(Np^{r+1}))$, and we have $a(p,f_{P,\zeta})=C\gamma^{ks}\zeta^s$. Take $\zeta=1$. Then the p-adic unit $a(p,f_{P,1})=C\gamma^{ks}$ is an algebraic number α. This shows that for any $\zeta\in\mu_{p^\infty}$, $a(p,f_{P,\zeta})=\alpha$ up to p-power roots of unity. Thus the field generated by $a(p, f_{P,\zeta})$ for all $\zeta \in \mu_{p^\infty}$ is a finite extension of $\mathbb{Q}[\mu_{p^\infty}]$. Then by Theorem 3.3 in [23] (see Strong horizontal theorem in Section 3.2 in the text), we conclude that $\mathcal{F}$ is a CM family.

Conversely, we suppose that $\mathcal{F}$ is a CM family. Then we find a Galois character $\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^\times$ for an imaginary quadratic field M such that $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \lambda$ (and hence $\rho_P = \mathrm{Ind}_M^{\mathbb{Q}} \lambda \bmod P$ for all $P \in \mathrm{Spec}(\mathbb{I})$) and λ is unramified at a unique factor $\mathfrak{p}|p$ in M. Then $a(p)$ is the value of the character $\lambda(Frob_{\mathfrak{p}})$ at the Frobenius element $Frob_{\mathfrak{p}}$ at $\mathfrak{p}$. Note that for the class number h of M, taking a generator α of $\mathfrak{p}^h$ and putting $\log_p(\mathfrak{p}) = \frac{1}{h}\log_p(\alpha)$ for the Iwasawa logarithm $\log_p$, we find that $\lambda(Frob_{\mathfrak{p}}) = t^{\log_p(\mathfrak{p})/\log_p(\gamma)}$ up to a root of unity. This shows the constancy of $\mathcal{L}$-invariant for the CM family (see [20, Section 5.3.3] for the description of the constant).

2. Recall of $\mathcal{L}$-invariant

For the completeness of the paper and the reader's convenience, we give a sketch of the proof of the fact that $\mathcal{L}(Ad(\rho_P))$ is proportional to the derivative of $a(p) \in W[\![T]\!]$ (the details can be found in [18]).

Let us start with some words on the history of the invariant. After Mazur, Tate and Teitelbaum [27], many number theorists have proposed diverse definitions of the $\mathcal{L}$-invariant which are expected to give the correction factor relating the conjectural arithmetic part of the leading term of the Taylor expansion of a given p-adic motivic L-function at an exceptional zero to its archimedean counterpart. For an elliptic curve $E_{/\mathbb{Q}}$ with multiplicative or ordinary good reduction modulo p, its p-adic L-function $L_p(s, E)$ has the following evaluation formula at $s = 1$:

$$L_p(1, E) = \left(1 - a_p^{-1}\right)\frac{L_\infty(1, E)}{\text{period}},$$

where $L_\infty(s, E)$ is the archimedean L-function of E, and a_p is the eigenvalue of the arithmetic Frobenius element at p on the unramified quotient of the p-adic Tate module $T(E)$ of E. If E has *split* multiplicative reduction, $a_p = 1$, $L_p(s, E)$ has zero at $s = 1$. This type of zero of a p-adic L-function resulting from the modification Euler p-factor is called an exceptional zero, and it is believed that if the archimedean L-value does not vanish, the order of the zero is the number e of such Euler p-factors; so, in this case, $e = 1$. Then $L'_p(1, E) = \frac{dL_p(s,E)}{ds}|_{s=1}$ is conjectured to be equal to the archimedean value $\frac{L_\infty(1,E)}{\text{period}}$ times an error factor $\mathcal{L}^{an}(E)$, the so-called $\mathcal{L}$-invariant:

$$L'_p(1, E) = \mathcal{L}^{an}(E)\frac{L_\infty(1, E)}{\text{period}}.$$

The problem regarding $\mathcal{L}$-invariants of motives is to find an explicit formula in terms of their p-adic realization without recourse to p-adic L-functions. For $E_{/\mathbb{Q}}$ split multiplicative at p, writing $E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times/q^{\mathbb{Z}}$ for the Tate period $q \in p\mathbb{Z}_p$, the solution conjectured by Mazur, Tate and Teitelbaum and proved by Greenberg and Stevens [8] is

$$\mathcal{L}^{an}(E) = \frac{\log_p(q)}{\mathrm{ord}_p(q)}.$$

Since E is modular, it is associated to an elliptic Hecke eigenform f_E of weight 2 with q-expansion $\sum_{n=1}^\infty a(n, f_E)q^n$. In particular, $a(p, f_E) = a_p = 1$ and $a(1, f_E) = 1$. We can lift f_E to a unique Λ-adic Hecke eigenform $\mathcal{F}$ for a finite flat extension Λ of $\mathbb{Z}_p[\![X]\!]$ (étale around $X = 0$) so that f_E is a specialization of $\mathcal{F}$ at $X = 0$. Then one of the key ingredients of their proof is the following formula:

$$\mathcal{L}^{an}(E) = -2\log_p(\gamma)\frac{da(p)}{dX}\bigg|_{X=0}.$$

Greenberg generalized in [7] the conjectural formula of his $\mathcal{L}$-invariant to general V when V is p-ordinary. We write $\mathcal{L}(V)$ for the $\mathcal{L}$-invariant of Greenberg. Suppose that V is a modular ordinary two-dimensional Galois representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\kappa(P))$. Recall the variable $X = \gamma^{-k}\zeta^{-1}t - 1$ for $\zeta = \varepsilon_P(\gamma)$. Then we have $P \cap W[\![X]\!] = (X)$. For the reader's convenience (and also for completeness), we give a sketch of the proof of the following theorem in [18]:

Theorem 2.1. *Let p be an odd prime, and suppose* Conjecture 2.2 *in the following section. Then we have*

$$\mathcal{L}\big(Ad(\rho_P)\big) = -2\log_p(\gamma)a_P(p)^{-1}\frac{da(p)}{dX}\bigg|_{X=0},$$

where $a_P(p) = (a(p) \bmod P) \in \overline{\mathbb{Q}}_p$.

2.1. Galois deformation

A main ingredient of the proof of Theorem 2.1 is the nearly ordinary Galois deformation theory. Let us recall one of its main results. Since ρ_P is irreducible and $\mathrm{Tr}(\rho_{\mathbb{I}}) \in \mathbb{I}$, by the theory of pseudo-representation, we can arrange $\rho_{\mathbb{I}}$ to have values in $\mathbb{I}_P$. Let $\widehat{\mathbb{I}}_P = \varprojlim_n \mathbb{I}_P/P^n\mathbb{I}_P$. It is known that $\widehat{\mathbb{I}}_P \cong \kappa(P)[\![X]\!]$ (see [20, Proposition 3.78]). The Galois character $\det(\rho_{\mathbb{I}})^{-1}\det(\rho_P)$ has values in the p-profinite group $1+\mathfrak{m}$ for the maximal ideal $\mathfrak{m}$ of $\mathbb{I}$, and hence we have its unique square root χ with values in $1+\mathfrak{m}$. Define a representation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\widehat{\mathbb{I}}_P)$ with $\det(\boldsymbol{\rho}) = \det(\rho_P)$ by $\boldsymbol{\rho}(\sigma) = \chi(\sigma)\rho_{\mathbb{I}}(\sigma)$. Note that $\boldsymbol{\rho} \equiv \rho_{\mathbb{I}} \bmod P$. Fix a decomposition subgroup $D_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at p. Normalize ρ_P so that $\rho_P|_{D_p} = \left(\begin{smallmatrix}\epsilon_P & *\\ 0 & \delta_P\end{smallmatrix}\right)$ with unramified δ_P. Then $\epsilon_P \neq \delta_P$ and ϵ_P is ramified. Note that $\bar{\epsilon} = (\epsilon_P \bmod \mathfrak{m})$ and $\bar{\delta} = (\delta_P \bmod \mathfrak{m})$ are characters (with values in $\mathbb{F}^\times = \kappa(\mathfrak{m})^\times$) of D_p well determined independent of P. We call $\boldsymbol{\rho}$ *p-distinguished* if $\bar{\delta} \neq \bar{\epsilon}$.

Simply write $\kappa := \kappa(P)$. Let S be the set of places of $\mathbb{Q}$ made up of all prime factors of Np and ∞. Consider the deformation functor into sets from the category of local artinian κ-algebras with residue field κ whose value at a local artinian κ-algebra A is given by the set of isomorphism classes of two-dimensional continuous Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(A)$ unramified outside S such that:

(D1) $(\rho_A \bmod \mathfrak{m}_A) \cong \rho_P$ for the maximal ideal $\mathfrak{m}_A$ of A;
(D2) Writing $\iota : \kappa \to A$ for the structure homomorphism of κ-algebras, we have the identity of the determinant characters:

$$\iota \circ \det(\rho_P) = \det(\rho_A);$$

(D3) $\rho_A|_{D_p} \cong \left(\begin{smallmatrix}\epsilon_A & *\\ 0 & \delta_A\end{smallmatrix}\right)$ with $\delta_A \equiv \delta_P \bmod \mathfrak{m}_A$.

The condition (D3) is the near ordinarity, and we call the character δ_A of D_p the *nearly ordinary character* of ρ_A. By the work started by Wiles and Taylor followed by Diamond, Fujiwara, Skinner and Wiles and Kisin, we know the following fact in almost all cases (cf. [20, Corollary 3.77 and Proposition 3.78]).

Conjecture 2.2. *If P is arithmetic, the above functor is pro-represented by the pair $(\widehat{\mathbb{I}}_P, \boldsymbol{\rho})$. Thus in particular, we have $\boldsymbol{\rho}|_{D_p} \cong \left(\begin{smallmatrix}\boldsymbol{\epsilon} & *\\ 0 & \boldsymbol{\delta}\end{smallmatrix}\right)$ with nearly ordinary character $\boldsymbol{\delta}$ of $\boldsymbol{\rho}$.*

Note here $\widehat{\mathbb{I}}_P \cong \kappa(P)[\![X]\!]$. The conjecture holds at least in the following cases:

(1) ρ_P is p-distinguished and $\overline{\rho}$ is absolutely irreducible over $\mathbb{Q}[\sqrt{p^*}]$ for $p^* = (-1)^{(p-1)/2}p$ (e.g., essentially the original $R = T$ theorem of Wiles and Taylor; see [5], [17, Theorem 5.29] and [20, Proposition 3.78]).

(2) This is via the theory of eigenvariety including the identification of the rigid-analytic full (dimension 3) Galois deformation space with the maximal spectrum of the full Hecke algebra (p-inverted) by Gouvêa and Mazur, Coleman and Mazur, Kisin and Chenevier (see [25, Theorem 8.2], [3] and [4]). Supposing that $\overline{\rho}$ is absolutely irreducible, the conjecture holds if the slope $< k(P)/2$, since the eigenvariety is étale around P (with absolutely irreducible $\overline{\rho}$) over the arithmetic point below P on the weight space; see also [24, Theorem 11.10].

(3) $\overline{\rho} \cong \chi_1 \oplus \chi_2$ for two Galois characters with $\chi_1|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])} \neq \chi_2|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])}$ [24, Theorem 11.10] and [25, Theorem 8.2].

The only remaining case is when $\overline{\rho}$ is reducible and $\chi_1|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])} = \chi_2|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])}$ (which means p-distinguishedness as the representation is odd). In this exceptional case, the result of Skinner and Wiles [32] applies (so, at least $\widehat{\mathbb{I}}_P$ is equal to the reduced part of the universal ring). The case (1) is the combination of the result in [5] and the method in [20, Chapter 3]. Indeed, if we have an $R = T$ theorem for one arithmetic point P, it extends to the entire connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ containing $\mathrm{Spec}(\mathbb{I})$. The specific one point P with $R = T$ is essentially covered either by [5] or [20, Chapter 3] (especially Section 3.2.4). In the cases (2) and (3), the full Galois deformation ring $\mathcal{R}$ around an arithmetic point is smooth over the weight space by the work quoted there. Since the full deformation ring does not have information of $a(p)$ (or $U(p)$), the natural projection $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{R})$ (for the ordinary deformation ring R) has image on one of the two leaves (of the infinite fern) crossing at P if the slope $\mathrm{ord}_p(a_P(p))$ is 0 (or more generally, $\mathrm{ord}_p(a_P(p))$ is less than $k(P)/2$); so, it is étale around the point $\widetilde{P} \in \mathrm{Spec}(R)$ coming from the arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$. Thus we have $\mathrm{Spec}(R_{\widetilde{P}}) \xrightarrow{\text{étale}} \mathrm{Spec}(\mathcal{R}_P) \xrightarrow{\text{smooth}} \mathrm{Spec}(W[\![T]\!]_P)$, which shows that the composite is étale, and hence $\widehat{R}_{\widetilde{P}} = \widehat{\mathbb{I}}_P = \kappa(P)[\![X]\!]$.

In the following sections, we shall start with a brief review of the definition by Greenberg of the Selmer group and the $\mathcal{L}$-invariant of the adjoint square of a two-dimensional modular ordinary p-adic Galois representation. After the review, we shall give a sketch of the proof of Theorem 2.1.

2.2. Selmer groups

We shall describe the definition due to Greenberg of his Selmer group associated to the adjoint square Galois representation. For simplicity, we assume that $S = \{p, \infty\}$; so, $N = 1$ (see [18] for the general case without this assumption). We may assume that κ has p-adic integer ring W. Let $\mathbb{Q}^S$ be the maximal extension unramified outside S. Let $M/\mathbb{Q}$ be a subfield of $\mathbb{Q}^S$. All Galois cohomology groups are continuous cohomology groups as defined in [17, 4.3.3]. We write $\mathfrak{p}$ for a prime of M over p and $\mathfrak{q}$ for general primes of M. Write $\mathfrak{G}_M^S = \mathrm{Gal}(\overline{\mathbb{Q}}^S/M)$ and $I_\mathfrak{p}$ for the inertia subgroup of the decomposition subgroup $D_\mathfrak{p} \subset \mathfrak{G}_M^S$.

Write now V for the two-dimensional vector space over κ with a continuous action of $\mathfrak{G}^S$ associated to the Hecke eigen cusp form f_P. We let $\mathfrak{G}^S$ act on $\mathrm{End}_\kappa(V)$ by conjugation and define $Ad(V) = \mathfrak{sl}(V) \subset \mathrm{End}_\kappa(V)$ by the trace 0 subspace of dimension 3.

We assume given a filtration as in (D3):

$$V \supsetneq F^+V \supsetneq \{0\} \tag{ord}$$

stable under the decomposition group D_p such that the inertia group $I_p \subset D_p$ acts on the quotient V/F^+V by δ_P. We take a basis of V compatible with the filtration (ord) so that $\rho_P|_{D_p}$ is upper triangular. Then $Ad(V)$ has the following three step filtration stable under D_p:

$$Ad(V) \supset F^-Ad(V) \supset F^+Ad(V) \supset \{0\}, \tag{F}$$

where

$$F^-Ad(V) = \{\phi \in Ad(V) \mid \phi(F^+V) \subset F^+V\} \quad \text{(upper triangular)},$$

$$F^+Ad(V)=\{\phi\in Ad(V)\mid\phi(F^+V)=0\}\quad\text{(upper nilpotent)}.$$

Note that D_p acts trivially on $F^-Ad(V)/F^+Ad(V)$; so, $F^-Ad(V)/F^+Ad(V)\cong\kappa$ as D_p-modules; so, the p-adic L-function of $Ad(V)$ has an exceptional zero at $s=1$.

For each prime $\mathfrak{p}|p$ of M, we put

$$U_{\mathfrak{p}}(Ad(V))=\mathrm{Ker}\left(\mathrm{Res}:H^1(D_{\mathfrak{p}},Ad(V))\to H^1\left(I_{\mathfrak{p}},\frac{Ad(V)}{F^+(Ad(V))}\right)\right).$$

Then we define

$$\mathrm{Sel}_M(Ad(V))=\mathrm{Ker}\left(H^1(\mathfrak{G}_M^S,Ad(V))\to\prod_{\mathfrak{p}}\frac{H^1(D_{\mathfrak{p}},Ad(V))}{U_{\mathfrak{p}}(Ad(V))}\right).\tag{2.1}$$

Replacing $U_{\mathfrak{p}}(Ad(V))$ by the bigger

$$U_{\mathfrak{p}}^-(Ad(V))=\mathrm{Ker}\left(\mathrm{Res}:H^1(D_{\mathfrak{p}},Ad(V))\to H^1\left(I_{\mathfrak{p}},\frac{Ad(V)}{F^-(Ad(V))}\right)\right)$$

for $\mathfrak{p}|p$, we define a bigger "minus" Selmer group $\mathrm{Sel}_M^-(Ad(V))\supset\mathrm{Sel}_M(Ad(V))$.

Taking the Tate-dual $Ad(V)^*(1)=\mathrm{Hom}_\kappa(Ad(V),\kappa)(1)$ with single Tate twist, and the filtration dual to (F), we define the dual Selmer group $\mathrm{Sel}_M(Ad(V)^*(1))$.

Assumed Conjecture 2.2 implies the following fact in [7] necessary to define $\mathcal{L}(Ad(V))$:

Lemma 2.3. *Suppose $(\widehat{\mathbb{I}}_P,\boldsymbol{\rho})$ is universal. We have $\mathrm{Sel}_{\mathbb{Q}}(Ad(V))=0$ and*

$$\mathrm{Sel}_{\mathbb{Q}}(Ad(V))=\mathrm{Sel}_{\mathbb{Q}}(Ad(V)^*(1))=0.\tag{V}$$

In the earlier article [18], the balanced Selmer group $\overline{\mathrm{Sel}}_{\mathbb{Q}}$ (see [7, (16)] and [20, Section 1.5.1]) is used to prove this type of result. However, by definition, we have $\mathrm{Sel}_{\mathbb{Q}}(Ad(V))\supset\overline{\mathrm{Sel}}_{\mathbb{Q}}(Ad(V))$ and by duality $\mathrm{Sel}_{\mathbb{Q}}(Ad(V)^*(1))\subset\overline{\mathrm{Sel}}_{\mathbb{Q}}(Ad(V)^*(1))$. On the other hand, by Greenberg [7, Proposition 2], we have

$$\dim\overline{\mathrm{Sel}}_{\mathbb{Q}}(Ad(V))=\dim\overline{\mathrm{Sel}}_{\mathbb{Q}}(Ad(V)^*(1)),$$

and hence, to prove the vanishing of all such Selmer groups, we only need to show $\mathrm{Sel}_{\mathbb{Q}}(Ad(V))=0$.

Proof of Lemma 2.3. Here is a sketch of the proof. For any derivation $\partial:\widehat{\mathbb{I}}_P\to\kappa$, consider $c_P:=(\partial\boldsymbol{\rho})\rho_P^{-1}:\mathfrak{G}^S\to\mathrm{End}(V)$. Applying ∂ to $\boldsymbol{\rho}(\sigma)\boldsymbol{\rho}(\tau)=\boldsymbol{\rho}(\sigma\tau)$, we verify c_∂ is cocycle. Since $\det(\boldsymbol{\rho})$ is constant, c_P has values in $Ad(V)$. Since $\boldsymbol{\rho}|_{D_p}$ is upper triangular, $[c_P]\in\mathrm{Sel}_{\mathbb{Q}}^-(Ad(V))$. By universality, any such cocycle is of the form c_∂. Thus the tangent space $\mathcal{T}_P\cong\kappa$ of $\mathrm{Spec}(\widehat{\mathbb{I}}_P)$ at P is isomorphic to $\mathrm{Sel}_{\mathbb{Q}}^-(Ad(V))$. Since the diagonal entry of c_∂ is nontrivial, $\mathrm{Sel}_{\mathbb{Q}}(Ad(V))$ is a proper subspace of one-dimensional subspace of $\mathrm{Sel}_{\mathbb{Q}}^-(Ad(V))$; so, it vanishes. The dual Selmer group corresponds to the value at the counter-part of the functional equation of $L(s,Ad(\rho_P))$ of the value corresponding to the original Selmer group. Indeed, Greenberg proved by cohomological computation that $\dim_\kappa\overline{\mathrm{Sel}}_{\mathbb{Q}}(Ad(V))=\dim_\kappa\overline{\mathrm{Sel}}_{\mathbb{Q}}(Ad(V)^*(1))$; so, the desired vanishing also follows for the dual. □

We write S for the set of ramified primes for V including p. We have the Poitou–Tate exact sequence (e.g., [17, Theorem 4.50 (5)]):

$$0\to\mathrm{Sel}_{\mathbb{Q}}(Ad(V))\to H^1(\mathfrak{G}_{\mathbb{Q}}^S,Ad(V))\to\frac{H^1(D_p,Ad(V))}{U_p(Ad(V))}\to\mathrm{Sel}_{\mathbb{Q}}(Ad(V)^*(1))^*.$$

Thus by (V), we have

$$H^1\big(\mathfrak{G}^S, Ad(V)\big) \cong \frac{H^1(D_p, Ad(V))}{U_p(Ad(V))}. \tag{I}$$

2.3. Greenberg's $\mathcal{L}$-invariant

By definition, for the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$, the restriction map $\mathrm{Res} : H^1(\mathfrak{G}^S, Ad(V)) \to H^1(\mathfrak{G}^S_{\mathbb{Q}_\infty}, Ad(V))$ brings the one-dimensional space $\mathrm{Sel}^-_{\mathbb{Q}}(Ad(V))$ into $\mathrm{Sel}_{\mathbb{Q}_\infty}(Ad(V))$, which is the cause of the existence of the exceptional zero for the characteristic power series of the Selmer group over $\mathbb{Q}_\infty$, and philosophically, one should be able to determine the $\mathcal{L}$-invariant via the cocycle giving a generator of $\mathrm{Sel}^-_{\mathbb{Q}}(Ad(V))$, which Greenberg did. Greenberg defined in [7] his invariant $\mathcal{L}(Ad(V))$ in the following way. Write $F^-H^1(D_p, Ad(V))$ for the image of $H^1(D_p, F^-Ad(V))$ in $H^1(D_p, Ad(V))$. By the definition of $U_p(Ad(V))$, the subspace $\frac{F^-H^1(D_p,Ad(V))}{U_p(Ad(V))}$ inside the right-hand side of (I) is isomorphic to $\mathrm{Sel}^-_{\mathbb{Q}}(Ad(V)) \cong \kappa$. Namely, we have

$$\mathrm{Sel}^-_{\mathbb{Q}}\big(Ad(V)\big) \xrightarrow[\mathrm{Res}]{\sim} \frac{F^-H^1(D_p, Ad(V))}{U_p(Ad(V))} \subset \frac{H^1(D_p, Ad(V))}{U_p(Ad(V))}.$$

By projecting down to $F^-Ad(V)/F^+Ad(V) \cong \kappa$ with trivial D_p-action, cocycles in $\mathrm{Sel}^-_{\mathbb{Q}}(Ad(V))$ give rise to a subspace L of

$$\mathrm{Hom}\big(D_p^{ab}, F^-Ad(V)/F^+Ad(V)\big) = \mathrm{Hom}\big(D_p^{ab}, \kappa\big).$$

Note that

$$\mathrm{Hom}\big(D_p^{ab}, \kappa\big) \cong \kappa \times \kappa$$

canonically by $\phi \mapsto (\frac{\phi([u,\mathbb{Q}_p])}{\log_p(u)}, \phi([p,\mathbb{Q}_p]))$ for any $u \in \mathbb{Z}_p^\times$ of infinite order. Here $[x, \mathbb{Q}_p]$ is the local Artin symbol (suitably normalized).

If a cocycle c representing an element in $\mathrm{Sel}^-_{\mathbb{Q}}(Ad(V))$ is unramified, it gives rise to an element in $\mathrm{Sel}_{\mathbb{Q}}(Ad(V))$. By the vanishing (V) of $\mathrm{Sel}_{\mathbb{Q}}(Ad(V))$, this implies $c = 0$; so, the projection of L to the first factor κ (via $\phi \mapsto \phi([u, \mathbb{Q}_p])/\log_p(u)$) is surjective. Thus this subspace L is a graph of a κ-linear map

$$\mathcal{L} : \kappa \to \kappa,$$

which is given by the multiplication by an element $\mathcal{L}(Ad(V)) \in \kappa$.

2.4. Proof of Theorem 2.1

Write $\boldsymbol{\rho}|_{D_p} \cong \left(\begin{smallmatrix} \boldsymbol{\epsilon} & * \\ 0 & \boldsymbol{\delta} \end{smallmatrix}\right)$ with nearly ordinary character $\boldsymbol{\delta}$. We know that c_∂ for $\partial = \frac{d}{dX}$ gives a nontrivial element in $\mathrm{Sel}^-(Ad(V))$. The image of c_∂ in $\mathrm{Hom}(D_p^{ab}, \kappa)$ is $\delta_P^{-1}\partial\boldsymbol{\delta}|_{X=0}$. We know that

$$\delta_P^{-1}\boldsymbol{\delta}\big([p, \mathbb{Q}_p]\big) = a_P(p)^{-1}a(p) \quad \text{and} \quad \delta_P^{-1}\boldsymbol{\delta}\big([u, \mathbb{Q}_p]\big) = \big(\zeta\gamma^k\big)^{-\log_p(u)/2\log_p(\gamma)} t^{\log_p(u)/2\log_p(\gamma)}$$

for $k = k(P)$ and $\zeta - \varepsilon_P(\gamma)$ by our construction. Then to get the desired result is just a simple computation (done in [18]).

2.5. $\mathbb{I}$-adic $\mathcal{L}$-invariant

We can go through a similar argument to define the $\mathcal{L}$-invariant of $Ad(\rho_{\mathbb{I}})$ which interpolate $\mathcal{L}(Ad(\rho_P))$. Let Q be the quotient field of $\mathbb{I}$, and regard $\boldsymbol{\rho}$ as a representation into $GL_2(Q)$, writing $\mathbb{V}$ for its space. Taking $\partial = t\frac{d}{dt}$, we can think of $\mathbf{c} = \boldsymbol{\rho}^{-1}\partial\boldsymbol{\rho} : \mathfrak{G}^S \to Ad(\mathbb{V})$. Fixing a basis of $\mathbb{V}$ so that $\boldsymbol{\rho}|_{D_p} = \left(\begin{smallmatrix}\epsilon & * \\ 0 & \delta\end{smallmatrix}\right)$, again we have three step filtration

$$Ad(\mathbb{V}) \supset F^- Ad(\mathbb{V}) \supset F^+ Ad(\mathbb{V}) \supset 0$$

in exactly the same manner as in (F). Note that $\mathbf{c}$ is a 1-cocycle of $\mathfrak{G}^S$ with values in $Ad(\mathbb{V})$ whose restriction to D_p has values in $F^- Ad(\mathbb{V})$. Note that the cohomology class of $\mathbf{c}$ depends only on $\rho_{\mathbb{I}}$ not the character twist $\boldsymbol{\rho}$ which is dependent on P. Then considering $\mathbf{c}_{ab} = \mathbf{c}|_{D_p} \bmod F^+ Ad(\mathbb{V})$, we get $\mathbf{c}_{ab} \in \mathrm{Hom}(D_p^{ab}, Q) \cong Q \times Q$. Then we define

$$\mathcal{L}\big(Ad(\rho_{\mathbb{I}})\big) = \mathcal{L}\big(Ad(\mathbb{V})\big) = \frac{\mathbf{c}_{ab}([p, \mathbb{Q}_p])}{\frac{\mathbf{c}_{ab}([\gamma, \mathbb{Q}_p])}{\log_p(\gamma)}}. \tag{L}$$

By definition, if P is arithmetic, the construction of $\mathcal{L}(Ad(\mathbb{V}))$ can be done inside $\mathbb{I}_P$, since $\boldsymbol{\rho}$ has values in $GL_2(\mathbb{I}_P)$. Thus $\mathcal{L}(Ad(\mathbb{V})) \in \mathbb{I}_P$, and hence $\mathcal{L}(Ad(\mathbb{V}))(P) = (\mathcal{L}(Ad(\mathbb{V})) \bmod P)$ is a well-defined number in the residue field $\kappa(P) \subset \overline{\mathbb{Q}}_p$ of P. If further $(\widehat{\mathbb{I}}_P, \boldsymbol{\rho})$ is universal, $\mathcal{L}(Ad(\mathbb{V}))(P) = \mathcal{L}(Ad(\rho_P))$. In any case, we can take (L) to be the definition of $\mathcal{L}(Ad(\rho_{\mathbb{I}}))$ (which does not require any assumption except for the existence of $\rho_{\mathbb{I}}$), and we get the following result from the proof of Theorem 2.1 without assuming any conjecture.

Theorem 2.4. *We have, for $a = a(p) \in \mathbb{I}$,*

$$\mathcal{L}\big(Ad(\rho_{\mathbb{I}})\big) = -2\log_p(\gamma)a^{-1}t\frac{da}{dt}.$$

This is the meaning of analytic continuation of the adjoint $\mathcal{L}$-invariant described before the theorem in the Introduction.

3. Motivation

The constancy of $\mathcal{L}(Ad(\rho_P))$ characterizes CM components. In this last section, we overview known characterization of CM components. This section is an attempt to convince the reader the importance of such characterization.

3.1. Congruence criterion and the $\mathbb{I}$-adic $\mathcal{L}$-invariant

We generalize our construction of $\mathbf{h}$ to cover modular forms including Eisenstein series. We repeat the definition in the Introduction replacing cusp forms by modular forms. Consider the space of modular forms $M_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ with $(p \nmid N, r \geqslant 0)$ (including Eisenstein series). Let the rings $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively. The Hecke algebra over $\mathbb{Z}[\psi]$ is $H = \mathbb{Z}[\psi][T(n) \mid n = 1, 2, \ldots] \subset \mathrm{End}(M_{k+1}(\Gamma_0(Np^{r+1}), \psi))$. We put $H_{k+1,\psi} = H_{k+1,\psi/W} = H \otimes_{\mathbb{Z}[\psi]} W$ for a p-adic discrete valuation ring $W \subset \overline{\mathbb{Q}}_p$ containing $\mathbb{Z}_p[\psi]$. Sometimes our $T(p)$ is written as $U(p)$ as the level is divisible by p. The ordinary part $\mathbf{H}_{k+1,\psi/W} \subset H_{k+1,\psi/W}$ is the maximal ring direct summand on which $U(p)$ is invertible. Let $\psi_1 = \psi_N \times$ the tame p-part of ψ. Then, we have a unique 'big' Hecke algebra $\mathbf{H} = \mathbf{H}_{\psi_1/W}$ such that

(1) $\mathbf{H}$ is free of finite rank over $\Lambda := W[\![T]\!]$ equipped with $T(n) \in \mathbf{H}$ for all n,
(2) if $k \geqslant 1$ and $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a character,

$$\mathbf{H}/\big(1+T-\psi(\gamma)\varepsilon(\gamma)\gamma^k\big)\mathbf{H} \cong \mathbf{H}_{k+1,\varepsilon\psi_k} \quad (\gamma = 1+p) \text{ for } \psi_k := \psi_1\omega^{1-k},$$

sending $T(n)$ to $T(n)$, where ω is the Teichmüller character.

A (normalized) Hecke eigenform in $M_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ has slope 0 if $f|U(p) = a \cdot f$ with $|a|_p = 1$. For each irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbf{H})$, again we have a semi-simple Galois representation $\rho_{\mathbb{I}}$ characterized by (Gal). The restriction of the Hecke operators $T(n)$ to the space of cusp forms produces a canonical Λ-algebra surjection $\mathbf{H} \twoheadrightarrow \mathbf{h}$ sending $T(n)$ to $T(n)$. Thus an irreducible component of $\mathrm{Spec}(\mathbf{h})$ can be regarded as an irreducible component of $\mathrm{Spec}(\mathbf{H})$. A component $\mathbb{I}$ is called *cuspidal* if $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$.

By abusing the language, we call a Galois representation ρ *abelian* if there exists an open subgroup $G \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the semi-simplification $(\rho|_G)^{ss}$ has abelian image. We call $\mathbb{I}$ an *abelian component* if $\rho_{\mathbb{I}}$ is abelian. Assuming that $\mathbb{I}$ is normal and that the connected component $\mathrm{Spec}(\mathbb{T}) \subset \mathrm{Spec}(\mathbf{h})$ containing $\mathrm{Spec}(\mathbb{I})$ is Gorenstein, as explained in [17, Section 5.3.6], we have a p-adic L-function

$$L_p = L_p\big(Ad(\rho_{\mathbb{I}})\big) := L_p\big(1, Ad(\rho_{\mathbb{I}})\big) = L_p\big(1, \rho_{\mathbb{I}}^{sym\otimes 2} \otimes \det(\rho_{\mathbb{I}})^{-1}\big) \in \mathbb{I}$$

interpolating

$$L_p(P) := P(L_p) = (L_p \mod P) = \frac{L(1, Ad(\rho_P))}{\text{period}} \quad \text{for all arithmetic } P.$$

Writing $\mathrm{Spec}(\mathbf{h}) = \mathrm{Spec}(\mathbb{I}) \cup \mathrm{Spec}(\mathbb{I}^\perp)$ for the complement $\mathbb{I}^\perp$, we have

$$\mathrm{Spec}(\mathbb{I}) \cap \mathrm{Spec}\big(\mathbb{I}^\perp\big) = \mathrm{Spec}\big(\mathbb{I} \otimes_{\mathbf{h}} \mathbb{I}^\perp\big) \cong \mathrm{Spec}\big(\mathbb{I}/(L_p)\big) \quad \text{(a congruence criterion).}$$

The Gorenstein-ness of $\mathbb{T}$ is known to be true if $\rho_{\mathfrak{m}}$ is irreducible with $\delta \neq \epsilon$, where $\rho_{\mathfrak{m}}|_{D_p} = \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ (see [29, Section 9] and [22, Section 4]).

Congruence modulo a prime l of a fixed Hecke eigenform $f \in S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ with another Hecke eigenform g in the same space is controlled by the canonical integer part $c(f)$ of the adjoint L-value $L(1, Ad(f))$ (i.e., basically $l|c(f) \Leftrightarrow$ the existence of g with $g \equiv f \mod \mathfrak{l}$ for a prime $\mathfrak{l}|l$). Thus $c(f)$ is a product of the prime factors of the discriminant/conductor of the Hecke algebra in $\mathbb{Q}(f)$ and more if we have g non-conjugate to f. This fact was first discovered by the author in [9] and [10] and developed into the congruence criterion in the above form in [12, (0.7)]. See [22, Sections 5 and 6] and [17, Section 5.3.6] for an up-to-date treatment of the criterion and the construction of L_p. The above criterion characterizes L_p and at this moment, L_p is well defined up to units in $\mathbb{I}$ (see Conjecture 3.1 how to normalize L_p).

The Gorenstein locus in $\mathrm{Spec}(\mathbb{T})$ is a nonempty open dense subscheme by [26, Theorem 24.6]. By the same argument in the above references, we have the following local version of the congruence criterion. If $\mathrm{Spec}(\mathbb{T}_0) \subset \mathrm{Spec}(\mathbb{T})$ is an open subscheme in the Gorenstein locus and $\mathrm{Spec}(\mathbb{I}_0) = \mathrm{Spec}(\mathbb{T}_0) \cap \mathrm{Spec}(\mathbb{I})$ is normal, we have L_p well defined in $\mathbb{I}_0$ and a local version

$$\mathrm{Spec}(\mathbb{I}_0) \cap \mathrm{Spec}\big(\mathbb{I}_0^\perp\big) = \mathrm{Spec}\big(\mathbb{I}_0 \otimes_{\mathbf{h}} \mathbb{I}_0^\perp\big) \cong \mathrm{Spec}\big(\mathbb{I}_0/(L_p)\big) \quad \text{(a local congruence criterion).}$$

Here $\mathrm{Spec}(\mathbb{I}_0^\perp) = \mathrm{Spec}(\mathbb{I}^\perp) \cap \mathrm{Spec}(\mathbb{T}_0)$. Therefore, if $P \in \mathrm{Spec}(\mathbb{I})$ is in the smooth locus over W and the localization $\mathbb{T}_P$ is Gorenstein, we have L_p well defined in $\mathbb{I}_P$ and an infinitesimal version

$$\mathrm{Spec}(\mathbb{I}_P) \cap \mathrm{Spec}\big(\mathbb{I}_P^\perp\big) = \mathrm{Spec}\big(\mathbb{I}_P \otimes_{\mathbf{h}} \mathbb{I}_P^\perp\big) \cong \mathrm{Spec}\big(\mathbb{I}_P/(L_p)\big) \quad \text{(an infinitesimal congruence criterion).}$$

See [22, Sections 5 and 6] and [17, Section 5.3.6] for the criterion and the construction of L_p.

If we interpolate the adjoint L-values including the cyclotomic variable, i.e., adding a variable s interpolating $L(s, Ad(\rho_P))$ moving s, we need to multiply the L-value by the modifying Euler p-factor. For this enlarged two-variable adjoint L-function, the modifying factor vanishes at $s=1$; so, $L_p(s, Ad(\rho_{\mathbb{I}}))$ has an exceptional zero at $s=1$. See [15, p. 97] where the explicit modifying factor $E_2(Q,P)$ is given, $E_2(Q,P)$ vanishes if $k(P)=k(Q)-1$ with $\eta(p)=1$ under the notation there, and this implies vanishing at the line $s=1$ of the two-variable adjoint p-adic L-function. Therefore, for an $\mathcal{L}$-invariant $0 \neq \mathcal{L}^{an}(Ad(\rho_{\mathbb{I}})) \in \mathbb{I}[\frac{1}{p}]$, we expect to have $L'_p(1, Ad(\rho_{\mathbb{I}})) \stackrel{?}{=} \mathcal{L}^{an}(Ad(\rho_{\mathbb{I}}))L_p$ up to units in $\mathbb{I}$ (in the style of Mazur, Tate and Teitelbaum). As we have seen in the above section, Greenberg proposed a definition of a number $\mathcal{L}(Ad(\rho_P))$ conjectured to be equal to $\mathcal{L}^{an}(Ad(\rho_P))$ for arithmetic P. We have interpolated Greenberg's $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho_P))$ over arithmetic P and got an analytic function $\mathcal{L}(Ad(\rho_{\mathbb{I}})) \neq 0$ in $\mathbb{I}[\frac{1}{p}]$ so that $\mathcal{L}(Ad(\rho_{\mathbb{I}}))(P) = \mathcal{L}(Ad(\rho_P))$ for all arithmetic P.

Conjecture 3.1. *We have*

$$\mathcal{L}\big(Ad(\rho_{\mathbb{I}})\big) = \mathcal{L}^{an}\big(Ad(\rho_{\mathbb{I}})\big)$$

for the $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho_{\mathbb{I}}))$ defined in (L) *in Section* 2.5 *up to units in $\mathbb{I}$.*

The element $L_p \in \mathbb{I}$ is only determined up to units in $\mathbb{I}$. We can normalize L_p insisting on exactitude of the above identity (removing the ambiguity of units) in the conjecture. In other words, we define L_p by $\frac{L'_p(1, Ad(\rho_{\mathbb{I}}))}{\mathcal{L}(Ad(\rho_{\mathbb{I}}))}$ and ask if the congruence criterion still holds for this choice.

3.2. Is characterizing abelian components important?

Here is a list of such characterizations (possibly conjectural):

- A cuspidal $\mathbb{I}$ is abelian $\Leftrightarrow$ cuspidal $\mathbb{I}$ is a CM component $\Leftrightarrow$ there exist an imaginary quadratic field $M = \mathbb{Q}[\sqrt{-D}\,]$ in which p splits into $\mathfrak{p}\overline{\mathfrak{p}}$ and a character $\Psi = \Psi_{\mathbb{I}} : G_M = \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^{\times}$ of conductor $\mathfrak{c}\mathfrak{p}^{\infty}$ for an ideal $\mathfrak{c}$ with $\mathfrak{c}\bar{\mathfrak{c}}D_M|N$ such that $\rho_{\mathbb{I}} = \mathrm{Ind}_M^{\mathbb{Q}}\Psi$, where D_M is the discriminant of M. This should be well known; see [23, (CM1–3) in Section 1]. Thus we called cuspidal abelian component a *CM component*. This implies $L_p = L_p(\Psi^-)L(0, (\frac{M/\mathbb{Q}}{}))$, where $\Psi^-(\sigma) = \Psi(c\sigma c^{-1}\sigma^{-1})$ for complex conjugation c, and $L_p(\Psi^-)$ is the *anticyclotomic* Katz p-adic L-function associated to Ψ^-. This is a base of the proof by Mazur and Tilouine (e.g., [28]; see also [11] as a precursor of the result of Mazur and Tilouine) of the anticyclotomic main conjecture (see [19] and [21] for a version for CM fields).
- $\mathbb{I}$ is abelian $\Leftrightarrow$ ρ_P is abelian for a single arithmetic prime P. By Ribet [31], if ρ_P is abelian, ρ_P has complex multiplication or Eisenstein. Then P has to be on a CM component or on an Eisenstein component (see [22, Sections 3 and 4]).
- $\mathbb{I}$ abelian $\Leftrightarrow$ $\rho_{\mathbb{I}}$ mod p is abelian. This is almost equivalent to the vanishing of the Iwasawa μ-invariant for $L_p(\Psi^-)$ (which is known if $\mathfrak{c}$ is made up of primes split over $\mathbb{Q}$). This is a main result in [22].
- (Strong vertical conjecture in [23].) Consider the field $\mathcal{V}_r(\mathbb{I}) \subset \overline{\mathbb{Q}}$ generated by $a_P(p)$ for all arithmetic P with level $\leqslant Np^{r+1}$ for a fixed $r \geqslant 0$. Then $\mathbb{I}$ is abelian $\Leftrightarrow$ $[\mathcal{V}_r(\mathbb{I}) : \mathbb{Q}] < \infty$. This was a question that L. Clozel asked me in the early 1990s. This holds true if the family contains weight 2 cusp form whose abelian variety has good ordinary reduction modulo p or more generally a weight $k \geqslant 2$ cusp form whose motive is potentially crystalline ordinary at p (see [23, Theorem 3.2]). Here a crystalline motive is ordinary if its Newton polygon of the crystalline Frobenius coincides with the Hodge polygon. By applying this crystalline-ordinary criterion, the family $\mathcal{F}_\Delta$ containing Ramanujan's Δ-function has $\mathcal{V}_0(\mathbb{I})$ of infinite degree over $\mathbb{Q}$. In the 1970s, Y. Maeda made a conjecture asserting that $\mathbb{Q}(a(p,f))$ for any normalized Hecke eigenform f in $S_k(SL_2(\mathbb{Z}))$ has degree equal to $d := \dim S_k(SL_2(\mathbb{Z}))$ with its Galois closure having Galois group isomorphic to the symmetric group $\mathfrak{S}_d$ of d letters. This conjecture is numerically verified up to large weight and large p (e.g., [1]) and implies our conjecture if $N=1$.

- (Strong horizontal theorem in [23].) Fix $k \geqslant 1$ and consider the field $\mathcal{H}_k(\mathbb{I})$ generated by $a_P(p)$ over $\mathbb{Q}(\mu_{p^\infty})$ for all arithmetic P with a fixed weight $k \geqslant 1$. Then $\mathbb{I}$ is abelian $\Leftrightarrow$ $[\mathcal{H}_k(\mathbb{I}) : \mathbb{Q}(\mu_{p^\infty})] < \infty$ (see [23, Theorem 3.3]).
- $\rho_{\mathbb{I}}$ restricted to the decomposition group at p is completely reducible $\Leftrightarrow$ $\mathbb{I}$ is abelian. This is the result of Ghate and Vatsal in [6].
- For cuspidal $\mathbb{I}$, $\mathcal{L}(Ad(\rho_{\mathbb{I}}))$ is a constant function over $\mathrm{Spf}(\mathbb{I})$ if and only if $\mathbb{I}$ is a CM component. This is a corollary of Strong horizontal theorem and is what we have proven in this paper.
- (Conjecture/Question.) Does a cuspidal component $\mathbb{I}$ have CM by an imaginary quadratic field M if

$$\mathcal{L}\big(Ad(f_P)\big) = \log_p(\mathfrak{p}) \quad \text{(up to algebraic numbers)}$$

  for one arithmetic P for a prime factor $\mathfrak{p}$ of p in M? Here taking a high power $\mathfrak{p}^h = (\alpha)$, $\log_p(\mathfrak{p}) = \frac{1}{h}\log_p(\alpha)$ for the Iwasawa logarithm $\log_p$.
- (A wild guess.) If $\mathbb{Q}(f_P) = \mathbb{Q}$ with $k(P) \geqslant 27$ for a cusp form f_P, then $\mathbb{I}$ has CM. As is well known, there exists a non-CM eigenform spanning the one-dimensional space $S_{26}(SL_2(\mathbb{Z}))$, and the question is if this is the highest weight rational Hecke eigenform without CM.

All statements seem to have good arithmetic consequences, and I am convinced of the importance of giving as many characterizations of abelian components as possible.

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