* Adjoint L-value as a period integral and the mass formula of Siegel–Shimura

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Abstract: For a quaternion algebra $D_{/\mathbb{Q}}$ and a quadratic field $E = \mathbb{Q}[\sqrt{\Delta}]_{/\mathbb{Q}}$, we compute as $L(1, Ad(\rho_F) \otimes \left(\frac{\Delta}{}\right))$ times the mass of Siegel–Shimura the period of Doi-Naganuma lift of an elliptic Hecke eigen new form F to the quaternionic Shimura variety associated to $D \otimes_{\mathbb{Q}} E$ over Shimura subvarieties associated to D. Here ρ_F is the compatible system of Galois representations associated to F.

§0. An idea of Waldspurger. For an elliptic cusp form F of level M, an idea of Waldspurger of computing the period of a theta lift of F for a quadratic space $V = W \oplus W^{\perp}$ over an orthogonal Shimura subvariety $S \times S^{\perp} \subset S_V$ is two-folds:
(S) Split $\theta(\phi)(\tau, h_0, h^{\perp}) = \theta(\phi_0)(\tau, h_0) \cdot \theta(\tau, \phi^{\perp})(h^{\perp})$ for a decomposition $\phi = \phi_0 \otimes \phi^{\perp}$ (ϕ and $\phi^{\perp}$ Schwartz–Bruhat functions on $W_{\mathbb{A}}$ and $W_{\mathbb{A}}^{\perp}$);
(R) For the theta lift $\theta^*(F)(h) = \int_X F(\tau)\theta(\phi)(\tau, h)d\mu$ with $X = X_0(M)$, the period P over the Shimura subvariety $S \times S^{\perp}$ is:

$$\int_{S\times S^{\perp}} \int_X F(\tau)\theta(\Phi)(\tau; h)d\mu dh \quad (d\mu = \eta^{-2} d\xi d\eta)$$
$$= \int_X F(\tau) \left(\int_{S^{\perp}} \theta(\phi^{\perp})(\tau; h^{\perp})dh^{\perp}\right) \cdot \left(\int_S \theta(\phi_0)(\tau; h_0)dh\right) d\mu.$$

Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\phi)$ and $E(\phi^{\perp})$, reaching Rankin-Selberg integral

$$P = \int_X F(\tau)E(\phi^{\perp})E(\phi_0)d\mu = L\text{-value}.$$

Interesting to explore what L-value shows up?

§**1. Choice of V:** For a $\mathbb{Q}$-vector space V and a $\mathbb{Q}$-algebra A, write $V_A := V \otimes_{\mathbb{Q}} A$. Let $E := \mathbb{Q}[\sqrt{\Delta}]$ or $\mathbb{Q} \times \mathbb{Q}$ with square-free $\Delta \in \mathbb{Z}$. Pick a quaternion algebra $D_{/\mathbb{Q}}$ and put $D_E := D \otimes_{\mathbb{Q}} E$. Let $1 \neq \sigma \in \mathrm{Gal}(E/\mathbb{Q})$ act on D through the factor E. Then

$$V = D_\sigma := \{x \in D_E | x^\sigma = x^\iota\} \quad \text{for } x^\iota = \mathrm{Tr}(x) - x.$$

We have $Q(x) = xx^\sigma = N(x) = s(x,x)/2 \in \mathbb{Q}$ for $s(x,y) = \mathrm{Tr}(x^\iota y)$. We have four Cases RM, RH, CM, CH of $(E_{\mathbb{R}}, D_{\mathbb{R}})$. Here the symbol "M" (resp. "C", "R" "H") indicates $D_{\mathbb{R}} = \mathbb{H}$ (resp. $E_{\mathbb{R}} = \mathbb{C}$, $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$, $D_{\mathbb{R}} = M_2(\mathbb{R})$). The splitting in (S) is

$$V = Z \oplus D_0, \quad Z = \mathbb{Q} \text{ with quadratic form } Q_Z(z) = z^2, \text{ and}$$

$$D_0 := \{v \in \sqrt{\Delta}D | \mathrm{Tr}(v) = 0\} \text{ with } Q_0(v) = vv^\sigma = N(v)$$

Signature of Z is positive and that of D_0 depends on the cases, write $\mathrm{SO}_0 := \mathrm{SO}_{D_0} \cong D^\times/\text{center}$ and $\mathrm{SO}_\sigma := \mathrm{SO}_{D_\sigma}$.

§2. **Schwartz–Bruhat functions of weight k.** On $Z = \mathbb{Q}$, for a Dirichlet character ψ as a function supported on $\widehat{\mathbb{Z}} \subset Z_{\mathbb{A}(\infty)}$. For $\mathbf{e}(\tau) = \exp(2\pi\sqrt{-1}\tau)$, this ψ produces theta series $\sum_{n\in\mathbb{Z}} \psi(n) n^j \mathbf{e}(n^2\tau)$. On D_0, take an Eichler order R_0 and take (C): $\boxed{\phi_0(v) = (\phi_{\widehat{L}}(v) - c^3\phi_{\widehat{L}}(c^{-1}v))/(1-c^3)}$ $(1 < c \in \mathbb{Z}$ fixed) for the characteristic function $\phi_{\widehat{L}}$ of $\widehat{L} := D_{0,\mathbb{A}} \cap \sqrt{\Delta}\widehat{R}_0$.

At ∞, $\phi(x_\infty) = H(x_\infty)\mathbf{e}(P(x_\infty)\sqrt{-1})$ for a harmonic polynomial H, a positive majorant $P = Q_Z \oplus P_0$ of the reduced norm and $H(z \oplus v) = (z + H_0(v))^k = \sum_{j=0}^{k} \binom{k}{j} z^j H_0(v)^{k-j}$ for linear $H_0(v)$.

$$\phi = \sum_{j=0}^{k} \binom{k}{j} \phi_j^Z \otimes \phi_{k-j}^0 \quad \text{on } D_{0,\mathbb{A}} \text{ with } \phi_{k-j}^0(0) = 0 \text{ unless } j = k.$$

with $\phi_j^Z(z_\infty) = z_\infty^j \mathbf{e}(z_\infty^2\sqrt{-1})$, $\phi_j^0(v_\infty) = H_0(v_\infty)^j \mathbf{e}(P_0(v_\infty)\sqrt{-1})$,

§**3. Theta kernel.** We have

$$\mathsf{SO}_\sigma(\mathbb{Q}) = \{h \in D_E^\times | N(h) \in \mathbb{Q}^\times\}/\mathbb{Q}^\times \subset D_E^\times/E^\times$$

and $h \in D_E^\times$ acts on D_σ by $x \mapsto h^{-1}xh^\sigma$. Thus $S_E = \mathsf{SO}_\sigma(\mathbb{Q})\backslash\mathsf{SO}_\sigma(\mathbb{A})$ is a Shimura surface in Case RM, S_E has dimension 0 in Case RH and S_E is real 3-dimensional in Case C.

Let $\mathsf{Mp}(\mathbb{A}) \twoheadrightarrow \mathsf{SL}_2(\mathbb{A})$ be the metaplectic cover, and $\mathbf{r}(g)$ be the Weil representation. The theta series for $g_\tau = \eta^{-1/2}\left(\begin{smallmatrix}\eta & \xi\\ 0 & 1\end{smallmatrix}\right)$ is

$$\theta(\tau; h) = \eta \sum_{\alpha \in D_\sigma} (\mathbf{r}(g_\tau)\phi)(h^{-1}\alpha h^\sigma) : \mathsf{Mp}(\mathbb{A}) \times \mathsf{SO}_\sigma(\mathbb{A}) \to \mathbb{C},$$

which is left $\mathsf{SL}_2(\mathbb{Q}) \times \mathsf{SO}_\sigma(\mathbb{Q})$-invariant. Assume $\boxed{\theta^*(F) \neq 0}$ for $\theta^*(F) := \int_X F(-\overline{\tau})\theta(\phi)(\tau; h)\eta^{k-2}d\xi d\eta$. The theta lift $\theta^*(F)$ is a weight (k, k) automorphic form on $D_E^\times$.

§4. **Theta differential form.** To compute the period on $S = \mathrm{SO}_0(\mathbb{Q})\backslash\mathrm{SO}_0(\mathbb{A}) \subset S_E$, we convert $\theta^*(F)$ into a sheaf valued differential d–form $\Theta^*(F)$ over S_E for $d = \dim_{\mathbb{R}} S$ in a canonical way of Eichler–Shimura and Hida. Similarly $\theta(\phi)(\tau; h)$ is converted to a differential d–form $\Theta(\phi)$.

The sheaf $L^*_E(n; A)$ $(n = k-2)$ comes from the $D^\times_E$-representation $g \mapsto g^{sym\otimes n} \otimes \sigma(g)^{sym\otimes n}$ and $L^*_E(n; A)$ has a canonical Clebsch-Gordan projection $\nabla : L^*_E(n; A)|_S \twoheadrightarrow A$. Any other component has vanishing period. The period is

$$P_1(F) := \int_S \nabla\Theta^*(F).$$

§5. **Siegel–Weil Eisenstein series.** By Weil, $g \mapsto (\mathbf{r}(g)\phi)(0)$ factors through $B(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})$ for the Borel subgroup $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset \mathrm{SL}_2$. Siegel–Weil Eisenstein series is

$$E(\phi)(g) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} (\mathbf{r}(\gamma g)\phi)(0)|\,\mathrm{Im}(\gamma_\infty g_\infty(\sqrt{-1}))|^s\Big|_{s=0}.$$

Note $\boxed{E(\phi^0_{k-j})|_{B(\mathbb{A})} = 0}$ unless $k = j$.

The Siegel–Weil formula by Kudla-Rallis/Sweet is

$$2E(\phi)(g) = \int_S \theta(\phi)(g;h)d\omega(h) \quad \text{for the Tamagawa measure } d\omega.$$

Our measure dh has volume 1 on $\widehat{R}_0^\times$; so, $dh \neq d\omega$. The ratio $\boxed{\mathfrak{m}_1(\zeta(2)/\pi^\epsilon) = \mathfrak{m}(R_0) = 2dh/d\omega}$ is the mass of Siegel–Shimura, which is an explicit rational number $\mathfrak{m}_1$ (computed by Shimura in 1999) times $\zeta(2)/\pi^\epsilon$ for $\epsilon = 1$ in Case M and $\epsilon = 2$ in Case H.

§6. **Conclusion in Case RM.** Note $\mathrm{SL}_2(\mathbb{Q}) = B(\mathbb{Q}) \sqcup B(\mathbb{Q})JB(\mathbb{Q})$ for $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})\widehat{\Gamma}_0(M)\mathrm{SO}_2(\mathbb{R})$ by strong approximation. Then $B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}) = \overline{B} \sqcup T(\mathbb{Q})\backslash JB(\mathbb{A})$ for $\overline{B} := B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})\mathrm{SO}_2(\mathbb{R}) \cong [0,1) \times \mathbb{R}_+^\times$ and the diagonal torus T. Thus for $X = \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})/\widehat{\Gamma}_0(M)\mathrm{SO}_2(\mathbb{R})$

$$P_1(F) = \mathfrak{m}(R_0) \int_X F(-\overline{\tau}) \sum_j \binom{k}{j} \theta(\phi_j^Z) E(\phi_0^{k-j}) \eta^{-2} d\xi d\eta$$

$$\overset{(C)}{=} \mathfrak{m}(R_0) \int_{\overline{B}} F(-\overline{\tau}) \theta(\phi_k^Z)(\mathbf{r}(g_\tau)\phi_0^0)(0)\eta^{-1} d\xi d\eta$$

$$= \mathfrak{m}(R_0) \int_0^\infty \int_0^1 \sum_{n \in \mathbb{Z}} \psi(n) n^k \mathbf{e}(n^2\tau) F(-\overline{\tau}) d\xi \eta^{k-1} d\eta$$

$$= c_E \mathfrak{m}_1 (2\pi)^{-k} \Gamma(k) L\left(1, Ad(F) \otimes \left(\frac{\Delta}{}\right)\right)$$

for a simple constant $0 \neq c_E \in \mathbb{Q}$ depending on E.

In the other cases, the Γ-factor changes slightly.

§**7. Case RH has interesting feature.** In this case, we have

$$P_1(F) = c_E \mathfrak{m}_1 2(4\pi)^{-k+1}\Gamma(k)(1, Ad(F) \otimes (\left(\frac{\Delta}{}\right))$$

Suppose $k = 2$ now for simplicity. Writing $S = \{x\}_{x \in Cl_D(\widehat{R}_0)}$, for $e_x = |x\widehat{R}_0^\times x^{-1} \cap D^\times|$,

$$\mathfrak{m}_1(\zeta(2)/\pi^2) = \sum_{x \in Cl_D(\widehat{R}_0)} e_x^{-1}$$ (Mass formula of Siegel–Shimura)

and

$$\mathfrak{m}_1(c_E 2(4\pi)^{-k+1}\Gamma(k)L(1, Ad(F) \otimes (\left(\frac{\Delta}{}\right))) = \sum_{x \in Cl_D(\widehat{R}_0)} e_x^{-1}\theta^*(F)(x)$$

(an adjoint mass formula).

The period formula is an adjoint analogue of the mass formula.

§8. Periods over Shimura curves other than S.

For each $\alpha \in D_\sigma \cap D_E^\times$, consider an involution σ_α of D_E given by $x \mapsto \alpha x^\sigma \alpha^{-1}$. Then $\alpha \mapsto D_\alpha = H^0(\langle \sigma_\alpha \rangle, D_E)$ is a parameterization of all quaternion subalgebras of D_E. If $\alpha \in Z$, then $D_\alpha = D$. Let $S_\alpha := \mathrm{SO}_{D_{\alpha,0}}(\mathbb{Q}) \backslash \mathrm{SO}_{D_{\alpha,0}}(\mathbb{A}) \hookrightarrow S_E$ be the Shimura subvariety associated to D_α.

Pick a Hecke eigen harmonic differential 2-form f with values in $L_E^*(n; E)$ on S_E. Then the S_α-period of f is defined by

$$P_\alpha(f) := \int_{S_\alpha} \nabla f.$$

So $P_1(\Theta^*(F)) = P_1(F)$.

§9. **Expansion of theta descent.** For the invariant pairing $(\cdot,\cdot) : L^*_E(n;\mathbb{C}) \times L^*_E(n;\mathbb{C}) \to \mathbb{C}$, $(f(h) \wedge \Theta(\phi)(\tau))|_{S_\alpha}$ is a harmonic differential 2-form on S_α. Define the theta descent by $\theta_*(f)(\tau) := \int_{S_E}(f \wedge \Theta(\phi)(\tau))$. Let Γ be the level group of $(f \wedge \Theta(\tau))$ in $D^\times_E$. Then

$$\theta_*(f) = c_? \sum_{\alpha \in D_\sigma/\Gamma; D_{\alpha,\mathbb{R}} \cong M_2(\mathbb{R})} \phi^{(\infty)}(\alpha) P_\alpha(f) \mathbf{e}(|N(\alpha)|\tau_?),$$

where $\tau_? = \tau$ in Case RH and $-\overline{\tau}$ otherwise, $c_? \neq 0$ is a simple constant depending on the cases.

Write $\mathbb{Q}(f)$ for the Hecke field of f, and assume that θ_* is Hecke equivariant. Consequences:

$P_\alpha(f) = 0$ if f is not a theta lift; transcendence of $P_\alpha(\Theta^*(F))$ is independent of D and α;

$P_\alpha(\Theta^*(F))$ is $P_1(F)$ replacing $\mathfrak{m}_1$ by a constant $\mathfrak{m}_\alpha \in \mathbb{Q}(F)$.

We call $\mathfrak{m}_\alpha$ an adjoint mass, which is not yet fully computed.