$\mathcal{L}$-INVARIANTS AND p-ORDINARY FAMILIES OF HILBERT MODULAR FORMS

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1. Introduction

Let $p > 2$ be a prime. After the conjecture of Mazur-Tate-Teitelbaum, number theorists have proposed diverse definitions of the $\mathcal{L}$–invariant which are expected to give the leading term of the Taylor expansion of a given motivic p–adic L–function at an exceptional zero. For an elliptic curve $E_{/\mathbb{Q}}$ with multiplicative reduction modulo p, its p–adic L–function $L_p(s,E)$ has the following expression at $s = 1$:

$$L_p(1,E) = (1-(a_p^{-1}p)p^{-s}|_{s=1})\frac{L_\infty(1,E)}{\text{period}}\ (s \in \mathbb{C}),$$

where $L_\infty(s,E)$ is the archimedean L–function of E, and a_p is the eigenvalue of the arithmetic Frobenius element at p on the unramified quotient of the p–adic Tate module $T(E)$ of E. Thus if E has *split* multiplicative reduction, $a_p = 1$, and $L_p(s,E)$ has zero at $s = 1$. This type of zero of a p–adic L–function resulted from the modification Euler p–factor is called an exceptional zero, and it is generally believed that if the archimedean L–values does not vanish, the order of the zero is the number e of such Euler p–factors; so, in this case, $e = 1$. Then $L'_p(1,E) = \frac{dL_p(s,E)}{ds}|_{s=1}$ is conjectured to be equal to the archimedean value $\frac{L_\infty(1,E)}{\text{period}}$ times an error factor $\mathcal{L}(E)$, the so-called $\mathcal{L}$–invariant:

$$L'_p(1,E) = \mathcal{L}(E)\frac{L_\infty(1,E)}{\text{period}}.$$

Writing $E(\mathbb{Q}_p) = \mathbb{Q}_p^\times/q^{\mathbb{Z}}$ for the Tate period $q \in p\mathbb{Z}_p$, the solution conjectured and proved by Greenberg-Stevens is

$$(\mathcal{L}) \qquad \mathcal{L}(E) = \frac{\log_p(q)}{\mathrm{ord}_p(q)}.$$

Since E is modular, it is associated to an elliptic Hecke eigenform f_E of weight 2 with $L(s,f_E) = L(s,E)$. In particular, $f_E|U(p) = a_pf$ for

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$a_p = 1$. We can lift f_E to a unique Λ–adic form $\mathcal{F}$ with $\mathcal{F}|U(p) = a(p)\mathcal{F}$ for a finite flat extension Λ of $\mathbb{Z}_p[[X]]$ so that f_E is a specialization of $\mathcal{F}$ at $X = 0$. Then one of the key ingredient in of the proof of $(\mathcal{L})$ is:

$$\mathcal{L}(E) = -2\log_p(\gamma)\frac{da(p)}{dX}\Big|_{X=0} = \log_p(\gamma)\frac{da(p)}{dx}\Big|_{x=0}\ (1+x = \sqrt{1+X}^{-1}),$$

where γ is the generator of $\Gamma = 1 + p\mathbb{Z}_p$ corresponding to $1 + x$ under the identification: $W[[\Gamma]] = W[[x]]$.

Greenberg has generalized the conjectural formula of his $\mathcal{L}$–invariant to general p-adic representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ when V is p–ordinary. Take a p-ordinary two dimensional Galois representation ρ associated to a Hilbert modular Hecke eigenform f of (parallel) weight $k \geq 2$ for a totally real field F. Let O be the integer ring of F. We write S for the set of all prime factors of p in F, and assume that $F/\mathbb{Q}$ is **unramified** at p. We consider $\Lambda = W[[x_\mathfrak{p}]]_{\mathfrak{p}\in S}$ and regard it as the Iwasawa algebra of $\prod_\mathfrak{p} N_{F_\mathfrak{p}/\mathbb{Q}_p}(1 + \mathfrak{p}O_\mathfrak{p})$ sending a generator $\gamma_\mathfrak{p} \in 1 + p\mathbb{Z}_p = N_{F_\mathfrak{p}/\mathbb{Q}_p}(1 + \mathfrak{p}O_\mathfrak{p})$ to $1 + x_\mathfrak{p}$. We have a unique Λ–adic Hecke eigenform $\mathcal{F}$ so that its specialization at a point of $Spf(\Lambda)$ over $(x_\mathfrak{p})_\mathfrak{p}$ gives f. Write $\mathcal{F}|U(p_\mathfrak{p}) = a(\mathfrak{p})\mathcal{F}$. We would like to prove

Theorem 1.1. *If the versal nearly p–ordinary deformation ring R of ρ with the fixed determinant* $\det(\rho)$*, after localization completion at ρ, is isomorphic to a local ring at ρ of the universal nearly p–ordinary Hecke algebra with the fixed central character, then we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho)) = \pm\left(\prod_\mathfrak{p}\log_p(\gamma_\mathfrak{p})\right)\det\left(\frac{da(\mathfrak{p})}{dx_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'\in S}\Big|_{x=0}.$$

If f gives an elliptic curve $E_{/F}$ with split multiplicative reduction at all $\mathfrak{p} \in S$, taking the Tate period $q_\mathfrak{p}$ with $E(F_\mathfrak{p}) = F_\mathfrak{p}^\times/q_\mathfrak{p}^{\mathbb{Z}}$, we have

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho)) = \prod_\mathfrak{p}\frac{\log_p(N_{F_\mathfrak{p}/\mathbb{Q}_p}(q_\mathfrak{p}))}{\mathrm{ord}_p(N_{F_\mathfrak{p}/\mathbb{Q}_p}(q_\mathfrak{p}))}.$$

The assumption of the theorem has been verified in many cases by Wiles, Taylor-Wiles, Diamond, Fujiwara and Skinner-Wiles.

I will try to give the proof of this fact in the rest of the talk, assuming for simplicity that ρ is **finitely** ramified outside p and ∞.

2. Selmer Groups

We recall the Greenberg's definition of Selmer groups. Let K be a finite extension of $\mathbb{Q}_p$. Fix a finite set Σ of rational primes containing p and write $\mathbb{Q}^{(\Sigma)}/\mathbb{Q}$ for the maximal extension unramified outside Σ

and ∞. We put $\mathfrak{G} = \mathrm{Gal}(\mathbb{Q}^{(\Sigma)}/\mathbb{Q})$ and $\mathfrak{G}_F = \mathrm{Gal}(\mathbb{Q}^{(\Sigma)}/F)$. Let V be a finite dimensional K-vector space with a continuous action of $\mathfrak{G}$.

Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We write $D = D_p$ for the decomposition group of the p–adic norm: $| * |_p = |i_p(*)|_p$.

Let $\chi : \mathfrak{G} \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character; so, $\zeta^\sigma = \zeta^{\chi(\sigma)}$ for $\zeta \in \mu_{p^\infty}$. We assume V to be p-ordinary; so, we have a filtration of the following type for integers a and b with $b \leq 0 < a$:

$$\text{(ord)}\quad V = F^bV \supset F^{b+1}V \supset \cdots \supset F^0V \supset F^1V \supset \cdots \supset F^{a+1}V = \{0\}$$

stable under the decomposition group D and the inertia group $I_p \subset D$ acts on each subquotient $F^iV/F^{i+1}V$ by χ^i. Once V satisfies (ord), its dual $V^*(1) = \mathrm{Hom}_K(V, K) \otimes \chi$ again satisfies (ord) for $F^{-i}V^*(1) = (F^iV)^\perp(1)$. Let W be the p–adic integer ring of K, and take a W–lattice T in V stable under $\mathfrak{G}$.

Let $M/\mathbb{Q}$ be a subfield of $\overline{\mathbb{Q}}$. We write $\mathfrak{p}$ for a prime of M over p and $\mathfrak{q}$ for general primes of M. Write $D_\mathfrak{q}$ for the decomposition group at $\mathfrak{q}$ in $\mathfrak{G}_M$ and $I_\mathfrak{q}$ for the inertia subgroup of $D_\mathfrak{q}$. We write $F^+V = F^1V$. For each prime $\mathfrak{q}$ of M, we put

$$L_\mathfrak{q}(V) = \begin{cases} \mathrm{Ker}(\mathrm{Res} : H^1(M_\mathfrak{q}, V) \to H^1(I_\mathfrak{q}, V)) & \text{if } \mathfrak{q} \nmid p, \\ \mathrm{Ker}(\mathrm{Res} : H^1(M_\mathfrak{p}, V) \to H^1(I_\mathfrak{p}, \frac{V}{F^+(V)})) & \text{if } \mathfrak{p}|p. \end{cases}$$

Then we define for the image $L_\mathfrak{q}(V/T)$ of $L_\mathfrak{q}(V)$ in $H^1(M_\mathfrak{q}, V/T)$

$$\text{(2.1)}\quad \mathrm{Sel}_M(A) = \mathrm{Ker}(H^1(M, A) \to \prod_\mathfrak{q} \frac{H^1(M_\mathfrak{q}, A)}{L_\mathfrak{q}(A)}) \ \text{ for } A = V, V/T.$$

The classical Selmer group of V is given by $\mathrm{Sel}_M(V/T)$, equipped with discrete topology. Write $\mathbb{Q}_\infty$ for the cyclotomic $\mathbb{Z}_p$–extension of $\mathbb{Q}$. The Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty}(V/T)$ is an Iwasawa module of co-finite type.

We define $F^{00}V \supset F^+V$ to be the largest subspace of V stable under D_p so that D_p acts trivially on $F^{00}V/F^+V$. Similarly let $F^{11}V$ be the smallest subspace of F^+V so that D_p acts on $F^+V/F^{11}V$ by χ. Then we write $F^{jj}V/T$ for the image of $F^{jj}V$ in V/T. For simplicity, referring to [G] Section 3 the treatment in more general cases, we assume

(∗) Either $F^{11}V = F^+V$ or $F^{00}V/F^{11}V$ does not have a direct summand isomorphic to K or $K(1)$.

Using the global Tate duality and the Poitou-Tate exact sequence, Greenberg has shown the following implication: ([G] proposition 2):

$$\text{(V)}\qquad |\mathrm{Sel}_\mathbb{Q}(V/T)| < \infty \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_{q \in \Sigma} \frac{H^1(\mathbb{Q}_q, V)}{L_q(V)}.$$

The definition of the Selmer group $\mathrm{Sel}_M(V/T)$ can be obviouly extended to a representation V of $\mathfrak{G}_F$ for a number field F and an extension M/F if V is ordinary at every p-adic place $\sigma \in S$ of F. In that case, we call V and T ordinary at p, and we freely speak of Selmer group $\mathrm{Sel}_M(V/T)$ and $\mathrm{Sel}_M(V)$ defined for such $\mathfrak{G}_F$-representations V. If $F/\mathbb{Q}$ is unramified at p and a $\mathfrak{G}_F$-representation V is ordinary at p, $\mathrm{Ind}_F^{\mathbb{Q}} V$ is ordinary at p as a $\mathfrak{G}$-representation.

3. Greenberg's $\mathcal{L}$–invariant

We suppose the following condition in addition to (V):

(S) D_p acts semi-simply on $gr^i(V) = F^iV/F^{i+1}V$ for all i.

Regard $\mathrm{Sel}_{\mathbb{Q}_\infty}(V/T)$ as an Iwasawa module and write its characteristic power series as $f_V(X) \in W[[X]]$. We set $e = \dim_K F^+V/F^{11}V + \dim_H F^{00}V/F^+V$. Then Greenberg proved $X^e | f(X)$ and conjectured in [G] Conjecture 2 that

(C1) $\mathrm{Sel}_{\mathbb{Q}_\infty}(V/T)$ is a co-torsion module over $W[[X]]$,

(C2) If the condition (V) is satisfied, $f_V(X)$ is factored as $f_V(X) = X^e g_V(X)$ with $g_V(0) \neq 0$.

Here is Greenberg's definition of $\mathcal{L}(V)$ giving $g_V(0)$: The long exact sequence of $F^{00}V/F^+V \hookrightarrow V/F^+V \twoheadrightarrow V/F^{00}V$ gives a homomorphism:

$$H^1(\mathbb{Q}_p, F^{00}V/F^+V) = \mathrm{Hom}(G^{ab}_{\mathbb{Q}_p}, F^{00}V/F^+V) \xrightarrow{\iota} H^1(\mathbb{Q}_p, V)/L_p(V).$$

Note that

$$\mathrm{Hom}(G^{ab}_{\mathbb{Q}_p}, F^{00}V/F^+V) \cong (F^{00}V/F^+V)^2$$

canonically by $\phi \mapsto (\frac{\phi([\gamma,\mathbb{Q}_p])}{\log_p(\gamma)}, \phi([p,\mathbb{Q}_p]))$. Here $[x,\mathbb{Q}_p]$ is the local Artin symbol (suitably normalized). Since

$$L_p(F^{00}V/F^+V) = \mathrm{Ker}(H^1(\mathbb{Q}_p, F^{00}V/F^+V) \xrightarrow{\mathrm{Res}} H^1(I_p, F^{00}V/F^+V)),$$

the image of ι is isomorphic to $F^{00}V/F^+V$. By (V), we have a unique subspace $\mathbb{T}$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\mathrm{Im}(\iota) \hookrightarrow \prod_{q \in \Sigma} \frac{H^1(\mathbb{Q}_q, V)}{L_q(V)}.$$

Then by the restriction, $\mathbb{T}$ gives rise to a subspace L of

$$\mathrm{Hom}(G^{ab}_{\mathbb{Q}_p}, F^{00}V/F^+V) \cong (F^{00}V/F^+V)^2$$

isomorphic to $F^{00}V/F^+V$. If a cocycle c representing an element in $\mathbb{T}$ is unramified, it gives rise to an element in $\mathrm{Sel}_{\mathbb{Q}}(V/T)$. By finiteness of $\mathrm{Sel}_{\mathbb{Q}}(V/T)$, this implies $c = 0$; so, the projection of L to the first factor $F^{00}V/F^+V$ (via $\phi \mapsto \phi([\gamma,\mathbb{Q}_p])/\log_p(\gamma)$) is surjective. Thus this subspace L is a graph of a K–linear map $\mathcal{L} : F^{00}V/F^{+}V \to F^{00}V/F^{+}V$. We then define $\mathcal{L}(V) = \det(\mathcal{L}) \in K$.

4. Proof of the theorem

For simplicity, we suppose that p splits completely in $F/\mathbb{Q}$. We take a p–ordinary Hecke eigenform f and its 2–dimensional Galois representation $\rho : \mathfrak{G}_F \to GL_2(W)$. We take $T = \mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho)$ and put $V = T \otimes_W K$. If ρ is p-ordinary at all p-adic places, V is also ordinary. For each $\sigma \in S$, identifying it with $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ and extending it to $\mathfrak{G}$, put $D_\sigma = \sigma D \sigma^{-1}$ and ${}_\sigma D = D_\sigma \cap \mathfrak{G}_F$.

For each ${}_\sigma D$, we can define $F_\sigma^{00} Ad(\rho)$. If we take a matrix form $\rho : \mathfrak{G} \to M_2(W)$ of the Galois representation T so that its restriction to ${}_\sigma D$ is given by $\rho(\tau) = \left(\begin{smallmatrix} \epsilon_\sigma(\tau) & \beta_\sigma(\tau) \\ 0 & \delta_\sigma(\tau) \end{smallmatrix}\right)$. We may identify $Ad(\rho)$ with the following subspace of $M_2(W)$:

$$\left\{ \xi \in M_2(W) \middle| \mathrm{Tr}(\xi) = 0 \right\}.$$

Then $F_\sigma^{00} Ad(\rho)$ is the subspace of $Ad(\rho)$ made up of upper triangular matrices, and $F_\sigma^{+} Ad(\rho)$ (on which ${}_\sigma D$ acts by $\epsilon_\sigma \delta_\sigma^{-1}$)) is made up of upper nilpotent matrices.

Identifying $\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho) = W[\mathfrak{G}] \otimes_{W[\mathfrak{G}_F]} Ad(\rho)$, we have

$$F^{?}T = \bigoplus_{\sigma \in S} \sigma^{-1} \left(F_\sigma^{?} Ad[\rho] \right) \quad \text{for } ? = 00 \text{ and } +.$$

By Shapiro's lemma, we have for a field M linearly disjoint from F

$$\mathrm{Sel}_M(V/T) \cong \mathrm{Sel}_{MF}(Ad(\rho) \otimes_W K/W),$$

and the following commutative diagram:

(4.1)

$$\begin{array}{ccccc} H^1(\mathfrak{G}, \mathrm{Ind}_F^{\mathbb{Q}} V) & \supset & \mathbb{T} & \xrightarrow{\mathrm{Res}} & H^1(\mathbb{Q}_p, \frac{F^{00}\,\mathrm{Ind}_F^{\mathbb{Q}} V}{\mathrm{Ind}_F^{\mathbb{Q}} F^{+}V}) \cong \left(\frac{F^{00}\,\mathrm{Ind}_F^{\mathbb{Q}} V}{\mathrm{Ind}_F^{\mathbb{Q}} F^{+}V} \right)^2 \\ \wr\downarrow & & \wr\downarrow & & \iota_p \downarrow \\ H^1(\mathfrak{G}_F, V) & \supset & \mathbb{T}_F & \xrightarrow{\mathrm{Res}} & \prod_{\sigma \in S} \sigma^{-1} \left(H^1(\mathbb{Q}_p, \frac{F_\sigma^{00} V}{F_\sigma^{+} V}) \right) \cong \prod_{\sigma \in S} \left(\frac{F_\sigma^{00} V}{F_\sigma^{+} V} \right)^2 \end{array}.$$

Then taking an inhomogeneous cocycle $c : \mathfrak{G}_F \to Ad(T)$ representing an element of $\mathbb{T}_F$, we may write $c(\tau) = \left(\begin{smallmatrix} a_\sigma(\tau) & b_\sigma(\tau) \\ 0 & -a_\sigma(\tau) \end{smallmatrix}\right)$ for $\sigma \in D_\sigma$. The cocycle c therefore gives rise to an infinitesimal nearly ordinary deformation $\widetilde{\rho}$ with $\det(\widetilde{\rho}) = \det \rho$:

$$\widetilde{\rho} : \mathfrak{G}_F \to GL_2(W[x]/(x^2)))$$

by $\widetilde{\rho}(\sigma) = \rho(\sigma) + c(\sigma)\rho(\sigma)x$ (see [MFG] 5.2.4).

Let $\boldsymbol{\rho} : \mathfrak{G}_F \to GL_2(R)$ be the versal nearly ordinary deformation over W for the minimal versal ring R. We thus have a W-algebra homomorphism $\varphi_\rho : R \to W[x]/(x^2)$ with $\varphi_\rho \circ \boldsymbol{\rho} \cong \widetilde{\rho}$. This φ_ρ may not be unique, but its differential $d\varphi_\rho$ from the tangent space $W\frac{\partial}{\partial x}$ of $Spec(W[x]/(x^2))$ to the tangent space T_ρ of $Spec(R_\rho)$ for the localization-completion R_ρ of R at ρ is injective (by the assumption $R_\rho \cong \mathbf{T}_\rho$). Sending c to $d\varphi_\rho\left(\frac{\partial}{\partial x}\right) \in T_\rho$, we get an injection $\mathbb{T}_F \hookrightarrow T_\rho$. By our assumption $R_\rho \cong \mathbf{T}_\rho \cong K[[x_\sigma]]_{\sigma \in S}$, T_ρ is generated by $\{\frac{\partial}{\partial x_\sigma}\}_{\sigma \in S}$, and thus T_ρ is $[F:\mathbb{Q}]$-dimensional over K. By the injectivity we have shown, $\mathbb{T}_F \cong T_\rho$, and we have

$$\begin{aligned} a_\sigma([p_\sigma, F_\sigma]) &= \sum_\tau c_\tau \frac{d\boldsymbol{\delta}_{\mathfrak{p}}([p_\sigma, \mathbb{Q}_p])}{dx_\tau}\Big|_{x=0} \\ a_\sigma([\gamma_\sigma, \mathbb{Q}_p]) &= \sum_\tau c_\tau \frac{d\boldsymbol{\delta}_\sigma([\gamma_\sigma, \mathbb{Q}_p])}{dx_\tau}\Big|_{x=0}, \end{aligned} \tag{4.2}$$

for the generator $\gamma_\sigma = \gamma$ of the σ-component of $1 + pO_p = (1 + p\mathbb{Z}_p)^S$. Thus

$$\mathcal{L}(V) = \pm \prod_\sigma \left(\log_p(\gamma_\sigma)\right) \det\left(\left(\frac{\partial\boldsymbol{\delta}([\gamma_s, \mathbb{Q}_p]}{\partial x_\tau}\right)^{-1}_{\sigma,\tau} \left(\frac{\partial\boldsymbol{\delta}([p_s, \mathbb{Q}_p])}{\partial x_\tau}\right)_{\sigma,\tau} \right).$$

This yields the desired formula, beacuse $\boldsymbol{\delta}_\sigma([\gamma_\sigma, \mathbb{Q}_p]^s) = (1 + x_\sigma)^s$ and $\mathcal{F}|U(p_\sigma) = \boldsymbol{\delta}_\sigma([p_\sigma, \mathbb{Q}_p)]\mathcal{F}$ for the Hecke operator $U(p_\sigma)$.

If f is associated to a split multiplicative abelian variety, it has beeen shown in my Israeli journal paper that $\left(\frac{\partial\boldsymbol{\delta}([p_s,\mathbb{Q}_p])}{\partial x_\tau}\right)_{\sigma,\tau}$ is diagonal. Then the the linear map L is the direct sum of local linear maps sending σ-component $F_\sigma = \mathbb{Q}_p$ into itself. Then an argument due to Greenberg shows tha formula relating $\mathcal{L}(V)$ to the product of $\frac{\log_p(q_\sigma)}{\mathrm{ord}_p(q_\sigma)}$ over $\sigma \in S$.

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