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The proof of the main theorem and its finer version Theorem 4.1 of the paper rely on the Tate conjecture proved by Faltings, which implies that if two simple rational abelian varieties have isomorphic compatible system, they are isogenous over $\mathbb{Q}$. Here is a remark in [EAI, page 109] about the present status of the Tate conjecture for general rank 2 motives defined over $\mathbb{Q}$ (see also [EAI, Remark 3.23]): "The Tate conjecture is not completely known for two given geometric rank 2 motives, though through modularity of two-dimensional compatible systems of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by Khare-Wintenberger [KhW] I Theorem 10.1 combined with the Tate conjecture for "modular" rank 2 motives proven by Brown and Ghate [BrG] Corollary 2.4.2, once two rank 2 Grothendieck motives are (both) realized in the cohomology of modular curves (not just as compatible systems), Galois endomorphisms comes from motivic endomorphisms (as Tate's conjecture predicts)."

We add some more remarks on the Tate conjecture for rank 2 motives:

(1) In [BrG] quoted above, the surjectivity of the map: $\mathrm{End}(M) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{End}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(M_l)$ for each modular rank 2 Scholl motive M with l-adic realization M_l is proven. The injectivity is proven in [GGQ, §3]. Their result would apply to two different Scholl motive M, N (i.e., we have $\mathrm{Hom}(M, N) \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(M_l, N_l)$). If M and N are non CM, the Galois image on $GL(M_l)$ and $GL(N_l)$ contain an open subgroup of $SL_2(\mathbb{Z}_l)$; so, the Tate conjecture would follow for all regular motives in the Tannakian category generated by Scholl motives?

(2) The Tate conjecture in the (possibly) bigger category generated by all rank 2 motives defined over $\mathbb{Q}$ (not just by Scholl motives) is not known yet, even if Serre's mod p modularity conjecture is proven by Khare–Wintenberger. For two such general regular rank 2 motives M, N, if one can prove the Mumford–Tate conjecture for $M \times N$, the Tate conjecture for rank 2 absolute-Hodge motives would follow. This might be doable under present technology?

References

Books

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Articles

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