$\mathcal{L}$–INVARIANTS OF ADJOINT SQUARE GALOIS REPRESENTATIONS

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1. Adjoint square Selmer group

Let p be an odd prime. We start with a totally real field F inside a fixed algebraic closure $\overline{\mathbb{Q}}$ and a Galois representation $\rho = \rho_F : \mathfrak{G}_F = \mathrm{Gal}(F^{(p)}/F) \to GL_2(A)$, where A is a local p-profinite Noetherian normal integral domain flat over $\mathbb{Z}_p$ with maximal ideal $\mathfrak{m}_A$ (A could be regular rings $\mathbb{Z}_p$ and $\mathbb{Z}_p[[x_1, \dots, x_r]]$) and $F^{(p)}/F$ is the maximal extension unramified outside p and ∞. We write W for the coefficient ring of A; so, W is the integral closure of $\mathbb{Z}_p$ in A and is a discrete valuation ring. We indicate by ρ also the representation space $\rho \cong A^2$ with the action given by ρ. We assume

- (unr) F is unramified at p;
- (det) $\det(\rho_F) = \psi\chi$ for the p-adic cyclotomic character χ and a totally even character $\psi : \mathfrak{G}_F \to W^\times$;
- (ord) For each prime $\mathfrak{p}|p$ of F, the representation space ρ_F has an exact sequence $0 \to \varepsilon_\mathfrak{p} \to \rho \to \delta_\mathfrak{p} \to 0$ with $\delta_\mathfrak{p} \cong A$ stable under the decomposition group $D_\mathfrak{p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of $\mathfrak{p}$ (Near ordinarity). Assume $\delta_\mathfrak{p} \not\equiv \varepsilon_\mathfrak{p} \mod \mathfrak{m}_A$;
- (lcy) The restriction of $\delta_\mathfrak{p}$ and $\varepsilon_\mathfrak{p}$ to the inertia subgroup $I_\mathfrak{p}$ factors through the cyclotomic Galois group $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$;
- (ai$_F$) $\rho_F \mod \mathfrak{m}_A$ is absolutely irreducible on $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

By (lcy), we have a Galois character $\delta_\mathfrak{p} : \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p}) \to A^\times$; so, for the Galois group $\Gamma_\mathfrak{p}$ of the cyclotomic $\mathbb{Z}_p$-extension of $F_\mathfrak{p}$, $\delta_\mathfrak{p}$ induces an algebra structure of A over $W[[\Gamma_\mathfrak{p}]]$. Fix a generator γ of $\Gamma_\mathfrak{p}$, we identify $W[[\Gamma_\mathfrak{p}]]$ with a power series ring $W[[x_\mathfrak{p}]]$ via $\gamma \mapsto 1 + x_\mathfrak{p}$.

The character $\delta_\mathfrak{p} : D_\mathfrak{p} \to A^\times$ is called the nearly ordinary character of ρ at $\mathfrak{p}$.

We define the adjoint square $Ad(\rho_F)$ by the action $Ad(\rho_F)(\sigma)(T) = \rho(\sigma)T\rho(\sigma)^{-1}$ on $\{T \in \mathrm{End}_A(\rho) | \mathrm{Tr}(T) = 0\}$. Then $Ad(\rho)$ has a three step filtration

$$\begin{array}{ccccc} Ad(\rho) & \supset & Z_\mathfrak{p}(\rho) & \supset & Ad^+_\mathfrak{p}(\rho) = \varepsilon_\mathfrak{p}\delta_\mathfrak{p}^{-1} \\ \cap & & \| & & \| \\ M_2(A) & \supset & \left\{\left(\begin{smallmatrix} a & * \\ 0 & -a \end{smallmatrix}\right) \middle| a \in A\right\} & \supset & \left\{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right) \middle| a \in A\right\} \end{array}$$

stable under $D_\mathfrak{p}$. The decomposition group $D_\mathfrak{p}$ acts trivially on $Z_\mathfrak{p}(\rho)/Ad^+_\mathfrak{p}(\rho)$. Writing A^* for the Pontryagin dual of A and taking an extension L/F inside $\overline{\mathbb{Q}}$, we define the Galois cohomologic Selmer group

$$\mathrm{Sel}_L(Ad(\rho)) = \mathrm{Ker}\left(H^1(\mathfrak{G}_L, Ad(\rho) \otimes_A A^*) \xrightarrow{\mathrm{Res}} \prod_{\mathfrak{P}|p} H^1(I_\mathfrak{P}, \frac{Ad(\rho) \otimes_A A^*}{Ad^+_\mathfrak{p}(\rho) \otimes_A A^*})\right).$$

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Here $\mathfrak{G}_L$ is the Galois group over L of the maximal extension unramified outside p and ∞, $\mathfrak{P}$ runs over prime factors of p in L, and $I_{\mathfrak{P}} \subset D_{\mathfrak{P}}$ is the inertia group at $\mathfrak{P}$. We equip with $\mathrm{Sel}_L(Ad(\rho))$ the discrete topology; so, its Pontryagin dual $\mathrm{Sel}^*_L(Ad(\rho))$ is a compact A-module. In particular, for the cyclotomic $\mathbb{Z}_p$-extension F_∞/F, $\mathrm{Sel}^*_{F_\infty}(Ad(\rho))$ is a $A[[\Gamma]]$-module of finite type, where $\Gamma = \mathrm{Gal}(F_\infty/F)$. Then $\mathrm{Sel}^*_{F_\infty}(Ad(\rho))$ has the characteristic power series $\mathcal{F}_\rho(x) = \mathrm{char}_{A[[x]]}(\mathrm{Sel}^*_{F_\infty}(Ad(\rho))$. Here we fix a generator $\gamma \in \Gamma$, and identify $A[[\Gamma]]$ with a power series ring $A[[x]]$ by $\gamma \mapsto 1 + x$.

Since $D_{\mathfrak{p}}$ acts trivially on $Z_{\mathfrak{p}}(\rho)/Ad^+_{\mathfrak{p}}(\rho)$, $\mathcal{F}_\rho(x)$ has trivial zero: $x|\mathcal{F}_\rho(x) \Leftrightarrow \mathcal{F}_\rho(0) = 0$. Here is a conjecture (basically due to R. Greenberg) in which all the identities are up to units in A:

Conjecture-Definition *Let g be the number of prime factors of p in F. Suppose that for a prime $P \in Spec(A)(W)$, $\rho_P = \rho_F \mod P$ is associated to a pure critical rank two motive with coefficients in T defined over F (so, W is a local factor of the integer ring of $T \otimes_{\mathbb{Q}} \mathbb{Q}_p$). Then $\mathcal{F}_\rho(x) = x^g\Phi_\rho(x)$ with $\Phi_\rho(0) = \frac{\mathcal{L}(Ad(\rho))}{\log_p(\gamma)^g}\mathrm{char}_A(\mathrm{Sel}_F(Ad(\rho)) \neq 0$ for an element $\mathcal{L}(Ad(\rho)) \in A$ (in particular, $\mathcal{F}_\rho(x) \neq 0$).*

2. Cyclotomic derivatives

In order to attack this conjecture, we relate the $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho))$ to a local derivative of an element in the Galois deformation ring of ρ. We consider all nearly ordinary deformations $\widetilde{\rho} : \mathfrak{G}_F \to GL_2(\widetilde{A})$ (for a local profinite W-algebra $\widetilde{A}$ with $W/\mathfrak{m}_W = \widetilde{A}/\mathfrak{m}_{\widetilde{A}}$) of $\overline{\rho}_F = \rho_F \mod \mathfrak{m}_A$ satisfying (lcy) (such deformations are called locally cyclotomic). Thus the deformation functor is given by

$$\widetilde{A} \mapsto \{\widetilde{\rho} : \mathfrak{G}_F \to GL_2(\widetilde{A}) \text{ with (lcy), (ord) and (det)}|\widetilde{\rho} \mod \mathfrak{m}_{\widetilde{A}} \cong \overline{\rho}\}/\cong.$$

We insist here that the reduction modulo $\mathfrak{m}_W$ of the nearly ordinary character of $\widetilde{\rho}$ is given by $\overline{\delta}_{\mathfrak{p}} = \delta_{\mathfrak{p}} \mod \mathfrak{m}_W$ for all $\mathfrak{p}|p$. Among them, we have a universal couple $(R_F, \boldsymbol{\rho}_F)$; thus, we have a unique W-algebra homomorphism $\varphi_{\widetilde{\rho}} : R_F \to \widetilde{A}$ for each $\widetilde{\rho}$ so that $\varphi_{\widetilde{\rho}} \circ \boldsymbol{\rho}_F \cong \widetilde{\rho}$. Again R_F is canonically an algebra over $W[[x_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then we have the derivative $\frac{\partial \mathbf{a}(\mathfrak{p})}{\partial x_{\mathfrak{p}'}} \in R_F$ for $\mathbf{a}(\mathfrak{p}) = \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])$ and the nearly ordinary character $\delta_{\mathfrak{p}} : \Gamma_{\mathfrak{p}} \to R_F^\times$.

If L/F is an extension in $\overline{\mathbb{Q}}$, then we have a base-change map $\varphi_{L/F} : R_L \to R_F$ given by $\varphi_{L/F} \circ \boldsymbol{\rho}_L \cong \boldsymbol{\rho}_F|_{\mathfrak{G}_L}$. For each $\sigma \in \mathrm{Aut}(L/F)$, $\boldsymbol{\rho}^\sigma_L(g) = \boldsymbol{\rho}_L(\widetilde{\sigma} g \widetilde{\sigma}^{-1})$ is a locally cyclotomic nearly ordinary deformation; so, we have a unique automorphism $[\sigma] : R_L \to R_L$ with $[\sigma] \circ \boldsymbol{\rho}_L \cong \boldsymbol{\rho}^\sigma_L$; so, $\mathrm{Aut}(L/F)$ acts on R_L, where $\widetilde{\sigma}$ is an extension of σ to $\overline{\mathbb{Q}}/F$. Define $J_{L/F} \subset R_L$ be the ideal generated by $[\sigma](r) - r$ $(r \in R_L)$. Then, under (ai_F), if L/F is a Galois extension of p-power degree,

$$R_L/J_{L/F} \cong R_F \text{ via } \varphi_{L/F}. \tag{BC}$$

By a standard argument due to Mazur, writing $\Gamma_L = \prod_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}$, we have a canonical isomorphism

$$\mathrm{Sel}^*_L(Ad(\rho)) \cong \Omega_{R_L/W[[\Gamma_L]]} \widehat{\otimes}_{R_L, \varphi_{\rho_L}} A$$

of $A[\mathrm{Gal}(L/F)]$-modules. Here for a closed subring B of R_L, $\Omega_{R_L/B}$ is the module of continuous (under the profinite topology) differentials over B. Note that $W[[\Gamma_L]] \cong W[[x_{\mathfrak{P}}]]_{\mathfrak{P}|p}$ if L/F is finite and $W[[\Gamma_{F_\infty}]] = W$.

3. Results

Here are some results due to myself and the work by Taylor-Wiles for $F=\mathbb{Q}$ and Fujiwara for general F:

Theorem 3.1. *Let the assumption be as in the conjecture. Suppose* $(\mathrm{ai}_{F[\mu_p]})$ *and that* ρ_P *is associated to a Hilbert modular form of level* p. *Let* L/F *be a totally real finite* p*-extension.*

1. *Every nearly* p*-ordinary deformation of* $\overline{\rho}_F$ *is* p*-adically modular with respect to* $GL(2)_{/L}$, *and* R_L *is a local complete intersection free of finite rank over* $W[[x_{\mathfrak{P}}]]_{\mathfrak{P}|p}$.
2. *We have* $\mathrm{char}_A(\mathrm{Sel}_L(Ad(\rho)))\neq 0$ *and* $\mathrm{char}_{A[[x]]}(\mathrm{Sel}_{F_\infty}(Ad(\rho)))\neq 0$.
3. *We have*
$$\frac{\mathcal{L}(Ad(\rho))}{\log_p(\gamma)^g}=a(p)^{-1}\det\left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'|p},$$
where $a(\mathfrak{p})=\delta_{\mathfrak{p}}([p,F_{\mathfrak{p}}])$, $a(p)=\prod_{\mathfrak{p}|p}a(\mathfrak{p})$ *and* $\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}}=\varphi_\rho(\frac{\partial \mathbf{a}(\mathfrak{p})}{\partial x_{\mathfrak{p}'}})$. *If further* A *is regular and* $\det\left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'|p}\neq 0$, *the conjecture holds.*
4. *If* ρ_P *is associated to an abelian variety (with real multiplication) having multiplicative reduction at* m *prime factors* $\mathfrak{p}|p$ *with* $m\geq g-1$ *and* $Spec(A)$ *is an irreducible component of* $Spec(R_F)$, *we have* $\det\left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'|p}\neq 0$; *so, for almost all specializations* ρ_Q $(Q\in Spec(A)(W))$ *of* ρ, $\mathcal{L}(Ad(\rho_Q))\neq 0$.
5. *If* $A=W$ *and* ρ *is associated to an elliptic curve* E *over* F *with split multiplicative reduction at all* $\mathfrak{p}|p$, *we have* $\mathcal{L}(Ad(\rho))=\prod_{\mathfrak{p}|p}\frac{\log_p(N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(q_{\mathfrak{p}}))}{\mathrm{ord}_p(N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(q_{\mathfrak{p}}))}$, *where* $q_{\mathfrak{p}}$ *is the Tate period of* E *at* $\mathfrak{p}$ $(E(F_{\mathfrak{p}})=F_{\mathfrak{p}}^\times/q_{\mathfrak{p}}^{\mathbb{Z}})$. *In particular, if* p *splits completely in* $F/\mathbb{Q}$, $\mathcal{L}(Ad(\rho))\neq 0$.
6. *Let* $\mu(L)$ *be the* μ*-invariant of* $\mathrm{char}_A(\Omega_{R_L/W}\otimes_{R_L}A)$. *Then* $\mathcal{L}(Ad(\rho))\neq 0$ *if the invariant* $\mu(L)$ *is bounded for any finite intermediate extension inside* F_∞/F.

Let us describe briefly why $\mathcal{L}(Ad(\rho))$ is related to the derivative determinant $\det\left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}}\right)$. For simplicity, we assume that $R_0=W[[x_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then $\Omega_{R_0/W}=\bigoplus_{\mathfrak{p}|p}R_0dx_{\mathfrak{p}}\cong R_0^g$. Let F_n/F be the fixed field of Γ^{p^n}. Write $(R_n,\boldsymbol{\rho}_n)$ for $(R_{F_n},\boldsymbol{\rho}_{F_n})$. By a slightly nontrivial argument, we have $(R_\infty,\boldsymbol{\rho}_\infty)=\varprojlim_n(R_n,\boldsymbol{\rho}_n)$.

Write $J_n=J_{F_n/F}$ for the kernel of the projection $R_n\to R_0$. We have a standard exact sequence of $A[[\Gamma]]$-modules (with a splitting section of A-modules)

$$0\to\frac{J_n}{J_n^2}\otimes_{\mathcal{R}_0}A\to\Omega_{R_n/W}\otimes_{R_n}A\xrightarrow{\pi_n}\Omega_{R_0/W}\otimes_{\mathcal{R}_0}A(\cong A^g)\to 0.$$

To make notation simple, we write S_n for $\Omega_{R_n/W}\otimes_{R_n}A$ and T_n for $J_n/J_n^2\otimes_{R_n}A$. By the above A-split exact sequence, we have

$$S_n=A^g\oplus T_n\quad\text{as }A\text{-modules.}$$

By (BC), we have $J_n=R_n([\gamma]-1)R_n$ and the following formula of the coinvariants:

$$H_0(\Gamma,S_n)=S_0\cong A^g.$$

Thus if $S_n\cong A^g\oplus T_n$ up to A-torsion as $A[[\Gamma]]$-modules, we get $\Phi_\rho(0)\neq 0$.

We try to construct an $A[[\Gamma]]$-linear section of $T_n \hookrightarrow S_n$. Since $\frac{J_n}{J_n^2}$ is an R_0-module and R_0 is fixed by the Γ-action, $\frac{J_n}{J_n^2}$ is an $A[[\Gamma]]$-module. Define $\delta : R_n \to \frac{J_n}{J_n^2}$ by $\delta(a) = [\gamma](a) - a$, which is an R_n^Γ-derivation, because

$$\delta(ab) = [\gamma](ab) - [\gamma](a)b + [\gamma](a)b - ab = [\gamma](a)\delta(b) - \delta(a)b$$

with $[\gamma](a)\delta(b) \equiv a\delta(b) \mod J_n^2$. This induces a surjective map $\varphi_n : \Omega_{R_n/W} \to \frac{J_n}{J_n^2}$ with $\delta = \varphi_n \circ d$, because $\frac{J_n}{J_n^2}$ is generated by $[\gamma](y) - y$ for $y \in R_n$. We put $\varphi_n^A = \varphi_n \otimes \mathrm{id} : S_n = \Omega_{R_n/W} \otimes_{R_n} A \to T_n$. Put $X_n = \mathrm{Ker}(\varphi_n^A)$, and we have an exact sequence of $A[[\Gamma]]$-modules:

$$0 \to X_n \to S_n \to T_n \to 0.$$

The natural map $\bigoplus_{\mathfrak{p}} Ad\mathbf{a}(\mathfrak{p}) \cong \Omega_{W[[\mathbf{a}(\mathfrak{p})]]_{\mathfrak{p}}/W} \otimes_{W[[\mathbf{a}(\mathfrak{p})]]} A \to X_\infty$ composed with π_∞ has matrix $\left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'|p}$, and if the determinant does not vanish, the sequence $T_\infty \hookrightarrow S_\infty \to A^g$ splits up to A-torsion; so, we get basically (3).

As for (6), the extension $T_n \hookrightarrow S_n \twoheadrightarrow A^g$ in $\mathrm{Ext}^1_{A[[\Gamma]]}(A^g, T_n) = H^1(\Gamma/\Gamma^{p^n}, T_n)^g$ is killed by p^n, and the obstruction of splitting the sequence is in the p-torsion of S_n which is bounded by the assumption; so, after tensoring $\mathbb{Q}_p$, $T_\infty \hookrightarrow S_\infty \to A^g$ splits as $A[[\Gamma]]$-modules, and hence the result.

The identity (5) then follows from (3) and a result of Greenberg. It is known by an analytic argument, $\log_p(q_{\mathfrak{p}}) \neq 0$ (the so-called theorem of St. Etienne).

There are still many open questions to ask; for example,

- Is R_∞ Noetherian? Or equivalently, $\mu(\Omega_{R_\infty/W} \otimes_{R_\infty} W) = 0$? Therefore, this is just asking when $\mu(\mathrm{char}(\mathrm{Sel}^*_{F_\infty}(Ad(\rho_P))) = 0$.
- By the assertion (2), the P-adic localization-completion R_P is Noetherian. What is the structure of R_P? For example, what is $\dim R_P$? Actually we know that R_n under the assumption of the theorem is a local complete intersection for all finite n.
- The module X_∞ contains $\Omega_{W[[\mathbf{a}(\mathfrak{p})]]/W} \otimes_{W[[\mathbf{a}(\mathfrak{p})]]} A$. Are they equal?