

LECTURES ON A PROOF OF THE ANTI-CYCLOTOMIC MAIN CONJECTURE

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1. INTRODUCTION

Let M be a CM field inside a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ with maximal real subfield F . The field F is totally real, and M is a totally imaginary quadratic extension of F . We fix a prime $p > 3$ *unramified* in $M/\mathbb{Q}$. We assume to have a p -ordinary CM type $\Sigma = \{\sigma : M \hookrightarrow \overline{\mathbb{Q}}\}$ of M . Namely, fixing embeddings $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, the embeddings $i_p \circ \sigma$ for $\sigma \in \Sigma$ induce exactly a half Σ_p of the p -adic places of M . We identify Σ_p with a subset of prime factors

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of p in M . Write c for complex conjugation on $\mathbb{C}$ and also the automorphism of M induced by c . Then c is the generator of $\text{Gal}(M/F)$, and the disjoint union $\Sigma_p \sqcup \Sigma_p^c$ for $\Sigma_p^c := \{\sigma \circ c \mid \sigma \in \Sigma\}$ gives the total set of prime factors of p in M . For a multi-index $e = \sum_{\mathfrak{P}|p} e(\mathfrak{P})\mathfrak{P} \in \mathbb{Z}[\Sigma_p \sqcup \Sigma_p^c]$, we write $\mathfrak{P}^e$ for $\prod_{\mathfrak{P}|p} \mathfrak{P}^{e(\mathfrak{P})}$.

A Hecke character $\psi : M^\times \backslash M_{\mathbb{A}}^\times \rightarrow \mathbb{C}^\times$ is called *anticyclotomic* if $\psi(x^c) = \psi(x)^{-1}$. Regarding ψ as a Galois character $\text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \mathbb{C}^\times$ by class field theory, this is equivalent to having $\psi(c\sigma c) = \psi(\sigma)^{-1}$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F)$. We call ψ has *split* conductor if the conductor of ψ is divisible only by primes split in M/F . We fix a continuous anticyclotomic character $\psi : \text{Gal}(\overline{F}/M) \rightarrow W^\times$ of finite order, where W is the completed p -adic integer ring of the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$. Note that W is the Witt vector ring with coefficients in an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}_p$ and hence is a discrete valuation ring. It is an easy consequence of class field theory (see [HMI, Lemma 5.31]) that we can always find another Hecke character $\varphi : M_{\mathbb{A}}^\times / M^\times M_\infty^\times \rightarrow \mathbb{C}^\times$ such that $\psi(x) = \varphi^-(x) = \varphi^{-1}(x)\varphi(x^c)$. Regarding φ and ψ as Galois characters, this is equivalent to $\psi(\sigma) = \varphi^{-1}(\sigma)\varphi(c\sigma c^{-1})$ for any complex conjugation c in $\text{Gal}(\overline{F}/F)$. We consider the following four conditions for an anti-cyclotomic character ψ of conductor $\mathfrak{c}\mathfrak{P}^e$ with $\mathfrak{c}$ prime to p :

- (h1) *The character ψ has order prime to p with exact conductor $\mathfrak{c}\mathfrak{P}^e$.*
- (h2) *The conductor $\mathfrak{c}$ is a product of primes split in M/F .*
- (h3) *The local character $\psi_{\mathfrak{P}}$ is non-trivial for all $\mathfrak{P} \in \Sigma_p$.*
- (h4) *The restriction ψ^* of ψ to $\text{Gal}(\overline{F}/M[\sqrt{p^*}])$ for $p^* = (-1)^{(p-1)/2}p$ is non-trivial.*

We study arithmetic of the unique $\mathbb{Z}_p^{[F:\mathbb{Q}]}$ -extension M_∞^- of M (unramified outside p and ∞) on which $c\sigma c^{-1} = \sigma^{-1}$ for all $\sigma \in \Gamma_M^- = \text{Gal}(M_\infty^-/M)$. The extension M_∞^-/M is called the anticyclotomic tower over M . Let $M(\psi)/M$ be the class field with ψ inducing the isomorphism $\text{Gal}(M(\psi)/M) \cong \text{Im}(\psi)$. Let $L_\infty/M_\infty^-M(\psi)$ be the maximal p -abelian extension unramified outside Σ_p . Each $\gamma \in \text{Gal}(L_\infty/M)$ acts on the normal subgroup $X = \text{Gal}(L_\infty/M_\infty^-M(\psi))$ continuously by conjugation, and by the commutativity of X , this action factors through $\text{Gal}(M(\psi)M_\infty^-/M)$. We have a canonical splitting $\text{Gal}(M(\psi)M_\infty^-/M) = \Gamma_M^- \times G_{\text{tor}}(\psi)$ for the maximal torsion subgroup $G_{\text{tor}}(\psi) \cong \text{Im}(\psi)$. Since ψ is of order prime to p , the decomposition is canonical. Then we look into the Γ_M^- -module: $X[\psi] = X \otimes_{\mathbb{Z}_p[G_{\text{tor}}(\psi)], \psi} W$.

As is well known, $X[\psi]$ is a $W[[\Gamma_M^-]]$ -module of finite type, and it is a torsion module by a result of Fujiwara (see [HMI, Theorem 5.33]) generalizing the fundamental Wiles–Taylor’s $R = \mathbb{T}$ theorem. Thus we can think of the characteristic element $\mathcal{F}^-(\psi) \in W[[\Gamma_M^-]]$ of the module $X[\psi]$. As we have seen in [HT93, Theorem II] and [HT94] (see also [H26, §6]), we have the anticyclotomic p -adic Hecke L -function $L_M^-(\psi) = L_p^-(\psi) \in W[[\Gamma_M^-]]$ (constructed by Katz). Here we use notation when relevance to the CM field M is important. Then we prove

Theorem I. *Assume conditions (h1) and (h3–4). We have the identity: $\mathcal{F}^-(\psi) = L_p^-(\psi)$ up to a unit in $W[[\Gamma_M^-]]$.*

This is proved in [H06] under (h1–4). It is described in [H26, §6] how to reduce the case without the assumption (h2) to the proof under (h1–4) by quadratic base-change; so, we give an exposition of a proof under (h1–4). The current proof is shorter than the original proof in [H06]. The simplification comes from the study

of an optimal embedding of the integer ring R of M into Eichler orders of quaternion algebras (see Remark 5.1).

The conditions $p > 3$ and (h4) are necessary because of technical restrictions of the Taylor–Wiles method (see [HMI, §3.2–3]).

The assertion (in the theorem) is called the anticyclotomic main conjecture for CM fields. The present idea of the proof is a refinement of those exploited in [HT93], [HT94] and [H26, Corollary 1.43], where we have proven $L_p^-(\psi)|\mathcal{F}^-(\psi)$ in $W[[\Gamma_M^-]]$. One of the main ingredients of the proof is the congruence power series $H(\varphi) \in W[[\Gamma_M]]$ of the CM -component of the universal nearly ordinary Hecke algebra $\mathbf{h}$ for $GL(2)/F$. In the joint works [HT93] and [HT94] with Tilouine and also in this paper, we take the Hecke algebra of (outside p) level $\mathfrak{N}' := N_{M/F}(\mathfrak{C})d(M/F)$ for the conductor $\mathfrak{C} = \mathfrak{C}(\varphi)$ of φ and the relative discriminant $d(M/F)$ of M/F . Write $\mathfrak{N} = \mathfrak{N}(\varphi)\mathfrak{N}'$ for the minimal (outside p) level of the p -ordinary family discussed in [H26, §6.1]. We take the nearly p -ordinary Hecke algebra $\mathbf{h}$ deforming to $\text{Ind}_M^F \varphi$ defined in [H26, Definition 1.35]) for the automorphic induction $\pi(\varphi)$ of φ . We write R for the nearly ordinary universal deformation ring over W of $\text{Ind}_M^F \varphi$ under an appropriate ramification data outside p in [H26, §6.5] representing the deformation functor $\mathcal{D}^{ord}$ in [H26, (6.15)]. The $R = \mathbb{T}$ theorem is formulated in [HMI, §3.2] using such level groups following Fujiwara [F06].

The space $\text{Spf}(\mathbb{T}) = \text{Spf}(R)$ has a unique irreducible component canonically isomorphic to $\text{Spf}(W[[\Gamma_M]])$ which carries the universal deformation of the form induced from a character of $\text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow W[[\Gamma_M]]^\times$. The projection $\mathbb{T} \rightarrow W[[\Gamma_M]]$ gives rise to a family of nearly ordinary Hecke eigenform $\{f(\lambda_P)\}_P$ indexed by arithmetic points $P \in \text{Spf}(W[[\Gamma_M]])$ whose Galois representation is given by $\text{Ind}_M^F \lambda_P$ for a Hecke character λ_P deforming φ . Writing $\text{Spf}(\mathbb{T}) = \text{Spf}(W[[\Gamma_M]]) \cup \text{Spf}(\mathbb{T}^\perp)$ for the complementary reduced closed subscheme $\text{Spf}(\mathbb{T}^\perp)$, $H(\varphi)$ is defined by the identity $W[[\Gamma_M]] \otimes_{\mathbb{T}} \mathbb{T}^\perp = W[[\Gamma_M]]/(H(\varphi))$ [H26, Definition 1.31]. It is known that $H(\varphi) \in W[[\Gamma_M^-]]$ [H26, Remark 1.36] up to units in $W[[\Gamma_M]]$.

Another important ingredient is the divisibility proven in [H26, Corollary 1.43]:

$$(L) \quad (h(M)/h(F))L_p^-(\psi^-)|H(\varphi) \text{ in } W[[\Gamma_M^-]].$$

Here $h(M)$ (resp. $h(F)$) is the class number of M (resp. F). On the other hand, the $R = \mathbb{T}$ -theorem implies

$$(F) \quad H(\varphi) = (h(M)/h(F))\mathcal{F}^-(\psi^-) \text{ up to units in } W[[\Gamma_M^-]]$$

(see [HMI, §3.1-3, §5.3]). Thus we need to prove:

$$(R) \quad H(\varphi)(\kappa)|(h(M)/h(F))L_p^-(\psi^-)(\kappa) \text{ in } W$$

for a (single) weight κ specialization, where $\Phi(\kappa)$ is the value of a power series $\Phi \in \overline{W}[[\Gamma_M^-]]$ at $\kappa \in \text{Spf}(W[[\Gamma_M^-]])(W)$. Each arithmetic point $P \in \text{Spf}(W[[\Gamma_M^-]])(W)$ has weight $\kappa = \kappa(P) = (\kappa_1 = \kappa_1(P), \kappa_2 = \kappa_2(P))$ with $\kappa_j \in \mathbb{Z}[I]$ and $\kappa_1 - \kappa_2 \geq I$ in the sense of [PAF, §4.3.1] (and [HMI, (2.3.4)]) where the role of κ_1 and κ_2 are reversed (i.e., κ gives the Hodge weight of the attached motive). In other words, we choose the Hodge type (p, q) with $p > q$ out of $\{(p, q), (q, p)\}$. Hereafter we take $\kappa = (I, 0)$ (the Hodge weight of abelian varieties). By (L) and Nakayama's lemma, the reverse divisibility (R) (specialized at κ) implies the theorem.

In the (finite dimensional) space $S_\kappa^{n, ord}(\mathfrak{N}_p^\infty, \varepsilon_\lambda; W)$ of nearly p -ordinary cusp forms of weight κ with coefficients in W and with a suitable Neben character ε_λ

[HMI, §4.3.9], we have a CM Hecke eigenform $f(\lambda)$ of a Hecke character $\lambda = \lambda_P$ of weight κ (regarded as a Galois character) such that λ^- factors through $\text{Gal}(M(\psi)M_\infty^-/M)$ with $\lambda^-|_{G_{\text{tor}(\psi)}} = \psi$ and $\lambda \equiv \varphi \pmod{\mathfrak{m}_W}$. We write $\mathfrak{N}(\lambda)$ ($\mathfrak{N}(\lambda)|\mathfrak{N}_p^\infty$) for the minimal level of $f(\lambda)$. This form studied in [H26, §6] is of minimal level (possibly of level smaller than that of the primitive form). Since the CM local ring $\mathcal{R}$ of $\mathbf{h}$ is a Gorenstein ring [HMI, 1.52 and 3.28], the number $H(\varphi)(\kappa)$ is the maximal denominator of the reduced fractions $\frac{(f(\lambda), f)}{(f(\lambda), f(\lambda))}$ in W as f running through all elements of $S_\kappa(\mathfrak{N}(\lambda), \varepsilon_\lambda; W)$ (see §4.1), where $(\cdot, \cdot)$ is the Petersson inner product of level $\mathfrak{N}(\lambda)$ divided by a canonical period.

As seen in [HT93, Theorem 7.1] and [H09, (3.5)] (see [H07, Proposition 5.7] for a p -adic version), we have $\pi^{\kappa_1 - \kappa_2 + \Sigma}(f(\lambda), f(\lambda)) = c_1(h(M)/h(F))L(1, \lambda^-)$ for an innocuous constant $c_1 \in W$ (for the constant c_1 , see (5.14)). The quotient $\frac{\pi^{2(\kappa_1 - \kappa_2)} W_p(\lambda^-)(f(\lambda), f(\lambda))}{\Omega^{2(\kappa_1 - \kappa_2)}}$ is then the value

$$(h(M)/h(F))L_p^-(\psi^-)(\kappa) \in W.$$

Here $W_p(\lambda^-)$ is the local Gauss sum of λ^- at p , Ω is the Néron period of the abelian variety of CM type Σ (defined over $\overline{\mathbb{Q}} \cap W$), and the exponent $\kappa_1 - \kappa_2$ is determined by the weight κ .

Since $H(\varphi)(\kappa)$ is the maximal denominator of reduced fractions $\frac{(f(\lambda), f)}{(f(\lambda), f(\lambda))}$ (see Lemma 4.1), what we need to show (to prove (R)) is the W -integrality of the numbers $\frac{\pi^{2(\kappa_1 - \kappa_2)} W_p(\lambda^-)(f(\lambda), f)}{\Omega^{2(\kappa_1 - \kappa_2)}}$ for all $f \in S_\kappa(\mathfrak{N}(\lambda), \varepsilon_\lambda; W)$. This we will show by a detailed analysis of the residue formulas of a Shimura series, defined on orthogonal groups of signature $(n, 2)$, introduced in [Sh80] and [Sh81], and we take those associated with a theta series of M and the determinant (quadratic form) of $M_2(F)$. The validity of the q -expansion principle is very important to show the W -integrality, because we write the Petersson inner product as a value of a modular form (with integral q -expansion) at a CM point of (the product of two copies of) the Hilbert modular variety. This modular form is obtained as the residue of a Shimura series. However in our fully isotropic case, the orthogonal similitude group of signature $(2, 2)$ over F is isogenous to the product $GL(2) \times GL(2)_F$; so, basically we are dealing with Hilbert modular forms, and the q -expansion principle [PAF, Corollary 4.23] is known by a work of Ribet (see [PAF, Theorem 4.21]).

Another important point is to write down every W -integral Hilbert cusp form as a W -integral linear combination of theta series of the definite quaternion algebra unramified at every finite (henselian) place. Such a problem over $\mathbb{Q}$ was first studied by Eichler (his basis problem; see [HMI, §2.6.2]) and then generalized to the Hilbert modular case by Shimizu and Jacquet-Langlands in different manners. We scrutinize the integrality of the Jacquet-Langlands-Shimizu correspondence (mainly using duality between Hecke algebras and their spaces of cusp forms; see [HMI, §2.3.4]).

At the last step of finalizing the W -integral correspondence, we again need a result of Fujiwara: Freeness theorem in [F06, Theorem 0.4] of quaternionic cohomology groups as Hecke modules, which is valid again under the assumptions (h1-4) for cusp forms with complex multiplication. The everywhere unramified definite quaternion algebra exists only when the degree $[F : \mathbb{Q}]$ is even; so, we will at the end reduce, by a base-change argument, the case of odd degree to the case of even degree.

In this paper, we write D (resp. $d(M/F)$) for the discriminant of $F/\mathbb{Q}$ (resp. M/F). We regard $D \in \mathbb{Z}$ but $d(M/F)$ is an ideal of F . We write R (resp. O) for the integer ring of M (resp. M). For a number field X , we write I_X for the set of complex embeddings of X and we put $I = I_F$. The module of formal linear combination of I_X is denoted by $\mathbb{Z}[I_X]$, and $A \subset I_X$ is identified $\sum_{\sigma \in A} \sigma \in \mathbb{Z}[I_X]$. This convention particularly applies to I and Σ .

2. DEFINITE QUATERNIONIC THETA SERIES SPAN INTEGRALLY MODULAR FORMS

We take a quaternion algebra B/F with a fixed isomorphism $B_{F_{\mathbb{A}}^{(\infty)}} \cong M_2(F_{\mathbb{A}}^{(\infty)})$ (so, B is unramified everywhere at finite places). We fix a maximal order O_B of B so that the above isomorphism identifies $\widehat{O}_B$ with $M_2(\widehat{O})$. We regard B as a quadratic space over F with the quadratic form $N : B \rightarrow F$ given by the reduced norm. Then the associated symmetric bilinear form is $S(x, y) = N(x + y) - N(x) - N(y) = \text{Tr}(xy^t)$ for the reduced trace Tr and the main involution ι of B given by $\text{Tr}(x) = x + x^t$.

We consider the following open compact subgroup of $GL_2(F_{\mathbb{A}}^{(\infty)})$ for a level ideal $\mathfrak{N}$:

$$(2.1) \quad U_0(\mathfrak{N}) = U_0^B(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{O}) \mid c \equiv 0 \pmod{\mathfrak{N}\widehat{O}} \right\} \subset B_{F_{\mathbb{A}}^{(\infty)}}^\times,$$

where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_{\ell} \mathbb{Z}_{\ell}$. Then we introduce the following semi-group $\Delta_0(\mathfrak{N}) = \Delta_0^B(\mathfrak{N})$ inside $B_{F_{\mathbb{A}}^{(\infty)}}$

$$(2.2) \quad \Delta_0^B(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_{\mathbb{A}}^{(\infty)}) \cap M_2(\widehat{O}) \mid c \equiv 0 \pmod{\mathfrak{N}\widehat{O}}, d_{\mathfrak{N}} \in O_{\mathfrak{N}}^\times \right\},$$

where $d_{\mathfrak{N}}$ is the projection of $d \in \widehat{O}$ to $\prod_{\mathfrak{l} \mid \mathfrak{N}} O_{\mathfrak{l}}$ for prime ideals $\mathfrak{l}$.

Take a quadratic CM extension M/F , and write R (resp. O) for the integer ring of M (resp. F). Since B is unramified everywhere at finite place and M is totally imaginary, the field M is embeddable into B . Write $\text{Gal}(M/F) = \langle c \rangle$. Suppose that $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(F_p)$ for $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and

- (unr) $F/\mathbb{Q}$ is unramified at p ;
- (ord) all prime factors of (p) in F splits in M/F .

Taking F -algebra embeddings $l, r : M \hookrightarrow B$, identify B with $M \otimes_F M \cong M \oplus M$ ($a \otimes b \mapsto (ab, a^c b)$) by $(a \otimes b)v = l(a)vr(b^c)$.

2.1. The $M \otimes M$ module B . When we regard B as an M -vector space, we agree to use right multiplication by $b \in M$ given by $bv = v \cdot r(b^c)$. Therefore $M \otimes_F M$ is identified with $M \oplus M$ by $a \otimes b \mapsto (ab, a^c b)$.

Take a basis M over F of the form $1, \delta$ with $\Delta := \delta^2 \in F$. Then $M \otimes_F M$ has F -basis $(1 \otimes 1, \delta \otimes 1, 1 \otimes \delta, \delta \otimes \delta)$. Under the identification as above of $M \otimes_F M$ with $M \oplus M$, for the idempotent $e_1 = (1, 0), e_2 = (0, 1) \in M \oplus M$, we have

$$e_1 = \frac{1 \otimes 1}{2} + \frac{\delta \otimes \delta}{2\Delta} \text{ and } e_2 = \frac{1 \otimes 1}{2} - \frac{\delta \otimes \delta}{2\Delta}.$$

First suppose that a prime $\mathfrak{q}$ of F splits in M . Taking

$$(2.3) \quad l(a) = r(a) = \text{diag}[a^c, a],$$

$$e_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \text{ and } e_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Then, for the identity matrix $1_2 = \text{diag}[1, 1]$, $v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $v_2 = 1_2$,

$$(2.4) \quad \begin{aligned} (a \otimes b)v_2 &= l(a)r(b^c) = \text{diag}[a^c b, ab^c] = r(ab^c) = a^c b v_2, \\ (a \otimes b)v_1 &= v_1 r((ab)^c) = (ab)v_1. \end{aligned}$$

This confirms that v_1 (resp. v_2) is the eigenvector of the left (resp. right) component of $M \oplus M$. We record this choice as

$$(2.5) \quad l(M_{\mathfrak{q}})\mathbf{b}r(M_{\mathfrak{q}}) = M_{\mathfrak{q}} \oplus M_{\mathfrak{q}} \quad \text{for } \mathbf{b} = v_1 + v_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now assume that $\mathfrak{q}$ is a non-split prime in M/F . Then there exist eigenvectors $v_j \in B_{\mathfrak{q}}$ such that $(a \otimes b)v_1 = (ab)v_1$ and $(a \otimes b)v_2 = (ab^c)v_2$ as $ab, ab^c \in M_{\mathfrak{q}}$ and B is an $M_{\mathfrak{q}}$ -vector space. Write $M_{\mathfrak{q}} = F_{\mathfrak{q}}[\sqrt{d_{\mathfrak{q}}}]$. If $B_{\mathfrak{q}} \cong M_2(F_{\mathfrak{q}})$, we may assume

$$(2.6) \quad l(a + b\sqrt{d_{\mathfrak{q}}}) = r(a + b\sqrt{d_{\mathfrak{q}}}) = \begin{pmatrix} a & b \\ bd_{\mathfrak{q}} & a \end{pmatrix},$$

where $d_{\mathfrak{q}}$ is either a $\mathfrak{q}$ -adic unit or its $\mathfrak{q}$ -adic valuation equal to 1. Then for $v_1 = \text{diag}[-1, 1]$ and $v_2 = \begin{pmatrix} 0 & 1 \\ d_{\mathfrak{q}} & 0 \end{pmatrix}$, we have

$$(2.7) \quad \begin{aligned} ((a + b\sqrt{d_{\mathfrak{q}}}) \otimes 1)v_1 &= v_1 r(a - b\sqrt{d_{\mathfrak{q}}}) = (a + b\sqrt{d_{\mathfrak{q}}})v_1, \\ ((a + b\sqrt{d_{\mathfrak{q}}}) \otimes 1)v_2 &= v_2 r(a + b\sqrt{d_{\mathfrak{q}}}) = (a - b\sqrt{d_{\mathfrak{q}}})v_2. \end{aligned}$$

In any case, write $B_i := e_i B = M v_i$ ($i = 1, 2$) for the idempotents $e_1 := (1, 0)$, $e_2 := (0, 1) \in M \oplus M$, we have $B = B_1 \oplus B_2$, and B_j is an eigenspace under the left action l of M .

Recall that we have identified $B_{\mathbb{A}(\infty)}$ with $M_2(\widehat{O})$. For a prime $\mathfrak{l}$ of F , we define the Eichler order of level $\mathfrak{l}^m$ by

$$(2.8) \quad \widehat{O}_0(\mathfrak{l}^m)_{\mathfrak{l}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F_{\mathfrak{l}}) \mid c \in \mathfrak{l}^m O_{\mathfrak{l}} \right\}.$$

Here $\widehat{O}_0(\mathfrak{l}^0)_{\mathfrak{l}}$ is the fixed maximal order $O_{B, \mathfrak{l}}$ of $B_{\mathfrak{l}}$. We then put for an O -ideal $\mathfrak{M} \neq 0$, $\widehat{O}_0(\mathfrak{M}) = \prod_{\mathfrak{l}} \widehat{O}_0(\mathfrak{l}^{e(\mathfrak{l})})$, where $\mathfrak{M} = \prod_{\mathfrak{l}} \mathfrak{l}^{e(\mathfrak{l})}$ is the prime decomposition of $\mathfrak{M}$ (for $\mathfrak{l} \nmid \mathfrak{M}$, we agree to put $e(\mathfrak{l}) = 0$).

Lemma 2.1. *Let $\mathfrak{l}$ be a prime of F either split in M or ramified in M . Let $L_{\mathfrak{l}} = z\widehat{O}_0(\mathfrak{l}^e)$ and $z \in F_{\mathfrak{l}}^{\times}$, and assume that $e \leq 1$ if $\mathfrak{l}$ is ramified in M . Then for the idempotent e_j of $M_{\mathfrak{l}} \otimes_{F_{\mathfrak{l}}} M_{\mathfrak{l}}$, $L_{\mathfrak{l}} = e_1 L_{\mathfrak{l}} \oplus e_2 L_{\mathfrak{l}}$.*

The assumption of the above lemma is actually a necessary and sufficient condition for $r = l$ bringing $R_{\mathfrak{l}}$ into $\widehat{O}_0(\mathfrak{l}^e)$ [QAL, Lemma 30.6.16] which is a key to prove the direct sum decomposition $L_{\mathfrak{l}} = e_1 L_{\mathfrak{l}} \oplus e_2 L_{\mathfrak{l}}$.

Proof. By multiplication by a central element z , we can transfer splitting of $L_{\mathfrak{l}}$ to $zL_{\mathfrak{l}}$; so, we may assume that $z = 1$. If $\mathfrak{l} = (\varpi)$ with $\varpi \in O_{\mathfrak{l}}$ splits into $\mathfrak{L}\overline{\mathfrak{L}}$ in M , identifying $M_{\mathfrak{l}} = F_{\mathfrak{l}} \oplus F_{\mathfrak{l}}$ and noting $\text{diag}[1, \varpi^e] = l(1, \varpi^e) = r(1, \varpi^e)$, we see $e_1 L_{\mathfrak{l}} = O_{\mathfrak{l}} \text{diag}[1, \varpi^e] v_1 = O_{\mathfrak{l}} v_1 \text{diag}[\varpi^e, 1]$ and $e_2 L_{\mathfrak{l}} = O_{\mathfrak{l}} v_2$. If $\mathfrak{l}$ is ramified in M , we may assume that $M_{\mathfrak{l}} = F_{\mathfrak{l}}[\sqrt{\varpi}]$ and r and l are as in (2.6) with $d_{\mathfrak{l}} = \varpi$. Then plainly, $r(R_{\mathfrak{l}}) = l(R_{\mathfrak{l}}) \subset \widehat{O}_0(\mathfrak{l})$, and hence $L_{\mathfrak{l}} = l(R_{\mathfrak{l}})v_1 \oplus l(R_{\mathfrak{l}})v_2 = v_1 r(R_{\mathfrak{l}}) \oplus v_2 r(R_{\mathfrak{l}})$. $\square$

Even if an O -lattice $L \subset B$ does not satisfies the assumption of Lemma 2.1, we can estimate the difference of L and $e_1 L \oplus e_2 L$. Set $L_i = L \cap B_i$ and $L^i = e_i L \subset B_i$. Then $L_i \subset L^i$ with $[L^i : L_i] < \infty$. Set $L^M = L^1 \oplus L^2 \supset L$ and $L_M := L_1 \oplus L_2 \subset L$. Since $e_1(B)$ is the eigenspace of $M \oplus M$ killed by the right factor M , we have

$$(2.9) \quad L_2 = \{x \in L \mid S(B_1, x) = 0\} \quad \text{and} \quad L_1 = \{x \in L \mid S(B_2, x) = 0\},$$

because multiplication by units in $(M \otimes_F M)^\times$ preserves the inner product $S(x, y) = \text{Tr}(xy')$ up to scalar similitude. Thus the multiplication by e_j is the orthogonal projection of B to B_j under the inner product S . We consider the two O -orders:

$$(2.10) \quad R_l = \{x \in M \mid l(x)L \subset L\}, \quad \text{and} \quad R_r = \{x \in M \mid Lr(x) \subset L\}.$$

We want to determine primes dividing the annihilator $\text{Ann}_O(L^M/L_M)$ of L^M/L_M . Note that $R \oplus R \subset M \oplus M$ is the maximal order of $M \oplus M$, and hence $R_l \otimes_O R_r \subset R \oplus R$. Here is the result:

Lemma 2.2. *Let $d(R_?/O)$ be the relative discriminant of $R_?/O$ for $? = l, r$. Then we have $d(R_l/O)d(R_r/O)L^M \subset L_M$.*

Proof. The process constructing L^M and L_M can be done at each localization $B_{\mathfrak{q}}$ for primes $\mathfrak{q}$ of O . Then $L_{i,\mathfrak{q}} = L_{\mathfrak{q}} \cap M_{\mathfrak{q}}L_i$ and $e_j(L_{\mathfrak{q}}) = e_j(L)_{\mathfrak{q}}$. If a prime $\mathfrak{q}$ of O is unramified in R_r and R_l , we have $R_{l,\mathfrak{q}} \otimes_{O_{\mathfrak{q}}} R_{r,\mathfrak{q}} = R_{\mathfrak{q}} \oplus R_{\mathfrak{q}}$ inside $M_{\mathfrak{q}} \oplus M_{\mathfrak{q}}$, and hence $L_{\mathfrak{q}}^M = L_{M,\mathfrak{q}}$ by definition. By the definition of the discriminant, taking a basis w_1, w_2 (resp. v_1, v_2) of $R_{\mathfrak{q}}$ (resp. $R_{?,\mathfrak{q}}$) over $O_{\mathfrak{q}}$, we have $(w_1, w_2)\delta = (v_1, v_2)$ for $\delta \in M_2(O_{\mathfrak{q}}) \cap \text{GL}_2(F_{\mathfrak{q}})$ (an explicit form of δ can be found in [H26, §5.3]). Then $d(R_?/O)_{\mathfrak{q}} = \det(S(v_i, v_j))_{i,j} = \det[\delta(S(w_i, w_j))_{i,j}^t \delta] = \det(\delta^2)d(R/O)$. Since $(\det(\delta)) = \text{Fitt}_{O_{\mathfrak{q}}}(R_{\mathfrak{q}}/R_{?,\mathfrak{q}})$ for the Fitting ideal $\text{Fitt}_{O_{\mathfrak{q}}}(X)$ of a $O_{\mathfrak{q}}$ -module X [EMI, §6.2.6], we have

$$d(R_l/O)d(R_r/O)(R \oplus R) \subset R_l \oplus R_r \subset R \oplus R \subset M \oplus M.$$

Thus $d(R_l/O)d(R_r/O)L^M \subset (R_l \oplus R_r)L \subset L_M$. This shows the desired assertion. $\square$

2.2. Splitting B -theta series into a product of M -theta series. Pick a Hecke character $\lambda : M_{\mathbb{A}}^\times/M^\times \rightarrow \mathbb{C}^\times$ such that $\lambda(x_\infty) = x_\infty^{-\Sigma}$ for a p -ordinary CM type Σ of prime-to- p conductor $\mathfrak{C}(\lambda) = \mathfrak{F}(\lambda)\mathfrak{F}_c(\lambda)\mathfrak{I}(\lambda)\mathfrak{R}(\lambda)$ with $\mathfrak{F}(\lambda) + \mathfrak{F}(\lambda)^c = R$ and $\mathfrak{F}(\lambda) \subset \mathfrak{F}_c(\lambda)^c$. Here $\mathfrak{R}(\lambda)$ is made of ramified primes and $\mathfrak{I}(\lambda)$ is made of unramified inert primes. Later in (5.6), we modify λ to have conductor supported by Σ_p without losing generality; so, we impose unramifiedness of λ at Σ_p^c here. Since ψ^- has split conductor (h2), we may also assume that $\lambda_{\mathfrak{q}} = \lambda_{F,\mathfrak{q}} \circ N_{M/F}$ at primes $\mathfrak{q} \mid \mathfrak{r}$ for $\mathfrak{i} = \mathfrak{i}(\lambda) = \mathfrak{I}(\lambda) \cap F$ and $\mathfrak{r} = \mathfrak{r}(\lambda) = \mathfrak{R}(\lambda) \cap F$. Write $\mathfrak{f} = \mathfrak{f}(\lambda) = \mathfrak{F}(\lambda) \cap F$.

We modify the identification of B_p with $M_2(F_p)$ so that r and l both bring $(x, y) \in M_p = M_{\Sigma_p} \times M_{\Sigma_p^c}$ onto $\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$ in $M_2(F_p)$. For $\mathfrak{l} \mid \mathfrak{f} = \mathfrak{F}(\lambda) \cap O$, we take the prime factor $\mathfrak{L} \mid \mathfrak{l}$ so that $\mathfrak{L} \mid \mathfrak{F}$, and we identify $B_{\mathfrak{l}}$ with $M_2(F_{\mathfrak{l}})$ bringing $(x, y) \in M_{\mathfrak{l}} = M_{\mathfrak{L}} \times M_{\overline{\mathfrak{L}}}$ to $\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$ in $M_2(F_{\mathfrak{l}})$. For $\mathfrak{l} \mid d(M/F)$, we embed $R_{\mathfrak{l}}$ by $r = l$ into the $O_{\mathfrak{l}}$ -order of $M_2(F_{\mathfrak{l}})$ generated by the scalar in $O_{\mathfrak{l}}$ and $\Delta_0^B(\mathfrak{l}^\alpha)$ for the exponent α of $\mathfrak{l}$ in $d(M/F)$, that is the Eichler order $\widehat{O}_0(d(M/F))_{\mathfrak{l}}$ of level $d(M/F)_{\mathfrak{l}}$.

We choose a Neben character set

$$(2.11) \quad \varepsilon = \varepsilon_\lambda = (\varepsilon_1, \varepsilon_2 : \widehat{O}^\times \rightarrow W^\times, \varepsilon_+ : F^\times \backslash F_{\mathbb{A}(\infty)}^\times \rightarrow W^\times)$$

as follows: Let $\mathfrak{f} = \mathfrak{f}(\lambda) := \mathfrak{F}(\lambda) \cap F$. Taking the p -adic avatar $\widehat{\lambda} : \text{Gal}(\overline{F}/M) \rightarrow W^\times$, consider $\rho_\lambda := \text{Ind}_M^F \widehat{\lambda}$. Then $\rho_\lambda|_{\text{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}})} \cong \lambda_{\mathfrak{q}} \oplus \lambda_{c,\mathfrak{q}}$ for $\mathfrak{q} \mid \mathfrak{f}$ with unramified $\lambda_{\mathfrak{q}}$. Regarding $\lambda_{\mathfrak{q}}$ and $\lambda_{c,\mathfrak{q}}$ as characters: $F_{\mathfrak{q}}^\times \rightarrow W^\times$ by local class field theory, we put $\varepsilon_{1,\mathfrak{q}} = \lambda_{c,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times} = 1$ and $\varepsilon_2 = \lambda_{\mathfrak{q}}|_{O_{\mathfrak{q}}^\times}$ and $\varepsilon_+ = \det(\rho_\lambda)/\mathcal{N}$ for the p -adic cyclotomic character $\mathcal{N}$; so, ε_+ is the p -adic avatar of the central character of the

automorphic induction $\pi(\lambda)$ of λ . For either a prime factor of p or non-split primes in $\mathfrak{N}' := N_{K/F}(\mathfrak{C})d(M/F)$,

$$\begin{aligned} \varepsilon_{1,\mathfrak{p}} &= \lambda_{c,\mathfrak{p}}|_{O_{\mathfrak{p}}^\times} = 1 \quad \text{and} \quad \varepsilon_2(x_p) = \lambda_{\mathfrak{p}}|_{O_{\mathfrak{p}}^\times}(x_p) \quad \text{for } \mathfrak{p}|p, \\ \varepsilon_{1,\mathfrak{q}} &= \varepsilon_{2,\mathfrak{q}} = \lambda_{F,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times} \quad \text{for } \mathfrak{q}|\mathfrak{i}, \\ \varepsilon_{1,\mathfrak{q}} &= \lambda_{F,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times} \quad \text{and} \quad \varepsilon_2(x_p) = \lambda_{F,\mathfrak{q}}|_{O_{\mathfrak{q}}^\times} \left(\frac{M_{\mathfrak{q}}/F_{\mathfrak{q}}}{} \right) \quad \text{for } \mathfrak{q}|\mathfrak{r}. \end{aligned}$$

Outside $\mathfrak{N}'$, we put $\varepsilon_1 = \varepsilon_2 = 1$. See [H26, Definition 1.33].

Let

$$(2.12) \quad \varepsilon^- := \varepsilon_1^{-1} \varepsilon_2 \quad \text{and} \quad \mathfrak{N} = \mathfrak{N}(\lambda) = \mathfrak{C}(\varepsilon^-) \quad (\text{the conductor of } \varepsilon^-).$$

Writing $\varepsilon \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \varepsilon_1(a) \varepsilon_2(d)$, since $\varepsilon^- = \varepsilon_1 \varepsilon_2^{-1}$ factors through $O/\mathfrak{N}$, we can extend the character ε of $U_0(\mathfrak{N}')$ to $U_0(\mathfrak{N})$ by putting $\varepsilon(u) = \varepsilon_2(\det(u)) \varepsilon^-(a)^{-1}$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{N})$. The character $\varepsilon(u)$ has another expression $\varepsilon(u) = \varepsilon_1(\det(u)) \varepsilon^-(d)$, because they induce the same character on $U_0(\mathfrak{N}')$ and on $U_0(\mathfrak{N}) \cap SL_2(\widehat{O})$. Hereafter we use the expression $\varepsilon(u) = \varepsilon_1(\det(u)) \varepsilon^-(d)$.

We now suppose

- (D) B is a totally definite quaternion algebra B_F with a fixed isomorphism $B_{F_{\mathbb{A}}^{(\infty)}} \cong M_2(F_{\mathbb{A}}^{(\infty)})$.

The existence of B as above requires $[F : \mathbb{Q}]$ to be even. We let $\delta \in \Delta_0^B(\mathfrak{N})$ act on $w \in W$ by $w \mapsto \varepsilon(\delta)w$ and write this $\Delta_0(\mathfrak{N})$ -module $W(\varepsilon)$. Using the underlying ring structure on $W = W(\varepsilon) = W(\varepsilon^{-1})$, we define a perfect pairing $[\cdot, \cdot] : W(\varepsilon) \times W(\varepsilon^{-1}) \rightarrow W$ by the product $[v, w] = vw$. Then $[\delta v, \delta w] = [v, w]$ for $\delta \in \Delta_0(\mathfrak{N})$.

Proposition 2.3. *Let the notation be as above; in particular, $l, r : M \hookrightarrow B$ is as in §2.1. Suppose the following three conditions:*

- (a) $p\mathfrak{N}/d(M/F)$ is prime to $d(M/F)$, $d(M/F)$ is odd (i.e., prime to (2)) and $\mathfrak{I}(\lambda) = \mathfrak{R}(\lambda) = (1)$;
- (b) $r(R)Ll(R) \subset L$ with $L_{\mathfrak{f}pd(M/F)} = \widehat{O}_0(\mathfrak{f}\mathfrak{p}^e d(M/F))_{\mathfrak{f}p} \subset B_{\mathfrak{f}p}$ for the conductor $\mathfrak{p}^e = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e(\mathfrak{p})}$ of ε_2 and $\mathfrak{f} = \mathfrak{F}(\lambda) \cap F$;
- (c) ε_1 is trivial on $\widehat{O}^\times$ and $\kappa = (I, 0)$ (so, $\varepsilon_2 = \varepsilon^-$).

Let $v \in W(\varepsilon)$ and $w \in W(\varepsilon^{-1})$. Then $\phi : \widehat{L} \rightarrow W$ given by $\phi(\gamma) = [\gamma v, w]$ is a W -integral linear combination of functions of the form $\phi_1 \otimes \phi_2$ for functions $\phi_j : L^j \rightarrow W$ such that

- (1) $\phi_1(y) = \phi_{1,p}(y_p) \phi_1^{(p)}(y^{(p)})$ (resp. $\phi_2(x) = \phi_{2,p}(x_p) \phi_2^{(p)}(x^{(p)})$), where we embed $x \in M$ into $M_p \times M^{(p)}$ by $x \mapsto (x_p, x^{(p)})$ and for a $\mathbb{Z}$ -module $X \subset B$, $X^{(p)} = X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ with $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$;
- (2) $\phi_{2,p}(\text{diag}[b, d]) = \varepsilon_2(d)$ if $(b, d) \in O_p^2$ and vanishes outside $O_p \times O_p^\times \subset O_p^2 = L_p^2$;
- (3) $\phi_{1,p}$ is the characteristic function of $L_p^1 \cong O_p \times \mathfrak{p}^e O_p$;
- (4) $\phi_2^{(p)} = \varepsilon_2^{(p)}(d)$ factors through the finite quotient $L^2/\mathfrak{f}d(M/F)L^2$ of $L^{2,(p)}$, and $\phi_1^{(p)}$ is the characteristic function of $\widehat{L}^{1,(p)} \cong \widehat{O}^{(p)} \times \widehat{\mathfrak{N}}^{(p)}$;
- (5) the function ϕ_j is supported on L^j and has values in W .

The assumption (b) implies $L = L^1 \oplus L^2$ for $L^j = e_j \widehat{L} \cap B$ by Lemma 2.1.

Proof. We regard ϕ as a function of $B_{\mathbb{A}}^{(\infty)} = B_p \times B_{\mathbb{A}}^{(p\infty)}$ supported on $\widehat{L}$ so that $\phi(b) = \phi_{p\mathfrak{f}}(b_{p\mathfrak{f}})\phi^{(p\mathfrak{f})}(b^{(p\mathfrak{f})})$ for $\phi_{p\mathfrak{f}} = \phi|_{B_{p\mathfrak{f}}}$ and $\phi^{(p\mathfrak{f})} = \phi|_{B_{\mathbb{A}}^{(p\mathfrak{f}\infty)}}$. Taking the left factor of F in (??) as $M_{\Sigma_p^c \mathfrak{F}^c}$, we identify B_p with

$$M_2(F_{p\mathfrak{f}}) = M_{p\mathfrak{f}} \oplus M_{p\mathfrak{f}} = \begin{pmatrix} M_{\Sigma_p^c \mathfrak{F}^c} & M_{\Sigma_p^c \mathfrak{F}^c} \\ M_{\Sigma_p \mathfrak{F}} & M_{\Sigma_p \mathfrak{F}} \end{pmatrix},$$

since we regard B as M -vector space by $\xi v = v r(\xi^c)$ for $v \in B$ and $\xi \in M$. Then $\phi_{p\mathfrak{f}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon_1(a)\varepsilon_2(d)[v, w]$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{O}_0(\mathfrak{p}^e \mathfrak{f})_{p\mathfrak{f}}$. This shows the desired assertion for ϕ_p , since $L_{p\mathfrak{f}} = L_{p\mathfrak{f}}^1 \oplus L_{p\mathfrak{f}}^2$.

As for the component outside $p\mathfrak{f}$ (i.e., component at a prime $\mathfrak{q}|d(M/F)$), For r, l as in (2.6), we find $\alpha(a, b) := l(a + b\sqrt{d_{\mathfrak{q}}})v_1 \in L_{\mathfrak{q}}^1$ and $\beta(a', b') := l(a' + b'\sqrt{d_{\mathfrak{q}}})v_2 \in L_{\mathfrak{q}}^2$, $\alpha(a, b) + \beta(a', b') = \begin{pmatrix} * & * \\ * & a+b'd_{\mathfrak{q}} \end{pmatrix}$; so, $\phi_{\mathfrak{q}} \begin{pmatrix} * & * \\ * & a+b'd_{\mathfrak{q}} \end{pmatrix} = \varepsilon_2(a + b'd_{\mathfrak{q}}) = \left(\frac{M_{\mathfrak{q}}/F_{\mathfrak{q}}}{a+b'd_{\mathfrak{q}}}\right) = \left(\frac{M_{\mathfrak{q}}/F_{\mathfrak{q}}}{a}\right)$. This shows the desired assertions. $\square$

2.3. Duality and Eichler's basis problem. We continue to assume $\kappa = (\kappa_1, 0)$. Let $\mathbf{h} := \mathbf{h}_{\kappa}(\mathfrak{N}(\lambda), \varepsilon; W)$ be the Hecke algebra acting on $S := S_{\kappa}(\mathfrak{N}(\lambda), \varepsilon; W)$ generated over W by the Hecke operators $T(y)$. Each Hilbert modular cusp form $f \in S$ has the following Fourier expansion [HMI, 2.26]:

$$f(y, x) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}} \sum_{0 \ll \xi \in F} a(\xi dy, f) \mathbf{e}_F(\sqrt{-1}\xi y_{\infty}) \mathbf{e}_F(\xi x),$$

where $\mathbf{e}_F(x_{\infty}) = \exp(2\pi\sqrt{-1}\sum_{\sigma \in I} x_{\sigma})$ and $y \mapsto a(y, f)$ is a function on $F_{\mathbb{A}}^{\times(\infty)}$ supported on $\widehat{O}$. Here d is an idele generator of the absolute different of F (so, $d\widehat{O} \cap F = \mathfrak{d}$). Then we define a W -bilinear pairing $\langle \cdot, \cdot \rangle : \mathbf{h} \times S \rightarrow W$ by $\langle h, f \rangle = a(1, f|h)$. This pairing induces a perfect pairing [HMI, 2.28]:

$$(2.13) \quad \text{Hom}_W(S, W) \cong \mathbf{h} \quad \text{and} \quad \text{Hom}_W(\mathbf{h}, W) \cong S.$$

The latter map is induced by

$$\phi \mapsto |y|_{\mathbb{A}} \sum_{0 \ll \xi \in \mathfrak{d}^{-1}} \phi(T(\xi dy)) \mathbf{e}_F(\sqrt{-1}\xi y_{\infty}) \mathbf{e}_F(\xi x) \in S.$$

By (2.13), we obtain

Proposition 2.4. *Let the notation be as above. Let V be an $\mathbf{h}$ -module and V' be an W -module of finite type with an W -bilinear product $\langle \cdot, \cdot \rangle : V \times V' \rightarrow W$. Then we have:*

(1) *The formal q -expansion for $v \in V$ and $w \in V'$:*

$$f(v \otimes w)(y, x) = |y|_{\mathbb{A}} \left\{ \sum_{\xi \gg 0} \langle v|T(\xi yd), w \rangle \mathbf{e}_F(\sqrt{-1}\xi y_{\infty}) \mathbf{e}_F(\xi x) \right\}$$

gives a unique element of S , where $\mathbf{e}_F(z) = \exp(2\pi\sqrt{-1}\sum_{\sigma} z_{\sigma})$ for $z = (z_{\sigma})_{\sigma \in I} \in \mathbb{C}^I$.

(2) *The map $v \otimes w \mapsto f(v \otimes w)$ gives an W -linear map of $V \otimes_W V'$ into S with $f((v|T(y)) \otimes w) = f(v \otimes w)|T(y)$. If further V' is an $\mathbf{h}$ -module and $\langle v|h, w \rangle = \langle v, w|h \rangle$ for all $v \in V$, $w \in V'$ and $h \in \mathbf{h}$, then the map f induces an $\mathbf{h}$ -linear map: $V \otimes_{\mathbf{h}} V' \rightarrow S_{\kappa}(\mathfrak{N}, \varepsilon; W)$.*

- (3) Suppose that $\mathbb{T}$ is an W -algebra direct summand of $\mathbf{h}$, and put $V(\mathbb{T}) = \mathbb{T} \cdot V$ and $S(\mathbb{T}) = \mathbb{T} \cdot S$. If $V(\mathbb{T})$ is $\mathbb{T}$ -free of finite rank and $\text{Hom}_W(V(\mathbb{T}), W)$ is embedded into V' by the pairing $\langle \cdot, \cdot \rangle$, then the map $f : V(\mathbb{T}) \otimes_W V' \rightarrow S(\mathbb{T})$ is surjective.

We just apply (2.13) to $\phi_{v \otimes w} \in \text{Hom}_W(\mathbf{h}, W)$ given by $\phi_{v \otimes w}(h) = \langle hv, w \rangle$.

Let $\mathbb{T}$ be the local ring of $\mathbf{h}$ through which the algebra homomorphism $\phi_\lambda \in \text{Hom}_W(\mathbf{h}, W)$ given by $f(\lambda)|T(y) = \phi_\lambda(T(y))f(\lambda)$ factors. Then under (h1–4), we have a Galois representation $\rho_{\mathbb{T}} : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \text{GL}_2(\mathbb{T})$ unramified outside $\mathfrak{N}'p$ for $\mathfrak{N}' := N_{M/F}(\mathfrak{C}(\lambda))d(M/F)$ such that $\text{Tr}(\rho_{\mathbb{T}}(\text{Frob}_{\mathfrak{l}})) = T(\varpi_{\mathfrak{l}})|_{\mathbb{T}}$ for a prime element $\varpi_{\mathfrak{l}}$ of $F_{\mathfrak{l}}$ for all primes $\mathfrak{l} \nmid \mathfrak{N}'p$. By Fujiwara [F06, Theorems 0.1] (see also [H26, Theorem 1.49]), under (h1–4), $(\mathbb{T}, \rho_{\mathbb{T}})$ is universal among p -nearly ordinary deformations over W of $\text{Ind}_M^F \overline{\rho} = \text{Ind}_M^F \overline{\lambda}$ satisfying the local property attached to ε (see [H06, (H1–7)] and [HMI, §3.2–3.3]) and $\mathbb{T} \cdot H^0(Y^B(\mathfrak{N}(\lambda)), W(\varepsilon)) \cong \mathbb{T}$ as $\mathbb{T}$ -modules [F06, Theorem 0.4], where $Y^B(\mathfrak{N}) = B^\times \backslash B_{\mathbb{A}}^\times / U_0^B(\mathfrak{N}(\lambda)) F_{\mathbb{A}}^\times B_{\mathbb{R}}^\times$. Thus we obtain from Proposition 2.4

Corollary 2.5. *Let*

$$V := \mathbb{T} \cdot H^0(Y^B(\mathfrak{N}(\lambda)), W(\varepsilon)), \quad V' := \mathbb{T} \cdot H^0(Y^B(\mathfrak{N}(\lambda)), W(\varepsilon^{-1})).$$

Then the map $V \otimes_W V' \ni v \otimes w \mapsto f(v \otimes w) \in \mathbb{T} \cdot S$ is a surjective $\mathbb{T}$ -linear map. In other words, the theta series of B span the $\mathbb{T}$ -component $\mathbb{T} \cdot S$ over W .

2.4. Theta series span Hilbert modular forms. We give here an explicit set of theta series which span $\mathbb{T} \cdot S$ in Corollary 2.5 in a slightly more general setting. Let $E = F \oplus F$ as a semi-simple algebra over F diagonally embedded. We deal with the quadratic space (B, N) with totally definite B . For simplicity, we assume that the weight is $\kappa = (I, 0)$ on GL_2 and (κ, κ) on $B_E^\times$ for $B_E = B \otimes_F E = B \times B$.

Let $Sh := B^\times \backslash B_{\mathbb{A}(\infty)}^\times / F_{\mathbb{A}(\infty)}^\times U_0^B(\mathfrak{N})$ and $Sh_E = Sh \times Sh$. By (D), Sh is a finite set. In the previous section, Sh is written as $Y_0^B(\mathfrak{N})$. We choose a complete set $\mathcal{S}$ of representatives for Sh so that for each $a \in \mathcal{S}$, the level component $a_{\mathfrak{N}'}$ belongs to $U_0(\mathfrak{N}')$. Fix Neben characters $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon^-, \varepsilon_+)$ and suppose $\mathfrak{N}' = \mathfrak{C}(\varepsilon_1)\mathfrak{C}(\varepsilon_2)$ and $\mathfrak{N} = \mathfrak{C}(\varepsilon_1) \cap \mathfrak{C}(\varepsilon_2)$ (so, $\mathfrak{N} = \mathfrak{C}(\varepsilon^-)$). Here we allow p -factors in $\mathfrak{N}$ and $\mathfrak{N}'$. For the class $[a] \in Sh$ of $a \in \mathcal{S}$, write $\overline{\Gamma}_a$ is the image of $(aU_0^B(\mathfrak{N})a^{-1} \cap B^\times)$ in $B^\times / F^\times$. Put $e_a := |\overline{\Gamma}_a|$.

We consider a finite order character $\varepsilon : U_0^B(\mathfrak{N}) \rightarrow A^\times$ coming from the Neben characters $(\varepsilon_1, \varepsilon_2, \varepsilon^-, \varepsilon_+)$. Let $S_\varepsilon(A)$ be the space of functions $f : B^\times \backslash B_{\mathbb{A}(\infty)}^\times \rightarrow A$ such that $f(gu) = \varepsilon(u)f(g)$ for $u \in U_0^B(\mathfrak{N})$ if ε is non-trivial, and $S(A) = S_1(A) := \{\mathcal{F} : Sh \rightarrow A \mid \sum_{[a] \in Sh} e_a^{-1} \mathcal{F}([a]) = 0\}$ if ε is trivial. When $\varepsilon = 1$, we define $M_1(A)$ by the space of all functions $Sh \rightarrow A$; so, $S_1(A) \subset M_1(A)$, and if $\varepsilon \neq 1$, we just put $S_\varepsilon(A) = M_\varepsilon(A)$. As long as $S_\varepsilon(A) \neq 0$ for a W -algebra $A \subset \mathbb{C}_p$, Γ_a acts trivially on $A(\varepsilon)$.

Set for a subring A of $\mathbb{C}$

$$\begin{aligned} M_{1,1}^E(A) &= \{f : Sh_E \rightarrow A\} = M_1(A) \otimes_A M_1(A), \\ (2.14) \quad S_{1,1}^E(A) &= S_1(A) \otimes_A S_1(A), \\ M_\varepsilon^E(A) &= S_{\varepsilon,\varepsilon}^E(A) = S_\varepsilon(A) \otimes_A S_{\varepsilon^{-1}}(A) \quad \text{as functions on } Sh_E \text{ if } \varepsilon \neq 1. \end{aligned}$$

Here for $\mathcal{F} \in S_\varepsilon(A)$ and $\mathcal{G} \in S_{\varepsilon^{-1}}(A)$, we define

$$\mathcal{F} \otimes \mathcal{G} : Sh_E \rightarrow A \text{ by } \mathcal{F} \otimes \mathcal{G}(a, b) = \mathcal{F}(a)\mathcal{G}(b).$$

In this way, the last line of (2.14) makes sense. Note that $(\mathcal{F} \otimes \mathcal{G})(au, bu') = \varepsilon(u)\varepsilon^{-1}(u')(\mathcal{F} \otimes \mathcal{G})(a, b)$ for $u, u' \in U_0^B(\mathfrak{N})$. If we need to emphasize the level $\mathfrak{N}$, we write $S_\varepsilon(\mathfrak{N}; A)$ and $M_\varepsilon(\mathfrak{N}; A)$ for $S_\varepsilon(A)$ and $M_\varepsilon(A)$, respectively.

Let e be the LCM of the order of all maximal torsion subgroups of $B^\times$. Since F is unramified over $\mathbb{Q}$ at p , e is prime to p . We define a perfect A -linear pairing $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_A : M_\varepsilon(\mathfrak{N}; A) \times M_{\varepsilon^{-1}}(\mathfrak{N}; A) \rightarrow A[1/e]$ by

$$(2.15) \quad \langle \mathcal{F}, \mathcal{G} \rangle = \sum_{a \in \mathcal{S}} e_a^{-1} \mathcal{F}(a) \mathcal{G}(a).$$

Here is our choice of a Bruhat function relative to ε . If $\varepsilon = 1$ and $\mathfrak{N} = O$, we take the Bruhat function $\phi^{(\infty)} = \phi_1^{(\infty)}$ which is the characteristic function of $\widehat{O}_0(\mathfrak{N})$ and if $\mathfrak{N} \neq O$, we take $\phi^{(\infty)}$ to be the characteristic function of $\Delta_0^B(\mathfrak{N})$ times ε^{-1} well defined on $\Delta_0^B(\mathfrak{N})$. Put $\phi_\infty(v) = \mathbf{e}_F(N(v)\tau)$.

We have Hecke operators $T(\mathfrak{n})$ acting on $\mathcal{F} \in S_\varepsilon(A)$ as follows: Let

$$\mathbf{T}(\mathfrak{n}) := \{b \in \Delta_0^B(\mathfrak{N}) | N(b)\widehat{O} = \widehat{\mathfrak{n}}\}$$

for an integral ideal $\mathfrak{n}$. Decompose $\mathbf{T}(\mathfrak{n}) = \bigsqcup_{c \in C(\mathfrak{n})} cU_0^B(\mathfrak{N})$. Then $\mathcal{F}|T(\mathfrak{n})(a) = \sum_{c \in C(\mathfrak{n})} \varepsilon^{-1}(c)\mathcal{F}(ac)$. The algebra $H_\varepsilon(A) = H_\varepsilon(\mathfrak{N}; A)$ is defined to be the A -subalgebra of $\text{End}_A(M_\varepsilon(\mathfrak{N}; A))$ generated by $T(\mathfrak{n})$ for all positive integers n .

Writing the variable $h \in B_{E_A}^\times = B_A^\times \times B_A^\times$ as $h = (h_L, h_R)$ and $\widehat{\Gamma}_E := U_0^B(\mathfrak{N}) \times U_0^B(\mathfrak{N})$, we define the theta kernel by

$$\theta(\tau; h_L, h_R) = \sum_{\xi \in B} \phi(h_L^{-1}\xi h_R).$$

By our choice of ϕ , we have $\theta(\tau; h_L u, h_R u') = \varepsilon(u)\varepsilon^{-1}(u')\theta(\tau; h_L, h_R)$ for $u, u' \in U_0^B(\mathfrak{N})$. In particular, $(h_L, h_R) \mapsto \mathcal{F}(h_L)\mathcal{G}(h_R)\theta(\tau; h_L, h_R)$ factors through Sh_E . Then define the theta descent $\theta_*(\mathcal{F} \otimes \mathcal{G})$ of $\mathcal{F} \otimes \mathcal{G}$ by

$$\begin{aligned} \theta_*(\mathcal{F} \otimes \mathcal{G})(\tau) &= \int_{Sh_E} \theta(\phi)(\tau; h) \mathcal{F}(h_L) \mathcal{G}(h_R) d\mu_h \\ &= \sum_{(a,b) \in \mathcal{S}^2} \mathcal{F}(a) \mathcal{G}(b) e_a^{-1} e_b^{-1} \theta(\phi_{a,b}), \end{aligned}$$

where $\theta(\phi_{a,b}) := \sum_{\xi \in B} \phi^{(\infty)}(a^{-1}\xi b) \mathbf{e}_F(N(\xi)\tau)$.

Theorem 2.6. *Let the notation and the assumptions be as above. Then we have, for $\mathcal{F} \in S_\varepsilon(\mathfrak{N}; A)$ and $\mathcal{G} \in S_{\varepsilon^{-1}}(\mathfrak{N}; A)$,*

$$\theta_*(\mathcal{F} \otimes \mathcal{G})(y, x) = |y|_A \sum_{0 \ll \xi \in F}^{\infty} \langle \mathcal{F}|T(\xi \mathfrak{d}\eta), \mathcal{G} \rangle \mathbf{e}_F(\sqrt{-1}y) \mathbf{e}_F(\xi x)$$

in $S_{(I,0)}(U_0(\mathfrak{N}), \varepsilon; A)$, where $T(\mathfrak{n}) = 0$ if $\mathfrak{n}$ is not integral. Moreover, if $\mathbb{T} \cdot \mathcal{S}_\varepsilon(\mathfrak{N}; W) \cong \mathbb{T}$ as $\mathbb{T}$ -modules, the map $\mathcal{F} \otimes \mathcal{G} \mapsto \theta_*(\mathcal{F} \otimes \mathcal{G})$ induces a surjective W -linear map:

$$\mathbb{T} \cdot \mathcal{S}_\varepsilon(\mathfrak{N}; W) \otimes \mathbb{T} \cdot \mathcal{S}_{\varepsilon^{-1}}(\mathfrak{N}; W) \twoheadrightarrow S_{(I,0)}(U_0(\mathfrak{N}), \varepsilon; A),$$

where $\mathbb{T}$ acts on $\mathcal{S}_{\varepsilon^{-1}}(\mathfrak{N}; W)$ by the adjoint action under $\langle \cdot, \cdot \rangle$. In particular, $\{\theta(\phi_{a,b})\}_{a,b \in \mathcal{S}}$ span $\mathbb{T} \cdot \mathcal{S}_\varepsilon(\mathfrak{N}; W)$ over W as long as $\mathbb{T} \cdot \mathcal{S}_\varepsilon(\mathfrak{N}; W) \cong \mathbb{T}$.

It appears that we need to assume $\frac{1}{e_a e_b} \in A$ to assure A -integrality. However the series $\theta(\phi_{a,b})$ has $e_a e_b$ overlapping terms, and hence $e_a^{-1} e_b^{-1} \theta(\phi_{a,b})$ is A -integral except for its constant term. We get an A -linear map $\theta_* : S_{\varepsilon, \epsilon}^E(A) \rightarrow S_2(U_0(\mathfrak{N}), \varepsilon; A)$ given by $\mathcal{F} \otimes \mathcal{G} \mapsto \theta_*(\mathcal{F} \otimes \mathcal{G})$, as $\theta_*(\mathcal{F} \otimes \mathcal{G})$ does not have constant term.

Proof. By definition,

$$\begin{aligned} \theta_*(\mathcal{F} \otimes \mathcal{G})(\tau) &= \int_{Sh_E} \theta(\phi)(\tau; h) \mathcal{F}(h_L) \mathcal{G}(h_R) d\mu_h \\ &= \sum_{(a,b) \in Sh \times Sh} \mathcal{F}(a) \mathcal{G}(b) e_a^{-1} e_b^{-1} \sum_{\xi \in B} \phi(a^{-1} \xi b) \mathbf{e}_F(N(\xi) \tau) \\ &= \sum_{(a,b) \in Sh \times Sh} e_a^{-1} e_b^{-1} \mathcal{F}(a) \mathcal{G}(b) \theta_{a,b}(\tau), \end{aligned}$$

where $\theta_{a,b} = \theta(\epsilon^{-1} \phi_{a,b})$ with the characteristic function $\phi_{a,b}^{(\infty)}$ of $a \Delta_0(\mathfrak{N}) b^{-1}$ and the measure $d\mu_h$ is chosen so that $\int_{B_E^\times \backslash B_E^\times(a,b) \hat{\Gamma}_E B_{E_\mathbb{R}}^\times / \hat{\Gamma}_E B_{E_\mathbb{R}}^\times} d\mu_h = e_a^{-1} e_b^{-1}$. By the Jacquet Langlands correspondence, we have $\theta_*(\mathcal{F} \otimes \mathcal{G}) \in M_{(I,0)}(\mathfrak{N}, \varepsilon; A)$.

For a decomposition $\mathbf{T}(\mathfrak{n}) = \bigcup_{c \in C(\mathfrak{n})} c U_0^B(\mathfrak{N})$, we have

$$\mathcal{F}|T(\mathfrak{n})(x) = \sum_{c \in C(\mathfrak{n})} \varepsilon(c)^{-1} \mathcal{F}(xc).$$

Writing $ac = \xi b_c u$ for $\xi \in \Gamma_a \backslash B^\times / \Gamma_b$, $b_c \in Sh$ and $u \in U_0^B(\mathfrak{N})$, $\mathcal{F}|T(\mathfrak{n})(a) = \sum_c \mathcal{F}(b_c)$ and $\xi \in B^\times \cap a \Delta_0(\mathfrak{N}) b_c^{-1}$. Then

$$\begin{aligned} \langle \mathcal{F}|T(\mathfrak{n}), \mathcal{G} \rangle &= \sum_{a \in S} e_a^{-1} \mathcal{G}(a) \sum_c \varepsilon(c)^{-1} \mathcal{F}(ac) \\ &= \sum_{a \in S} e_a^{-1} \mathcal{G}(a) \sum_c \varepsilon(c)^{-1} \mathcal{F}(b_c) \\ &\stackrel{\phi = \varepsilon^{-1}}{=} \sum_{a \in S} e_a^{-1} \mathcal{G}(a) \sum_c \phi(a^{-1} \xi b_c) \mathcal{F}(b_c) \end{aligned}$$

This shows $|y|_{\mathbb{A}}^{-1} \theta_*(\mathcal{F} \otimes \mathcal{G})(y, x)$ taken off the constant term is equal to

$$\sum_{\mathfrak{n}} \sum_{a \in S} e_a^{-1} \mathcal{G}(a) \sum_c \phi(a^{-1} \xi b_c) \mathcal{F}(b_c) = \sum_{\xi \gg 0} \langle \mathcal{F}|T(\xi y d), \mathcal{G} \rangle \mathbf{e}_F(\sqrt{-1} y) \mathbf{e}_F(\xi x).$$

The constant term is given by $\sum_{a,b} e_a^{-1} e_b^{-1} \mathcal{F}(a) \mathcal{G}(b) = \sum_a e_a^{-1} \mathcal{F}(a) \sum_b e_b^{-1} \mathcal{G}(b) = 0$; so, we forget about it. Applying Proposition 2.4 (3) to $V = S_\varepsilon(W)$ and $V' = S_{\varepsilon^{-1}}(W)$, we get the last two assertions. $\square$

3. SHIMURA SERIES

We give a short review of Shimura series, which we use to show integrality of the Petersson inner product: $\frac{(\theta(\lambda), f)}{\text{period}} \in W$ with a canonical period for all $f \in S_\kappa^{n, \text{ord}}(\mathfrak{N}, \varepsilon; W)$ for $\kappa = (I, 0)$.

Define, for $z, w \in \mathfrak{H}$, $v \in M_2(\mathbb{C})$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$(3.1) \quad p(z, w) := - \begin{pmatrix} z & 1 \\ 1 & \end{pmatrix} (w, 1) J = \begin{pmatrix} z & -wz \\ 1 & -w \end{pmatrix} \quad \text{and} \quad [v; z, w] = \text{Tr}(p(z, w) v^t).$$

By computation, $[v; z, w]$ never vanishes for $0 \neq v \in M_2(\mathbb{R})$, and for $\alpha, \beta \in \mathrm{GL}_2(\mathbb{R})$, we have

$$(3.2) \quad \begin{aligned} \alpha p(z, w) \beta^t &= p(\alpha(z), \beta(w)) j(\alpha, z) j(\beta, z), \\ [\alpha v \beta^t; z, w] &= j(\alpha^t, z) j(\beta^t, w) [v; \alpha^{-1}(z), \beta^{-1}(w)], \end{aligned}$$

where $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$. For $v \in M_2(F_\infty)$, $z, w \in \mathfrak{H}^I$ and $k \in \mathbb{Z}[I]$, put $[v; z, w]^k := \prod_{\sigma \in I} [v_\sigma; z_\sigma, w_\sigma]^{k_\sigma}$ and $||[v; z, w]|^{sI} := \prod_{\sigma \in I} ||[v_\sigma; z_\sigma, w_\sigma]|^s$ ($s \in \mathbb{C}$) for the σ -component indicated by the subscript σ . Though we give the definition of $p(z, w)$ outright without explanation, there is a recipe for defining this type of a standard equivariant map p from the symmetric space of $\mathrm{SO}_V(\mathbb{R})$ into the complexification of quadratic spaces V due to Shimura [ALT, §3.2].

The Shimura series for $GL(2) \times GL(2)/_F$ is defined for $0 < k \in \mathbb{Z}[I]$ and $0 \leq m \in \mathbb{Z}[I]$ in [Sh81, II (4.11)] by

$$(3.3) \quad \begin{aligned} H(z, w; s) &= H_{k, m}(z, w; s; \phi^{(\infty)}, f) \\ &= [U] \sum_{\alpha} \phi^{(\infty)}(\alpha) a(-\det(\alpha), f) |\det(\alpha)|^m [\alpha; z, w]^{-k} ||[\alpha; z, w]|^{-2sI} \end{aligned}$$

for $(z, w) \in \mathfrak{H}^I \times \mathfrak{H}^I$, where U is a subgroup of finite index of the group $O_+^\times$ of all totally positive units for which each term of the above sum is invariant, $\alpha \neq 0$ runs over $M_2(F)/U$ and $[U] = [O_+^\times : U]^{-1}$. When $m = 0$, we simply write H_k for $H_{k, 0}$. The positivity of k means that $k \geq 0$ and $k_\sigma > 0$ for at least one $\sigma \in I$. Here f is a Hilbert modular form given by the Fourier expansion: $\sum_{\xi \in F} a(\xi, f) \mathbf{e}_F(\xi \tau)$ for $\tau \in \mathfrak{H}^I$ of weight ℓ ($\mathbf{e}_F(\xi \tau) = \exp(2\pi i \sum_{\sigma} \xi^\sigma \tau_\sigma)$) with $a(\xi, f) = 0$ if $\xi^\sigma < 0$ for some $\sigma \in I$, U is a subgroup of finite index of the group $O_+^\times$ of all totally positive units for which each term of the above sum is invariant, $[U] := [O_+^\times : U]^{-1}$ and $\phi^{(\infty)} : M_2(F_{\mathbb{A}(\infty)}) \rightarrow \mathbb{C}$ is a locally constant compactly supported function (a Bruhat function). To have invariance of the terms under the unit group U , we need to assume

$$(3.4) \quad k - \ell - 2m = [k - \ell - 2m]I \quad (I = \sum_{\sigma \in I} \sigma) \text{ for an integer } [k - \ell - 2m].$$

The series (3.3) converges absolutely and locally uniformly with respect to all variables s, z, w if

$$(3.5) \quad \mathrm{Re}(s) > n + 2 + 2\theta(f) - [k - \ell - 2m]$$

as was shown in [Sh81, I.5.1, I.5.2], where $\theta(f) = -1$ when f is a constant, and otherwise, $\theta(f) = \theta \geq -\frac{1}{2}$ with $|a(\xi, f) \xi^{-\ell/2}| = O(|N(\xi)|^\theta)$ for the norm map $N = N_{F/\mathbb{Q}}$. This series is a generalization of Eisenstein series, because if we take $f = 1$ (so $\ell = 0$ and $m = 0$), the series gives an Eisenstein series for $GL(2) \times GL(2)$ over F .

We are going to describe an analytic continuation of the series and compute its q -expansion at holomorphic evaluation points.

3.1. $\mathrm{PGL}(2)$ as an orthogonal group. The construction of Shimura series comes from identifying $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ with an orthogonal group of signature $(2, 2)$. We briefly describe this point. Here is how to realize the product $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ (modulo diagonally embedded $\{\pm 1\}$) as an orthogonal group.

Let $V = M_2(\mathbb{R})$, and consider the symmetric bilinear form $S : V \times V \rightarrow \mathbb{R}$ given by $S(x, y) = \text{Tr}(xy^t)$, where $yy^t = y^t y = \det(y)$ for 2×2 matrices y . We let $(a, b) \in \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})$ act on V by $x \mapsto axb^t$. Then

$$\begin{aligned} S(axb^t, ayb^t) &= \text{Tr}(axb^t by^t a^t) = \det(b) \text{Tr}(axy^t a^t) \\ &= \det(b) \text{Tr}(xy^t a^t a) = \det(a) \det(b) S(x, y). \end{aligned}$$

Thus we have an isomorphism

$$(\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})) / \{\pm(1, 1)\} \hookrightarrow \text{GO}(\mathbb{R})$$

with the similitude norm $\nu(a, b) = \det(a) \det(b)$. Since the symmetric space of $\text{GO}(\mathbb{R})$ has dimension 2 over $\mathbb{C}$, the above isomorphism induces an isomorphism of the identity connected components

$$(3.6) \quad (\text{GL}_2^+(\mathbb{R}) \times \text{GL}_2^+(\mathbb{R})) / \{\pm(1, 1)\} \cong \text{GO}^+(\mathbb{R}).$$

Since $\text{GO}(\mathbb{R})$ has eight connected components:

$$\text{GO}(\mathbb{R}) = \bigsqcup_{j=0}^1 \bigsqcup_{\epsilon=\pm 1, \epsilon'=\pm 1} \text{GO}^+(\mathbb{R}) \iota^j(\text{diag}[1, \epsilon], \text{diag}[1, \epsilon']),$$

the cokernel of above morphism is generated by the main involution $\iota : x \mapsto \text{Tr}(x) - x$

$$(3.7) \quad \text{Coker}((\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})) / \{\pm(1, 1)\} \hookrightarrow \text{GO}(\mathbb{R})) = \langle \iota \rangle.$$

Write $\mathfrak{Z}$ for the symmetric domain of the identity connected component $\text{GO}^+(\mathbb{R})$ of $\text{GO}(\mathbb{R})$. Since the symmetric domain of $\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})$ is given by $\mathfrak{H}_z \times \mathfrak{H}_w$ for the upper half complex plane $\mathfrak{H}_z = \mathfrak{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, we find that $\mathfrak{Z} \cong \mathfrak{H}_z \times \mathfrak{H}_w$.

We are going to make this isomorphism: $\mathfrak{Z} \cong \mathfrak{H}_z \times \mathfrak{H}_w$ explicit. Since $V_{\mathbb{C}} = M_2(\mathbb{C})$, writing $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$, we put

$$\mathcal{Z} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \mid ad = bc, \overline{a}d - b\overline{c} + d\overline{a} - c\overline{b} < 0 \right\}.$$

Pick $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{Z}$, and suppose that $c = 0$. Then by the defining equation of $\mathcal{Z}$, $ad = 0 \Rightarrow 0 = \overline{a}d + d\overline{a} < 0$, which is a contradiction. Thus $c \neq 0$; so, we define for v as above, $z = \frac{a}{c}$ and $w = -\frac{d}{c}$. Then $-zw = \frac{b}{c}$, and hence (see [Sh81, II (4.6)])

$$(3.8) \quad v = cp(z, w) \quad \text{with} \quad p(z, w) = \begin{pmatrix} z & -wz \\ 1 & -w \end{pmatrix} = -{}^t(z, 1)(w, 1)\varepsilon,$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By computation,

$$(3.9) \quad S(p(z, w), \overline{p(z, w)}) = (w - \overline{w})(z - \overline{z}) = -z\overline{w} + zw - \overline{z}w + \overline{z}\overline{w} < 0.$$

From this, it is clear that $\mathcal{Z} \cong \mathbb{C}^\times \times (\mathfrak{H}^2 \sqcup \overline{\mathfrak{H}}^2)$. By this isomorphism, for $\alpha \in \text{GO}(\mathbb{R})$, we can define its action $\alpha(z, w) \in (\mathfrak{H}^2 \sqcup \overline{\mathfrak{H}}^2)$ and a factor $\mu(\alpha; z, w) \in \mathbb{C}^\times$ of automorphy by

$$\alpha \cdot p(z, w) = p(\alpha(z, w))\mu(\alpha; z, w).$$

By a direct computation, writing $j(v, z) = cz + d$ for $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $v(z) = \frac{az+b}{cz+d}$, we have, for $(\alpha, \beta) \in \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})$,

$$(3.10) \quad \alpha p(z, w)\beta^t = p(\alpha(z), \beta(w))j(\alpha, z)j(\beta, w).$$

Thus

$$(\alpha, \beta)(z, w) = (\alpha(z), \beta(w)) \quad \text{and} \quad \mu((\alpha, \beta); (z, w)) = j(\alpha, z)j(\beta, w).$$

Our description here is plainly ad hoc. If you want to know the background behind these identifications, see [ALT, §3.2.1–2].

3.2. Integral expression. We recall a standard theta series of the indefinite quadratic space $(M_2(F), N = \det)$. A standard Schwartz function on $M_2(F_\infty)$ which gives rise to the theta series we study is given by

$$u \mapsto \Psi_0(u) = \eta^I \mathbf{e}_F(\det(u)\tau + \frac{\eta}{2yt} |[u; z, w]|^2)$$

for $\mathbf{e}_F(x_\infty) := \mathbf{e}(\sum_{\nu \in I} x_\nu)$ with $\mathbf{e}(v) = \exp(2\pi\sqrt{-1}v)$, $\tau = \xi + i\eta$, $z = x + yi$ and $w = r + ti$ with $\xi, x, r \in F_\infty$ and $\eta, y, t \in F_{\infty+}^\times$ [ALT, §3.2]. Here $F_{\infty+}^\times$ is the identity connected component of $F_\infty^\times$. We define for $\tau \in \mathfrak{H}^I$

$$\Psi_k(u) = \prod_{\nu} \Psi_{k_{\nu}, \nu}^{\tau, z, w}(u_{\nu}) \quad (0 \leq k = \sum_{\nu} k_{\nu} \nu \in \mathbb{Z}[I]), \quad (3.11)$$

$$\Psi_{k_{\nu}, \nu}^{\tau, z, w}(u_{\nu}) = \eta_{\nu}^{k_{\nu}+1} [u_{\nu}; z_{\nu}, w_{\nu}]^{k_{\nu}} \mathbf{e}(\det[u_{\nu}] \tau_{\nu} + i \frac{\eta_{\nu}}{2y_{\nu}t_{\nu}} |[u_{\nu}; z_{\nu}, w_{\nu}]|^2).$$

Often we write $\Psi_{k_{\nu}, \nu}(u_{\nu})$ for $\Psi_{k_{\nu}, \nu}^{\tau, z, w}(u_{\nu})$. Let $\phi = \phi^{(\infty)} \phi_{\infty}$ be a Schwartz-Bruhat function on $M_2(F_{\mathbb{A}})$. We take $\phi_{\infty} = \Psi_k^{z, w, \tau}$ studied in the previous subsection. Then we define

$$\Theta(\phi) = \Theta_k(\phi) = \sum_{\ell \in V} \phi(\ell). \quad (3.12)$$

Then $\Theta(\phi)$ is an automorphic form of $(\tau, z, w) \in \mathfrak{H}^I \times \mathfrak{H}^I \times \mathfrak{H}^I$ having an automorphic property described in [H06, §2.4] (see also [ALT, Theorem 4.19]).

Let Γ be a congruence subgroup of $SL_2(F)$ which leaves $\Theta_k(\phi)$ and f fixed; thus, $\Gamma \subset \Gamma^{\tau}(\phi^*)$ for $\Gamma^{\tau}(\phi^*)$ in [H06, 2.3]. The stabilizer Γ_{∞} of the infinity cusp has the following canonical exact sequence:

$$0 \rightarrow \mathfrak{a} \longrightarrow \Gamma_{\infty} \longrightarrow U \rightarrow 1 \quad (3.13)$$

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto \epsilon$$

for a fractional ideal $\mathfrak{a}$ and a subgroup $U \subset O^\times$ of finite index. By shrinking Γ a little, we may assume that $U \subset O_+^\times$. We recall the integral expression of the Shimura series involving Siegel's theta series given in [Sh81, I (7.2), II (6.5b)]:

$$[U]N(\mathfrak{a})^{-1} \sqrt{|D|}^{-1} \int_{F_{\infty+}^\times / U^2} \left(\int_{F_{\infty}/\mathfrak{a}} \Theta(\phi) d^m f(\tau) d\xi \right) \eta^{(s-1)I} d^\times \eta, \quad (3.14)$$

where $d^m = \prod_{\sigma} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \tau_{\sigma}} \right)^{m_{\sigma}}$, $\phi(u) = \phi^{(\infty)}(u^{(\infty)}) \Psi_k^{\tau, z, w}(u_{\infty})$ and $d^\times \eta$ is the multiplicative Haar measure given by $\otimes_{\sigma} (\eta_{\sigma}^{-1} d\eta_{\sigma})$. By a standard Rankin convolution done in [H06, (3.7)], for $\Gamma_F(k) := \prod_{\sigma} \Gamma(k_{\sigma})$ with the Gamma function $\Gamma(s)$, we conclude

$$\begin{aligned} & 2^{1-[F:\mathbb{Q}]} \pi^{-k-sI} \Gamma_F(k+sI) y^{k+sI} t^{k+sI} \overline{H_{k,m}(z, w; \bar{s}; \bar{\phi}^{(\infty)}, f_c)} \\ &= [U]N(\mathfrak{a})^{-1} \sqrt{|D|}^{-1} \int_{F_{\infty+}^\times / U^2} \int_{F_{\infty}/\mathfrak{a}} \Theta(\phi) d^m f(\tau) \eta^{(s-1)I} d\xi d^\times \eta, \end{aligned}$$

where $f_c(z) = \overline{f(-\bar{z})}$. In other words, by taking complex conjugation, we have, for $\tilde{\phi} = \overline{\phi}^{(\infty)} \phi_\infty$,

$$(3.15) \quad 2^{1-[F:\mathbb{Q}]} \pi^{-k-sI} \Gamma_F(k+sI) y^{k+sI} t^{k+sI} H_{k,m}(z, w; s; \phi^{(\infty)}, f) \\ = [U] N(\mathfrak{a})^{-1} \sqrt{|D|}^{-1} \int_{F_{\infty+}^\times / U^2} \int_{F_\infty / \mathfrak{a}} \overline{\Theta(\tilde{\phi}) d^m f_c(\tau)} \eta^{(s-1)I} d\xi d^\times \eta.$$

Since $\Theta(\phi)$ with $k > 0$ (resp. $k = 0$) is rapidly decreasing (resp. slowly increasing) [ALT, §3.2.4] and f is rapidly decreasing towards cusps, we can analytically continue the function H to a meromorphic function on the whole s -plane (see [Sh81] I Section 7). We recall the process. We assume that $m = 0$ (the case $m > 0$ is treated in [H06, §3.1]). Since $\Gamma_\infty \backslash \mathfrak{H}^I \cong (F_{\infty+}^\times / U^2) \times (F_\infty / \mathfrak{a})$, we can rewrite the above integral as

$$(3.16) \quad 2^{1-[F:\mathbb{Q}]} \pi^{-k-sI} \Gamma_F(k+sI) y^{k+sI} t^{k+sI} H_k(z, w; s; \phi^{(\infty)}, f) \\ = N(\mathfrak{a})^{-1} \sqrt{|D|}^{-1} \int_{\Gamma \backslash \mathfrak{H}^I} \overline{\Theta(\tilde{\phi}) f_c(\tau) E(\tau; \bar{s})} d\mu(\tau),$$

where $d\mu(\tau)$ is the invariant measure $\eta^{-2I} d\xi d\eta$ on $\mathfrak{H}^I$, and

$$(3.17) \quad E(\tau; s) = E_{k-\ell}(\tau; s) = \eta^{sI} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \tau)^{\ell-k} |j(\gamma, \tau)|^{-2sI}.$$

The integral (3.16) (in general for $m \geq 0$) is convergent for all $s \in \mathbb{C}$ except for s giving rise to a singularity of the Eisenstein series $E(\tau; s)$, because $\Theta(\phi)$ for $k > 0$ does not have constant term at any cusp; so, it is rapidly decreasing. Thus the integral of (3.16) converges absolutely for any slowly increasing automorphic form $f(\tau)$ as long as $E(\tau; s)$ is finite. This is the proof of the analytic continuation given in [Sh80, §13]. This proof works well even when $k = 0$ for cusp forms f .

3.3. Fourier expansion. We assume that $m = 0$ and recall the q -expansion computed in [CAL, Theorem 4.95] and [H06, 3.2] for $B = M_2(F)$. Write V for $M_2(F)$ when we regard it as a quadratic space with the quadratic form “det”. The corresponding bilinear form is written as $S(x, y) = \text{Tr}(xy^t)$.

Although in our application, we take $k = \ell = I$, we first state a general result. First we define the partial Fourier transform of ϕ for $\phi : V_\mathbb{A} \rightarrow \mathbb{C}$ following [H06, (2.33)]:

$$(3.18) \quad \phi^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{F_\mathbb{A}^2} \phi \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \mathbf{e}_\mathbb{A}(ab' - ba') da' db',$$

where $\mathbf{e}_\mathbb{A} : F_\mathbb{A}/F \rightarrow \mathbb{C}^\times$ is the additive character with $\mathbf{e}_\mathbb{A}(x_\infty) = \mathbf{e}_F(x_\infty)$ for $x_\infty \in F_\infty$. Then as computed in [H06, §3.2]:

Theorem 3.1. *Suppose that f is a holomorphic cusp form of weight $k > 0$. Let Γ be a congruence subgroup of $SL_2(F)$ fixing $f(\tau)\Theta(\phi)(\tau)$. Then we have*

$$(-1)^{[F:\mathbb{Q}]} |D| \int_{\Gamma \backslash \mathfrak{H}^I} \Theta(\phi)(\tau; z, w) f(\tau) d\mu(\tau) \\ = (-2i)^k t^k y^k \sum_{\alpha \in \Gamma \backslash M_2(F); \det(\alpha) \ll 0} \phi^{*(\infty)}(\alpha) \mathbf{e}_F(\det(\alpha) \bar{z}) f|_k \alpha(\bar{w}),$$

where $f|\alpha(\overline{w}) = \det(\alpha)^{k-I} f(\alpha(\overline{w}))j(\alpha, \overline{w})^{-k}$ for $\alpha \in M_2(F)$ with totally negative determinant.

We can apply the above theorem to the following integral:

$$\int_{\Gamma \backslash \mathfrak{H}^I} \Theta(\tilde{\phi})(s-1) E_{k-\ell}(\tau; s) \overline{f}_c(\tau) d\mu(\tau)$$

at $s = 1$ when $k = \ell$, because $E(\tau, s)$ has a simple pole at $s = 1$ whose residue is a constant $c_\Gamma \neq 0$ (independent of τ). We then have

Corollary 3.2. *Take $k = \ell$. We have, if f is a holomorphic cusp form of weight ℓ ,*

$$\begin{aligned} \text{Res}_{s=1} H_k(z, w; s; \phi^{(\infty)}, f) &= c_\Gamma (-i)^{[F:\mathbb{Q}]} 2^{-1} [U] |D|^{-3/2} N(\mathfrak{a})^{-1} \frac{(2\pi i)^{k+I}}{\Gamma_F(k+I)} \\ &\quad \times y^{-I} t^{-I} \sum_{\alpha \in \Gamma \backslash M_2(F); \det(\alpha) \gg 0} \phi^{*(\infty)}(\epsilon \alpha) \mathbf{e}_F(\det(\alpha) z) f|_k \alpha(w), \end{aligned}$$

where $c_\Gamma = \text{Res}_{s=1} E(w; s)$.

For the exact value of the constant $c_\Gamma \neq 0$ (which we do not need), see [H99, (RES3), p.173].

3.4. CM points. Let (z_0, w_0) be a point in $\mathfrak{H}^I$ such that $\sigma(M) = F[z_{0,\sigma}]$ and $\tau(L) = F[w_{0,\tau}]$ for embeddings $\sigma : M \hookrightarrow \mathbb{C} \in \Sigma$ and $\tau : L \hookrightarrow \mathbb{C} \in \Sigma_L$ for a CM type Σ_L of L are totally imaginary quadratic extensions of F (so, CM fields). Write $\sigma_0 := \sigma|_F \in I$ for $\sigma \in \Sigma$. Let $Y = M \otimes_F L$, and we embed Y into $M_2(F) \otimes_F M_2(F) = M_4(F)$ by $(a, b) \mapsto \rho_M(a) \otimes \rho_L(b)$ with

$$\begin{pmatrix} z_0 a \\ a \end{pmatrix} = \rho_M(a) \begin{pmatrix} z_0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_0 b \\ b \end{pmatrix} = \rho_L(b) \begin{pmatrix} w_0 \\ 1 \end{pmatrix}.$$

We see easily that $\rho_M(a)^\iota = \rho_M(a^c)$ and $\rho_L(b)^\iota = \rho_L(b^c)$ for complex conjugation c . We regard $V = M_2(F)$ as a Y -module for the multiplicative semi-group Y via $\rho_M \otimes \rho_L$; in other words, $(a, b)v = \rho_M(a)v\rho_L(b)^\iota$.

We have four distinct Y -eigenvectors $p(z_{0,\sigma}, w_{0,\sigma})$, $p(z_{0,\sigma}, \overline{w}_{0,\sigma})$, $p(\overline{z}_{0,\sigma}, w_{0,\sigma})$ and $p(\overline{z}_{0,\sigma}, \overline{w}_{0,\sigma})$ in $M_2(\mathbb{C}) = V \otimes_{F,\sigma} \mathbb{C}$, whose eigenvalues of (a, b) are $(a^\sigma b^\sigma)$, $(a^\sigma b^{\sigma c})$, $(a^{\sigma c} b^{\sigma c})$ and $(a^{\sigma c} b^\sigma)$, respectively. Since $V \otimes_{F,\sigma} \mathbb{C}$ is free of rank 1 over $Y_\sigma = Y \otimes_{F,\sigma} \mathbb{C}$, $V = M_2(F)$ is free of rank 1 over Y (because $\mathbb{C}$ is faithfully flat over F). Thus

$$(3.19) \quad \text{we find } v \in V \text{ such that } V = Yv.$$

Suppose now

$$(3.20) \quad L = M.$$

Then $Y \cong M \oplus Y_0$ with $Y_0 \cong M$, the first projection to M is given by $a \otimes b \mapsto ab^c$ and the second to Y_0 is given by $a \otimes b \mapsto ab$. Since c is an automorphism of M , $p(z, w)$ and $p(\overline{z}_{0,\sigma}, \overline{w}_{0,\sigma})$ belongs to $M_\sigma = M \otimes_{F,\sigma} \mathbb{C} \subset Y \otimes_{F,\sigma} \mathbb{C}$. The vectors $p(z_{0,\sigma}, \overline{w}_{0,\sigma})$ and $p(\overline{z}_{0,\sigma}, w_{0,\sigma})$ belongs to $Y_{0,\sigma} = Y_0 \otimes_F \mathbb{C}$ and are orthogonal to M_σ .

Thus $\Sigma = \sum_{\sigma_0 \in I} \sigma$ gives rise to a CM type of M (with $\Sigma \sqcup \Sigma^c$ giving all complex embeddings of M). This shows: writing $V \ni \alpha = av \oplus bv$ with $a \in M$ and $b \in Y_0$

$$\begin{aligned}
 (\alpha^\sigma; z_0, \sigma, w_0, \sigma) &= S(\alpha^\sigma, p(z_0, \sigma, w_0, \sigma)) = S(av_\sigma, p(z_0, \sigma, w_0, \sigma)) \\
 &= S(v_\sigma, \rho_M(a^c) p(z_0, \sigma, w_0, \sigma)) \stackrel{(3.2)}{=} S(v_\sigma, a^c p(z_0, \sigma, w_0, \sigma)) \\
 (3.21) \quad &\stackrel{(*)}{=} a^{\sigma^c} [v_\sigma; z_0, \sigma, w_0, \sigma], \\
 &[\alpha; z_0, w_0]^{-k} |[\alpha; z_0, w_0]|^{-2sI} = C^{-k\Sigma} |C^\Sigma|^{-2s} a^{-k\Sigma^c} N(a)^{-s},
 \end{aligned}$$

where $C = (C_\sigma)_{\sigma \in \Sigma}$ with $C_\sigma = [v_\sigma; z_0, \sigma, w_0, \sigma]$, $N(a)$ is the absolute norm of $a \in M$, and at $(*)$, we regard S as a $\mathbb{C}$ -bilinear form as $p(z, w)$ is an element of $V \otimes_F \mathbb{C}$ after scalar extension to $\mathbb{C}$. Here we have written $k\Sigma = \sum_{\sigma_0 \in I} k_\sigma \sigma$ and $k\Sigma^c = \sum_{\sigma_0 \in I} k_\sigma \sigma^c$.

Since $p(z_0, \sigma, w_0, \sigma)$ and $p(\bar{z}_0, \sigma, \bar{w}_0, \sigma)$ span a scalar extension to $\mathbb{C}$ of a subspace on which S_σ is negative definite, S is totally positive definite on $W = Y_0 v$, because every vector in W is orthogonal to $p(z_0, \sigma, w_0, \sigma)$ and $p(\bar{z}_0, \sigma, \bar{w}_0, \sigma)$. We write S_{W_σ} (resp. $S_{W_\sigma^\perp}$) for the restriction of S to $W = Mv_\sigma \otimes_{F, \sigma} \mathbb{R}$ (resp. the orthogonal complement $W_\sigma^\perp = Y_0 v_\sigma^\perp \otimes_{F, \sigma} \mathbb{R}$ of W_σ in $V \otimes_{F, \sigma} \mathbb{R}$). Thus we can write $S_{W_\sigma}(x, y) = -\text{Tr}_{\mathbb{C}/\mathbb{R}}(\delta_M^\sigma x \bar{y})$ (resp. $S_{W_\sigma^\perp}(x, y) = \text{Tr}_{\mathbb{C}/\mathbb{R}}(\delta_0^\sigma x \bar{y})$) for $x, y \in W$ and $0 \ll \delta_M \in F$ (resp. $0 \ll \delta_0 \in F$). By this fact, δ_M is a totally positive element of F ; so, we may assume that $\delta_M = \delta_0 = \frac{1}{2}$ by changing v if necessary.

3.5. Special values of Shimura series. As we have explained already, we choose v in (3.19) as in the previous subsection so that

$$(3.22) \quad S_Y((a, b), (a', b')) = \frac{1}{2} \text{Tr}_{M/F}(-aa'^c + bb'^c).$$

We see, supposing

$$(3.23) \quad \phi^{(\infty)}(u) = \phi_M \otimes \phi_0$$

for Bruhat functions $\phi_M : M_{\mathbb{A}}^{(\infty)} v \rightarrow \mathbb{C}$ and $\phi_0 : Y_{0, \mathbb{A}}^{(\infty)} v \rightarrow \mathbb{C}$,

$$\begin{aligned}
 C^{k\Sigma} |C^\Sigma|^{2s} H_k(z_0, w_0; s; \phi^{(\infty)}, f) \\
 = [U] \sum_{\alpha \in M/U} \phi_M(\alpha v) \sum_{\beta \in Y_0/U} \phi_0(\beta v) a(\alpha \alpha^c - \beta \beta^c, f) \alpha^{-k\Sigma^c} N(\alpha)^{-s},
 \end{aligned}$$

where C is as in (3.21). We now set $\theta(\phi_0) = \sum_{\beta \in Y_0} \phi_0(\beta v) \mathbf{e}_F(\beta \beta^c z)$. Then for $f'(z) = \theta(\phi_0) f(z) = \sum_{\xi \in F} a(\xi, f') \mathbf{e}_F(\xi z)$, we have

$$a(\xi, f') = \sum_{\beta \in Y_0/U} a(\xi - \beta \beta^c, f) \phi_0(\beta v),$$

which is a finite sum because $\{x \in Y_0 \otimes_F \mathbb{R} | x^\sigma x^{\sigma^c} < \xi^\sigma \forall \sigma\}$ is a compact set. Thus we have, under (3.22) and (3.23)

$$\begin{aligned}
 (3.24) \quad C^{k\Sigma} |C^\Sigma|^{2s} H_k(z_0, w_0; s; \phi^{(\infty)}, f) \\
 = [U] \sum_{\alpha \in M/U} \phi_M(\alpha v) a(\alpha \alpha^c, f') \alpha^{-k\Sigma^c} N(\alpha)^{-s}.
 \end{aligned}$$

The series (3.24) is the Rankin convolution of f' and the theta series $\theta(\phi_{k, M})$ of the norm form of M for $\phi_{k, M}(\alpha) = \alpha^{k\Sigma} \phi_M(\alpha v)$ (see (3.29)).

3.6. Petersson inner product formula. For a given theta series $\theta_M(\phi)$ of weight $k+I$ of a CM field M/F , we are going to write down the inner product $\langle \theta_M(\phi), f'_c \rangle$ for a special value of a modular form on $GL(2) \times GL(2)$, taking $f' = f_{\theta_M(\phi')}$ for another theta series $\theta_M(\phi') = \theta(\phi_0)$ with $\phi' = \phi_0$ of weight I of M . Here $f'_c(z) = \overline{f'(-\bar{z})}$; so, f'_c is a holomorphic modular form whose Fourier coefficients (at the infinity) are the complex conjugate of those of f' . The modular form is given by, up to an explicit constant,

$$\text{Res}_{s=1} H_k(z, w; s; \phi' \otimes \phi^{(\infty)}, f).$$

Let f and g be Hilbert modular forms on $\Gamma \subset SL_2(F)$ with Fourier expansion $f = \sum_{\xi \in F} a(\xi, f) \mathbf{e}_F(\xi \tau)$ and $g = \sum_{\xi \in F} a(\xi, g) \mathbf{e}_F(\xi \tau)$ for $z \in \mathfrak{H}^I$. We take the ideal $\mathfrak{a} \subset F$ and the unit group $U \subset O_+^\times$ as in (3.13). Let ℓ and κ be the weights of f and g respectively. We suppose that one of f and g is a cusp form so that $\bar{f}g$ is rapidly decreasing towards cusps.

We let $\epsilon \in U$ act on $\mathfrak{H}^I$ by $\tau \mapsto \epsilon^2 \tau$. Then $f(\epsilon^2 \tau) = \epsilon^{-\ell} f(\tau)$ and $g(\epsilon^2 \tau) = \epsilon^{-\kappa} g(\tau)$. Then the function $\bar{f}g(\tau) \eta^{(\ell+\kappa)/2}$ is U -invariant. We then consider

$$D(s; f, g) = [U^2] N(\mathfrak{a})^{-1} \sqrt{|D|}^{-1} \int_{F_\infty/\mathfrak{a}} \int_{F_\infty^\times/U^2} \bar{f}(\tau) g(\tau) \eta^{sI+(\ell+\kappa)/2} d\xi d^\times \eta.$$

We now assume the equal parity condition that

$$(3.26) \quad \ell \equiv \kappa \pmod{2\mathbb{Z}[I] + \mathbb{Z} \sum_{\sigma \in I} \sigma}.$$

Thus we find $m \in \mathbb{Z}[I]$ such that $\ell - \kappa - 2m \in \mathbb{Z}I$ for $I = \sum_{\sigma} \sigma$; so, we write $\ell - \kappa - 2m = [\ell - \kappa - 2m]I$ with $[\ell - \kappa - 2m] \in \mathbb{Z}$. Replacing Γ by

$$\left\{ \gamma \in \Gamma \mid (\bar{f}g \eta^{(\ell+\kappa)/2} \circ \gamma)(\tau) = (\bar{f}g \eta^{(\ell+\kappa)/2})(\tau) j(\gamma, \bar{\tau})^\ell j(\gamma, \tau)^\kappa |j(\gamma, \tau)|^{-\ell-\kappa} \right\}$$

if necessary, we have

$$(3.27) \quad [U^2]^{-1} N(\mathfrak{a}) \sqrt{|D|} D(s; f, g) = \int_{\Gamma \backslash \mathfrak{H}^I} \bar{f}(\tau) g(\tau) \eta^{\ell-m} E_{[\ell-\kappa-2m]I, m}(\tau; s+1 - \frac{[\ell-\kappa-2m]}{2}) d\mu(\tau),$$

where

$$E_{nI, m}(\tau; s) = \eta^{sI} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\frac{j(\gamma, \bar{\tau})}{j(\gamma, \tau)} \right)^m j(\gamma, \tau)^{-nI} |j(\gamma, \tau)|^{-2sI}.$$

When $m = 0$, we write simply E_{nI} for $E_{nI, 0}$ ($n \in \mathbb{Z}$). Since $E_{nI, m}(\tau; s)$ has meromorphic continuation on the whole s -plane as a slowly increasing function (outside its singularity in s), the above integral gives an analytic continuation of $D(s; f, g)$ to the whole complex s -plane. In particular if $\ell = \kappa$, the L -function $D(s; f, g)$ can have a pole at $s = 0$ because in that case, we can choose $m = 0$ and $E_0(\tau, s)$ has a simple pole at $s = 1$ with a constant residue.

By the same calculation as in §3.2, we have an absolutely and locally uniformly convergent series

$$(3.28) \quad \begin{aligned} & 2^{[F:\mathbb{Q}]-1} (4\pi)^{sI+(\ell+\kappa)/2} D(s; f, g) \\ &= [U^2] \Gamma_F(sI + (\ell + \kappa)/2) \sum_{0 \ll \xi \in F^\times / U^2} \overline{a(\xi, f) \xi^{-\ell/2}} a(\xi, g) \xi^{-\kappa/2} N(\xi)^{-s} \end{aligned}$$

if $\operatorname{Re}(s) > \theta(f) + \theta(g) + 1$, where $\theta(?) = -1$ when $?$ is a constant, and otherwise, $\theta(?) = \theta \geq -\frac{1}{2}$ with $|a(\xi, ?) \xi^{-\ell/2}| = O(|N(\xi)|^\theta)$ for the norm map $N = N_{F/\mathbb{Q}}$.

Let us recall the theta series:

$$\theta(\phi_{k,M}) = \sum_{a \in M} \phi_{k,M}(a) \mathbf{e}_F(aa^c \tau)$$

for $\phi_{k,M}(a) = a^{k\Sigma} \phi_M(a)$ as in (3.24). This theta series has weight $k + I$ [HMI, §2.5.4] (or [CAL, §4.2]). We compute $D(s; f'_c, \theta(\phi_{k,M}))$ for a cusp form f' of weight $\ell + I$:

$$\begin{aligned} & 2^{[F:\mathbb{Q}]-1} \frac{(4\pi)^{sI+(\ell+k+2I)/2}}{\Gamma_F(sI + (\ell + k + 2I)/2)} D(s; f'_c, \theta(\phi_{k,M})) \\ &= [U^2] \sum_{0 \ll \xi \in F^\times / U^2} a(\xi, f') \xi^{-(\ell+I)/2} a(\xi, \theta(\phi_{k,M})) \xi^{-(k+I)/2} N(\xi)^{-s} \\ &= [U^2] \sum_{\alpha \in M^\times / U} \phi_M(\alpha) a(\alpha \alpha^c, f') \alpha^{k\Sigma} (\alpha \alpha^c)^{-(k+\ell+2I)/2} N(\alpha)^{-s} \\ &= [U^2] \sum_{\alpha \in M^\times / U} \phi_M(\alpha) a(\alpha \alpha^c, f') \alpha^{k\Sigma} (\alpha^{-k\Sigma} \alpha^{-k\Sigma}) N(\alpha)^{-s-1+(k-\ell)/2} \\ &= [U^2] \sum_{\alpha \in M^\times / U} \phi_M(\alpha) a(\alpha \alpha^c, f') \alpha^{-k\Sigma} N(\alpha)^{-s-1+(k-\ell)/2} \end{aligned}$$

Recall $C = (C_\sigma)_{\sigma \in \Sigma}$ with $C_\sigma = [v_\sigma; z_{0,\sigma}, w_{0,\sigma}]$ in (3.21). From this, we get under the notation and the assumption of (3.24)

$$(3.29) \quad \begin{aligned} & 2^{1-[F:\mathbb{Q}]} C^{k\Sigma} |C^\Sigma|^2 (4\pi)^{-sI-k} \Gamma_F(sI+k) H_k(z_0, w_0; s; \phi^{(\infty)}, f) \\ &= [U : U^2] D(s-1 + \frac{[k-\ell]}{2}; f'_c, \theta(\phi_{k,M})), \end{aligned}$$

where ℓ is the weight of f (so, weight of f'_c is $\ell + I$). Note here that $[U : U^2] = 2^{[F:\mathbb{Q}]-1}$.

Since $E_0(\tau; s)$ has a simple pole at $s = 1$ with constant residue $c_\Gamma \neq 0$, if $k = \ell$ and $\phi^{(\infty)} = \phi_M \otimes \phi_0$, we have from (3.27)

$$(3.30) \quad \begin{aligned} & 4^{1-[F:\mathbb{Q}]} C^{k\Sigma} |C^\Sigma|^2 (4\pi)^{-k-I} \Gamma_F(k+I) \operatorname{Res}_{s=1} H_k(z_0, w_0; s; \phi^{(\infty)}, f) \\ &= \operatorname{Res}_{s=1} D(s-1; f'_c, \theta(\phi_{k,M})) \\ &= \operatorname{Res}_{s=1} D(s-1; f'_c, \theta(\phi_{k,M})) \\ &= [U^2] N(\mathfrak{a})^{-1} \sqrt{|D|}^{-1} c_\Gamma \langle \theta(\phi_{k,M}), f'_c \rangle_\Gamma, \end{aligned}$$

where

$$\langle g, f \rangle_\Gamma = \int_{\Gamma \backslash \mathfrak{H}^I} g(\tau) \overline{f(\tau)} \eta^k d\mu(\tau).$$

Let $\Psi(z, w) = \Psi_f(z, w)$ be the modular form on $GL(2) \times GL(2)$ given by the Fourier expansion:

$$(3.31) \quad \Psi(z, w) = \Psi_f(z, w) = \sum_{\alpha \in \Gamma \backslash M_2(F), \det(\alpha) \gg 0} \phi^{*(\infty)}(\epsilon \alpha) \mathbf{e}_F(\det(\alpha)z) f|_k \alpha(w)$$

as in Corollary 3.2. Then taking Γ sufficiently small and combining Corollary 3.2 and (3.30), we get the following explicit formula:

Theorem 3.3. *Let f be a Hilbert modular cusp form of weight k . Then we have*

$$\langle \theta(\phi_{k,M}), f'_c \rangle_\Gamma = 2^{-k-2I} |D|^{-1} C^{k\Sigma} |C^\Sigma|^{2i^k} \text{Im}(z_0)^{-I} \text{Im}(w_0)^{-I} \Psi_f(z_0, w_0)$$

under the notation of (3.24). Here $C = (C_\sigma)_{\sigma \in \Sigma}$ with $C_\sigma = [v_\sigma; z_{0,\sigma}, w_{0,\sigma}]$ in (3.21).

4. PREPARATION FOR THE PROOF

We choose λ with $\psi = \lambda^-$. We shall prove the theorem in the introduction under the assumptions (h1–4). We first prove the theorem in the case where the degree $[F : \mathbb{Q}]$ is even. The odd degree case will be reduced to the even degree case via quadratic base change.

4.1. Congruence power series divides self inner product. We first remark the following fact. Fix a W -algebra homomorphism $\phi : \mathbf{h}_{\kappa, \varepsilon_\lambda} := \mathbf{h}_{\kappa}^{n.\text{ord}}(\mathfrak{N}, \varepsilon_\lambda; W) \rightarrow W$ factoring through a reduced local ring $\mathbb{T}$ of $\mathbf{h}_{\kappa, \varepsilon_\lambda}$. Here ε_λ is as in (2.11). Let $f = f_\phi$ be the corresponding normalized Hecke eigenform by duality (2.13) with $f|T(\mathfrak{l}) = \phi(T(\mathfrak{l})f)$. Pick a Hecke hermitian non-degenerate inner product $(\cdot, \cdot) : S_{\kappa, \varepsilon_\lambda} \otimes_W S_{\kappa, \varepsilon_\lambda} \rightarrow W$ with $(f_\phi, f_\phi) \neq 0$. Here $S_{\kappa, \varepsilon_\lambda} := S_\kappa(\mathfrak{N}, \varepsilon_\lambda; W)$ and by saying “Hecke hermitian”, we mean $(g|T(\mathfrak{l}), g') = (g, g'|T^*(\mathfrak{l}))$ for an involution $h \mapsto h^*$ of $\mathbf{h}_{\kappa, \varepsilon_\lambda}$. We extend this bilinear form linearly to the field of fraction $\text{Frac}(W)$ of W . Regarding ϕ as an algebra homomorphism of $\mathbf{h}_{\kappa}(\mathfrak{N}, \varepsilon_\lambda; W)$ factoring through $\mathbf{h}_{\kappa}^{n.\text{ord}}(\mathfrak{N}, \varepsilon_\lambda; W)$, we consider a linear form

$$\ell = \ell_\phi : S_\kappa(\mathfrak{N}, \varepsilon_\lambda; \text{Frac}(W)) \rightarrow \text{Frac}(W)$$

$$\text{given by } \ell_\phi(g) = \frac{(f_\phi, g)}{(f_\phi, f_\phi)}.$$

Lemma 4.1. *Let the notation be as above. Let $H(\phi) = H(\phi)_{\kappa, \varepsilon_\lambda}$ be the congruence number of ϕ . Suppose that $(\cdot, \cdot)$ is non-degenerate with values in W on $S_{\kappa, \varepsilon_\lambda} \otimes_W S_{\kappa, \varepsilon_\lambda}$. Then $H(\phi)|(f_\phi, f_\phi)$.*

Proof. Since $\mathbb{T}$ is reduced, we can write $g = cf_\phi + g'$ for g' orthogonal to f with $c \in \text{Frac}(W)$. Since $(f_\phi, g) = (f_\phi, cf_\phi + g') = c(f_\phi, f_\phi)$, we have $\ell_\phi(g) = \frac{(f_\phi, g)}{(f_\phi, f_\phi)} = c$. The congruence ideal C of W for λ is given by

$$C := \{x \in W \mid x\ell_\phi(g) \in W \text{ for all } g \in S_\kappa(\mathfrak{N}, \varepsilon_\lambda; W)\}.$$

Then $H(\phi)$ is by definition a generator of C . For all $g \in S_\kappa(\mathfrak{N}, \varepsilon_\lambda; W)$, we have $c = \ell_\phi(g) = \frac{(f_\phi, g)}{(f_\phi, f_\phi)}$, which implies $(f_\phi, f_\phi)\ell_\phi(g) \in W$ and hence $(f_\phi, f_\phi) \in C = W \cdot H(\phi)$. Thus $H(\phi)|(f_\phi, f_\phi)$. $\square$

4.2. Integrality of values of modular forms. Now we specialize κ to be $(I, 0)$ as explained in the introduction. For $f \in S_{(I,0)}(\mathfrak{N}, \varepsilon; W)$, choosing $u \in GL_2(F_\infty)$ with $u(\mathbf{i}) = \tau$ for each $\tau \in \mathfrak{H}^I$, define $f_x(\tau) = f(xu_\infty)J(u_\infty, \mathbf{i})$ with $J(g, z) = \det(g)^{-I}j(g, z)^{2I}$ for each $x \in GL_2(F_{\mathbb{A}(\infty)})$. Then f_x is a holomorphic weight $2I$ modular form on $\mathfrak{H}^I$ for all x [HMI, (2.3.5)]. By the strong approximation theorem,

$$GL_2(F) \backslash GL_2(F_{\mathbb{A}}^{(\infty)}) / U_0(\mathfrak{N}) \cong F^\times \backslash F_{\mathbb{A}(\infty)}^\times / \det(U_0(\mathfrak{N})) \cong Cl_F$$

via $y \mapsto \det(y)$ for the class group Cl_F of F . From this, writing $\mathfrak{y} = y\hat{O} \cap F$ for the ideal corresponding to the idele y and setting $\tilde{y} = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, $f \in S_{(I,0)}(\mathfrak{N}, \varepsilon; W)$ is determined by the q -expansions $\{f(y) := f_{\tilde{y}}(q) = \sum_{0 \leq \xi \in \mathfrak{y}^*} a(\xi, f)q^\xi\}_y$. Here $\mathfrak{y}^* = \mathfrak{y}^{-1}\mathfrak{d}^{-1}$ for the absolute different $\mathfrak{d}$ of F and $f(y)$ is the q -expansion at the Tate AVRMs $\text{Tate}_{\mathfrak{y}^*, O}(q)$ (in [K78, 1.1] and [HMI, §4.2.5]) of the classical modular form $f_{\tilde{y}}$ (of [HMI, (SA2)]) of weight $k = 2I$ on the following congruence subgroup:

$$(4.1) \quad \Gamma_0(\mathfrak{N}; \mathfrak{y}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \mid a, d \in O, b \in \mathfrak{y}, c \in \mathfrak{y}^{-1}\mathfrak{N}, ad - bc \in O_+^\times \right\},$$

where $O_+^\times$ is the subgroup of totally positive units.

A classical modular form with q -expansion coefficients in W on a slightly smaller Γ_1 -type congruence subgroup:

$$(4.2) \quad \Gamma(\mathfrak{N}; \mathfrak{y}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N}; \mathfrak{y}) \mid a \equiv d \equiv 1 \pmod{\mathfrak{N}} \right\}$$

has a moduli theoretic interpretation, which we recall in the following paragraph. We write $S_{2I}(\Gamma(\mathfrak{N}; \mathfrak{y}); A)$ for the space of the classical cusp forms on $\Gamma(\mathfrak{N}; \mathfrak{y})$ of weight $2I$ with q -expansion coefficients in A . Hereafter we write simply S_2 for S_{2I} .

Let A be a valuation ring with residual characteristic p . An abelian scheme $X_{/A'}$ over an A -algebra A' is called an abelian variety with real multiplication by O (AVRM) if it has an embedding: $O \hookrightarrow \text{End}(X_{/A'})$ of algebras such that $H^0(X, \Omega_{X/O}) = (O \otimes_{\mathbb{Z}} A')\omega$ for a nowhere vanishing differential ω . Here we have used the unramifiedness of F at p (otherwise, we need to formulate this condition as $H^0(X, \Omega_{X/O}) = (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} A')\omega$). Each Hilbert modular form $f \in S_2(\Gamma(\mathfrak{N}; \mathfrak{y}); A)$ can be regarded as a function of quintuples: $(X, \Lambda, i, \omega, A')$ made up of an A -algebra A' , an AVRMs X over A' , a polarization Λ whose polarization ideal is given by $\mathfrak{y}^*$, an embedding $i : \mu_{\mathfrak{N}} \hookrightarrow X$ of group schemes over A' and a differential ω as above (see, for more details of AVRMs, [HMI, §4.2] and [PAF, §4.1]). Here $\mu_{\mathfrak{N}}$ is the group scheme made up of $\mathfrak{N}$ -torsion points of $\mathbb{G}_m \otimes \mathfrak{d}^{-1}$, that is, $\mu_{\mathfrak{N}}(A) = \{\zeta \in \mathbb{G}_m \otimes \mathfrak{d}^{-1}(A) \mid \mathfrak{N}\zeta = 0\}$, regarding $\mathbb{G}_m \otimes \mathfrak{d}^{-1}(A)$ as an additive group. Every ingredient of the quintuple has to be defined over A' . As a function of $(X, \Lambda, i, \omega)_{/A'}$, f satisfies the following conditions (see [HMI, §4.2.7]):

- (M1) $f((X', \Lambda', i', \omega')_{/C}) = \rho(f(X, \Lambda, i, \omega)_{/A'})$ if $\rho : A' \rightarrow C$ is an A -algebra homomorphism and $(X', \Lambda', i', \omega')_{/C} \cong (X, \Lambda, i, \omega) \times_{A', \rho} C$. Here “ $\cong$ ” implies $\phi : X \times_{A'} C \cong X'_{/C}$ as AVRMs, ${}^t\phi \circ \Lambda' \circ \phi = \Lambda \times_{A'} C$, $\phi \circ i = i'$ and $\phi^*\omega' = \omega$.
- (M2) f vanishes at all cusps, that is, the q -expansion of f at every Tate quintuple vanishes at $q = 0$.
- (M3) $f((X, \Lambda, i, \alpha\omega)_{/A'}) = \alpha^{-k}f((X, \Lambda, i, \omega)_{/A'})$ for $\alpha \in (A' \otimes_{\mathbb{Z}} O)^\times$.

The “Neben” character $\varepsilon : U_0(\mathfrak{N}) \rightarrow \overline{\mathbb{Q}}^\times$ restricted to $U_0^1(\mathfrak{N}) = U_0(\mathfrak{N}) \cap SL_2(\hat{O})$ factors through $U_0^1(\mathfrak{N})/U^1(\mathfrak{N})$ for $U^1(\mathfrak{N}) = U(\mathfrak{N}) \cap SL_2(\hat{O})$ (the conductor of

ε^- is $\mathfrak{N}$), because $\varepsilon(u) = \varepsilon_1(\det(u))\varepsilon^-(d)$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus to evaluate $f \in S_{(I,0)}(\mathfrak{N}, \varepsilon; A)$ at an AVR M X of CM type, we only need to specify $\mu_{\mathfrak{N}} \hookrightarrow X$.

Let M/F be the CM quadratic extension as in the introduction. Recall the decomposition: $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}\mathfrak{R}$ in §2.2 of the conductor of the Hecke character λ such that $\mathfrak{F} + \mathfrak{F}_c = R$ with $\mathfrak{F} \subset \mathfrak{F}_c^c$, $\mathfrak{R}$ is a product of ramified primes in M/F , and $\mathfrak{I}$ is made up of inert unramified primes in M/F . Recall from (2.11) and (2.12) that $\mathfrak{N}(\lambda)$ is the conductor of $\varphi^- \left(\frac{M/F}{\cdot} \right)$. Then by (h2), the prime factors of $\mathfrak{N}(\lambda)$ are either split or ramified over F . If $\mathfrak{l} \mid \mathfrak{N}(\lambda)$ and $\mathfrak{l} = \mathfrak{L}\overline{\mathfrak{L}}$ (with $\mathfrak{L} \neq \overline{\mathfrak{L}}$) in M , we may choose $\mathfrak{L}$ so that $\mathfrak{L} \supset \mathfrak{F}$. The exponent of $\mathfrak{l}$ in $\mathfrak{N}(\lambda)$ is less than or equal to that of $\mathfrak{L}$ in $\mathfrak{F}$. Thus to evaluate $f(\lambda)$ at a CM point, we need to specify the level structure for the level $d(M/F)\mathfrak{f}$ ($\mathfrak{f} = \mathfrak{F} \cap F$). Actually we later need the level structure at other primes non-split in M/F ; so, we first specify level structure for split primes and then extend the definition to non-split primes. We shall do this first for an abelian variety of CM type Σ with multiplication by R . Hereafter $\mathfrak{F}$ is an integral ideal of R with $\mathfrak{F} + \mathfrak{F}_c = R$ and prime to p (because we need to be more careful for primes dividing p).

Let W be the Witt vector ring with coefficients in $\overline{\mathbb{F}}$ as in the introduction. Define $\mathcal{W} = i_p^{-1}(W) \subset \overline{\mathbb{Q}}$, which is a discrete valuation ring unramified over $\mathbb{Z}_{(p)}$ with algebraically closed residue field $\overline{\mathbb{F}}$. Recall the integer ring R of M . We consider $X(R)_{/\mathbb{C}}$ to be the algebraization of the complex torus $\mathbb{C}^\Sigma/R^\Sigma$, where $R^\Sigma = \{(a^\sigma)_{\sigma \in \Sigma} \mid a \in R\}$ and $\mathbb{C}^\Sigma = R^\Sigma \otimes_{\mathbb{Z}} \mathbb{R}$. Since $X(R)$ has complex multiplication, it can be defined over $\overline{\mathbb{Q}}$ and hence over a number field (see [ACM, 12.4]). By the main theorem of complex multiplication (see [ACM, 18.6]), $X(R)$ and its ℓ -divisible group for any prime ℓ outside p are actually defined over an infinite extension K of $\mathbb{Q}$ unramified at p . By the criterion of good reduction by unramifiedness of ℓ -power torsion points (see [ST68]), we find a model $X(R)_{/\mathcal{W}}$ of $X(R)_{/K}$.

By choosing $\delta \in M$ with $\text{Im}(\sigma(\delta)) > 0$ for $\sigma \in \Sigma$, we have a polarization pairing $(x, y) \mapsto \text{Tr}_{M/\mathbb{Q}}(\delta xc(y))$. This pairing identifies $R \wedge_O R$ with $\mathfrak{h}$ for a suitable choice of a fractional ideal $\mathfrak{h} \subset F$ (prime to p) and induces a $\mathfrak{h}^*$ -polarization $\lambda = \lambda(R)$. Thus we have the CM-quadruple $(X(R), \lambda(R), i(R), \omega(R))_{/\mathcal{W}}$, choosing $\omega(R)$ so that $H^0(X(R), \Omega_{X(R)/\mathcal{W}}) = (O \otimes_{\mathbb{Z}} \mathcal{W})\omega(R)$.

Since W has algebraically closed residue field, for any integer m prime to p , we have $X(R)[m] = \{x \in X(R)(W) \mid mx = 0\} \cong (\mathbb{Z}/m\mathbb{Z})^{[M:\mathbb{Q}]}$ and $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$ as group schemes over W . Thus we define the level $\mathfrak{f}$ -structure to be

$$\mu_{\mathfrak{f}} \cong O/\mathfrak{f} \cong X(R)[\mathfrak{F}] = \{x \in X(R)(W) \mid \mathfrak{F}x = 0\}.$$

Since the Frobenius map of $\overline{\mathbb{F}}_p$ acts by multiplication by p (times a unit) on $\Omega_{X(R)/W}$, the p -divisible group $X(R)[\mathfrak{P}^\infty]_{/\mathcal{W}}$ for $\mathfrak{P} \in \Sigma_p$ is connected. Since the residue field of $\mathcal{W}$ is algebraically closed, we see that $X(R)[\mathfrak{P}^e] \cong \mu_{\mathfrak{P}^e}$ over $\mathcal{W}$ (for $e = (e(\mathfrak{P}))_{\mathfrak{P} \in \Sigma_p}$), which gives rise to the level $\mathfrak{p}^e$ -structure we need.

Since $X(R)$ determines polarization ideal $\mathfrak{h}^*$, we have $R \wedge R \cong \mathfrak{h}$. Thus we can choose a base w_1 and w_2 of R so that $R = Ow_1 + \mathfrak{h}w_2$. For any integral ideal $\mathfrak{q}$ prime to p , we choose a generator $\varpi_{\mathfrak{q}}$ of $\mathfrak{q}O_{\mathfrak{q}}$. Fixing an isomorphism $O/\mathfrak{q} \cong \mathfrak{h}/\mathfrak{q}$, we embed $O/\mathfrak{q} \cong \mathfrak{h}/\mathfrak{q}/\mathfrak{h} \hookrightarrow \mathfrak{q}^{-1}R_{\mathfrak{q}}/R_{\mathfrak{q}} \cong X(R)[\mathfrak{q}]$ by sending x to $\varpi_{\mathfrak{q}}^{-1}xw_2 \in M_{\mathfrak{q}}/R_{\mathfrak{q}}$, which gives the level $\mathfrak{q}$ -structure on $X(R)$. We choose the base $w = (w_1, w_2)$ so that the level $\mathfrak{p}^e\mathfrak{f}$ -structure we have chosen coincides with the one for $\mathfrak{q}$ if $\mathfrak{p}^e\mathfrak{f} + \mathfrak{q}$ is non-trivial. We may always choose w so that $w_0 = w_1/w_2 \in \mathfrak{H}^I$. Therefore

choosing the basis (w_1, w_2) is almost equivalent to the choice of a point $w_0 \in \mathfrak{H}^I$ modulo $\Gamma(\mathfrak{N}, \mathfrak{h})$ for $\mathfrak{N} = \mathfrak{q} \cap \mathfrak{fp}^e$.

We write the level structure as $i(R) : \mu_{\mathfrak{N}} \hookrightarrow X(R)[\mathfrak{N}]$. The above definition of the quadruple $x(R) = (X(R), \Lambda(R), i(R), \omega(R))_{/\mathcal{W}}$ can be generalized to ideals of an O -order of R . Let $\mathfrak{e}$ be an integral ideal of F prime to $p\mathfrak{f}$. Let $R' = O + \mathfrak{e}R$ be the O -order of M of conductor $\mathfrak{e}$. We take a proper fractional ideal $\mathfrak{A}$ of R' prime to $p\mathfrak{f}q d(M/F)$.

A fractional R' -ideal $\mathfrak{A}$ is called *R' -proper* if $\{x \in M | x\mathfrak{A} \subset \mathfrak{A}\} = R'$. The polarization pairing on R (so on M) induces the polarization $\Lambda(\mathfrak{A})$ on $\mathfrak{A}$. We identify $\mathfrak{A} \wedge \mathfrak{A}$ with a fractional ideal $\eta(\mathfrak{A})$ of F under this pairing. It is easy to verify $\eta(\mathfrak{A}) = \eta(R)\mathfrak{e}N_{M/F}(\mathfrak{A})$. Then we can choose a basis w of $\mathfrak{A}$ so that $\mathfrak{A} = Ow_1 + \eta(\mathfrak{A})w_2$ and $w_0(\mathfrak{A}) = w_1/w_2 \in \mathfrak{H}^I$. This choice w gives rise to the level structure $i(\mathfrak{A}) : \mu_{\mathfrak{N}} \hookrightarrow X(\mathfrak{A})[\mathfrak{N}]$. We can always find an étale constant subgroup $C \cong O/\mathfrak{c}$ ($\mathfrak{c}$ prime to $\mathfrak{f}q d(M/F)p$) in $X(R)$ such that the étale quotient $X(\mathfrak{A}) = X(R)/C$ over $\mathcal{W}$ (e.g. [GME, §1.8.3]) gives a model over $\mathcal{W}$ of $\mathbb{C}^{\Sigma}/\mathfrak{A}^{\Sigma}$. Since $\mathfrak{c}$ is prime to $p\mathfrak{f}$, the level structure $i(R)$ and the differential $\omega(R)$ induce a unique level structure and a unique differential $\omega(\mathfrak{A})$ on $X(\mathfrak{A})$. We make a choice w so that the two level structures (one coming from $i(R)$ and another from the base w) coincide at primes where the two are well defined. Thus we have a unique point $w_0(\mathfrak{A}) \in \mathfrak{H}^I/\Gamma_1(\mathfrak{N}, \mathfrak{h})$. Having w is equivalent to having the quadruple $x(\mathfrak{A}) = (X(\mathfrak{A}), \Lambda(\mathfrak{A}), i(\mathfrak{A}), \omega(\mathfrak{A}))$ over $\mathbb{C}$.

Supposing that $f \in S_2(\Gamma(\mathfrak{N}; \mathfrak{h}); \mathcal{W})$ (and regarding f as a complex modular form), we may interpret the value $f(x(\mathfrak{A}))$ in terms of evaluation at a CM point $w_0(\mathfrak{A}) \in \mathfrak{H}^I$. For each $z = (z_1, z_2)$ with $z_0 := \frac{z_1}{z_2} \in \mathfrak{H}^I$, we consider the lattice $L_z = L_z^{\eta} = 2\pi i(Oz_1 + \eta z_2) \subset F_{\mathbb{C}} = F \otimes_{\mathbb{Q}} \mathbb{C}$. We define a pairing $\langle \cdot, \cdot \rangle : F_{\mathbb{C}} \times F_{\mathbb{C}} \rightarrow \mathbb{R}$ by $\langle 2\pi i(az_1 + bz_2), 2\pi i(cz_1 + dz_2) \rangle = ad - bc$, which induces a η^* -polarization $\lambda_z = \lambda_z^{\eta}$ on the complex torus $X_z = X_z^{\eta} = F_{\mathbb{C}}/L_z$. Thus we can algebraize X_z to an abelian variety $X_{z/\mathbb{C}}$. We have a canonical level $\mathfrak{N}$ -structure $i_z : (\mathfrak{o}^{-1} \otimes O/\mathfrak{N}) \cong 2\pi i(\eta z_2 \otimes O/\mathfrak{N}) \subset X_z(\mathbb{C})$ as long as η is prime to $\mathfrak{N}$. Then the analytic value of f at z is given by

$$(4.3) \quad z_2^{-2I} f((z_0, 1)) = f(z) = f(x_z^{\eta}) \quad \text{for } x_z^{\eta} = (X_z, \lambda_z, i_z, du),$$

where u is the variable $(u_{\sigma})_{\sigma \in I}$ with $u_{\sigma} \in \mathbb{C}$ identifying $F_{\mathbb{C}}$ with $\mathbb{C}^I$ as $\mathbb{C}$ -algebras.

Defining the canonical period $\Omega \in F_{\mathbb{C}}^{\times} = (\mathbb{C}^{\times})^{\Sigma}$ by

$$(4.4) \quad \omega(R) = \Omega du$$

and choosing η so that $R = (2\pi i)^{-1} L_{z_0}^{\eta}$, we find $x(\mathfrak{A}) \cong x_z^{\eta(\mathfrak{A})}$ and

$$(4.5) \quad f(x(\mathfrak{A})) = \frac{(2\pi i)^{2I} f(z)}{\Omega^{2I}} \in \mathcal{W} \quad \text{up to units in } \mathcal{W},$$

because $\omega(\mathfrak{A})/\omega(R) \in (O \otimes_{\mathbb{Z}} \mathcal{W})^{\times}$ (see [ACM, §32] and [K78, II]). Here writing $\Omega = (\Omega_{\sigma}) \in \mathbb{C}^{\Sigma}$,

$$\Omega^{2I} = \prod_{\sigma \in \Sigma} \Omega_{\sigma}^2 \quad \text{and} \quad (2\pi i)^{2I} = \prod_{\sigma \in \Sigma} (2\pi i) = (2\pi i)^{2[F:\mathbb{Q}]}.$$

Since $\mathcal{W}$ -integral modular forms $f(z, w)$ of weight (I, I) for the product of congruence subgroups: $\Gamma(\mathfrak{N}; \mathfrak{h}) \times \Gamma(\mathfrak{N}'; \mathfrak{h}')$ classify the pairs of test objects: $(x_z^{\eta}, x_w^{\eta'})$,

the same formula is valid (by the same proof given in [K78]): up to units in $\mathcal{W}$,

$$(4.6) \quad f(x(\mathfrak{A}), x(\mathfrak{B})) = \frac{(2\pi i)^{2I} f(z, w)}{\Omega^{2I}}.$$

4.3. Error terms of integral decomposition. Let B be a quaternion algebra over F everywhere unramified at finite places. First we take $B = M_2(F)$. We choose two fractional R -ideals $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of M . Then we decompose $\mathfrak{B}_1 = \mathfrak{b}_1 z_1 + O z_2$ and $\mathfrak{B}_2 = \mathfrak{b}_2 w_1 + O w_2$ with $z_0 = z_1/z_2 \in \mathfrak{H}^I$ and $w_0 = w_1/w_2 \in \mathfrak{H}^I$. The regular representation l of R on $\mathfrak{B}_1$ given by $l(\alpha) \begin{pmatrix} z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} z_0 \alpha \\ \alpha \end{pmatrix}$ gives an embedding of R into

$$O_L^l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in O, b \in \mathfrak{b}_1, c \in \mathfrak{b}_1^{-1} \right\}.$$

Similarly we define an embedding $r : R \hookrightarrow O_L^r$ replacing z_0 by w_0 , where

$$O_L^r = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in O, b \in \mathfrak{b}_2, c \in \mathfrak{b}_2^{-1} \right\}.$$

We consider the tensor product: $\mathfrak{B}_1 \otimes_O \mathfrak{B}_2$ and $L = l(\mathfrak{B}_1) v \cdot r(\mathfrak{B}_2) \subset M_2(F)$ for a suitable $v \in M_2(F)$.

We want to determine the factors of L_M and L^M . Since L^1 is the projection of L to the first factor M of $M \otimes_F M = M \oplus M$, writing the projection to the first factor as $(a \otimes b) \mapsto a^c b$ (so the projection to the second factor is given by $(a \otimes b) \mapsto a^c b^c$), we have $L^1 \cong \mathfrak{B}_1^c \mathfrak{B}_2$ and $L^2 \cong \mathfrak{B}_1^c \mathfrak{B}_2^c$.

Since $R \otimes_O R$ can be identified with

$$\{(a, b) \in R \oplus R \mid a \equiv b \pmod{\mathfrak{d}(M/F)}\}$$

inside $R \oplus R \subset M \otimes_F M$ for the relative different $\mathfrak{d}(M/F)$ for M/F , we see that $L_1 \cong \mathfrak{B}_1^c \mathfrak{B}_2 \mathfrak{d}(M/F)$ and $L_2 \cong \mathfrak{B}_1^c \mathfrak{B}_2^c \mathfrak{d}(M/F)$.

Remark 4.2. We analyze the choice of v locally at primes $\mathfrak{p} \mid p$ of F when $\mathfrak{B}_{j,\mathfrak{p}} = R_{\mathfrak{p}}$ for $j = 1, 2$. Since the prime ideal $\mathfrak{p}$ is split into $\mathfrak{P} \mathfrak{P}^c$ with $\mathfrak{P} \in \Sigma_p$ in M , by choosing the base (e_1, e_2) for $e_1 = (1, 0), e_2 = (0, 1)$ of $R_{\mathfrak{p}} = R_{\mathfrak{P}^c} \oplus R_{\mathfrak{P}}$ over $O_{\mathfrak{p}}$, we may assume that $l(\alpha) = r(\alpha) = \begin{pmatrix} \alpha^c & 0 \\ 0 & \alpha \end{pmatrix}$. Recall $\mathbf{b}$ in (2.5) for $\mathfrak{q} = \mathfrak{p}$ given by $\mathbf{b} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. By computation, we have

$$l(\alpha) \mathbf{b} r(\beta^c) = \begin{pmatrix} \alpha^c & 0 \\ 0 & \alpha \end{pmatrix} \mathbf{b} \begin{pmatrix} \beta & 0 \\ 0 & \beta^c \end{pmatrix} = \begin{pmatrix} \alpha^c \beta & \alpha^c \beta^c \\ \alpha \beta & \alpha \beta^c \end{pmatrix}.$$

This shows that $l(R_{\mathfrak{p}}) \mathbf{b} r(R_{\mathfrak{p}}) = M_2(O_{\mathfrak{p}})$, and regarding $M_2(O_{\mathfrak{p}})$ as an $R_{\mathfrak{p}}$ -module via $\alpha x = l(\alpha)x$, we find

$$M_2(O_{\mathfrak{p}}) = \begin{pmatrix} R_{\mathfrak{P}^c} & R_{\mathfrak{P}^c} \\ R_{\mathfrak{P}} & R_{\mathfrak{P}} \end{pmatrix}.$$

Take $O_{\mathfrak{p}}$ basis $w = (w_1, w_2)$ and $z = (z_1, z_2)$ of $R_{\mathfrak{p}}$ in M so that $w \equiv z \equiv (e_1, e_2) \pmod{\mathfrak{p}^m}$ for $m \geq e(\mathfrak{p})$ for $e(\mathfrak{p})$ as in Proposition 2.3. We define $p_H(z, w) = z_2 w_2 p(z_0, w_0)$ and $[u; z, w] = S(u, p_H(z, w))$ (the homogeneous form of $[u; z_0, w_0]$). Then we find $[\mathbf{b}; z, w] = (z_1 - z_2)(w_2 - w_1)$ and that $[\mathbf{b}; z, w]$ is a $\mathfrak{p}$ -adic unit.

4.4. CM theta series. Pick an arithmetic Hecke character $\lambda : M_{\mathbb{A}}^{\times}/M^{\times} \rightarrow \mathbb{C}^{\times}$. By (h2) and $\lambda^{-1}|_{G_{\text{tor}}(\psi)} = \psi$, the automorphic induction $\pi(\lambda)$ is in the principal series everywhere at finite places. See [CAL, §4.2.6] for an elementary construction the primitive form of the automorphic induction $\pi(\lambda)$. Write $\pi(\lambda)^{(\infty)} = \widehat{\otimes}_{\mathfrak{l}} \pi(\pi_{1,\mathfrak{l}}, \pi_{2,\mathfrak{l}})$ for characters $\pi_{j,\mathfrak{l}} : F_{\mathfrak{l}}^{\times} \rightarrow \mathbb{C}^{\times}$. Then we have [H26, Definition 1.33]

$$(4.7) \quad \mathfrak{N}(\lambda) = \prod_{\mathfrak{l} \mid p} (\mathfrak{C}(\pi_{1,\mathfrak{l}}) \cap \mathfrak{C}(\pi_{2,\mathfrak{l}})),$$

which is well defined as $\mathfrak{C}(\pi_{1,\mathfrak{l}}) \cap \mathfrak{C}(\pi_{2,\mathfrak{l}}) = O_{\mathfrak{l}}$ for almost all $\mathfrak{l}$. Assume $\lambda(x_{\infty}) = x_{\infty}^{-\Sigma}$ to get cups form of weight $\kappa = (I, 0)$. As before, we write $\mathfrak{C}(\lambda) = \mathfrak{F}\mathfrak{F}_c\mathfrak{J}\mathfrak{R}$ with $\mathfrak{F} \subset \mathfrak{F}_c^c$. Let

$$(4.8) \quad \Xi := \{\mathfrak{L} : \text{prime of } M \mid \mathfrak{L} \supset \mathfrak{F}\mathfrak{P}^{\Sigma_p}, \mathfrak{L} \mid \mathfrak{N}(\lambda)\}.$$

We shall compute the q -expansion of classical modular forms associated to the minimal form $f(\lambda)$ in the automorphic induction $\pi(\lambda)$. Pick $y \in F_{\mathbb{A}}^{\times}$ with $y_p = y_{\infty} = 1$. Then, we get the following formula of the complex Fourier coefficients:

$$a(\xi y d, f(\lambda)) = \sum_{xx^c = \xi y d, x_{\Xi} = 1} \lambda(xR),$$

where $xR = F \cap x\widehat{R}$ and x runs over $(\widehat{R} \cap M_{\mathbb{A}(\infty)}^{\times})/(\widehat{R}(\Xi))^{\times}$ [H06, (6.5)] (see also [CAL, Theorem 4.60]). This shows that, for the corresponding classical form $f_{\bar{y}}$ of weight $2I$ as in [HMI, (SA2)],

$$f_{\bar{y}} = N(\mathfrak{y})^{-1} \sum_{\mathfrak{A}; \mathfrak{A}\mathfrak{A}^c \sim \mathfrak{y}\mathfrak{d}} \lambda(\mathfrak{A})\theta(\lambda; \mathfrak{A}),$$

where $\mathfrak{A}$ runs over a complete representative set for ideal classes of M with $\mathfrak{A}\mathfrak{A}^c = \alpha\mathfrak{y}\mathfrak{d}$ ($\mathfrak{y} = y\widehat{O} \cap F$) for a totally positive $\alpha \in F$ and

$$(4.9) \quad \theta(\lambda; \mathfrak{A}) = \sum_{\xi \in \mathfrak{A}^{-1}/\mu(R)} \lambda(\xi^{(\infty)\Xi}) q^{\alpha\xi\xi^c},$$

where $\mu(R)$ is the subgroup of $R^{\times}$ made of all roots of unity. Here we regard λ as an idele character $\lambda : M_{\mathbb{A}}^{\times}/M^{\times} \rightarrow \mathbb{C}^{\times}$ by putting

$$\lambda(x) = \lambda(xR)x_{\infty}^{-\Sigma},$$

and ξ runs over elements in $\mathfrak{A}^{-1}$ such that $\xi\mathfrak{A}$ is outside Ξ for Ξ as in (4.8). As a locally constant function on $\widehat{\mathfrak{A}}^{-1}$, the p -component of $\phi'_1 : \xi \mapsto \lambda(\xi^{(\Xi)})$ is given by λ_p^{-1} restricted to $\mathfrak{A}_p^{-1}$ by the following reason: ϕ'_1 is the characteristic function of $\mathfrak{A}_{\mathfrak{l}}^{-1}$ for $\mathfrak{l}$ outside the conductor of λ , and taking $\xi \in \mathfrak{A}^{-1}$ with $\xi \equiv 1 \pmod{\mathfrak{C}(\lambda)}$, we see that $\phi'_1(\xi) = \lambda(\xi^{(\Xi)}) = \lambda(\xi^{(p)}) = \lambda(\xi_p)^{-1}$.

The modular form $\theta(\lambda; \mathfrak{A})$ is of weight $(I, 0)$ with Neben character ε_{λ} of level $\mathfrak{N} = \mathfrak{N}(\lambda)$ on

$$\widetilde{\Gamma}_0(\mathfrak{n}) = \widetilde{\Gamma}_0(\mathfrak{N}(\lambda); \mathfrak{y}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} O & \mathfrak{y} \\ \mathfrak{N}\mathfrak{y}^{-1} & O \end{pmatrix} \mid ad - bc \in O_+^{\times} \right\}.$$

Here $O_+^{\times}$ is the group of totally positive units in F , and the Neben character ε_{λ} sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\widetilde{\Gamma}_0(\mathfrak{N}(\lambda); \mathfrak{y})$ to $(\varepsilon_1(ad - bc)\varepsilon^-(d))^{-1}$ for $\varepsilon^- = \varepsilon_1^{-1}\varepsilon_2$ [HMI, (2.3.5)]. In [HMI], the role of ε_1, κ_1 and ε_2, κ_2 are reversed here (following the notation of [H06]). Since the factor “ $\varepsilon_1(ad - bc)$ ” disappears on $\Gamma_0(\mathfrak{N}) = \Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y}) :=$

$\tilde{\Gamma}_0(\mathfrak{N}(\lambda); \mathfrak{h}) \cap \mathrm{SL}_2(F)$, $\{f(\lambda)(\mathrm{diag}[a, 1]x)\}_{a \in F_{\mathbb{A}(\infty)}^\times}$ with x running over $\mathrm{SL}_2(F_{\mathbb{A}})$ determines $f(\lambda)$.

Remark 4.3. *Locally at a prime $\mathfrak{l}|\mathfrak{N}'$, the Neben character set $\varepsilon = (\varepsilon_{1,\mathfrak{l}}, \varepsilon_{2,\mathfrak{l}}, \varepsilon_{+,\mathfrak{l}})$ is the twist by $\varepsilon_1 \circ \det$ of the Neben character set $(1, \varepsilon_{\mathfrak{l}}^-, \varepsilon'_{+,\mathfrak{l}})$. Taking an automorphic representation π of minimal level $\mathfrak{N} = \mathfrak{N}(\lambda)$ with a minimal vector of Neben-type ε , this means that the local vector $w(g)$ of the Whittaker model of $\pi_{\mathfrak{l}} = \pi(\pi_{1,\mathfrak{l}}, \pi_{2,\mathfrak{l}})$ at $\mathfrak{l}$ of Neben-type ε as a function on $\mathrm{GL}_2(F_{\mathfrak{l}})$, after having the extension $\pi_{1,\mathfrak{l}}$ of $\varepsilon_{1,\mathfrak{l}}$ to $F_{\mathfrak{l}}^\times$, $w \otimes \tilde{\pi}_{1,\mathfrak{l}}^{-1}(g) := w(g)\pi_{1,\mathfrak{l}}^{-1}(\det(g))$ gives rise to a vector with Neben character set $(1, \varepsilon_{\mathfrak{l}}^-, \varepsilon'_{+,\mathfrak{l}})$ for $\varepsilon'_{+,\mathfrak{l}} = \tilde{\varepsilon}_{1,\mathfrak{l}}^{-2}\varepsilon_{+,\mathfrak{l}}$. Thus as long as $\pi = \pi(\lambda)$ is automorphic everywhere principal at finite places (i.e., under (h2)), $f(\lambda)$ restricted to $\mathrm{diag}[a, 1]\mathrm{SL}_2(F_{\mathbb{A}})$ for a running through a complete representative set of $F_{\mathbb{A}}^\times / F_+^\times \hat{O}^\times \cong \mathrm{Cl}_F^+$, determines $f(\lambda)$. This is the reason that we can work with $\mathrm{SL}(2)$ -modular forms on $\Gamma_0(\mathfrak{N}(\lambda); \mathfrak{h})$ for which contribution of inert or ramified factor disappears under the assumption that $\mathfrak{c}$ is a product of primes split in M/F . Since $f(\lambda)(\epsilon x) = f(\lambda)(x)$ for $\epsilon \in O^\times$ and $[\tilde{\Gamma}_0(\mathfrak{N}) : O^\times \Gamma_0(\mathfrak{N})]$ is a 2-power, $f(\lambda)$ is automorphic under the bigger $O^\times \Gamma_0(\mathfrak{N})$, and the computation of Petersson inner product over slightly different $\tilde{\Gamma}_0(\mathfrak{N}) \backslash \mathfrak{H}^I$ and $\Gamma_0(\mathfrak{N}) \backslash \mathfrak{H}^I = O^\times \Gamma_0(\mathfrak{N}) \backslash \mathfrak{H}^I$ does not affect the proof of the p -integrality of $(*) := \frac{\pi^{2(\kappa_1 - \kappa_2)} W_p(\lambda^-)(f(\lambda), f(\lambda))}{\Omega^{2(\kappa_1 - \kappa_2)}}$ as p is odd. In addition, to see the p -integrality of $(*)$, the trace map from $O^\times \Gamma_0(\mathfrak{N})$ to $\tilde{\Gamma}_0(\mathfrak{N})$ preserves p -integrality by the q -expansion principle. So hereafter, we work with level $\mathfrak{N}$ and the single Neben character ε^- forgetting about the larger level $\mathfrak{N}'$.*

5. PROOF OF THEOREM I UNDER (H1–4)

In this proof, the level $\mathfrak{N}$ and $\mathfrak{N}'$ are as in the introduction which are the (outside p) minimal level and the conductor of $f(\lambda)$ in §4.9. They only depend on the blanch character φ . We first suppose that $[F : \mathbb{Q}]$ is even. Then we have a totally definite quaternion algebra everywhere unramified at finite places. We first assume that B is totally definite everywhere unramified at finite places (so, $[F : \mathbb{Q}]$ is even) and that $\mathfrak{N}$ is a product of primes split in M/F and $d(M/F)$. By this assumption, we can decompose $O_0(\mathfrak{N})$ as a sum of two ideals of M by Lemma 2.1.

Recall the fixed maximal order O_B of B and the identification $\hat{O}_B = M_2(\hat{O})$. We described in [H26, §6.3] to reduce the general case of φ to the case where φ has split conductor in [H26, §6.3] via quadratic base-change. In any case, we will give details of the base-change argument. Thus for the moment, we assume here

(h2) *The conductor $\mathfrak{c}$ of ψ is a product of primes outside p split in M/F .*

As explained in §4.9, the automorphic induction $\pi(\lambda)$ generated by $f(\lambda)$ is everywhere principal, and $f(\lambda)$ on $\mathrm{GL}_2(F_{\mathbb{A}})$ and its image $f^B(\lambda)$ on $B_{\mathbb{A}}^\times$ under the Jacquet–Langlands correspondence has Neben character $U_0(\mathfrak{N}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varepsilon^-(d_{\mathfrak{N}})\epsilon_1(ad - bc) \in W^\times$. For an embedding $\iota : R \hookrightarrow B$ of O -algebras, we have $\mathrm{Tr}(\iota(a)) = \mathrm{Tr}_{M/F}(a)$ for the reduced trace Tr of B and $\iota(a)\iota(a)^\iota = N_{M/F}(a) = aa^c$. This shows $\iota(a^c) = \iota(a)^\iota$ for the main involution ι of B .

We work with $O^\times \Gamma_0(\mathfrak{N}; \mathfrak{h})$, and the contribution of $\mathfrak{N}'/\mathfrak{N}$ and ε_1 disappears as explained in Remark 4.3.

5.1. Theta series supported translates of $O_0(\mathfrak{N})$. We write G/O for the group scheme given by $G(A) = (O_B \otimes_O A)^\times$ for an O -algebra A . Write $U_0^B(\mathfrak{N}) = \widehat{O}_0(\mathfrak{N})^\times$. We fix a complete representative set $\mathcal{A} := \{a_1, \dots, a_h\}$ for

$$G(F) \backslash G(F_{\mathbb{A}}) / U_0^B(\mathfrak{N}) G(F_\infty) F_{\mathbb{A}}^\times = G(F) \backslash G(F_{\mathbb{A}}^{(\infty)}) / U_0^B(\mathfrak{N}) F_{\mathbb{A}(\infty)}^\times$$

with $a_{i,\mathfrak{N}_p} = a_{i,\infty} = 1$, and we pick a complete representative set $\mathcal{Z}$ for $Cl_F = F_{\mathbb{A}(\infty)}^\times / F^\times \widehat{O}^\times$. We regard $\mathcal{Z} \subset G(F_{\mathbb{A}}^{(\infty)})$. We consider, for $i, j = 1, 2, \dots, h$ and $z \in \mathcal{Z}$,

$$\begin{aligned} \Delta_{ijz}(\mathfrak{N}) &= a_i^{-\iota} z \cdot \Delta_0^B(\mathfrak{N}) a_j^\iota \cap B = a_i z \cdot \Delta_0^B(\mathfrak{N}) a_j^{-1} \cap B, \\ (5.1) \quad O_{ijz}(\mathfrak{N}) &= a_i^{-\iota} z \cdot \widehat{O}_0(\mathfrak{N}) a_j^\iota \cap B = a_i z \cdot \widehat{O}_0(\mathfrak{N}) a_j^{-1} \cap B \\ \Gamma_0^i(\mathfrak{N}) &= G^1(F) \cap a_i U_0^B(\mathfrak{N}) a_i^{-1} G(F_\infty), \end{aligned}$$

where $G^1(A) = \{g \in G(A) | gg^\iota = 1\}$. We write simply $O_i(\mathfrak{N})$ for $O_{ii}(\mathfrak{N})$. Thus $\Delta_{ijz}(\mathfrak{N}) \subset O_{ijz}(\mathfrak{N})$.

Since B is totally definite, the image $\overline{\Gamma}_0^i(\mathfrak{N})$ of $\Gamma_0^i(\mathfrak{N})$ modulo center is a finite group. We then put $e_i = |\overline{\Gamma}_0^i(\mathfrak{N})|$. Let $\phi \in H^0(Y_0^B(\mathfrak{N}), W(\varepsilon))$; so, we may regard $\phi : G(F_{\mathbb{A}}) \rightarrow W$ with $\phi(\gamma x u) = \varepsilon(u) \phi(x)$ for $u \in U_0^B(\mathfrak{N}) G(F_\infty) F_{\mathbb{A}}^\times$ and $\gamma \in G(F)$. Similarly, we choose $\phi' \in H^0(Y_0^B(\mathfrak{N}), W(\varepsilon^{-1}))$. We take $\langle \cdot, \cdot \rangle_W$ in (2.15) as $(\cdot, \cdot)$ in §4.1. Then

$$(\phi, \phi')_W := \langle \phi, \phi' \rangle_W = \sum_{i=1}^h e_i^{-1} \phi(a_i) \phi'(a_i) \in W.$$

As already remarked, we know $e_i^{-1} \in W$ by the unramifiedness of $F/\mathbb{Q}$.

Pick $y \in F_{\mathbb{A}}^\times$ with $y_p \mathfrak{N}_\infty = 1$. Supposing $\xi y d$ is integral, we consider $T(\xi y d)$ for $0 \ll \xi \in F$. By (unr), we assume $d_p = 1$. We choose a decomposition

$$U_0^B(\mathfrak{N}) \begin{pmatrix} \xi y d & 0 \\ 0 & 1 \end{pmatrix} U_0^B(\mathfrak{N}) = \bigsqcup_{\beta} \beta U_0^B(\mathfrak{N}).$$

Here we can choose β so that $\beta \beta^\iota = \xi y d$, because

$$U \backslash U \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} U / U = U_0^B(\mathfrak{N}) \backslash U_0^B(\mathfrak{N}) \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} U_0^B(\mathfrak{N}) / U_0^B(\mathfrak{N})$$

writing $U = \{u \in U_0^B(\mathfrak{N}) | uu^\iota = 1\}$. By the definition of the Hecke operator action in [HMI, page 167]

$$\phi | T(\xi y d)(x) = \sum_{\beta} \beta^{-1} \phi(x \beta).$$

In [HMI], the right coset representatives y is used; so, the formula is $\sum_{\beta^\iota} \beta \phi(x \beta^\iota)$ (and we have made a variable change $\beta \mapsto \beta^\iota$ to make the original right coset decomposition to the left one).

Since $a_i \beta \in \sqcup_{j,z} G(F) a_j z^{-1} \cdot U_0^B(\mathfrak{N})$ in $G(F_{\mathbb{A}}^{(\infty)})$, we can write $a_i \beta = \gamma_i a_j u_i z^{-1}$ for $\gamma_i \in \Delta_{ijz}(\mathfrak{N})$ and $u_i \in U_0^B(\mathfrak{N})$. Thus we have, writing $\mathfrak{a}_i = N_{B/F}(a_i) \widehat{O} \cap F$ and $\mathfrak{y} = y \widehat{O} \cap F$, $\mathfrak{a}_i \mathfrak{y} \mathfrak{d} \mathfrak{z}^2 \xi = N(\gamma_i^{-1}) \mathfrak{a}_j$, $\mathfrak{z} = z \widehat{O} \cap F$; in other words, $\mathfrak{y}^{-1} \mathfrak{d}^{-1} \mathfrak{a}_i^{-1} \mathfrak{a}_j \mathfrak{z}^{-2}$ is generated by a totally positive element $\alpha_{ijz} \in F$ prime to $p\mathfrak{N}$. Thus we have

$$\xi = \alpha_{ijz} \gamma_i \gamma_i^\iota \text{ up to totally positive units.}$$

Then we see, up to totally positive units,

$$\phi | \mathbb{T}(\xi y d)(a_i) = \sum_{\beta} \beta^{-1} \phi(a_i \beta) = \sum_{\gamma_i} \gamma_i^{-1} \phi(a_j).$$

Here, extending $\varepsilon : U_0(\mathfrak{N}) \rightarrow \overline{\mathbb{Q}}^\times$ to $\varepsilon : U_0(\mathfrak{N})F_{\mathbb{A}(\infty)}^\times \rightarrow \overline{\mathbb{Q}}^\times$ by ε_+ so that $\varepsilon(zu) = \varepsilon(u)\varepsilon_+(z)$ for $u \in U_0(\mathfrak{N})$ and $z \in F_{\mathbb{A}(\infty)}^\times$. Note that $\varepsilon_+ = \widehat{\varepsilon}_+$ for the p -adic avatar $\widehat{\varepsilon}_+ : F_{\mathbb{A}}^\times/F^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ of the central character ε_+ , since ε_+ has finite order by the assumption $\kappa = (I, 0)$. Then we have

$$\gamma_i^{-1}\phi(a_j) = \varepsilon(\gamma_i^{-1})\phi(a_j),$$

which is p -integral if $\phi(a_j)$ is in $W(\varepsilon)$.

Defining a pairing $[\cdot, \cdot] : W(\varepsilon) \times W(\varepsilon^{-1}) \rightarrow W$ by $[v, w] = vw$ (since $W(\varepsilon) = W(\varepsilon^{-1}) = W$ as W -modules) and

$$\Theta_{ijz}(v, w) = \frac{1}{e_i e_j} \sum_{\gamma \in \Delta_{ijz}(\mathfrak{N}) \cap \text{Supp}(\varepsilon)} [\gamma v, w] q^{\alpha_{ijz} \gamma \gamma^\iota}$$

for $v \in H^0(\Gamma_0^j(\mathfrak{N}), W(\varepsilon))$ and $w \in H^0(\Gamma_0^i(\mathfrak{N}), W(\varepsilon^{-1}))$ (and rewriting $\gamma_i^{-\iota}$ as γ), we find for $y \in F_{\mathbb{A}}^\times$ with $y_p = 1$

$$(5.2) \quad f(\phi \otimes \phi')(y) = N(y)^{-1} \sum_{i,j,z; \mathfrak{a}_i \mathfrak{a}_j^{-1} \mathfrak{z}^{-2} \sim \eta \mathfrak{d}} \Theta_{ijz}(\phi(a_j), \phi'(a_i)),$$

where $\mathfrak{a} \sim \mathfrak{b}$ indicates that the two ideals belong to the same strict class in F . Here Θ_{ijz} is a theta series of the O -lattice $O_{ijz}(\mathfrak{N})$ and is a Hilbert modular form of weight $\kappa \varepsilon$ on $\Gamma_0(\mathfrak{N}; \eta)$ for $\eta = F \cap y\widehat{O}$. Since $[\varsigma \gamma \zeta v, w] = [\gamma \zeta v, \varsigma w] = [\gamma v, w]$ for $\zeta \in \Gamma_0^j(\mathfrak{N})$ and $\varsigma \in \Gamma_0^i(\mathfrak{N})$, the theta series $\Theta_{ijz}(v, w)$ has integral q -expansion coefficients (except for the constant term) as $N(\gamma) = \gamma \gamma^\iota$ has $e_i e_j$ overlaps. Since the pairing: $[v, w] = vw$ is p -integral valued on $W(\varepsilon) \times W(\varepsilon^{-1})$ and α_{ijz} is prime to $p\mathfrak{N}$, the theta series has p -integral Fourier coefficients (except possibly for the constant term). The constant term does not show up if $\phi \in H_{n, \text{ord}}^0(Y_0^B(\mathfrak{N}), W(\varepsilon)) \subset H_{\text{cusp}}^0(Y_0^B(\mathfrak{N}), W(\varepsilon))$. Thus restricting ϕ to the ordinary part, $f(\phi \otimes \phi')$ has to be cuspidal (cf. [H88, Theorem 6.2]) and hence, the constant term vanishes. We may forget about the integrality problem stemming from the denominator: $e_i e_j$.

We choose an ideal $\mathfrak{A}$ of M with $N_{M/F}(\mathfrak{A}) \sim \eta$. We choose $\alpha \gg 0$ with $\eta^{-1} \mathfrak{d}^{-1} N_{M/F}(\mathfrak{A}) = (\alpha)$. Then we consider the theta series defined in (4.9):

$$\theta(\lambda; \mathfrak{A}) = \sum_{\gamma \in \mathfrak{A}^{-1}} \lambda(\gamma) q^{\alpha \gamma \gamma^c}$$

for a Hecke character λ of conductor $\mathfrak{C}$ with $\lambda(\alpha) = \alpha^\Sigma$ if $\alpha \equiv 1$ modulo conductor of λ . Strictly speaking, we need to divide the above series by $|\mu(R)|$ (see (4.9)), but $|\mu(R)|$ is prime to p by the unramifiedness of p in $M/\mathbb{Q}$. So we forget about $|\mu(R)|$. Here we have freedom of choosing $\mathfrak{A}$ in its ideal class (by changing $\alpha \gg 0$ suitably).

We define the reversed Petersson inner product $(f, g) = \langle g, f \rangle = \overline{\langle f, g \rangle}$ to make it linear with respect to the right variable g . By the variable change $z \mapsto -\bar{z}$, we have

$$(5.3) \quad (f, g) = \langle f_c, g_c \rangle \quad \text{for } f_c(z) = \overline{f(-\bar{z})}.$$

Under (h3), we have proven in [H07, Proposition 5.7] the following equality up to units in W :

$$(MT) \quad (h(M)/h(F))L_p(\widehat{\lambda}^-) = \frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(f(\lambda), f(\lambda))_{\mathfrak{N}}}{\Omega^{2\Sigma}} \in W,$$

where $W_p(\lambda) = \prod_{\mathfrak{p} \in \Sigma_p} W(\lambda_{\mathfrak{p}})$ and recalling a prime element $\varpi_{\mathfrak{p}}$ of $M_{\mathfrak{p}}$,

$$W(\lambda_{\mathfrak{p}}) = N(\mathfrak{P}^{-e(\mathfrak{P})})\lambda(\varpi_{\mathfrak{p}}^{-e(\mathfrak{P})}) \sum_{u \in (R/\mathfrak{P}^{e(\mathfrak{P})})^\times} \lambda_{\mathfrak{p}}(u) \mathbf{e}_M \left(\frac{u}{\varpi_{\mathfrak{p}}^{e(\mathfrak{P})}} \right)$$

if $e(\mathfrak{P}) > 0$ and $W(\lambda_{\mathfrak{p}}) = 1$ otherwise.

Recall the local component $\mathbb{T}$ of $\mathbf{h}$ in Corollary 2.5 through which ϕ_λ defined below Proposition 2.4 factors through. For $V := \mathbb{T} \cdot H_{n,ord}^0(Y_0^B(\mathfrak{N}), W(\varepsilon))$ and $V^* := \mathbb{T} \cdot H_{n,ord}^0(Y_0^B(\mathfrak{N}), W(\varepsilon^{-1}))$, we would like to show

$$(GL) \quad \frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(\theta(\lambda; \mathfrak{A}), \Theta_{ijz}(\phi(a_j), \phi'(a_j)))_\Gamma}{\Omega^{2\Sigma}} \in W$$

for $\Gamma = \Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y})$ and the optimal CM period Ω defined in (4.4), as long as $\phi \in V$ and $\phi' \in V^*$.

Remark 5.1. Recall $O_i(\mathfrak{N}) = O_{iiz}(\mathfrak{N})$ with $z = 1$ in (5.1) for a product $\mathfrak{N}$ of split primes in M/F . We choose an embedding $\iota : M \hookrightarrow B$. We may then realize B as $B = \left\{ \begin{pmatrix} a^c & b^c \\ b\eta & a \end{pmatrix} \mid a, b \in M \right\}$ with O_B containing $\begin{pmatrix} a^c & b^c \\ b\eta & a \end{pmatrix}$ if $a, b \in R$. We first normalize $\iota(a) = \begin{pmatrix} a^c & 0 \\ 0 & a \end{pmatrix} \in B$; thus, ι induces $R \hookrightarrow O_B$. We now renormalize locally ι to achieve $\iota(R) \subset O_1(\mathfrak{N})$. By [QAL, §30.6], this is possible if $d(M/F)$ is odd with $\mathfrak{C}$ prime to $d(M/F)$, and local modification can be patched together to get a global embedding $\iota : R \hookrightarrow O_1(\mathfrak{N})$ as long as this condition is satisfied.

By the above remark, for a while we assume also

$$(odd) \quad d(M/F) \text{ is odd.}$$

At the end, by a quadratic base-change, we reduce the general case with even $d(M/F)$ to the case of odd $d(M/F) \neq (1)$. This assumption, allowing the use of Lemma 2.1, makes our argument quite simple compared in the original argument in [H06].

We identify $M_{\mathbb{A}}^\times$ with the image in $G(F_{\mathbb{A}})$ under ι . Since the theta series θ for a fractional R -ideal of M/F is supported over

$$\mathrm{GL}_2(F)\{x \in \mathrm{GL}_2(F_{\mathbb{A}}) \mid \det(x) \in N_{M/F}(M_{\mathbb{A}}^\times)\}U_0(\mathfrak{M})$$

for a suitable level $\mathfrak{M}$ (see Remark 5.5), if M/F is everywhere unramified (at finite places), the support is a “half” of $\mathrm{GL}_2(F_{\mathbb{A}})$. Because of this, we separate our argument into two cases: the case with M/F ramified somewhere at finite places and another everywhere unramified.

Case with $d(M/F) \neq (1)$: For a while, we assume that M ramifies over F at some prime ideals of F . The Hilbert class field over F is linearly disjoint from M over F , and the norm map $N_{M/F} : Cl_M = M_{\mathbb{A}}^\times / M^\times \widehat{R}^\times \rightarrow F_{\mathbb{A}}^\times / F^\times \widehat{O}^\times F_{\infty+}^\times = Cl_F^+$ is surjective by class field theory, which induces a surjection $M_{\mathbb{A}}^\times / M^\times \widehat{R}^\times \rightarrow F_{\mathbb{A}}^\times / \widehat{O}^\times (F_{\mathbb{A}}^\times)^2$. This in turn induces the surjection we have a surjection:

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / U_0^B(\mathfrak{N}) G(F_\infty) F_{\mathbb{A}}^\times \twoheadrightarrow F_+^\times \backslash F_{\mathbb{A}}^\times / \widehat{O}^\times (F_{\mathbb{A}}^\times)^2 =: Cl_F^{++}.$$

Recall $\mathfrak{N}' = \mathfrak{N}(\varphi) = N_{M/F}(\mathfrak{C})d(M/F)$ given in the introduction. Then we find a finite set $\mathcal{B} := \{b_{j'}\}_{j'}$ in $(M_{\mathbb{A}}^{\mathfrak{N}'p\infty})^\times$ so that $N_{M/F}(b_{j'})$ gives a complete representative set for $F^\times \backslash F_{\mathbb{A}}^\times / \widehat{O}^\times(F_{\mathbb{A}}^\times)^2$. By ι , we regard $\mathcal{B} \subset G(F_{\mathbb{A}})$. By the reduced norm map: $N_{B/F} : G(F_{\mathbb{A}}) \rightarrow F_{\mathbb{A}}^\times$, we have a surjection:

$$\overline{N} : G(F) \backslash G(F_{\mathbb{A}}) / U_0^B(\mathfrak{N})G(F_\infty)F_{\mathbb{A}}^\times \twoheadrightarrow Cl_F^{++}.$$

For each $b_{j'}$, writing the class of $N_{M/F}(b_{j'}) \in Cl_F^{++}$ as $[b_{j'}]_F$, we decompose $\overline{N}^{-1}([b_{j'}]_F) = \bigsqcup_k G(F)b_{j'}s_{j',k}U_0^B(\mathfrak{N})F_{\mathbb{A}}^\times G(F_\infty)$ for finitely many $s_k = s_{j',k} \in G(F_{\mathbb{A}(\infty)})$.

Lemma 5.2. *Let the notation be as above. Assume $d(M/F) \neq (1)$, and choose a prime $\mathfrak{q}$ of F outside $\mathfrak{N}'p$. We can choose $\mathcal{S}$ inside $G(F_{\mathfrak{q}}) \subset G(F_{\mathbb{A}})$. Moreover we may assume $N_{B/F}(s) = 1$ for all $s \in \mathcal{S}$.*

Proof. Since B is unramified everywhere at finite places, we can apply the strong approximation theorem applied ignoring the factor at $\mathfrak{q}$ [ALT, Theorem 1.13], and hence, noting $N(G(F_{\mathfrak{q}})) = F_{\mathfrak{q}}^\times$, the reduced norm map induces an isomorphism

$$G(F) \backslash G(F_{\mathbb{A}}) / G(F_{\mathfrak{q}})UG(F_\infty) \cong F_{\mathbb{A}}^\times / F^\times N(U)F_{\infty+}^\times$$

for each open compact subgroup U of $G(F_{\mathbb{A}(\infty)})$. Since $N_{B/F}(U_0^B(\mathfrak{N})) = \widehat{O}^\times$, if $d(M/F) \neq (1)$, for each $x \in G(F_{\mathbb{A}})$, we have b for some j' such that $a \in \overline{N}^{-1}([b]_F)$. This shows that we may choose $s'_{j',k} \in G(F_{\mathfrak{q}})$. Since $N(G(F_{\mathfrak{q}})) = F_{\mathfrak{q}}^\times$ and $N_{B/F}(G(F)) = F^\times \cap F_{\infty+}^\times$, we may choose $N_{B/F}(s_{j',k}) = 1$. $\square$

Corollary 5.3. *Choosing $b_{j'} \in G(F_{\mathbb{A}}^{(\mathfrak{q}\infty)})$, we can choose $\{s_{j',k}\}_k$ independent of j' inside $G^1(F_{\mathfrak{q}})$ for $G^1 := \text{Ker}(G/F \xrightarrow{N_{B/F}} \mathbb{G}_{m/F})$.*

Proof. The conjugation $x \mapsto b_{j'}^{-1}xb_j$ preserves the derived group G^1 of G and the identity map at $\mathfrak{q}$, $\{s_{j',k}\}_k$ concentrating at $\mathfrak{q}$ can be chosen independently of j' . $\square$

By the corollary, we drop the index j' from $s_{j',k}$ and write it simply as $\mathcal{S} := \{s_k\}_k$. Since $b \in \mathcal{B}$ and $s \in \mathcal{S}$ have disjoint support over primes of F , they commute. Thus we can choose a complete representative set $\mathcal{A} = \{a_j\}_j$ for $G(F) \backslash G(F_{\mathbb{A}(\infty)}) / U_0^B(\mathfrak{N})F_{\mathbb{A}(\infty)}^\times$ so that $\mathcal{A} = \mathcal{B} \cdot \mathcal{S} = \mathcal{S} \cdot \mathcal{B}$ with $\mathcal{A} = \{b_{i'}s_k = s_k b_{i'}\}_{i',k}$ for the finite set $\mathcal{S} := \{s_k\}_k$, where we regard $b_j \in \widehat{O}_1(\mathfrak{N}) \cap G(F_{\mathbb{A}})$ by ι_1 . We assume that $b_1 = 1$ and $s_1 = 1$ and write $a_j = s_{j''}b_{j'}$ (i.e., $i \leftrightarrow (i', i'')$ and $j \leftrightarrow (j', j'')$).

Again by the strong approximation theorem, we can write $s_k = \gamma_k u_k$ with $u_k \in U_0^B(\mathfrak{N})G(F_{\mathfrak{q}})$ and $\gamma_k \in G^1(F)$. Since b_j commutes with $\iota_1(R_1)$, conjugation by b_j does not alter ι_1 . Then defining $\iota_j : M \rightarrow B$ by $\gamma_{j''}\iota_1\gamma_{j''}^{-1}$ and putting R_j to be the inverse image under ι_j of $\iota_j(M) \cap O_j(\mathfrak{N})$, we find $R_j = R$, because $u_{j''}^{-1}\widehat{O}_1(\mathfrak{N})u_{j''} = \widehat{O}_1(\mathfrak{N})$ and $b_{j'}^{-1}\iota_1(R)b_{j'} = \iota_1(R)$. Moreover, since $\gamma_{j''} \in G(F)$, recalling the lattice $O_{ijz}(\mathfrak{N})$ of B from (5.1), we have, for each $z \in \mathcal{Z}$,

$$(5.4) \quad \iota_i(\xi)O_{ijz}(\mathfrak{N})\iota_j(\eta) = O_{ijz}(\mathfrak{N}) \text{ for all } \xi, \eta \in R \text{ and } i, j = 1, \dots, h,$$

since $z \in \mathcal{Z}$ is in the center of $G(F_{\mathbb{A}}^{(\infty)})$.

Case with $d(M/F) = (1)$: In this case, the image of $M_{\mathbb{A}}^\times$ in the class group Cl_F^{++} under the norm map is of index two with cokernel isomorphic to $\text{Gal}(M/F)$ by class

field theory; so, we need to expand $\mathcal{B}$: Choose the prime $\mathfrak{q}$ similarly to Lemma 5.2 inert unramified in M/F outside $\mathfrak{N}'p$. Pick an element $b' \in G(F_{\mathfrak{q}})$ with $N_{M/F}(b')$ generating $\mathfrak{q}O_{\mathfrak{q}}$ (so, $b' \notin N_{M/F}(M_{\mathfrak{q}})$), and set $\tilde{\mathcal{B}} = \mathcal{B} \sqcup b'\mathcal{B} = \{b_{i'}, b'b_{i'}\}_{i'}$. Here we still choose $\mathcal{B}$ inside $G(F_{\mathbb{A}}^{(\mathfrak{q}\infty)})$ so that elements in $\mathcal{B}$ commute under multiplication. Then the representatives $\mathcal{A}$ can be chosen as $\mathcal{S} \cdot \tilde{\mathcal{B}}$. In this case, $\sigma \in \mathcal{S}$ commutes with $b \in \mathcal{B}$ but possibly not with $b'b \in b'\mathcal{B}$. Thus, by the same argument as above, we find again an O -order R_j of $\mathfrak{q}$ -power conductor and an embedding $\iota_j : R_j \hookrightarrow O_j(\mathfrak{N})$. Thus we obtain

Lemma 5.4. *Let the notation be as above. Assume $d(M/F) = (1)$, and choose a prime $\mathfrak{q}$ of F outside $\mathfrak{N}'p$. Then we can choose $\mathcal{A}$ so that $\mathcal{A} = \mathcal{S} \cdot \tilde{\mathcal{B}}$. We can choose $\mathcal{S}$ inside $G^1(F_{\mathfrak{q}}) \subset G(F_{\mathbb{A}})$.*

Remark 5.5. *Continue to suppose $d(M/F) = (1)$. In this case, writing $\mathfrak{A} = A\hat{O}_M \cap M$ for $A \in M_{\mathbb{A}}^{\times}$ with $A_{\infty} = 1$, by definition, the lift of $\theta(\lambda; \mathfrak{A})$ to $\mathrm{GL}_2(F_{\mathbb{A}})$ is supported on $\mathrm{GL}_2(F) \mathrm{diag}[N_{M/F}(M_{\mathbb{A}}^{\times}), 1]U_0(\mathfrak{N})F_{\mathbb{A}}^{\times}\mathrm{GL}_2^{+}(F_{\infty})$ [CAL, Definition 4.59], we can ignore the components of the form:*

$$\mathrm{GL}_2(F) \mathrm{diag}[N_{M/F}(b_j b'), 1]U_0(\mathfrak{N})F_{\mathbb{A}}^{\times}\mathrm{GL}_2^{+}(F_{\infty}).$$

Thus limiting ourselves to the open sunset:

$$\begin{aligned} \bigcup_{A \in M_{\mathbb{A}}^{\times}} \mathrm{GL}_2(F) \mathrm{diag}[A, 1]U_0(\mathfrak{N})F_{\mathbb{A}}^{\times}\mathrm{GL}_2^{+}(F_{\infty}) \\ = \bigcup_{j'} \mathrm{GL}_2(F) \mathrm{diag}[N_{M/F}(b_{j'}), 1]U_0(\mathfrak{N})F_{\mathbb{A}}^{\times}\mathrm{GL}_2^{+}(F_{\infty}), \end{aligned}$$

we may ignore $b'\mathcal{B}_0$. Therefore the situation is the same as in the case of $d(M/F) \neq (1)$, and hereafter, we write $\mathcal{A} := \mathcal{S} \cdot \mathcal{B}$ which solely contributes to the theta series. We rewrite $\mathcal{A} = \{a_1, \dots, a_{h_0}\}$ for $h_0 = h/2$.

We have now proven under the notation in Remark 5.5: for each $z \in \mathcal{Z}$

$$(5.5) \quad \iota_i(\xi)O_{ijz}(\mathfrak{N})\iota_j(\eta) = O_{ijz}(\mathfrak{N}) \quad \text{for all } \xi, \eta \in R \text{ and } i, j = 1, \dots, h_0 = h/2.$$

There is no need to choose a single prime $\mathfrak{q}$ as a support of elements of $\mathcal{S}$. What we want is to show that we can choose the support of $\mathcal{S}$ away from any finite set of primes. Since $\mathcal{B}$ only matters by Remark 5.5 if $d(M/F) = (1)$, we ignore $b'\mathcal{B}$ and only work with $\mathcal{B}$ when $d(M/F) = (1)$. Under this simplification, each $b \in \mathcal{B}$ commutes with every $s \in \mathcal{S}$.

We regard $L_{ijz} = O_{ijz}(\mathfrak{N})$ as $R \otimes_O R$ module by $l = \iota_i$ and $r = \iota_j$: $(\alpha, \beta)b = \iota_i(\alpha)b\iota_j(\beta^c)$. Since our modification takes place only at the prime $\mathfrak{q}$ outside $\mathfrak{N}'p$, we may assume that r and l at primes $\mathfrak{l} \nmid \mathfrak{N}'p$ are intact as in (2.3) and (2.6). Since $M \otimes_F M = M \oplus M$, writing e_1 and e_2 the idempotent of left and right factors, we split $O_{ijz}(\mathfrak{N}) = L_{ijz}^1 \oplus L_{ijz}^2$ for $L_{ijz}^? = e_{?}L_{ijz}$ by Lemma 2.1 under the conditions (h2) and (odd). Then as studied in §4.3, we can write Θ_{ijz} of level $\mathfrak{N}$ as a p -integral linear combination of $\theta(\phi_1)\theta(\phi_2)$ of theta series of L_{ijz}^1 and L_{ijz}^2 , respectively. The functions $\phi_{?}$ ($? = 1, 2$) can be chosen to be p -integral.

5.2. $\theta(\phi_?)$ has optimal level. We now bound the level of $\theta(\phi_?)$. Recall $\mathcal{B} \subset G(F_{\mathbb{A}}^{(\mathfrak{N}'\mathfrak{q}p^\infty)})$ and $\mathcal{S} \subset G^1(F_{\mathfrak{q}})$ for a prime $\mathfrak{q} \nmid \mathfrak{N}'p$. As before, we write $a_i = sb_{i'}$ and $a_j = s'b_{j'}$ for $s, s' \in \mathcal{S}$. Note that $b_{i'} \cdot O_0(\mathfrak{N})b_{j'}^{-1} = \mathfrak{b}_{i'} \otimes \mathfrak{b}_{j'}^{-1}$ as $R \otimes_O R$ -modules for $\mathfrak{b}_{i'} = (b_{i'}\widehat{R} \cap M)$, we find from the discussion at the end of the previous section that $L_{ijz}^1 = \mathfrak{z}\mathfrak{b}_{i'}^c\mathfrak{b}_{j'}^{-1}$ and $L_{ijz}^2 = \mathfrak{z}\mathfrak{b}_{i'}^c\mathfrak{b}_{j'}^{-c}$, where $\mathfrak{z} = F \cap z\widehat{O}$. Thus we find that $\eta\mathfrak{d} = \mathfrak{a}_i^{-1}\mathfrak{a}_j = N_{M/F}(L_{ijz}^1) = N_{M/F}(L_{ijz}^2)$.

As explained in the introduction, we take φ with $\psi = \varphi^-$. We then take a weight κ member $f(\lambda)$ of the p -adic family (associated with φ : $\hat{\lambda}|_{G_{\text{tor}}(\mathfrak{C})} = \varphi$) with complex multiplication by M . We may assume that the weight κ of $f(\lambda)$ is $(I, 0)$. Then $(\mathfrak{N}, \mathfrak{N}')$ is also the minimal level and the conductor of $f(\lambda)$ outside p . By (h3) and the choice of Σ_p , ε is non-trivial at all $\mathfrak{P} \in \Sigma_p$. Replacing φ by $\varphi\eta$ for a finite order character $\eta : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow W^\times$ does not alter the anticyclotomic part φ^- . By a theorem of Chevalley ([C51]), we can choose η so that $\eta_{\mathfrak{l}} = \lambda_{\mathfrak{l}}^{-1}$ on the inertia group at $\mathfrak{l}$ for every prime $\mathfrak{l}$ in any given finite set of prime ideals. Thus as indicated in the introduction, we may assume

$$(5.6) \quad \lambda \text{ has conductor prime to } \Sigma_p^c.$$

Recall $\mathfrak{N} = \mathfrak{N}(\lambda) = \mathfrak{N}(\varphi)$ and $\mathfrak{f} = \mathfrak{F}(\lambda) \cap F = \mathfrak{F}(\varphi) \cap F$. Under this assumption (5.6), $\kappa = \kappa^*$, $\varepsilon^* = \varepsilon^{-1}$ and $[v, w] = vw$ by identifying $L(\kappa\varepsilon; W) = W$ (on which $\Delta_0(\mathfrak{N})$ acts via multiplication by ε) and $L(\kappa^*\varepsilon^*; W) = W$. Then

$$[\gamma\phi(a_j), \phi'(a_i)] = \varepsilon(\gamma_{\mathfrak{f}p})\phi(a_j)\phi'(a_i).$$

Regarding the character $\varepsilon : \Delta_0^B(\mathfrak{N})_{\mathfrak{f}p} \rightarrow W^\times$ as a function ε_{ijz} of $B \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$ supported on $\widehat{\Delta}_{ijz}(N) = a_i^{-1}z\Delta_0^B(\mathfrak{N})a_j$ by $\varepsilon_{ijz}(x) = \varepsilon(x_{\mathfrak{f}p})$ ($\widehat{\Delta}_{ijz}(N)_{\mathfrak{f}p} = \Delta_0^B(N)_{\mathfrak{f}p}$), the function $\chi_{ijz} : \gamma \mapsto \varepsilon(\gamma)[\phi(a_j), \phi'(a_i)]$ is the function ε_{ijz} multiplied by the p -integral constant: $\phi(a_j)\phi'(a_i)$. Write down χ_{ijz} as a sum $\chi_{ijz} = \sum_{\phi_1, \phi_2} \phi_1 \otimes \phi_2$ for finitely many p -integral locally constant functions $\phi_1 : L_{ijz}^1 \rightarrow W$ and $\phi_2 : L_{ijz}^2 \rightarrow W$. By Proposition 2.3, $\phi_{2,p}(x_p) = \lambda_{\Sigma_p}(x_{\Sigma_p})$ on $R_{\Sigma_p^c} \times R_{\Sigma_p}^\times$ and is supported by $(R_{\Sigma_p^c} \times R_{\Sigma_p}^\times) \subset L_{ij,p}^2$ (and $\phi_{1,p}$ is the characteristic function of $L_{ij,p}^1 = R_{\Sigma_p^c} \times \mathfrak{p}^e R_{\Sigma_p}$).

Recall $\Gamma_0^{i,B}(\mathfrak{N}) = G^1(F) \cap a_i U_0^B(\mathfrak{N})a_i^{-1}$ and $\Gamma_0^i(\mathfrak{N}) = \text{SL}_2(F) \cap a_i U_0(\mathfrak{N})a_i^{-1}$. Then as seen in Remark 4.3, we have $\varepsilon|_{O^\times \Gamma_0^{i,B}(\mathfrak{N})} = \varepsilon|_{O^\times \Gamma_0^i(\mathfrak{N})} = (\mathbf{1}, \varepsilon_2, \varepsilon_+)$ (i.e., ε_1 disappears and $\varepsilon^- = \varepsilon_2$). Applying Proposition 2.3 to $\varepsilon \otimes \varepsilon_1^{-1} := (\mathbf{1}, \varepsilon_2, \varepsilon_+)$ as in Remark 4.3, we find that $\phi_?$ ($? = 1, 2$) supported on $L_{ijz}^?$. Thus $\theta(\phi_2)$ (resp. $\theta(\phi_1)$) is automorphic with respect to the congruence subgroup $O^\times \Gamma_0^i(\mathfrak{N}) = O^\times \Gamma_0^i(\mathfrak{N}; \eta)$ for suitable η with Neben character set $\varepsilon \otimes \varepsilon_1^{-1}$ (resp. $(\mathbf{1}, \left(\frac{M/F}{\cdot}\right), \left(\frac{M/F}{\cdot}\right))$) on $O^\times \Gamma_0^i(\mathfrak{N})$. Then we need to take trace map from $O^\times \Gamma_0^i(\mathfrak{N})$ to $\widetilde{\Gamma}_0^i(\mathfrak{N}) = \text{GL}_2^+(F) \cap a_i U_0(\mathfrak{N})a_i^{-1}$ with Neben character set ε . This trace operation is p -integral (as its denominator is a 2-power as remarked in Remark 4.3), we ignore it and compute Petersson inner product with respect to $O^\times \Gamma_0^i(\mathfrak{N}) \backslash \mathfrak{H}^I = \Gamma_0^i(\mathfrak{N}) \backslash \mathfrak{H}^I$. This follows from the facts $L_{ijz} = L_{ijz}^1 \oplus L_{ijz}^2$, $L_{ijz}^1 = \mathfrak{z}\mathfrak{b}_{j'}^c\mathfrak{b}_{i'}^{-c}$, $L_{ijz}^2 = \mathfrak{z}\mathfrak{b}_{j'}\mathfrak{b}_{i'}^{-1}$ under the left R -module structure of $L_{ijz}^?$ in (2.4) and $\theta(\phi_?) = \sum_{\xi \in M} \phi_?(\xi) q^{\alpha_{ijz} \xi \xi^c}$ has the level as described above.

As seen in (4.9), the η -component of $f(\lambda)$ is given by a p -integral finite sum $|\mu(R)|^{-1} \sum_{\mathfrak{A}} \lambda(\mathfrak{A}) \theta(\lambda; \mathfrak{A})$ of theta series of the form:

$$\theta(\lambda; \mathfrak{A}) = \sum_{\xi \in \mathfrak{A}^{-1}} \lambda(\xi^{(\Xi\infty)}) q^{\alpha \xi \xi^c},$$

where $\mathfrak{A}\mathfrak{A}^c = \alpha\eta\mathfrak{d}$ (with $\alpha \gg 0$ in F). Here the sum $\sum_{\mathfrak{A}} \lambda(\mathfrak{A}) \theta(\lambda; \mathfrak{A})$ is over ideal classes of M whose norm is equivalent to $\eta\mathfrak{d}$. By choosing $v \in M_2(F)$ and $(z_0, w_0) \in \mathfrak{H}^I \times \mathfrak{H}^I$ as in Section 3.4, we identify $M_2(F)$ with $M \oplus M$. Then we choose $\mathcal{L} = \mathfrak{A}^{-1} \oplus L_{ijz}^2$ as an O -lattice of $M_2(F)$. Since we have freedom of changing $\mathfrak{A}$ in its ideal class, we may assume that the p -adic completion $\mathcal{L}_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is equal to $M_2(O_p)$ in $M_2(F_p) = B_p$, because $L_{ijz}^2 = O_p \oplus O_p$. Then $\mathcal{L}^1 = \mathfrak{A}^{-1}$ and $\mathcal{L}^2 = L_{ijz}^2$. We take $\phi'_1 : \mathcal{L}^1 \rightarrow W$ so that $\theta(\phi'_1) = \theta(\lambda; \mathfrak{A})$. Then $\phi'_1(\xi) = \lambda(\xi^{(\Xi\infty)})$ and $\phi'_{1, \Sigma_p} = \lambda_{\Sigma_p}^{-1}(\xi_{\Sigma_p})$, and ϕ'_{1, Σ_p^c} is the characteristic function of $R_{\Sigma_p^c}$.

(C) We choose two ideals $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of M and $v \in M_2(F)$ very *close* p -adically (resp. $d(M/F)$ -adically) to $\mathbf{b} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(F_{p\mathfrak{f}})$ in (2.5) (resp. $\mathbf{b} = \begin{pmatrix} -1 & 1 \\ d_q & 1 \end{pmatrix} \in M_2(F_{d(M/F)})$), taking $\mathfrak{q}$ in (2.5) to be $\mathfrak{q}$ in Lemma ?? splitting in M/F and Remark 4.2 so that $L = l(\mathfrak{B}_1)v \cdot r(\mathfrak{B}_2) \subset \mathcal{L}$ with $\mathcal{L}/L$ killed by a power of $d(M/F)\mathfrak{q}$.

Now we need to introduce another prime $\mathfrak{q}$ outside p and any given finite set of primes, because $\mathcal{L}^2$ is stable only by an O -order in M of $\mathfrak{q}$ -power conductor. We choose the base $\mathfrak{B}_1 = Oz_1 + \eta z_2$ and $\mathfrak{B}_2 = Ow_1 + \eta w_2$ again as in Remark 4.2. Let $z = (z_1, z_2)$ and $w = (w_1, w_2)$. Thus $z_0 = z_1/z_2$ and $w_0 = w_1/w_2$ are both in $\mathfrak{H}^I$. Thus we have from Theorem 3.3 and (5.3) that:

$$(\theta(\phi'_1), \theta(\phi_1)\theta(\phi_2))_{\Gamma} = \langle \theta(\phi'_1)_c, \theta(\phi_1 \otimes \phi_2)_c \rangle_{\Gamma} = C\Psi(z, w)$$

for a constant $C \in W^{\times}$ and a congruence subgroup

$$(5.7) \quad \Gamma = \Gamma_0(\mathfrak{N}(\lambda); \eta).$$

Here $\Psi(z', w')$ with $(z', w') \in \mathfrak{H}^I \times \mathfrak{H}^I$ is the homogeneous version of the modular form in (3.31):

$$(5.8) \quad \Psi(z', w') = \sum_{0 \ll \alpha \in \Gamma \backslash M_2(F)} \phi^{*(\infty)}(\epsilon\alpha) \mathbf{e}_F(\det(\alpha)z') \theta(\phi_1)|_1 \alpha(w')$$

for the partial Fourier transform ϕ^* of $\phi = \overline{\phi'_1} \circ c \otimes \phi_2$, because $\theta(\phi)_c(z) = \overline{\theta(\phi)(-\overline{z})} = \theta(\overline{\phi} \circ c)$. The constant C is prime to p (that is, $i_p(C) \in W^{\times}$) because of the following reason: Since Ψ is of weight (I, I) , the homogeneous form is given by $z_2^I w_2^I \Psi(z, w) = \Psi(z_0, w_0)$. Since v is very close p -adically to $\mathbf{b}$ in (C), we may assume that $v = \mathbf{b}$. Then by Theorem 3.3 and $[\mathbf{b}; z, w] = (z_1 - z_2)(w_2 - w_1)$, we have

$$\begin{aligned} C &= z_2^I w_2^I \operatorname{Im}(z_0)^{-I} \operatorname{Im}(w_0)^{-I} [v; z_0, w_0]^I |[v; z_0, w_0]|^{2I} \\ &= \frac{(z_1 - z_2)(w_2 - w_1) |(z_1 - z_2)(w_2 - w_1)|^2}{(z_1 \overline{z_2} - \overline{z_1} z_2)(w_1 \overline{w_2} - \overline{w_1} w_2)}, \end{aligned}$$

whose image under i_p is in $W^{\times}$ (by our choice of $v = \mathbf{b}$ in (C) of the base z and w as in Remark 4.2).

The local partial Fourier transform preserves p -integral Schwartz-Bruhat functions on $M_2(F_l)$ as long as $l \nmid p$. Since $M_p = M_{\Sigma_p} \oplus M_{\Sigma_p^c}$, we find

$$M_2(F_p) = M_p \oplus M_p = \begin{pmatrix} M_{\Sigma_p^c} & M_{\Sigma_p^c} \\ M_{\Sigma_p} & M_{\Sigma_p} \end{pmatrix}.$$

The first column is the factor M_p carrying $\overline{\phi}'_{1,p} \circ c$. The function $\overline{\phi}'_{1,p} \circ c$ is supported on R_p . Since complex conjugation interchanges a and c (see Proposition 2.3), we see from (4.9) that $(\overline{\phi}'_{1,p} \circ c) \begin{pmatrix} a \\ c \end{pmatrix} = \lambda_{\Sigma_p}(a)$ (because we have assumed that λ has conductor prime to $\Sigma_p c$: (5.6)). Similarly, $\phi_{2,p} \begin{pmatrix} b \\ d \end{pmatrix} = \lambda_{\Sigma_p}(d)$ for $b \in R_{\Sigma_p^c}$ and $d \in R_{\Sigma_p}$. Thus $\phi_p(a, b)$ equals to $\chi(a, b)\lambda_{\Sigma_p}(a)$ for the characteristic function χ of $R_p = O_p^\times \times O_p$. The partial Fourier transform is with respect to the variables “ $(a^{(p)}, b^{(p)})$ ” keeps p -integrality by the Fourier inversion formula. Thus we may concentrate on the p -component. Define for each $\mathfrak{P} \in \Sigma$, $\Phi_\lambda(x)$ to be equal to $\lambda_\mathfrak{P}(x)$ if $x \in R_\mathfrak{P}^\times$ and 0 outside $R_\mathfrak{P}^\times$. Then the Fourier transform of Φ_λ is given by $W(\lambda_\mathfrak{P})\lambda_\mathfrak{P}(\varpi_\mathfrak{P}^{e(\mathfrak{P})})\Phi_{\overline{\lambda}}(\varpi_\mathfrak{P}^{e(\mathfrak{P})}x)$ (see [BNT, Proposition 13 in VII.7]). Thus we need to prove $W(\lambda_\mathfrak{P}^-)W(\lambda_\mathfrak{P})\lambda(\varpi_\mathfrak{P}^{e(\mathfrak{P})}) \in W$. This can be done as follows: writing $e = e(\mathfrak{P})$ and $\varpi = \varpi_\mathfrak{P}$,

$$(5.9) \quad \begin{aligned} W(\lambda_\mathfrak{P}^-)\lambda(\varpi^e)W(\lambda_\mathfrak{P}) &= N(\mathfrak{P}^{-2e})\lambda(\varpi^e)\lambda(\varpi^{-ce})G(\lambda_\mathfrak{P}^{-1})G(\lambda_\mathfrak{P}) \\ &= \lambda_\mathfrak{P}(-1)\lambda(\varpi^{-ce})\lambda(\varpi^e)N(\mathfrak{P}^{-e}), \end{aligned}$$

where $G(\chi) = \sum_{u \in R/\mathfrak{P}^e} \chi(u)\mathbf{e}_M(\frac{u}{\varpi^e})$ for the conductor $\mathfrak{P}^e$ of χ . Note here that the infinity type of λ is $-\Sigma$, and hence $\lambda(\varpi^e)$ is up to unit equal to $\varpi^{e\Sigma}$ which is equal to $N(\mathfrak{P}^e)$ up to units in W . This shows the desired integrality.

Since the partial Fourier transform with respect to the character $\mathbf{e}_\mathbb{A}(ab' - ba')$ interchanges (a, b) , the support of $\overline{\phi}'_p$ is contained in

$$\begin{pmatrix} O_p & \varpi^{-e}O_p^\times \\ O_p & O_p \end{pmatrix} = \tau^{-1} \begin{pmatrix} O_p & O_p \\ \mathfrak{p}^e O_p & O_p^\times \end{pmatrix} \subset M_2(F_p),$$

where $\tau = \begin{pmatrix} 0 & -1 \\ \varpi^e & 0 \end{pmatrix}$.

The function $\phi_{1,p}$ is the characteristic function of $R_{\Sigma_p^c} \times \mathfrak{p}^e R_{\Sigma_p}^\times$. Since τ normalizes $U_0(\mathfrak{p}^e)_p$, we can choose complete representative set $\mathcal{R}$ for

$$U_0(\mathfrak{p}^e)_p \setminus \left(\begin{pmatrix} O_p & O_p \\ \mathfrak{p}^e O_p & O_p^\times \end{pmatrix} \times GL_2(F_{\mathbb{A}(p^\infty)}) \right)$$

such that $\alpha \in \mathcal{R}$ can be written as $\tau^{-1}\beta$ with p -component β_p is upper triangular (e.g. [MFG, 3.1.6]) with p -adic unit at the lower bottom corner. The Hecke operator UxU for $x \in \mathcal{R}$ preserves the p -integral structure of $S_{(I,0)}(\Gamma; W)$ (the space of cusp forms on Γ with W -integral Fourier coefficients). This fact follows, for example, [H88, Theorem 4.11], and actually, if $x \in GL_2(F)$ has upper triangular p -component with p -adic unit at the lower bottom corner, the action of $\theta \mapsto \theta|_1 x$ on modular forms preserves p -integrality since it is basically given by $\theta(z) \mapsto \theta(az)$ for totally positive a . Thus the action of β : $\theta(\phi_1)|_1 \tau^{-1} \mapsto \theta(\phi_1)|_1 \tau^{-1}\beta$ in (5.8) preserves the p -integrality [H88, Theorem 4.9], and $\theta(\phi_1)|_1 \tau^{-1}\beta$ has p -integral q -expansion with respect to the variable w if $\theta(\phi_1)|_1 \tau^{-1}$ is p -integral. Thus we need to prove that $\theta(\phi)|_1 \tau^{-1}$ has p -integral q -expansion coefficients, in order to show $\Psi(z, w)$ in (5.8) has p -integral q -expansion. Since $\theta(\phi)|_{\tau'}$ for $\tau' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is given

by $\theta(\widehat{\phi}_1)$ for the Fourier transform $\widehat{\phi}_1$ of ϕ_1 regarding it as a function on $M_{\mathbb{A}}$. The p -integrality only depends on the p -part $\phi_{1,p}$ of ϕ_1 . By computation, $\widehat{\phi}_{1,p}$ is $N(\mathfrak{p}^{-e})$ times the characteristic function of $R_{\Sigma_p^c} \times \mathfrak{p}^{-e} R_{\Sigma_p}$. Taking ϖ^{-e} in O , we find that $\theta(\phi_1)|\tau^{-1}$ is equal to $\theta(\widehat{\phi}_1)|_1 \begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-e} \end{pmatrix} (w) = \varpi^e \theta(\widehat{\phi}_1)(\varpi^e w)$ up to a p -adic unit. Since $\varpi^e N(\mathfrak{p}^{-e})$ is a p -adic unit, we get the desired integrality.

Write $(\theta(\lambda; \mathfrak{A}), f^{\mathfrak{y}})_{\Gamma_0(\mathfrak{N}, \mathfrak{y})}$ for the inner product $(\theta(\lambda; \mathfrak{A}), f^{\mathfrak{y}})$ for the $\mathfrak{y}$ -component $f^{\mathfrak{y}}$ of $f \in S_{(I,0)}(\mathfrak{N}, \varepsilon; W)$ with respect to $\Gamma_0(\mathfrak{N}; \mathfrak{y})$. By the q -expansion principle, we conclude from (4.6)

$$(5.10) \quad \frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(\theta(\lambda; \mathfrak{A}), \theta(\phi_1)\theta(\phi_2))_{\Gamma_0(\mathfrak{N}, \mathfrak{y})}}{\Omega^{2\Sigma}} \in W$$

for Γ in (5.7). This shows

$$\frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(\theta(\lambda; \mathfrak{A}), \Theta_{ij}^{\mathfrak{y}})_{\Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y})}}{\Omega^{2\Sigma}} \in W.$$

Since any $f \in S_{(I,0)}(\mathfrak{N}(\lambda), \varepsilon; W)$ is a W -linear combination of Θ_{ij} by Corollary 2.5, we conclude for all $f \in S_{(I,0)}(\mathfrak{N}(\lambda), \varepsilon; W)$

$$(5.11) \quad \frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(\theta(\lambda; \mathfrak{A}), f^{\mathfrak{y}})_{\Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y})}}{\Omega^{2\Sigma}} \subset W.$$

with $f(\lambda)^{\mathfrak{y}} = \theta(\lambda; \mathfrak{A})$.

we get, for all $f \in S_{(I,0)}(\mathfrak{N}(\lambda), \varepsilon; W)$,

$$(5.12) \quad \frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(\theta(\lambda; \mathfrak{A}), f^{\mathfrak{y}})_{\Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y})}}{\Omega^{2\Sigma}} \subset W.$$

Summing over ideal classes $\mathfrak{A}$ and $\mathfrak{y}$ in tandem, by Remark 5.6, we find

$$(5.13) \quad \frac{(2\pi i)^{2\Sigma} W_p(\lambda^-)(f(\lambda), S_{(I,0)}(\mathfrak{N}(\lambda), \varepsilon; W))_{\mathfrak{N}(\lambda)}}{\Omega^{2\Sigma}} \subset W.$$

Thus we conclude

$$H(\varphi) \Big| \frac{h(M)}{h(F)} L^-(\varphi)$$

as we explained in the introduction. Here $H(\varphi)$ is the congruence power series with respect to the nearly ordinary Hecke algebra $\mathbf{h}(\mathfrak{N}(\varphi), \varepsilon_{\varphi}; W)$ interpolating $h_{(I,0)}^{n,ord}(\mathfrak{N}(\lambda), \varepsilon_{\lambda}; W)$ (for all $\kappa \varepsilon \in \mathcal{A}(\mathbb{I})$). Thus $H(\varphi)$ divides the congruence power series H in [HT93] but could be smaller if $\mathfrak{C} \cap \mathfrak{C}^c$ contains non-trivial prime factor. In [HT93], we had an extra factor $\Delta(M/F; \mathfrak{C})$ which is equal to the product of the Euler factors of $L(s, \alpha) L(s, \varphi^{-1} \varphi_c)$ for primes outside p in $\mathfrak{C} \cap \mathfrak{C}^c$. This comes from the formula of the inner product of $\theta(\lambda)$ in [HT93] Theorem 7.1. After doing the same computation for $f(\lambda)$ of smaller level instead of $\theta(\lambda)$ and writing $k = \kappa_1 - \kappa_2 + I$ (see [H07] (5.5)), writing $(\cdot, \cdot)_{\mathfrak{N}}$ for the Petersson inner product of level $\mathfrak{N}$ defined in [H91, p.342–343] and D for the discriminant of $F/\mathbb{Q}$, we get the exact formula, if λ^- has split conductor:

$$(5.14) \quad (f(\lambda), f(\lambda))_{\mathfrak{N}(\lambda)} \doteq (f(\lambda)^u, f(\lambda)^u)_{\mathfrak{N}(\lambda)} \doteq \pi^{-(k+I)} \Gamma_F(k+I) L(1, \text{Ad}(f(\lambda)))$$

under the terminology of [HT93] Section 7 without any error terms, where “ $\doteq$ ” means up to p -adic units.

Remark 5.6. *The Petersson inner product $(f, g)_{\mathfrak{M}}$ in [H91, p.342–343] is an integration over $X_0(\mathfrak{M}) := \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(F_{\mathbb{A}}) / U_0(\mathfrak{M}) F_{\mathbb{A}}^{\times} C_{\infty}$ for the stabilizer $C_{\infty} = F_{\infty}^{\times} \mathrm{SO}_2(F_{\infty})$ under the invariant measure of volume 1 over $U_0(\mathfrak{M})$, while we computed the product $(f, g)_{\mathfrak{M}}'$ over the covering space $X'_0(\mathfrak{M}) := \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(F_{\mathbb{A}}) / U_0(\mathfrak{M}) F_{\infty}^{\times} C_{\infty}$. The fiber of $X'_0(\mathfrak{M}) \rightarrow X_0(\mathfrak{M})$ is finite induced by the center action of $Cl_F = F_{\mathbb{A}(\infty)} / F^{\times} \widehat{O}^{\times}$ and hence if $f\bar{g}$ factors through $X_0(\mathfrak{M})$, we have a simple relation $|Cl_F^+|(f, g)_{\mathfrak{M}} = (f, g)_{\mathfrak{M}}'$. However in our computation, $\theta(\lambda; \mathfrak{A})$ for the component corresponding $z = 1$, the component corresponding non-trivial z is $\theta(\lambda; \mathfrak{A})| \langle z \rangle = \theta(\lambda; \mathfrak{z}\mathfrak{A})$, where $\langle z \rangle$ is induced by the central action on $\mathrm{GL}_2(F_{\mathbb{A}})$. Since the lattice for Θ_{ijz} is $zO_{ij}(\mathfrak{M})$, and hence $(\theta(\lambda; \mathfrak{z}\mathfrak{A}), \Theta_{ijz})_{\mathfrak{N}} = (\theta(\lambda; \mathfrak{A}), \Theta_{ij1})_{\mathfrak{N}}$. Thus $|Cl_F|(f(\lambda), \sum_z \Theta_{ijz})_{\mathfrak{N}} = |Cl_F|(\theta(\lambda; \mathfrak{A}), \Theta_{ij1})_{\mathfrak{N}}$, and the class number cancels out. Therefore integrality in (5.13) is sufficient under the condition $d(M/F) \neq (1)$. If $d(M/F) = (1)$, $f(\lambda)$ is supported over the half of the fiber corresponding to $N_{M/F}(Cl_M) \subset Cl_F$ (see Remark 5.5); so, the formula is $(|Cl_F|/2)(f, g)_{\mathfrak{M}} = (f, g)_{\mathfrak{M}}'$ in this case. Still (5.13) is sufficient.*

5.3. Eliminating the condition (odd). Here is how to remove the condition (odd). We choose an appropriate totally real quadratic extension F'/F and put $E = F'M$. Since $\mathrm{Gal}(E/F) = \mathrm{Gal}(F'/F) \times \mathrm{Gal}(M/F) \cong (\mathbb{Z}/2\mathbb{Z})^2$, there exists a third quadratic extension M'/F in E/F . Note that M' is a CM field. In summary, we have a diagram

$$\begin{array}{ccccc} & & M' & & \\ & \nearrow & & \searrow & \\ F & \hookrightarrow & F' & \hookrightarrow & E. \\ & \searrow & & \nearrow & \\ & & M & & \end{array}$$

We can arrange F'/F so that

- (F1) for any prime $\mathfrak{r}$ of F ramifying in M , the inertia group of $\mathfrak{r}$ in $\mathrm{Gal}(E/F)$ is given by $\mathrm{Gal}(E/M')$ (so, any prime factor of $\mathfrak{r}$ in F' splits in E/F'),
- (F2) for any prime $\mathfrak{q}|2$ is unramified in E/F' (i.e., E/F' satisfies (odd)),
- (F3) E/M ramifies at least one finite place.

We consider the base-change $\psi_E = \psi \circ N_{E/M}$ which has split conductor relative to E/F . By the main theorem proven under (odd), we have $L_E^-(\psi_E) = \mathcal{F}_E^-(\psi_E)$ up to units in $\overline{W}[[\Gamma_E^-]]$. We have the restriction map $\mathrm{Res} : \Gamma_E^- \rightarrow \Gamma_M^-$. Since p is odd, Res is a surjection. In [H09, §4], we gave a detailed argument proving $\mathrm{Res}(\mathcal{F}_E^-(\psi_E)) = \mathcal{F}_M^-(\psi) \mathcal{F}_M^-(\psi\alpha)$ for $\alpha = \left(\frac{E/M}{\cdot}\right)$ and $\mathrm{Res}(L_E^-(\psi_E)) = L_M^-(\psi) L_M^-(\psi\alpha)$ up to units in $\overline{W}[[\Gamma_M^-]]$. Note that the condition [H09, (F2)] only applies to the prime factor of 2 in $d(M/F)$ as we proved the equality under (odd) in this paper. Thus

$$L_M^-(\psi) L_M^-(\psi\alpha) = \mathcal{F}_M^-(\psi) \mathcal{F}_M^-(\psi\alpha)$$

up to units in $\overline{W}[[\Gamma_M^-]]$. Note here that $\psi\alpha$ remains anticyclotomic (because α has order 2). As proven in [H26, (C) in §6.3], $L_M^-(\psi) | \mathcal{F}_M^-(\psi)$ and $L_M^-(\psi\alpha) | \mathcal{F}_M^-(\psi\alpha)$, we conclude the individual identity:

$$L_M^-(\psi) = \mathcal{F}_M^-(\psi) \quad \text{and} \quad L_M^-(\psi\alpha) = \mathcal{F}_M^-(\psi\alpha)$$

up to units in $\overline{W}[[\Gamma_M^-]]$. This finishes the proof of Theorem I.

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