Lecture slide No.4 for Math 207c
Structure of the universal ring when $k = 1$.

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We fix a theta series $f = \sum_{\mathfrak{a}} \varphi(\mathfrak{a}) q^{N(\mathfrak{a})}$ of weight $k = 1$ of a real quadratic field F. Here φ is a character of order a $\varphi : Cl_F^+(\mathfrak{f}) \to \mu_a(\overline{\mathbb{Q}})$ with conductor $\mathfrak{f}\infty$, where $\infty : F \hookrightarrow \mathbb{R}$ is a fixed embedding. In Case U$_+$ with $\wp^\varsigma \nmid \mathfrak{f}$, the universal deformation ring $\mathbb{T}$ is bigger than Λ. We try to determine the algebra as explicitly as possible. In this case, $\rho = \rho_f = \mathrm{Ind}_F^{\mathbb{Q}} \varphi$ regarding φ as a Galois character by class field theory. Let $\overline{\varphi} = \overline{\varphi}_{\mathfrak{p}} = (\varphi \bmod \mathfrak{p})$ for each prime $\mathfrak{p}$ of $\mathbb{Z}[\mu_a] = \mathbb{Z}[f]$. If $\mathfrak{p} \nmid a$, the order of $\overline{\varphi}_{\mathfrak{p}}$ is equal to a and $F(\overline{\rho}) = F(\rho)$. We write $G = \mathrm{Gal}(F^{(p)}(\rho)/\mathbb{Q})$ and put $H = \mathrm{Gal}(F^{(p)}(\rho)/F)$. Pick $\varsigma \in G$ inducing a non-trivial automorphism of $F_{/\mathbb{Q}}$. Define $\phi_\varsigma(g) = \phi(\varsigma^{-1} g \varsigma)$ for any character $\phi : H \to A^\times$. By irreducibility of ρ, $\varphi \neq \varphi_\varsigma$ (Mackey's theorem). We write ε for the fundamental unit of F.

§**4.1. Decomposition of $Ad(\mathrm{Ind}_F^{\mathbb{Q}}\phi)$.** In the standard form of $\varrho = \mathrm{Ind}_F^{\mathbb{Q}}\phi$ in §3.6, $\varrho(g)$ is either diagonal or anti-diagonal; so, the diagonal subalgebra $\mathfrak{t} := \{\mathrm{diag}[x, -x] | x \in A\} \subset Ad(\varrho) = \mathfrak{sl}_2(A)$ and the subspace $\mathfrak{a}$ of anti-diagonal matrices in $Ad(\varrho)$ is stable under G. Thus $Ad(\varrho) = \mathfrak{t} \oplus \mathfrak{a}$ as an $A[G]$-module. Plainly G acts on $\mathfrak{t}$ by $\alpha := \left(\frac{F/\mathbb{Q}}{}\right)$. Since $Ad(\varrho)|_H$ acts on the upper nilpotent matrices $\mathfrak{n}_+$ by $\varphi^- := \varphi\varphi_\varsigma^{-1}$, by Shapiro's lemma, $\mathrm{Hom}_{A[H]}(\varphi^-, Ad(\varrho)) = \mathrm{Hom}_{A[G]}(\mathrm{Ind}_F^{\mathbb{Q}}\varphi^-, Ad(\varrho))$, we find $\mathfrak{a} = \mathrm{Ind}_F^{\mathbb{Q}}\phi^-$. Note that $\phi_\varsigma^- = (\phi^-)^{-1}$; so, unless ϕ^- has order ≤ 2, $\mathfrak{a}$ is irreducible. If ϕ^- is quadratic, $\phi^- = \phi_\varsigma^-$ extends to a character $\tilde{\phi}^- : G \to A^\times$ and $\boxed{\mathrm{Ind}_F^{\mathbb{Q}}\phi^- = \tilde{\phi}^- \oplus \alpha\tilde{\phi}^-}$. In summary,

$$Ad(\mathrm{Ind}_F^{\mathbb{Q}}\phi) = \begin{cases} \alpha \oplus \mathrm{Ind}_F^{\mathbb{Q}}\phi^- & \text{if } (\phi^-)^2 \neq 1, \\ \alpha \oplus \tilde{\phi}^- \oplus \alpha\tilde{\phi}^- & \text{if } (\phi^-)^2 = 1. \end{cases}$$

§4.2. Action of σ on Sel$(Ad(\varrho))$.

Let $\pi : R^{ord} \to A$ with $\rho_A = \pi \circ \boldsymbol{\rho}^{ord}$. Suppose we have $\sigma_A \in$ Aut(A) such that $\sigma_A \circ \pi = \pi \circ \sigma$. Recall $j(\boldsymbol{\rho}^{ord} \cdot \alpha)j^{-1} = (\boldsymbol{\rho}^{ord})^\sigma$ for $j := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For each 1-cocycle $u : G \to Ad(\rho_A)^*$, we define $u^{[\sigma]}(g) = ju(g)^{\sigma_A}j^{-1}$. From $u(gh) = Ad(\rho_A)(g)u(h) + u(g)$, we find

$$\begin{aligned} u^{[\sigma]}(gh) &= j\rho_A^{\sigma_A}(g)jju(h)^{\sigma_A}jj\rho_A(g^{-1})^{\sigma_A}j + ju(g)^{\sigma_A}j \\ &= Ad(j\rho_A^{\sigma_A}j)(g)u^{[\sigma]}(h) + u^{[\sigma]}(g) = Ad(\rho_A \cdot \chi)(g)u^{[\sigma]}(h) + u^{[\sigma]}(g) \\ &= Ad(\rho_A)(g)u^{[\sigma]}(h) + u^{[\sigma]}(g). \end{aligned}$$

Since the conjugation of j preserves the upper triangular p-decomposition subgroup and p-inertia subgroup of Gal$(F(\boldsymbol{\rho}^{ord})/\mathbb{Q})$, in this way, σ acts on Sel$(Ad(\rho_A))$. In particular, if σ_A is trivial (i.e., $\rho_A = \mathrm{Ind}_F^{\mathbb{Q}} \phi$), $[\sigma]$ is just a conjugate action of j.

§4.3. Decomposition theorem of Sel(Ad(Ind$_F^{\mathbb{Q}}\phi$)).

Define, for $\varrho_\phi^- :=$ Ind$_F^{\mathbb{Q}}\phi^-$ and $M := F(\phi^-)$,

$$\text{Sel}(\alpha_A) := \text{Hom}(Cl_F \otimes_{\mathbb{Z}} A, A^\vee) \cong (Cl_F \otimes_{\mathbb{Z}} A)^\vee,$$
$$\text{Sel}(\varrho_\phi^-) := \text{Hom}(Cl_M(\wp^\infty)/\langle \mathfrak{P}^\varsigma \rangle_{\mathfrak{P}|\wp} \otimes_{\mathbb{Z}[H]} \phi^-, A^\vee).$$

Theorem 4.3: *We have* Sel(Ad(Ind$_F^{\mathbb{Q}}\phi$)) $\cong$ Sel(α_A) $\oplus$ Sel(ϱ_ϕ^-).

Proof for the α-factor: Write $\varrho :=$ Ind$_F^{\mathbb{Q}}\phi$. Then $H^1(G, Ad(\varrho))$ is isomorphic to

$$H^1(G, \alpha_A^*) \oplus H^1(G, (\varrho_\phi^-)^*) = H^1(G, \alpha_A^*) \oplus H^1(H, (\phi^-)^*).$$

The identity of the second factors is by Shapiro's lemma. Since α_A is realized on the diagonal matrix, by the definition of Sel($Ad(\varrho)$), it is unramified everywhere; so, it factors through $Cl_F \otimes_{\mathbb{Z}} A$ over H. Since G/H has order 2, the restriction map to H is an isomorphism, and the result follows. □

§**4.4. Proof for the** $\mathrm{Ind}_F^{\mathbb{Q}}\phi^-$**-factor.** Write $\rho_A^- := \mathrm{Ind}_F^{\mathbb{Q}}\phi^-$. The Shapiro's isomorphism is realized by the restriction map

$$H^1(G,(\rho_A^-)^*) \xrightarrow[\sim]{\text{Res}} H^1(H,(\rho_A^-)^*)^G = (H^1(H,(\phi^-)^*) \oplus H^1(H,(\phi_\varsigma^-)^*))^G,$$

which is an isomorphism. In the last factor, G acts on cocycles $u(g) \mapsto u_\varsigma(g) = \varsigma u(\varsigma^{-1}g\varsigma)$; so, interchanges the two factors. Therefore $H^1(G,(\rho_A^-)^*) \cong H^1(H,(\phi^-)^*)$. If $U : G \to (\rho_A^-)^*$ is a Selmer cocycle, we have $U(h) = \begin{pmatrix} 0 & u \\ u_\varsigma & 0 \end{pmatrix}$ for a cocycle $u : H \to (\phi^-)^*$. Note $M = F(Ad(\rho_A)) = F(\phi^-)$. By Selmer condition that $U|_{D_{\wp^\varsigma}}$ is lower triangular, we have $u|_{G_M}$ factors through $Cl_M(\wp^\infty)/\langle\mathfrak{P}^\varsigma\rangle_{\mathfrak{P}|\wp}$. Note that $H^q(H/G_M,\phi^-) = 0$ as $\phi^- \not\equiv 1 \mod \mathfrak{m}_A$. Since it is a $\mathbb{Z}[H]$-morphism into ϕ^-, it actually factors through $Cl_M(\wp^\infty)/\langle\mathfrak{P}^\varsigma\rangle_{\mathfrak{P}|\mathfrak{p}} \otimes_{\mathbb{Z}[H]} \phi^-$. Reversing the argument, it is an isomorphism. □

§**4.5.** $\pm$**-eigenspace of** σ **in** Sel$(Ad(\rho_A))$**.**

Lemma 4.5: The involution σ acts on Sel$(\mathrm{Ind}_F^{\mathbb{Q}}\phi^-)$ (resp. Sel(α_A)) by -1 (resp. $+1$). Here α_A is the character α regarded to have values in $A^\times$.

Proof: In the decomposition of Theorem 4.3, α_A is realized on the subspace $\mathfrak{t}$ of diagonal matrices and $\mathrm{Ind}_F^{\mathbb{Q}}\phi^-$ is realized on anti-diagonal matrices $\mathfrak{a} \subset \mathrm{Ad}(\overline{\rho})$. Since j acts by $+1$ on $\mathfrak{t}$ and -1 on $\mathfrak{a}$ and the action of σ on cocycle is conjugation by j as seen in §4.2, the action of σ on Sel(α_A) is by $+1$ and on Sel$(\mathrm{Ind}_F^{\mathbb{Q}}\phi^-)$ is by -1. $\square$

§**4.6. Set-up.** Hereafter, we write $M := F(Ad(\overline{\rho})) = F(\overline{\varphi}^-)$. Consider the following conditions:

(H0) the local character $\overline{\varphi}^-|_{D_p}$ is non-trivial (irreducibliity of $\overline{\rho}$).

(H1) $\wp^S \nmid \mathfrak{f}$ (ordinarity in Case U$_+$; $C = N_{F/\mathbb{Q}}(\mathfrak{c})D$).

(H2) the p-quotient $Cl_M \otimes_{\mathbb{Z}[H]} \overline{\varphi}^- = 0$ (this follows if the class number of M is prime to p), and the local character $\overline{\varphi}^-|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is different from the reduction $\overline{\omega}$ modulo p of the Teichmüller character $\omega = \omega_p$ acting on μ_p.

(H3) $h_F = |Cl_F|$ is prime to p ($Cl_M \otimes_{\mathbb{Z}[G]} Ad(\overline{\rho}) = 0 \Leftrightarrow$ (H2–3)).

Replacing φ by the Teichmüller lift of $\overline{\varphi}$, we assume the order of φ is prime to p.

In this real induced case, by J. Thorne, Taylor–Wiles condiction is removed; so, $R^{ord}_{\mathfrak{p}} \cong \mathbb{T}_{\mathfrak{p}}$ under (H0). We put $\mathbb{T}^{\pm}_{\mathfrak{p}} = R^{ord}_{\pm}$.

§**4.7. Presentation theorem again.** Assume (H0). Let $r_+ := \dim_{\mathbb{F}} \mathrm{Sel}(\overline{\alpha})$ for $\overline{\alpha} = \alpha_{\mathbb{F}}$ and $r_- = \mathrm{Sel}(\mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi})$.

Theorem 4.7: $\mathbb{T}_{\mathfrak{p}} \cong \Lambda[X_1^+, \ldots, X_{r_+}^+, X_1^-, \cdots, X_{r_-}^-]/(S_1, \ldots, S_r)$ *for* $r = r_+$ *so that* σ *fixes the image of* X_j^+ *in* $\mathbb{T}_{\mathfrak{p}}$ *and acts by* -1 *on the image in* $\mathbb{T}_{\mathfrak{p}}$ *of* X_i^-*.*

Proof: Assuming $r_+ = 0$ ($\Leftrightarrow p \nmid h_F = |Cl_F|$) and $r_- = 1$, we now prove this fact. So σ acts on $t^*_{\mathbb{T}/\mathbb{T}^+}$ by -1. We can choose a generator Θ so that $\sigma(\Theta) = -\Theta$. Then $x \mapsto \Theta x$ is a Λ-linear map of $\mathbb{T} \cong \Lambda^e$. Writing this map as a $d \times d$ matrix form L and define $D(X) = \det(X 1_e - L)$. Then $\mathbb{T} = \Lambda[X]/D(X)$ and $\mathbb{T}$ is a local complete intersection with $2|e$ as σ acts non-trivially on $\mathbb{T}_{/\Lambda}$. □

The principal ideal (Θ) is the relative different $\mathbb{T}(\sigma-1)T$ of $\mathbb{T}/\mathbb{T}^+$.

§**4.8. Structure theorem.** Assume (H0–3). Write $A = \mathbb{T}_{\mathfrak{p}}$ or $\mathbb{T}_{\mathfrak{p}}^{+}$. Let $e = \text{rank}_{\Lambda} \mathbb{T}$. Then the following four assertions hold:

(1) If $\langle\varepsilon\rangle - 1$ is a prime in Λ, then the ring A is isomorphic to a power series ring $W[[x]]$ of one variable over W; hence, A is a regular local domain and is factorial;

(2) The ring A is an integral domain fully ramified at $(\langle\varepsilon\rangle - 1)$;

(3) If p is prime to $e = \text{rank}_{\Lambda} A$, the ramification locus of $A_{/\Lambda}$ is given by $\text{Spec}(\Lambda_{\varepsilon})$ for $\Lambda_{\varepsilon} := \Lambda/(\langle\varepsilon\rangle - 1))$, the different for A/Λ is principal and generated by Θ^{a-1} and A is a normal integral domain of dimension 2 unramified outside $(\langle\varepsilon\rangle - 1)$ over Λ;

(4) If $p|e$, $\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{Q}$ is a Dedekind domain unramified outside $(\langle\varepsilon\rangle - 1)$ over $\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$, and the relative different for $\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{Q}/\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$ is principal and generated by Θ^{e-1};

(5) If $e = 2$, $\mathbb{T}^{+} = \Lambda$ and $\mathbb{T} = \Lambda[\sqrt{1 - \langle\varepsilon\rangle}]$.

Conjecture: $\boxed{e = 2}$ under (H0–3)?

§4.9. **Wall–Sun–Sun primes.** If $\langle\varepsilon\rangle - 1$ is not a prime ($\Leftrightarrow \varepsilon^{p-1} \equiv 1 \mod \wp^2$), by the existence of ambiguous classes, $\mathbb{T}$ cannot be factorial. Perhaps there is no example known of a prime $p \geq 5$ split in $F = \mathbb{Q}[\sqrt{5}]$ such that $\langle\varepsilon\rangle - 1$ is not a prime in $\mathbb{Z}_p[[T]]$. Consider $F = \mathbb{Q}[\sqrt{d}]$ with square-free $0 < d \in \mathbb{Z}$ and describe how to decide if $\wp^2|\varepsilon^{k-1} - 1$. Since $p > 2$, $\wp^2|(\varepsilon^{p-1} - 1) \Leftrightarrow \wp^2|(\varepsilon^{2(p-1)} - 1)$. On the other hand, $\varepsilon^{2(p-1)} - 1 = \varepsilon^{2(p-1)} - \varepsilon^{p-1}\varepsilon^{\varsigma(p-1)} = \varepsilon^{p-1}(\varepsilon^{p-1} - \varepsilon^{\varsigma(p-1)})$. Define $\alpha \in \mathbb{Z}$ so that $\varepsilon^2 - \alpha\varepsilon \pm 1 = 0$. Consider the corresponding Fibonacci type recurrence relation $f_n = \alpha f_{n-1} \mp f_{n-2}$. For the solution f_n with initial values $f_0 = 0$ and $f_1 = 1$, we have $f_n = \frac{\varepsilon^n - \varepsilon^{n\varsigma}}{\varepsilon - \varepsilon^\varsigma}$. Thus we have $\frac{\varepsilon^{p-1} - \varepsilon^{\varsigma(p-1)}}{\sqrt{d}} = f_{p-1}C$ for $C = \frac{\varepsilon - \varepsilon^\varsigma}{\sqrt{d}}$. If $d = 5$, we have $C = 1$.

$\langle\varepsilon\rangle - 1$ is not a prime in $\Lambda \Leftrightarrow p^2|f_{p-1}C$. (Wall–Sun–Sun primes)

For $F = \mathbb{Q}[\sqrt{1}]$, $p = 191, 643$ are such primes. It is conjectured infinity? of Wall-Sun-Sun primes (perhaps density 0).

§**4.10. Proof of (1).** Put $\mathcal{J} = \Theta\mathbb{T}$ and $\mathcal{J}^0 = \mathbb{T}$. For all $0 \neq u \in \mathbb{T}$, $[u] : x \mapsto ux$ induces the linear endomorphism $\mathrm{gr}(u)$ of the corresponding graded algebra $\mathrm{gr}_{\mathcal{J}}(\mathbb{T}) := \bigoplus_{n=0}^{\infty} \mathcal{J}^n/\mathcal{J}^{n+1}$ (with $\mathcal{J}^0 = \mathbb{T}$). Then $[u]$ is injective if $\mathrm{gr}(u)$ is injective [BCM, III.2.8, Corollary 1]. We have $\mathrm{gr}_{\mathcal{J}}(\mathbb{T}) \cong \Lambda_\varepsilon[x]$ for the polynomial ring $\Lambda_\varepsilon[x]$ where the variable x corresponds to the image $\overline{\Theta}$ of Θ in the first graded piece $\mathcal{J}/\mathcal{J}^2$. Take n so that $u \in \mathcal{J}^n$ but $u \notin \mathcal{J}^{n+1}$. Then $\mathrm{gr}(u) : \mathrm{gr}_{\mathcal{J}}(\mathbb{T}) \to \mathrm{gr}_{\mathcal{J}}(\mathbb{T})$ is multiplication by a polynomial of degree n. Assume that $\langle\varepsilon\rangle - 1$ is a prime; so, $(\langle\varepsilon\rangle - 1) = (T)$ in Λ and $\Lambda_\varepsilon = W$. Then $\mathrm{gr}_{\mathcal{J}}(\mathbb{T})$ is an integral domain isomorphic to the polynomial ring $W[x]$; so, if $u \neq 0$, $\mathrm{gr}(u)$ is injective, and hence, $[u]$ is injective; so, u is not a zero divisor. We conclude that $\mathbb{T}$ is an integral domain and $\mathbb{T} = \varprojlim_n \mathbb{T}/\mathcal{J}^n \cong W[[x]]$ by sending Θ to x. A power series ring over a discrete valuation ring is a unique factorization domain and is regular; so, we get the assertion (1).

§**4.11. Proof of (2–4).** $(D(0)) = (\langle\varepsilon\rangle - 1)$ follows from

$\Lambda/(\langle\varepsilon\rangle - 1) \cong \mathbb{T}/(\Theta) = \Lambda[[X]]/(X, D) = \Lambda/(D(0))$ (§3.27),

Thus $(D(0))$ is square-free. Let $P|(\langle\varepsilon\rangle - 1)$ be a prime factor; so, the localization Λ_P and its completion $\widehat{\Lambda}_P = \varprojlim_n \Lambda_P/P^n\Lambda_P$ are discrete valuation rings. Then $\widehat{\mathbb{T}}_P = \mathbb{T} \otimes_\Lambda \widehat{\Lambda}_P = \widehat{\Lambda}_P[[X]]/(D(X))$, and by Weierstrass preparation theorem $D(X) = D_P(X)U_P(X)$ for a distinguished polynomial $D_P(X) \in \widehat{\Lambda}_P[X]$ with respect to P and a unit $U_P(X) \in \widehat{\Lambda}_P[[X]]$. Since $\deg(D_P(X)) = \mathrm{rank}_{\widehat{\Lambda}_P} \widehat{\mathbb{T}}_P = \mathrm{rank}_\Lambda \mathbb{T} = \deg(D(X))$, we have $D(X) = D_P(X)$; so, $D(X)$ is an Eisenstein polynomial. Then $\widehat{\mathbb{T}}_P$ (resp. $\mathbb{T}_P$) is a discrete valuation rings fully ramified over $\widehat{\Lambda}_P$ (resp. Λ_P). Since $\mathbb{T} \hookrightarrow \mathbb{T}_P$, $\mathbb{T}$ is an integral domain. Writing $D(X) = X^e + a_1X^{e-1} + \cdots + a_0$, we have $(a_0) = (\langle\varepsilon\rangle - 1)$ and $(\langle\varepsilon\rangle - 1)|a_i$. Thus for $D'(X) = \frac{dD}{dX} = eX^{e-1} + \cdots + a_1$, we find $D'(\Theta)\mathbb{T} = \Theta^{e-1}\mathbb{T}$, and for the relative different $\mathfrak{d} = (D'(\Theta))$ we have $e\Theta^{e-1}\mathbb{T} \subset \mathfrak{d} \subset \Theta^{e-1}\mathbb{T}$, which shows (3–4). The proof of (5) is an exercise.

§4.12. **Local indecomposability conjecture.**

Conjecture G (R. Greenberg): For a p-ordinary Hecke eigenform f of weight $k \geq 2$, if f has no CM (not induced from a quadratic field), then $\rho_{f,\mathfrak{p}}|_{I_p}$ is indecomposable.

For a cusp form $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \psi)_{/W}$, we define $\theta := q\frac{d}{dq}$ as a differential operator on $W[[q]]$. It is well known that $\theta^m f$ is a p-adic limit of classical cusp forms (why?). Assume $p \nmid N$. If f is a $\mathfrak{p}$-ordinary Hecke eigenform with $f|T(n) = \lambda(T(n))f$, then we can distinguish two roots α, β of $X^2 - \lambda(T(p))X + \chi(p) = 0$ so that $|\alpha|_p = 1$ and $|\beta|_p = p^{1-k}$ (i.e., $p^{k-1}\|\beta$). We have two p-stabilizations $f^{ord}|U(p) = \alpha f^{ord}$ and $f^{crit}|U(p) = \beta f^{crit}$.

Conjecture C (R. Coleman): $f^{crit} = \theta^{k-1} g$ for a p-adic limit $g \in W[[q]]$ of cusp forms if and only if f has CM.

It is known that G $\Leftrightarrow$ C by Breuil–Emerton (Asterisque, (331):255–315, 2010). Try prove "$\Leftarrow$" of Conjecture C.

§**4.13. A theorem of Iwasawa.** Let $k = M_\wp$ with $\overline{D} = \mathrm{Gal}(k/\mathbb{Q}_p)$, k_∞/k be the unramified $\mathbb{Z}_p$-extension and F_∞/k be the cyclotomic $\mathbb{Z}_p$-extension $\subset k_\infty[\mu_{p^\infty}]$ with $\Gamma := \mathrm{Gal}(F_\infty/k) = \gamma^{\mathbb{Z}_p}$. Let $\mathcal{L}$ be the maximal abelian p-extension of $\mathcal{F}_\infty := F_\infty k_\infty$. Set $\mathcal{X} := \mathrm{Gal}(\mathcal{L}/\mathcal{F}_\infty)$ and $\Upsilon := \mathrm{Gal}(k_\infty F_\infty/F_\infty) = \upsilon^{\mathbb{Z}_p}$. Take $\widetilde{\gamma} \in \mathrm{Gal}(\mathcal{L}/k)$ with $\widetilde{\gamma}|_{F_\infty} = \gamma$. The commutator $\tau := [\upsilon, \widetilde{\gamma}]$ acts on $\mathcal{X}$ by conjugation, and $(\tau - 1)x := [\tau, x] = \tau x \tau^{-1} x^{-1}$ for $x \in \mathcal{X}$ is independent of the choice of $\widetilde{\gamma}$ and υ. Define $L \subset \mathcal{L}$ by the fixed field of $(\tau - 1)\mathcal{X}$. Let $X = \mathrm{Gal}(L/\mathcal{F}_\infty) = \mathcal{X}/(\tau - 1)\mathcal{X}$. Note $p \nmid [k : \mathbb{Q}_p]$.

Theorem 4.13; For the character $\eta : \mathrm{Gal}(k/\mathbb{Q}_p) \to \mathbb{Z}_p[\eta]^\times$, $X[\eta] = X \otimes_{\mathbb{Z}_p[\overline{D}]} \eta$ is a cyclic $\mathbb{Z}_p[\eta][[\Gamma \times \Upsilon]]$-module, where $\overline{D}$ acts on $\mathbb{Z}_p[\eta]$ by η.

This is essentially a theorem of Iwasawa; see, Proposition A.4.1 in a paper posted in Hida's web page ([CWE]: Appendix to a joint work with Castella and Wang-Erickson).

§**4.14. Some notation.** Pick $\phi_0 \in D_\wp$ so that $\overline{\rho}(\phi_0) = \left(\begin{smallmatrix} \overline{a} & 0 \\ 0 & \overline{b} \end{smallmatrix}\right)$ with $\overline{a} \neq \overline{b}$. Define $\phi = \lim_{n\to\infty} \phi_0^{q^n}$ $(q = |\mathbb{F}|)$. We can normalize $\rho_{\mathbb{T}}$ so that $j(\rho_{\mathbb{T}} \cdot \chi)j^{-1} = \rho_{\mathbb{T}}^{\sigma}$ (an exercise [CWE, A.3.1]), $\rho_H := \rho_{\mathbb{T}}|_H$ has values in $E := \left(\begin{smallmatrix} \mathbb{T}^+ & \mathbb{T}^- \\ \mathbb{T}^- & \mathbb{T}^+ \end{smallmatrix}\right)$, which is a $\mathbb{T}^+$-subalgebra of $M_2(\mathbb{T})$. Here $\rho_{\mathbb{T}}(\phi)$ is diagonal and by conjugation, it acts on upper (resp. lower) nilpotent part of E by ab^{-1} (resp. $a^{-1}b$). Let $I = \overline{I}_\wp$ (resp. $D = \overline{D}_\wp$) be the wild $\wp$-inertia (resp. $\wp$-decomposition) subgroup of Gal$(F(\rho_{\mathbb{T}})/F(\overline{\rho}))$ for $\wp$.

Note $\boldsymbol{\kappa}([p, \mathbb{Q}_p]) = \det(\rho_{\mathbb{T}}([p, \mathbb{Q}_p])) = 1$ since $\boldsymbol{\kappa}(g) = t^{\log_p \nu_p(g)/\log_p(\gamma)}$. Regard $\upsilon := [p, \mathbb{Q}_p]^f \in D$ for the residual degree f of $\mathfrak{P} = \wp \cap K(\overline{\rho})$, and recall $\varphi' := \rho_{\mathbb{T}}([p, \mathbb{Q}_p]^f) = \left(\begin{smallmatrix} u^{-f} & * \\ 0 & u^f \end{smallmatrix}\right)$ with $u^f \in \mathbb{T}_+$. Let W_1 be the subalgebra of $\overline{\mathbb{Q}}_p$ generated by the values of φ over $D_\wp$. Put $\Lambda_0 := \mathbb{Z}_p[[T]] \subset \Lambda_1 := W_1[[T, a]] \subset \mathbb{T}$ for $a = u^{2f} - 1 \in \mathfrak{m}_{\Lambda_1}$, which is the image of $W_1[[\Gamma \times \Upsilon]]$ for $k = M_\wp$. Note $\Upsilon = \upsilon^{\mathbb{Z}_p}$.

§4.15. Inertia theorem:

Suppose (H0) *and minimality of* $\mathbb{T}$*. Then,*

(1) after choosing I suitably in its conjugacy class, we have an exact sequence $\mathcal{U} \hookrightarrow I \twoheadrightarrow t^{\mathbb{Z}_p}$ with $\rho_{\mathbb{T}}(\mathcal{U})$ made of unipotent matrices,

(2) there exists a non-zero divisor $\theta \in \mathbb{T}^-$ satisfying $\theta^\sigma = -\theta$ and $\mathcal{U} = \Lambda_1\theta$; in other words, we have $\rho_{\mathbb{T}}(I) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in t^{\mathbb{Z}_p}, b \in \theta\Lambda_1 \right\}$.

We are going to show $\theta \doteqdot \Theta$ for Θ in §4.7 after proving this theorem.

§**4.16. Proof of (1):** From the definition of Λ-algebra structure of $\mathbb{T}$ and p-ordinarity, we know $\rho_{\mathbb{T}}(I) \subset M(\mathbb{T}) \cap E$ for the mirabolic subgroup $M(\mathbb{T}) := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \middle| a \in \mathbb{T}^\times, b \in \mathbb{T} \right\}$. Since $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) = [p, \mathbb{Q}_p]^{\widehat{\mathbb{Z}}} \ltimes \mathbb{Z}_p^\times$ for the maximal abelian extension $\mathbb{Q}^{ab}/\mathbb{Q}$ and the local Artin symbol $[p, \mathbb{Q}_p]$, we find

$$\rho_{\mathbb{T}}(I) \subset \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \middle| a \in t^{\mathbb{Z}_p}, b \in \mathbb{T}_- \right\},$$

and $\det(\rho_{\mathbb{T}}(I)) = \mathcal{T} := t^{\mathbb{Z}_p} \subset \Lambda^\times$. Thus we have an extension $1 \to \mathcal{U} \to \rho_{\mathbb{T}}(I) \to \mathcal{T} \to 1$. Recall $\phi_0 \in D_\wp$ with $\overline{\rho}(\phi_0) = \left(\begin{smallmatrix} \overline{a} & 0 \\ 0 & \overline{b} \end{smallmatrix} \right)$ ($\overline{a} \neq \overline{b}$) and $\phi = \lim_{n\to\infty} \phi_0^{q^n}$ inside $\mathrm{Gal}(F(\rho_{\mathbb{T}})/F)$. This extension is split by the conjugation action of ϕ_0 with $\mathcal{U}$ characterized to be an eigenspace on which ϕ_0 acts by ab^{-1} for the Teichmüller lift a, b of $\overline{a}, \overline{b}$; so, we may assume to have a section $s : \mathcal{T} \hookrightarrow \rho_{\mathbb{T}}(I)$ identifying $\mathcal{T}$ with $\left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \middle| a \in t^{\mathbb{Z}_p} \right\}$. Thus $\mathcal{U}$ is made of unipotent matrices. Here we used the assumption (H0). $\square$

§4.17. Known facts: non-triviality of $\mathcal{U}$.
Since $\Lambda \hookrightarrow \mathbb{T}$, $\Gamma \subset \mathbb{T}^\times$. Two known facts:

(a) For a W-algebra homomorphism $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$, if $\lambda|_\Gamma : \Gamma \to \overline{\mathbb{Q}}_p^\times$ coincides with ν_p up to a finite order character ϵ, $f := \sum_{n=1}^\infty \lambda(T(n))q^n$ is a weight 2 cusp form in $S_2(Cp^r, \psi_2\epsilon)$ [LFE, §7.3];
(b) The Galois representation $\rho_{\lambda,\mathfrak{p}} = \rho_{f,\mathfrak{p}}$ is locally indecomposable (Bin Zhao, Ann. L'inst. Fourier **64** (2014), 1521–1560).

By (b) and $\bigcap_\lambda \mathrm{Ker}(\lambda) = 0$, $\mathcal{U}$ contains non-zero divisor of $\mathbb{T}^-$. Thus it is "highly" non-zero.

§**4.18. Proof of (2):** We have $\mathcal{U} \subset \mathbb{T}_-$ and regard φ^- as an abelian irreducible $\mathbb{Z}_p$-representation acting on W regarded as a $\mathbb{Z}_p$-module.

Apply Iwasawa's theorem to the splitting field k of $\varphi^-|_{D_p}$ under the notation in §4.13. Then the Galois group $X'[\varphi^-]$ is cyclic over $W_1[[\Gamma \times \Upsilon]]$ ($\Gamma = \gamma^{\mathbb{Z}_p} \cong t^{\mathbb{Z}_p}$) and surjects onto $\mathcal{U}$. Since the action of $W_1[[\Gamma \times \Upsilon]]$ factors through Λ_1, $\mathcal{U}$ is cyclic over Λ_1; so, we have $\mathcal{U} \cong \Lambda_1$. Thus we conclude $\rho_H(I_1) = \mathcal{U} = \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right) \middle| a \in \theta\Lambda_1 \right\}$ inside $\rho_H(H)$ (for a suitable choice of $\theta \in \mathbb{T}_-$).

By the facts in §4.17, θ is a non-zero divisor. □
By (H2), $\mathbb{T}_- = \Theta\mathbb{T}^+$. Since $\theta \in \mathbb{T}^-$, we can write $\theta = u\Theta$ ($u \in \mathbb{T}$).

§**4.19. Theorem:** Θ/θ is a unit under (H0–2).

Proof: We have an exact sequence $\mathfrak{d} \hookrightarrow \mathbb{T} \twoheadrightarrow W[C_p]$ in §3.27. Taking σ-invariant subspace (indicated superscript "+"), $\mathbb{T}^+/\mathfrak{d}^+ \cong W[C_p]$. Recall the universal character $\Phi : \mathrm{Gal}(H_p/F) \to W[C_p] = \mathbb{T}^+/\mathfrak{d}^+$. Write $\rho_H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and put $a = A \mod \mathfrak{d}^+ = \Phi$, $d = D \mod \mathfrak{d}^+ = \Phi_\varsigma$, $b = B \mod \mathfrak{d}^+ : H \to \mathbb{T}^-/\mathfrak{d}^+\mathbb{T}^-$ and $c = C \mod \mathfrak{d}^+ : H \to \mathbb{T}^-/\mathfrak{d}^+\mathbb{T}^-$. If b has image in $\mathfrak{m}_{\mathbb{T}^+}(\mathbb{T}^-/\mathfrak{d}^+\mathbb{T}^-)$, by $c(g) = \varphi(\varsigma^2)b(\varsigma^{-1}g\varsigma)$, c has also. This implies $\rho_H \mod \mathfrak{m}_{\mathbb{T}^+}\mathfrak{d}^+$ is diagonal; so, $\rho_H^\sigma = j\rho_H j^{-1}$ which implies $\rho_{\mathbb{T}}^\sigma \mod \mathfrak{m}_{\mathbb{T}}\mathfrak{d} \cong \rho_{\mathbb{T}} \otimes \chi$, a contradiction as $\mathfrak{d}$ is the maximal ideal for which the identity holds. Thus b is onto. Replacing ρ_H by $\rho' := \xi^{-1}\rho_H\xi$ for $\xi := \begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix}$, ρ' has values in $\mathrm{GL}_2(\mathbb{T}^+)$ and $\rho' \mod \mathfrak{m}_{\mathbb{T}^+} = \begin{pmatrix} \overline{\varphi} & \overline{b} \\ 0 & \overline{\varphi}_\varsigma \end{pmatrix}$ with $\overline{b} = b/\Theta \mod \mathfrak{m}_{\mathbb{T}^+} \neq 0$. If u is a non-unit, b is unramified at $\wp$ (which is unramified also at $\wp^\varsigma$); so, everywhere unramified over $F(\overline{\varphi}^-)$, contradicting $Cl_{F(\overline{\varphi}^-)} \otimes_{\mathbb{Z}[H]} \overline{\varphi}^- = 0$. □

§4.20. Local indecomposability.

Corollary 4.20: If f is a Hecke eigenform belonging to $\mathbb{T}$ of weight $k \geq 2$, $\rho_{f,\mathfrak{p}}$ is indecomposable over $I_{\mathfrak{p}}$ under (H0–2).

This follows from the fact that (Θ) is exactly over $(\langle\varepsilon\rangle - 1)$, and hence for any height 1 prime P outside $(\langle\varepsilon\rangle - 1)$, $\Theta \mod P \neq 0$, and hence $\theta \mod P \neq 0$ by Theorem 4.19.

For the companion form case, the exceptional Artin representations and induced representations in Cases U_- and D, p-local indecomposability question is still open.

§4.21. Concluding remarks.

• Actually we can prove $\Lambda[\theta] \subset \mathbb{T}$ is an integral domain fully ramified over $(\langle\varepsilon\rangle - 1)$ similar to the structure theorem in §4.8 under (H0–1).

• Indecomposability as in Corollary 4.20 also holds under (H0–1) when F is real. Without assuming (H2), $\mathbb{T}^{-}$ is generated more than one element over $\mathbb{T}_{+}$; so, no single Θ. Obviously, the key point is to show $(\theta) \cap \Lambda = (\langle\varepsilon\rangle - 1)$.

• When F is an imaginary quadratic field, in Case U_{+} in the imaginary version, under (H0–2), local indecomposability holds for $\rho_{f,\mathfrak{p}}$ as long as f does not have CM (this is the main result of [CWE]).

• The inertia theorem is always true unless all f belonging to $\mathbb{T}$ have CM, though θ could be a zero-divisor if $\overline{\rho}$ is induced from an imaginary quadratic field.